Ann. Inst. H. Poincaré
Anal. Non Linéaire 41 (2024), 159–185
DOI 10.4171/AIHPC/77

Scaling-invariant Serrin criterion via one velocity component for the Navier–Stokes equations

Wendong Wang, Di Wu, and Zhifei Zhang

Abstract. The classical Ladyzhenskaya–Prodi–Serrin regularity criterion states that if the Leray weak solution u of the Navier–Stokes equations satisfies $u \in L^q(0,T;L^p(\mathbb{R}^3))$ with $\frac{2}{q}+\frac{3}{p}\le 1$, $p>3$, then it is regular in $\mathbb{R}^3\times(0,T)$. In this paper, we prove that the Leray weak solution is also regular in $\mathbb{R}^3\times(0,T)$ under the scaling-invariant Serrin condition imposed on one component of the velocity, i.e., $u_3\in L^{q,1}(0,T;L^p(\mathbb{R}^3))$ with $\frac{2}{q}+\frac{3}{p}\le 1$, $3<p<+\infty$. This result means that if the solution blows up at a time, then all three components of the velocity have to blow up simultaneously.

1. Introduction

In this paper we study the incompressible Navier–Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u\cdot\nabla u + \nabla\pi = 0, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0, \end{cases} \tag{NS}$$

where $(u(x,t),\pi(x,t))$ denote the velocity and the pressure of the fluid respectively.

In the pioneering work [23], Leray introduced the concept of weak solutions to (NS) and proved global existence for initial data $u_0\in L^2(\mathbb{R}^3)$. Kato [17] initiated the study of (NS) with initial data belonging to the space $L^3(\mathbb{R}^3)$ and obtained global existence in a subspace of $C([0,\infty),L^3(\mathbb{R}^3))$ provided the norm $\|u_0\|_{L^3(\mathbb{R}^3)}$ is small enough. The existence result for initial data small in the Besov space $\dot{B}^{-1+(3/p)}_{p,q}(\mathbb{R}^3)$ for $p\in[1,\infty)$ and $q\in[1,\infty]$ can be found in [4, 11]. The function spaces $L^3(\mathbb{R}^3)$ and $\dot{B}^{-1+(3/p)}_{p,q}(\mathbb{R}^3)$ for $(p,q)\in[1,\infty)^2$ both guarantee the existence of a local-in-time solution for any initial data. Koch and Tataru [19] showed that global well-posedness holds as well for small initial data in the space $\mathrm{BMO}^{-1}(\mathbb{R}^3)$. On the other hand, it has been shown by Bourgain and Pavlović [2] that the Cauchy problem with initial data in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ is ill posed no matter how small the initial data is.

2020 Mathematics Subject Classification. Primary 35Q30; Secondary 76D03, 35Q35, 76D05.
Keywords. Navier–Stokes equation, Serrin criterion, weak solution.

In two spatial dimensions, the Leray weak solution is unique and regular. In three spatial dimensions, the regularity and uniqueness of a weak solution is an outstanding open problem in mathematical fluid mechanics. It was known that if the weak solution u of (NS) satisfies a so-called Ladyzhenskaya–Prodi–Serrin-type (LPS) condition

$$u \in L^q(0,T;L^p(\mathbb{R}^3)) \quad \text{with } \frac{2}{q}+\frac{3}{p} \le 1,\ p \ge 3,$$

then it is regular in $\mathbb{R}^3 \times (0,T)$, see [10, 14, 29, 30], where the regularity in the class $L^\infty(0,T;L^3(\mathbb{R}^3))$ was proved by Escauriaza, Seregin and Šverák [10]. In [12], based on [18], Gallagher, Koch and Planchon gave an alternative proof of the result in [10] by the method of profile decomposition. In [13], they extended the method in [12] to release the space from L^3 to the Besov space with negative power. See [1, 36] for further extensions. Recently, Tao [31] proved the blow-up rate of the solution u of (NS) if the solution u blows up in finite time. We should mention that in the case of $\frac{2}{q}+\frac{3}{p}=1$, the function space $L^q_tL^p_x$ is invariant under the Navier–Stokes scaling:

$$u(x,t) \mapsto u^\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \quad \forall \lambda > 0, \tag{1.1}$$

where u^λ is still a solution to (NS) with initial data $u_0^\lambda := \lambda u_0(\lambda x)$.

Concerning the partial regularity of a weak solution satisfying the local energy inequality, initiated by Scheffer [28], Caffarelli, Kohn and Nirenberg [3] showed that the one-dimensional Hausdorff measure of the possible singular set is zero. One could check Lin [24] and Ladyzhenskaya and Seregin [22] for a simplified proof and improvements. Please refer to [15, 20, 32–35] and references therein for more relevant works.

Starting in [27], there are many interesting works devoted to a new LPS-type criterion, which only involves one component of the velocity. Neustupa, Novotný and Penel [26] proved the LPS-type criterion for one component $u_3 \in L^q(0,T;L^p(\mathbb{R}^3))$ with $\frac{2}{q}+\frac{3}{p} \le \frac{1}{2}$. Later, this condition was improved by Kukavica and Ziane [21] to

$$\frac{2}{q}+\frac{3}{p}=\frac{5}{8}, \quad p > \frac{24}{5},\ \frac{16}{5} \le q < +\infty;$$

and by Cao and Titi [5] to

$$\frac{2}{q}+\frac{3}{p} \le \frac{2}{3}+\frac{2}{3p}, \quad p > \frac{7}{2},$$

and then by Zhou and Pokorný [37] up to

$$\frac{2}{q}+\frac{3}{p} \le \frac{3}{4}+\frac{1}{2p}, \quad p > \frac{10}{3}.$$

However, these conditions are not scaling invariant. Recently, Chemin and Zhang [7] obtained a blow-up criterion via one velocity component in a scaling-invariant space $L^p_t(\dot{H}_x^{\frac{1}{2}+\frac{2}{p}})$ with $4 < p < 6$. Later, Chemin, Zhang and Zhang [8] released the restriction on p to $4 < p < \infty$ and Han et al. [16] extended the range of p to $2 \le p < +\infty$. However, as stated in [25], *the question whether the condition $u_3 \in L^q(0,T;L^p(\mathbb{R}^3))$ for p and q, basically satisfying the condition $\frac{2}{q}+\frac{3}{p}\le 1$, is sufficient for regularity of a solution u in $\mathbb{R}^3\times(t_1,t_2)$, is still open.*

Very recently, Chae and Wolf [6] made important progress and obtained the regularity of a solution to (NS) under the almost-LPS-type condition

$$u_3 \in L^q(0,T;L^p(\mathbb{R}^3)), \quad \frac{2}{q}+\frac{3}{p}<1,\ 3<p<\infty.$$

The aim of this paper is to obtain the LPS criterion via one velocity component with $\frac{2}{q}+\frac{3}{p}\le 1$. Now let us state our main result.

Theorem 1.1. *Let $u_0\in L^2(\mathbb{R}^3)\cap L^3(\mathbb{R}^3)$ and (u,π) be a Leray weak solution of (NS) in $\mathbb{R}^3\times(0,T)$. If u satisfies the condition*

$$u_3 \in L^{q,1}(0,T;L^p(\mathbb{R}^3)) \tag{1.2}$$

for some (p,q) with $\frac{2}{q}+\frac{3}{p}\le 1, 3<p<\infty$, then u is regular in $\mathbb{R}^3\times(0,T)$. Here $L^{q,1}$ denotes the Lorentz space with respect to the variable t.

Theorem 1.1 is a corollary of the following Theorem 1.2 and [36, Theorem 1.4]. The proof will be presented in Section 4.

Definition 1.1. Let $\Omega\subset\mathbb{R}^3$ and $T>0$. We say that (u,π) is a suitable weak solution of (NS) in $\Omega_T=\Omega\times(-T,0)$ if

(1) $u\in L^\infty(-T,0;L^2(\Omega))\cap L^2(-T,0;H^1(\Omega))$ and $\pi\in L^{\frac{3}{2}}(\Omega_T)$;

(2) (u,π) satisfies (NS) in the sense of distributions;

(3) a local energy inequality holds: for any nonnegative $\phi\in C_c^\infty(\mathbb{R}^3\times\mathbb{R})$ vanishing in a neighborhood of the parabolic boundary of Ω_T,

$$\int_\Omega |u(x,t)|^2\phi\,dx + 2\int_{-T}^t\int_\Omega |\nabla u|^2\phi\,dx\,ds \le \int_{-T}^t\int_\Omega |u|^2(\partial_s\phi+\triangle\phi)+u\cdot\nabla\phi(|u|^2+2\pi)\,dx\,ds \tag{1.3}$$

for any $t\in[-T,0]$.

For $z_0=(x_0,t_0)$, we denote $Q_r(z_0)=B_r(x_0)\times(t_0-r^2,t_0)$ with $z_0=(x_0,t_0)$ and $Q_r=Q_r((0,0))$.

Theorem 1.2. *Let (u,π) be a suitable weak solution of (NS) in $\mathbb{R}^3\times(-1,0)$. If u satisfies the condition*

$$u_3\in L^{q,1}(-1,0;L^p(\mathbb{R}^3)), \quad \frac{2}{q}+\frac{3}{p}\le 1 \text{ and } 3<p<\infty,$$

then it holds that

$$r^{-2}\|u\|^3_{L^3(Q_r(z_0))} \le C$$

for any $0 < r < \frac{1}{2}$ *and any* $z_0 \in \mathbb{R}^3 \times (-\frac{1}{2}, 0]$.

Let us make some remarks about our result.

(1) The initial data in $L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ in Theorem 1.1 implies the local-in-time regularity of weak solutions using weak–strong uniqueness theory (see [9] for example). Thus, the weak solution is actually a suitable weak solution, which is defined as above.

(2) Compared with the result in [6], our main contribution is that condition (1.2) with the equality is invariant under the Navier–Stokes scaling, which seems to be the first scaling-invariant regularity criterion in terms of one velocity component in the space $L^q_t L^p_x$. Due to the inclusion $L^{q,1} \subsetneq L^q$ for $q > 1$, the regularity of the weak solution under the condition $u_3 \in L^q(0, T; L^p(\mathbb{R}^3))$ with $\frac{2}{q} + \frac{3}{p} = 1$ is still open.

(3) Our result means that if the solution blows up at time T, then three components of the velocity will blow up at the same time. However, it remains open whether three components (or two components) of the velocity blow up at the same time and same position. For this, we need to establish a new local interior regularity criterion; see [34] for partial progress.

(4) Our key idea comes from an intuitive analysis for a toy model. Rigorous analysis is based on the introduction of a new iterative scheme, a new local anisotropic energy estimate and the atomic decomposition of the Lorentz space (see Section 2 for a detailed explanation).

The rest of this paper is organized as follows. Section 2 is devoted to presenting an intuitive argument, which helps us to show our main idea in a perspective view. In Section 3, we show the local anisotropic energy estimates. In Section 4, we prove Theorems 1.1 and 1.2.

2. An intuitive argument and main ideas

In this section, let us first present an intuitive argument to show the regularity condition via one velocity component. This argument explains the reason why the regularity criterion via one component is reasonable. Moreover, it explains that the requirement of the time variable in the Lorentz space seems critical.

For $x_h = (x_1, x_2)$, we introduce

$$U(x_3, t) = \int_{\mathbb{R}^2} |u(x_h, x_3, t)|^2 \, dx_h.$$

Then $U(t, x_3)$ satisfies

$$\frac{1}{2}\partial_t U - \frac{1}{2}\partial_{x_3}^2 U + \int_{\mathbb{R}^2} |\nabla u(x,t)|^2\, dx_h = -\int_{\mathbb{R}^2} u\cdot\nabla u\cdot u\, dx_h - \int_{\mathbb{R}^2} \nabla\pi\cdot u\, dx_h.$$

By integration by parts and $\nabla\cdot u = 0$, we get

$$\begin{aligned}
&\frac{1}{2}\partial_t U - \frac{1}{2}\partial_{x_3}^2 U + \int_{\mathbb{R}^2} |\nabla u(x,t)|^2\, dx_h \\
&\quad = \int_{\mathbb{R}^2} -\partial_{x_3} u_3 \frac{1}{2}|u|^2\, dx_h - \int_{\mathbb{R}^2} u_3\partial_{x_3}\frac{1}{2}|u|^2\, dx_h - \int_{\mathbb{R}^2} \partial_{x_3}\pi u_3\, dx_h \\
&\qquad - \int_{\mathbb{R}^2} \pi\,\partial_{x_3} u_3\, dx_h \\
&\quad = -\partial_{x_3}\int_{\mathbb{R}^2} u_3\frac{1}{2}|u|^2\, dx_h - \partial_{x_3}\int_{\mathbb{R}^2} \pi u_3\, dx_h.
\end{aligned}$$

Since each nonlinear term on the right-hand side includes a velocity component u_3, this simple argument shows that there is a chance to establish the regularity criterion via one velocity component.

Next let us motivate our result via the following toy equation:

$$\partial_t U - \partial_{x_3}^2 U = -\partial_{x_3}\int_{\mathbb{R}^2} u_3|u|^2\, dx_h.$$

Then we have

$$U(x_3,t) = e^{t\partial_{x_3}^2} U_0 - \int_0^t e^{(t-s)\partial_{x_3}^2}\partial_{x_3}\left(\int_{\mathbb{R}^2} u_3|u|^2\, dx_h\right) ds.$$

Using the estimate of the heat kernel, we obtain

$$\|U(\cdot,t)\|_{L^\infty} \le \|U(\cdot,t-\delta)\|_{L^\infty} + C\int_{t-\delta}^t (t-s)^{-\frac{1}{2}}\|u_3(s)\|_{L^\infty}\|U(\cdot,s)\|_{L^\infty}\, ds,$$

which gives

$$\begin{aligned}
\sup_{s\in[t-\delta,t]} \|U(\cdot,s)\|_{L^\infty} &\le \|U(\cdot,t-\delta)\|_{L^\infty} \\
&\quad + C\int_{t-\delta}^t (t-s)^{-\frac{1}{2}}\|u_3(s)\|_{L^\infty}\, ds \sup_{s\in[t-\delta,t]}\|U(\cdot,s)\|_{L^\infty}.
\end{aligned}$$

Therefore, if $u_3 \in L^{2,1}((0,T);L^\infty)$, then we have

$$\begin{aligned}
\int_{t-\delta}^t (t-s)^{-\frac{1}{2}}\|u_3(s)\|_{L^\infty}\, ds &\le \|(t-s)^{-\frac{1}{2}}\|_{L^{2,\infty}(t-\delta,t)}\big\|\|u_3(s)\|_{L^\infty}\big\|_{L^{2,1}(t-\delta,t)} \\
&\le C\big\|\|u_3(s)\|_{L^\infty}\big\|_{L^{2,1}(t-\delta,t)}.
\end{aligned}$$

This shows that

$$\sup_{s\in[t-\delta,t]} \|U(\cdot,s)\|_{L^\infty} \le \|U(\cdot,t-\delta)\|_{L^\infty} + C\big\|\|u_3(s)\|_{L^\infty}\big\|_{L^{2,1}(t-\delta,t)} \sup_{s\in[t-\delta,t]} \|U(\cdot,s)\|_{L^\infty}.$$

Thus, if δ is small enough, we conclude that

$$\sup_{s\in[t-\delta,t]} \|U(\cdot,s)\|_{L^\infty} \le 2\|U(\cdot,t-\delta)\|_{L^\infty}.$$

In particular, this argument implies that $u \in L^\infty_{t,x_3}(L^2_{x_h})$, which is a scaling-invariant estimate under the Navier–Stokes scaling.

Because of the nonlocality of the pressure, this argument seems difficult to apply to the original Navier–Stokes equations. To overcome this difficulty, we adapt the local energy method introduced in [6]. A key difference from [6] is that we only make the localization in the variables x_3 and t. More precisely, inserting $\phi = \Phi_n\eta$ in (1.3), we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3} |u(\cdot,t)|^2\Phi_n(\cdot,t)\eta(\cdot,t)\,dx + \int_{-1}^t\int_{\mathbb{R}^3} |\nabla u|^2\Phi_n\eta\,dx\,ds \\ &\quad\le \frac{1}{2}\int_{-1}^t\int_{\mathbb{R}^3} |u|^2(\partial_t+\triangle)(\Phi_n\eta)\,dx\,ds + \frac{1}{2}\int_{-1}^t\int_{\mathbb{R}^3} |u|^2u\cdot\nabla(\Phi_n\eta)\,dx\,ds \\ &\quad\quad + \int_{-1}^t\int_{\mathbb{R}^3} \pi u\cdot\nabla(\Phi_n\eta)\,dx\,ds, \end{aligned} \tag{2.1}$$

where Φ_n stands for the shifted fundamental solution to the backward heat equation in one spatial dimension, i.e.,

$$\Phi_n(x_3,t) = \frac{1}{\sqrt{4\pi(-t+r_n^2)}}e^{-\frac{x_3^2}{4(-t+r_n^2)}}, \quad (x_3,t)\in\mathbb{R}\times(-\infty,0),$$

with $r_n = 2^{-n}$, $n\in\mathbb{N}$. Moreover, $\eta = \eta(x_3,t) \in C_c^\infty((-1,1)\times(-1,0])$ denotes a cut-off function such that $0\le\eta\le 1$ in $\mathbb{R}\times(-1,0]$ and $\eta = 1$ on $(-\frac{1}{2},\frac{1}{2})\times(-\frac{1}{4},0)$.

Remark 2.1. Note that one can also take a cut-off function $\psi = \psi(x_h) \in C_0^\infty(\mathbb{R}^2)$ with $0\le\psi\le 1$ satisfying

$$\psi(x_h) = \psi(|x_h|) = \begin{cases} 1 & \text{in } B'(R) = \{x_h, |x_h| < R\}, \\ 0 & \text{in } \mathbb{R}^2\setminus B'(2R), \end{cases} \tag{2.2}$$

and

$$|D\psi| \le \frac{C}{R}, \quad |D^2\psi| \le \frac{C}{R^2}.$$

Inserting $\phi = \Phi_n\eta\psi$ in (1.3) and taking $R\to\infty$ implies formula (2.1) immediately, since the a priori estimates (3.1) and $\pi\in L^{3/2}(\Omega_T)$ hold and all derivatives of $\Phi_n\eta$ are bounded.

Next we introduce

$$\begin{aligned} U_n &= U(r_n) = \mathbb{R}^2 \times (-r_n, r_n), \\ Q_n &= U_n \times (-r_n^2, 0), \\ A_n &= \mathbb{R}^2 \times A_n^*, \end{aligned}$$

where

$$A_n^* = Q_n^* \setminus Q_{n+1}^*, \quad Q_n^* = (-r_n, r_n) \times (-r_n^2, 0).$$

Clearly, there exist absolute constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ and $j = 1, \ldots, n$, it holds that

$$\Phi_n \le c_2 r_j^{-1}, \quad |\partial_3 \Phi_n| \le c_2 r_j^{-2} \quad \text{in } A_j, \tag{2.3}$$

$$c_1 r_n^{-1} \le \Phi_n \le c_2 r_n^{-1}, \quad |\partial_3 \Phi_n| \le c_2 r_n^{-2} \quad \text{in } Q_n. \tag{2.4}$$

Given $n \in \mathbb{N}_0$, we introduce

$$\begin{aligned} E_n &= \sup_{t \in (-r_n^2, 0)} \int_{U_n} |u(t)|^2\, dx + \int_{-r_n^2}^0 \int_{U_n} |\nabla u|^2\, dx\, ds, \\ \mathcal{E} &= \sup_{t \in (-1, 0)} \int_{\mathbb{R}^3} |u(t)|^2\, dx + \int_{-1}^0 \int_{\mathbb{R}^3} |\nabla u|^2\, dx\, ds, \end{aligned}$$

and the main focus of this paper is to prove the boundedness of the scaling-invariant anisotropic quantity

$$r_n^{-1} E_n \le C \quad \forall n \ge 1.$$

To this end, we have to introduce many new ideas as follows.

(1) **New iteration scheme**. Recall the discrete iteration scheme in [6]:

$$\begin{aligned} \sum_{n=0}^{N} (r_n^{-\lambda} E_n(\rho)) &\le \frac{1}{2} \sum_{n=0}^{N} r_n^{1-\lambda} \sum_{i=0}^{n} (r_i^{-1} E_i(R)) + \text{L.O.T} \\ &\le \frac{1}{2} \sum_{i=0}^{N} (r_i^{-1} E_i(R)) \left(\sum_{n=i}^{N} r_n^{1-\lambda} \right) + \text{L.O.T}, \end{aligned}$$

which fails in the endpoint case $\lambda = 1$ (the critical case), since constant series cannot be summed. The condition $\lambda < 1$ implies the necessity of the subcritical condition of u_3. The symbol "L.O.T" represents some lower-order terms in the form $\frac{C(\mathcal{E})}{(R-\rho)^4}$. Our new iteration comes from the discrete inequality

$$y_n \le C_0 + \sum_{j=0}^{n-1} C_j y_j, \quad n \ge 1 \text{ and } y_0 \le C_0,$$

where $\{C_j\}_{j \in \mathbb{N}}, \{y_j\}_{j \in \mathbb{N}}$ are nonnegative series. Then $\{y_n\}$ is uniformly bounded if the infinite sum of $\sum_{j=0}^{+\infty} C_j$ is convergent (see Lemma A.1 for details).

(2) **New key local anisotropic energy estimate**:

$$y_n \le C\mathcal{E} + C\mathcal{E}^{\frac{3}{2}} + C\sum_{i=0}^{n-1} C_i y_i,$$

where $y_i = r_i^{-1}E_i$ and C_i is

$$C_i = \sum_{k=i}^{\infty}\Big(\int_{-r_k^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\Big)^{\frac{2p-3}{2p}} r_k^{-1} e^{-\frac{r_i^2}{32r_k^2}} + \cdots.$$

We also mention that like our argument for the above toy model, C_i is mainly deduced by the heat kernel via the Young inequality. To achieve it, we have to make a very subtle decomposition and summation argument for the nonlinear term and the pressure (see the proof of Proposition 3.1).

(3) **Atomic decomposition of the Lorentz space.** To show that $\sum_{j=0}^{+\infty} C_j < +\infty$, we need to introduce the atomic decomposition of the Lorentz space. See Lemmas A.2 and A.3 for the details.

3. Local anisotropic energy estimates

This section is devoted to proving the following key local anisotropic energy estimate.

Proposition 3.1. *Let (u,π) be a suitable weak solution of (NS) in $\mathbb{R}^3\times(-1,0)$. Suppose that (u,π) satisfies the assumptions of Theorem 1.2. Then there exists a positive series $\{C_i\}_{i\in\mathbb{N}}$ with $\sum_{i=0}^{\infty} C_i \le \|u_3(\cdot,s)\|_{L_t^{q,1}L_x^p}$ such that for any $n\in\mathbb{N}$ we have*

$$r_n^{-1}E_n \le C\mathcal{E} + C\mathcal{E}^{\frac{3}{2}} + C\sum_{i=0}^{n-1}(r_i^{-1}E_i)C_i.$$

This proposition can be directly deduced from the following Lemmas 3.1–3.3. Technically, it is delicate to construct the sequence $\{C_i\}$ bounded by $\|u_3(\cdot,s)\|_{L_t^{q,1}L_x^p}$, as each C_i is related to the value of u_3 on small cubes or cylinders.

Let us recall the following embedding inequality, which will be used frequently:

$$\|u\|^2_{L^m(-r_n^2,0;L^l(U_n))} \le CE_n \quad \forall 2\le m\le\infty,\ \frac{2}{m}+\frac{3}{l}=\frac{3}{2}. \tag{3.1}$$

3.1. Estimates for nonlinear terms

Lemma 3.1. *Let (u,π) be a suitable weak solution of (NS) in $\mathbb{R}^3\times(-1,0)$. Suppose that (u,π) satisfies the assumptions of Theorem 1.2. Then we have*

$$\int_{-1}^{t}\int_{U_0} |u|^2(\partial_t+\triangle)(\Phi_n\eta)\,dx\,ds \le C\mathcal{E}, \tag{3.2}$$

and there exists a positive series $\{B_i\}_{i\in\mathbb{N}}$ *with* $\sum_{i=0}^{\infty} B_i \leq \|u_3\|_{L_t^{q,1}L_x^p}$ *such that for any* $n\in\mathbb{N}$ *we have*

$$\int_{-1}^{t}\int_{U_0} |u|^2 u\cdot\nabla(\Phi_n\eta)\,dx\,ds \leq C\sum_{i=0}^{n}(r_i^{-1}E_i)B_i + C\mathcal{E}^{\frac{3}{2}}. \tag{3.3}$$

Proof. Let (u,π) be the solution satisfying the condition in Lemma 3.1. For the proof of (3.2), we notice that

$$\begin{aligned}&\int_{-1}^{t}\int_{U_0} |u|^2(\partial_t+\triangle)(\Phi_n\eta)\,dx\,ds\\ &\qquad=\int_{-1}^{t}\int_{U_0} |u|^2(\Phi_n\partial_t\eta+2\partial_3\Phi_n\partial_3\eta+\Phi_n\partial_{33}\eta)\,dx\,ds,\end{aligned}$$

which along with (2.3) and (3.1) implies

$$\int_{-1}^{t}\int_{U_0} |u|^2(\partial_t+\triangle)(\Phi_n\eta)\,dx\,ds \leq C\mathcal{E}.$$

This proves (3.2).

Next we turn to the proof of (3.3). Notice that

$$\begin{aligned}&\int_{-1}^{t}\int_{U_0} |u|^2 u\cdot\nabla(\Phi_n\eta)\,dx\,ds\\
&\leq \sum_{i=0}^{n-1}\int_{A_i}|u|^2|u_3|\,|\partial_3\Phi_n|\eta\,dx\,ds+\int_{Q_n}|u|^2|u_3|\,|\partial_3\Phi_n|\eta\,dx\,ds\\
&\quad+\int_{Q_0}|u|^2|u_3|\,|\partial_3\eta|\Phi_n\,dx\,ds\\
&\leq C\sum_{i=0}^{n-1}\int_{A_i}|u|^2|u_3|\frac{|x_3|}{(\sqrt{(-s+r_n^2)})^3}e^{-\frac{x_3^2}{4(-s+r_n^2)}}\eta\,dx\,ds\\
&\quad+C\int_{Q_n}|u|^2|u_3|\frac{|x_3|}{(\sqrt{(-s+r_n^2)})^3}e^{-\frac{x_3^2}{4(-s+r_n^2)}}\eta\,dx\,ds\\
&\quad+\int_{Q_0}|u|^2|u_3|\,|\partial_3\eta|\Phi_n\,dx\,ds = I_{21}+I_{22}+I_{23}.\end{aligned}$$

The last term I_{23} can be easily controlled as

$$I_{23}\leq\int_{Q_0}|u|^2|u_3|\,dx\,ds\leq C\mathcal{E}^{\frac{3}{2}}. \tag{3.4}$$

Before presenting the estimates for I_{21} and I_{22}, we introduce B_i as

$$B_i=\sum_{k=i}^{\infty}\Big(\int_{-r_k^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\Big)^{\frac{2p-3}{2p}} r_k^{-1}e^{-\frac{r_i^2}{32r_k^2}}+r_i^{-1}\Big(\int_{-r_i^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\Big)^{\frac{2p-3}{2p}}.$$

For I_{21} we have

$$\begin{aligned} I_{21} &\le C\sum_{i=0}^{n-1}\int_{A_i\cap\{r_{i+1}\le|x_3|\le r_i,-r_{i+1}^2\le s\le 0\}}|u|^2|u_3|\frac{|x_3|}{(\sqrt{(-s+r_n^2)})^3}e^{-\frac{x_3^2}{4(-s+r_n^2)}}\eta\,dx\,ds\\ &\quad + C\sum_{i=0}^{n-1}\int_{A_i\cap\{|x_3|\le r_i,-r_i^2\le s\le -r_{i+1}^2\}}|u|^2|u_3|\frac{|x_3|}{(\sqrt{(-s+r_n^2)})^3}e^{-\frac{x_3^2}{4(-s+r_n^2)}}\eta\,dx\,ds\\ &= I_{211}+I_{212}. \end{aligned}$$

First, by (3.1) we have

$$\|u\|^2_{L^{\frac{4p}{3}}(-r_i^2,0;L^{2p'}(U_i))}\le CE_i,$$

as $\frac{3}{2p}+\frac{3}{2p'}=\frac{3}{2}$. For the first term I_{211}, we have

$$\begin{aligned} I_{211} &\le C\sum_{i=0}^{n-1}\int_{-r_{i+1}^2}^0\|u(\cdot,s)\|^2_{L^{2p'}(U_i)}\|u_3(\cdot,s)\|_{L^p}\frac{1}{(-s+r_n^2)}e^{-\frac{r_i^2}{32(-s+r_n^2)}}\,ds\\ &\le C\sum_{i=0}^{n-1}E_i\left(\int_{-r_{i+1}^2}^0\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\frac{1}{(-s+r_n^2)^{\frac{2p}{2p-3}}}e^{-\frac{pr_i^2}{(32p-12)(-s+r_n^2)}}\,ds\right)^{\frac{2p-3}{2p}}. \end{aligned}$$

On the other hand, for any $s\in[-r_{i+1}^2,0]$,

$$\frac{1}{(-s+r_n^2)^{\frac{p}{2p-3}}}e^{-\frac{pr_i^2}{(32p-12)(-s+r_n^2)}+\frac{r_i^2}{32(-s+r_n^2)}}\le Cr_i^{-\frac{2p}{2p-3}}.$$

Gathering the above two estimates, we obtain

$$I_{211}\le C\sum_{i=0}^{n-1}(r_i^{-1}E_i)\left(\int_{-r_{i+1}^2}^0\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\frac{1}{(-s+r_n^2)^{\frac{p}{2p-3}}}e^{-\frac{r_i^2}{32(-s+r_n^2)}}\,ds\right)^{\frac{2p-3}{2p}}.$$

Before going further, we give our attention to the term

$$\left(\int_{-r_{i+1}^2}^0\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\frac{1}{(-s+r_n^2)^{\frac{p}{2p-3}}}e^{-\frac{r_i^2}{32(-s+r_n^2)}}\,ds\right)^{\frac{2p-3}{2p}},$$

which is actually controlled by B_i. Indeed, we notice that for any $0\le i\le n-1$,

$$\begin{aligned} &\left(\int_{-r_{i+1}^2}^0\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\frac{1}{(-s+r_n^2)^{\frac{p}{2p-3}}}e^{-\frac{r_i^2}{32(-s+r_n^2)}}\,ds\right)^{\frac{2p-3}{2p}}\\ &\quad=\left(\int_{-r_{i+1}^2-r_n^2}^{-r_n^2}\|u_3(\cdot,s+r_n^2)\|_{L^p}^{\frac{2p}{2p-3}}\frac{1}{(-s)^{\frac{p}{2p-3}}}e^{-\frac{r_i^2}{-32s}}\,ds\right)^{\frac{2p-3}{2p}}\\ &\quad\le\left(\int_{-r_i^2}^0\|\chi_n(s)u_3(\cdot,s+r_n^2)\|_{L^p}^{\frac{2p}{2p-3}}\frac{1}{(-s)^{\frac{p}{2p-3}}}e^{-\frac{r_i^2}{-32s}}\,ds\right)^{\frac{2p-3}{2p}}, \end{aligned} \tag{3.5}$$

where $\chi_n(s) = 1 - \mathbf{1}_{(-r_n^2,0]}(s)$. Now we denote $J_k = (-r_k^2, -r_{k+1}^2]$, and then

$$\begin{aligned}&\left(\int_{-r_i^2}^0 \|\chi_n(s)u_3(\cdot, s+r_n^2)\|_{L^p}^{\frac{2p}{2p-3}} \frac{1}{(-s)^{\frac{p}{2p-3}}} e^{-\frac{r_i^2}{-32s}}\,ds\right)^{\frac{2p-3}{2p}}\\&\quad\le \sum_{k=i}^{\infty}\left(\int_{J_k} \|\chi_n(s)u_3(\cdot, s+r_n^2)\|_{L^p}^{\frac{2p}{2p-3}} \frac{1}{(-s)^{\frac{p}{2p-3}}} e^{-\frac{r_i^2}{-32s}}\,ds\right)^{\frac{2p-3}{2p}}\\&\quad\le \sum_{k=i}^{\infty}\left(\int_{J_k} \|\chi_n(s)u_3(\cdot, s+r_n^2)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\right)^{\frac{2p-3}{2p}} r_{k+1}^{-1} e^{-\frac{r_i^2}{32r_k^2}}.\end{aligned}\tag{3.6}$$

On the other hand, we notice that for any $k \ge n$,

$$\int_{J_k} \|\chi_n(s)u_3(\cdot, s+r_n^2)\|_{L^p}^{\frac{2p}{2p-3}}\,ds = 0,$$

and for any $i \le k \le n-1$,

$$\begin{aligned}\int_{J_k} \|\chi_n(s)u_3(\cdot, s+r_n^2)\|_{L^p}^{\frac{2p}{2p-3}}\,ds &= \int_{-r_k^2+r_n^2}^{-r_{k+1}^2+r_n^2} \|u_3(\cdot, s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\\&\le \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds,\end{aligned}$$

which along with (3.5) and (3.6) implies

$$\begin{aligned}&\left(\int_{-r_{i+1}^2}^0 \|u_3(\cdot, s)\|_{L^p}^{\frac{2p}{2p-3}} \frac{1}{(-s+r_n^2)^{\frac{p}{2p-3}}} e^{-\frac{r_i^2}{32(-s+r_n^2)}}\,ds\right)^{\frac{2p-3}{2p}}\\&\quad\le 2\sum_{k=i}^{\infty}\left(\int_{-r_k^2}^0 \|u_3(\cdot, s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\right)^{\frac{2p-3}{2p}} r_k^{-1} e^{-\frac{r_i^2}{32r_k^2}} \le CB_i.\end{aligned}$$

Therefore, we obtain

$$I_{211} \le C\sum_{i=0}^{n-1}(r_i^{-1}E_i)B_i.\tag{3.7}$$

Similarly, for the second term I_{212}, we have

$$\begin{aligned}I_{212} &\le C\sum_{i=0}^{n-1}\left(\int_{-r_i^2}^{-r_{i+1}^2} \|u_3(\cdot, s)\|_{L^p}^{\frac{2p}{2p-3}} \frac{1}{(-s+r_n^2)^{\frac{2p}{2p-3}}}\,ds\right)^{\frac{2p-3}{2p}} \|u\|^2_{L^{\frac{4p}{3}}(-r_i^2,-r_{i+1}^2;L^{2p'}(U_i))}\\&\le C\sum_{i=0}^{n-1}(r_i^{-1}E_i)\left(\int_{-r_i^2}^{-r_{i+1}^2} \|u_3(\cdot, s)\|_{L^p}^{\frac{2p}{2p-3}} \frac{r_i^{\frac{2p}{2p-3}}}{(-s+r_n^2)^{\frac{2p}{2p-3}}}\,ds\right)^{\frac{2p-3}{2p}}.\end{aligned}$$

Due to $s \in [-r_i^2, -r_{i+1}^2]$, we have

$$\frac{r_i^{\frac{2p}{2p-3}}}{(-s+r_n^2)^{\frac{2p}{2p-3}}} \le C \frac{1}{(-s+r_n^2)^{\frac{p}{2p-3}}},$$

which along with the above estimate for I_{212} implies

$$\begin{aligned} I_{212} &\le C \sum_{i=0}^{n-1} (r_i^{-1} E_i) \left(\int_{-r_i^2}^{-r_{i+1}^2} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \frac{1}{(-s+r_n^2)^{\frac{p}{2p-3}}} \, ds \right)^{\frac{2p-3}{2p}} \\ &\le C \sum_{i=0}^{n-1} (r_i^{-1} E_i) r_i^{-1} \left(\int_{-r_i^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \, ds \right)^{\frac{2p-3}{2p}} \le C \sum_{i=0}^{n-1} (r_i^{-1} E_i) B_i. \end{aligned} \tag{3.8}$$

Therefore, we obtain

$$I_{21} \le C \sum_{i=0}^{n-1} (r_i^{-1} E_i) B_i. \tag{3.9}$$

Similarly, we have

$$\begin{aligned} I_{22} &= \int_{Q_n} |u|^2 |u_3| \, |\partial_3 \Phi_n| \eta \, dx \, ds \\ &\le C r_n^{-1} E_n \left(\int_{-r_n^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \frac{1}{(\sqrt{-s+r_n^2})^{\frac{2p}{2p-3}}} \, ds \right)^{\frac{2p-3}{2p}} \\ &\le C r_n^{-1} E_n B_n. \end{aligned} \tag{3.10}$$

Combining (3.9), (3.10) and (3.4), we finally have

$$\int_{-1}^{t} \int_{U_0} |u|^2 u \cdot \nabla(\Phi_n \eta) \, dx \, ds \le C \sum_{i=0}^{n} (r_i^{-1} E_i) B_i + C \mathcal{E}^{\frac{3}{2}}. \tag{3.11}$$

This finishes the proof of (3.3). We are left with the proof of the fact that

$$\sum_{i=0}^{\infty} B_i \le \|u_3\|_{L_t^{q,1} L_x^p}.$$

We notice that

$$\begin{aligned} \sum_{i=0}^{\infty} B_i \le& \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \left(\int_{-r_k^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \, ds \right)^{\frac{2p-3}{2p}} r_k^{-1} e^{-\frac{r_i^2}{32 r_k^2}} \\ &+ \sum_{i=0}^{\infty} r_i^{-1} \left(\int_{-r_i^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \, ds \right)^{\frac{2p-3}{2p}}. \end{aligned} \tag{3.12}$$

For the first term on the right-hand side, we have

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\Big(\int_{-r_k^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\Big)^{\frac{2p-3}{2p}} r_k^{-1}e^{-\frac{r_i^2}{32r_k^2}}\\
&\quad\le \sum_{k=0}^{\infty}\Big(\int_{-r_k^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\Big)^{\frac{2p-3}{2p}}\sum_{i=0}^{k}\frac{1}{r_k}e^{-\frac{r_i^2}{32r_k^2}}\\
&\quad\le C\sum_{k=0}^{\infty}\frac{1}{r_k}\Big(\int_{-r_k^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\Big)^{\frac{2p-3}{2p}}\\
&\quad\le C\sum_{k=0}^{\infty}{r_k}^{1-\frac{3}{p}-\frac{2}{\tilde q}}\Big(\int_{-r_k^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\tilde q}\,ds\Big)^{\frac{1}{\tilde q}},
\end{aligned}$$

where

$$\frac{2p}{2p-3}\le \tilde q<q=\frac{2p}{p-3},\quad 3<p<\infty.$$

By Lemma A.3 we obtain

$$\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\Big(\int_{-r_k^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\Big)^{\frac{2p-3}{2p}} r_k^{-1}e^{-\frac{r_i^2}{32r_k^2}}\le C\|u_3(\cdot,s)\|_{L_t^{q,1}L_x^p}. \tag{3.13}$$

On the other hand, the estimate of the second term of (3.12) is obvious, since

$$\sum_{i=0}^{\infty} r_i^{-1}\Big(\int_{-r_i^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\Big)^{\frac{2p-3}{2p}}\le C\|u_3(\cdot,s)\|_{L_t^{q,1}L_x^p}$$

by using the Hölder inequality and Lemma A.3 again.

Hence, we obtain

$$\sum_{i=0}^{\infty}B_i\le C\|u_3(\cdot,s)\|_{L_t^{q,1}L_x^p}.$$

The proof of this lemma is completed. ■

3.2. Estimate for the pressure

This subsection is devoted to showing the estimates for the third term on the right-hand side of (2.1), which is related to the control of the pressure π.

We first decompose the pressure π as $\pi=\pi_0+\pi_h$, where

$$-\triangle\pi_0=\partial_i\partial_j(u_iu_j\chi_{Q_0})\quad \text{in } \mathbb{R}^3\times(-1,0).$$

Hence π_h is harmonic in Q_0. Then we have

$$\begin{aligned}
&\int_{-1}^{t}\int_{U_0}\pi u\cdot\nabla(\Phi_n\eta)\,dx\,ds\\
&\quad=\int_{-1}^{t}\int_{U_0}\pi_0 u\cdot\nabla(\Phi_n\eta)\,dx\,ds+\int_{-1}^{t}\int_{U_0}\pi_h u\cdot\nabla(\Phi_n\eta)\,dx\,ds
\end{aligned}$$

$$= \int_{-1}^{t}\int_{U_0} \pi_0 u_3 \partial_3 \Phi_n \eta \, dx\, ds + \int_{-1}^{t}\int_{U_0} \pi_0 \Phi_n u_3 \partial_3 \eta \, dx\, ds - \int_{-1}^{t}\int_{U_0} \nabla \pi_h \cdot u(\Phi_n \eta) \, dx\, ds.$$

Next we deal with three terms on the right-hand side of the above equation.

Lemma 3.2. *Let (u,π) be a suitable weak solution of (NS) in $\mathbb{R}^3 \times (-1,0)$. Suppose that (u,π) satisfies the assumptions of Theorem 1.2. Then there exists a positive series $\{C_i\}_{i\in\mathbb{N}}$ with $\sum_{i=0}^{\infty} C_i \le \|u_3\|_{L_t^{q,1}L_x^p}$ such that for any $n \in \mathbb{N}$ we have*

$$\int_{-1}^{t}\int_{U_0} u_3 \pi_0 (\partial_3 \Phi_n \eta) \, dx\, ds \le C \sum_{i=0}^{n} (r_i^{-1} E_i) C_i.$$

We can also represent π_0 in the following way. For any $f_{ij} \in L^p(Q_0)$ with $1 < p < \infty$ and $i, j = 1,2,3$, we define

$$T(f)(x,t) = \text{P.V.} \int_{\mathbb{R}^3} K(x-y) : f(y,t) \chi_{U_0}(y) \, dy, \quad (x,t) \in \mathbb{R}^3 \times (-1,0),$$

where ":" stands for tensor contraction and the kernel is

$$K_{ij} = \partial_i \partial_j \Big(\frac{1}{4\pi |x|} \Big), \quad i, j = 1,2,3.$$

Then we have $\pi_0 = T(u_i u_j \chi_{Q_0})$.

Proof of Lemma 3.2. Let (u,π) be the solution satisfying the condition in Lemma 3.2. We first introduce the following notation. For $j \in \mathbb{N}_0$ let $\chi_j = \chi_{Q_j}$. Moreover, we set

$$\phi_j = \begin{cases} \chi_j - \chi_{j+1} & \text{if } j = 0,1,\ldots,n-1, \\ \chi_n & \text{if } j = n. \end{cases}$$

It is clear that

$$\sum_{j=0}^{n} \phi_j = (\chi_0 - \chi_1) + \cdots + \chi_n = 1 \Rightarrow f = \sum_{j=0}^{n} f\phi_j \quad \text{in } Q_0.$$

Taking $f = u_i u_j \chi_{Q_0}$, it holds that

$$\pi_0 = T(f) = \sum_{j=0}^{n} T(f\phi_j) = \sum_{j=0}^{n} \pi_{0,j}.$$

Then we have

$$\int_{-1}^{t}\int_{U_0} u_3 \pi_0 (\partial_3 \Phi_n \eta) \, dx\, ds = \sum_{k=0}^{n} \int_{-1}^{t}\int_{U_0} \pi_0 u_3 \partial_3 \Phi_n \phi_k \eta \, dx\, ds$$

$$
\begin{aligned}
&= \sum_{j=0}^{n}\sum_{k=0}^{n}\int_{-1}^{t}\int_{U_0}\pi_{0,j}u_3\partial_3\Phi_n\phi_k\eta\,dx\,ds\\
&= \sum_{k=0}^{n}\sum_{j=k}^{n}\int_{-1}^{t}\int_{U_0}\pi_{0,j}u_3\partial_3\Phi_n\phi_k\eta\,dx\,ds\\
&\quad+\sum_{j=0}^{n}\sum_{k=j+1}^{n}\int_{-1}^{t}\int_{U_0}\pi_{0,j}u_3\partial_3\Phi_n\phi_k\eta\,dx\,ds\\
&= \mathrm{II}_1+\mathrm{II}_2.
\end{aligned}
$$

Now we deal with the term II_1. By the definitions of the cut-off function ϕ_i and the singular operator T, II_1 can be written as

$$
\mathrm{II}_1=\sum_{k=0}^{n}\int_{-1}^{t}\int_{U_0}\Pi_{0,k}u_3\partial_3\Phi_n\phi_k\eta\,dx\,ds,
$$

where

$$
\Pi_{0,k}=\begin{cases}\pi_0 & \text{if } k=0;\\ T(\chi_k f) & \text{if } k=1,\dots,n.\end{cases}
$$

We first notice that

$$
\begin{aligned}
\mathrm{II}_1 &\le C\sum_{i=0}^{n-1}\int_{A_i\cap\{r_{i+1}\le|x_3|\le r_i,-r_{i+1}^2\le s\le 0\}}|\Pi_{0,i}|\,|u_3|\frac{|x_3|}{(\sqrt{(-s+r_n^2)})^3}e^{-\frac{x_3^2}{4(-s+r_n^2)}}\eta\,dx\,ds\\
&\quad+C\sum_{i=0}^{n-1}\int_{A_i\cap\{|x_3|\le r_i,-r_i^2\le s\le -r_{i+1}^2\}}|\Pi_{0,i}|\,|u_3|\frac{|x_3|}{(\sqrt{(-s+r_n^2)})^3}e^{-\frac{x_3^2}{4(-s+r_n^2)}}\eta\,dx\,ds\\
&\quad+C\int_{Q_n}|\Pi_{0,n}|\,|u_3|r_n^{-2}\,dx\,ds\\
&\le C\sum_{i=0}^{n-1}\int_{-r_{i+1}^2}^{0}\|u(\cdot,s)\|_{L^{2p'}(U_i)}^2\|u_3(\cdot,s)\|_{L^p}\frac{1}{(-s+r_n^2)}e^{-\frac{r_i^2}{32(-s+r_n^2)}}\,ds\\
&\quad+C\sum_{i=0}^{n-1}r_i^{-2}\Big(\int_{-r_i^2}^{-r_{i+1}^2}\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\Big)^{\frac{2p-3}{2p}}\|u\|^2_{L^{\frac{4p}{3}}(-r_i^2,-r_{i+1}^2;L^{2p'}(U_i))}\\
&\quad+Cr_n^{-2}\Big(\int_{-r_n^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\Big)^{\frac{2p-3}{2p}}\|u\|^2_{L^{\frac{4p}{3}}(-r_n^2,0;L^{2p'}(U_n))}
\end{aligned}
$$

provided that the kernel of T is a Calderón–Zygmund kernel such that for any $t\in(-1,0)$,

$$
\|\Pi_{0,k}(\cdot,t)\|_{L^{p'}(\mathbb{R}^3)}\le C\|u_iu_j(\cdot,t)\chi_k(\cdot,t)\|_{L^{p'}}\le\|u(\cdot,t)\|^2_{L^{2p'}}.
$$

By a similar argument leading to (3.7) and (3.8), we obtain

$$\mathrm{II}_1 \le C \sum_{i=0}^{n} (r_i^{-1} E_i) B_i. \tag{3.14}$$

Now we turn to the estimate of II_2, which is much more complicated. We have

$$\begin{aligned}\mathrm{II}_2 &= \sum_{j=n-2}^{n} \sum_{k=j}^{n} \int_{-1}^{t} \int_{U_0} \pi_{0,j} u_3 \partial_3 \Phi_n \phi_k \eta \, dx \, ds \\ &+ \sum_{j=0}^{n-3} \sum_{k=j}^{j+3} \int_{-1}^{t} \int_{U_0} \pi_{0,j} u_3 \partial_3 \Phi_n \phi_k \eta \, dx \, ds \\ &+ \sum_{j=0}^{n-3} \sum_{k=j+4}^{n} \int_{-1}^{t} \int_{U_0} \pi_{0,j} u_3 \partial_3 \Phi_n \phi_k \eta \, dx \, ds = \mathrm{II}_{21} + \mathrm{II}_{22} + \mathrm{II}_{23}.\end{aligned}$$

Using the property of a singular operator T and a similar argument to above, we get

$$\begin{aligned}\mathrm{II}_{21} &\le C \sum_{i=n-2}^{n} \sum_{k=i}^{n} \int_{Q_k} |\pi_{0,i}| \, |u_3| \frac{|x_3|}{(\sqrt{(-s+r_n^2)})^3} e^{-\frac{x_3^2}{4(-s+r_n^2)}} \eta \, dx \, ds \\ &\le C \sum_{i=n-2}^{n} \sum_{k=i}^{n} \Big(\int_{-r_k^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \, ds \Big)^{\frac{2p-3}{2p}} r_n^{-2} \|u\|^2_{L^{\frac{4p}{3}}(-r_k^2,0;L^{2p'}(U_i))} \\ &\le C \sum_{i=0}^{n} (r_i^{-1} E_i) B_i\end{aligned}$$

and

$$\begin{aligned}\mathrm{II}_{22} &\le C \sum_{i=0}^{n-3} \sum_{k=i}^{i+3} \int_{\substack{A_k \cap \{r_{k+1} \le |x_3| \le r_k, \\ -r_{k+1}^2 \le s \le 0\}}} |\pi_{0,i}| \, |u_3| \frac{|x_3|}{(\sqrt{(-s+r_n^2)})^3} e^{-\frac{x_3^2}{4(-s+r_n^2)}} \eta \, dx \, ds \\ &+ C \sum_{i=0}^{n-3} \sum_{k=i}^{i+3} \int_{\substack{A_k \cap \{|x_3| \le r_k, \\ -r_k^2 \le s \le -r_{k+1}^2\}}} |\pi_{0,i}| \, |u_3| \frac{|x_3|}{(\sqrt{(-s+r_n^2)})^3} e^{-\frac{x_3^2}{4(-s+r_n^2)}} \eta \, dx \, ds \\ &\le C \sum_{i=0}^{n-3} \sum_{k=i}^{i+3} \int_{-r_{k+1}^2}^{0} \|u(\cdot,s)\|^2_{L^{2p'}(U_i)} \|u_3(\cdot,s)\|_{L^p} \frac{1}{(-s+r_n^2)} e^{-\frac{r_k^2}{32(-s+r_n^2)}} \, ds \\ &+ C \sum_{i=0}^{n-3} \sum_{k=i}^{i+3} \Big(\int_{-r_k^2}^{-r_{k+1}^2} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \frac{1}{(-s)^{\frac{2p}{2p-3}}} \, ds \Big)^{\frac{2p-3}{2p}} \|u\|^2_{L^{\frac{4p}{3}}(-r_k^2,-r_{k+1}^2;L^{2p'}(U_i))} \\ &\le C \sum_{i=0}^{n} (r_i^{-1} E_i) \Big(\int_{-r_{i+1}^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \frac{1}{(-s+r_n^2)^{\frac{p}{2p-3}}} e^{-\frac{r_i^2}{32(-s+r_n^2)}} \, ds \Big)^{\frac{2p-3}{2p}} \\ &+ C \sum_{i=0}^{n} (r_i^{-1} E_i) \Big(\int_{-r_i^2}^{-r_{i+1}^2} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \frac{1}{(-s)^{\frac{p}{2p-3}}} \, ds \Big)^{\frac{2p-3}{2p}}.\end{aligned}$$

Hence, as in (3.7) and (3.8), we obtain

$$\mathrm{II}_{21} + \mathrm{II}_{22} \le C \sum_{i=0}^{n} (r_i^{-1} E_i) B_i . \tag{3.15}$$

Finally, we estimate the term II_{23} as

$$\mathrm{II}_{23} = \sum_{j=0}^{n-3} \sum_{k=j+4}^{n} \int_{-1}^{t} \int_{U_0} \pi_{0,j} u_3 \partial_3 \Phi_n \phi_k \eta \, dx \, ds.$$

By the definition of $\pi_{0,j}$, which is harmonic in $\mathbb{R}^2 \times (-r_{j+2}, r_{j+2}) \times (-r_{j+2}^2, 0)$, with the help of Lemma A.4 (passing $R \to \infty$), we get

$$\|\pi_{0,j}(\cdot,s)\|_{L^{p'}(U_k)} \le C r_k^{\frac{1}{p'}} r_j^{\frac{2}{p'} - \frac{3}{\ell}} \|\pi_{0,j}(\cdot,s)\|_{L^\ell(\mathbb{R}^3)}.$$

Hence, it follows that

$$\begin{aligned}
\mathrm{II}_{23} &\le C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n-1} \int_{\substack{A_k \cap \{r_{k+1} \le |x_3| \le r_k, \\ -r_{k+1}^2 \le s \le 0\}}} |\pi_{0,i}| \, |u_3| \frac{|x_3|}{(\sqrt{(-s + r_n^2)})^3} e^{-\frac{x_3^2}{4(-s+r_n^2)}} \eta \, dx \, ds \\
&\quad + C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n-1} \int_{\substack{A_k \cap \{|x_3| \le r_k, \\ -r_k^2 \le s \le -r_{k+1}^2\}}} |\pi_{0,i}| \, |u_3| \frac{|x_3|}{(\sqrt{(-s + r_n^2)})^3} e^{-\frac{x_3^2}{4(-s+r_n^2)}} \eta \, dx \, ds \\
&\quad + C \sum_{i=0}^{n-3} \int_{Q_n} |\pi_{0,i}| \, |u_3| \frac{|x_3|}{(\sqrt{(-s + r_n^2)})^3} \eta \, dx \, ds \\
&\le C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n-1} \int_{-r_{k+1}^2}^{0} r_k^{\frac{1}{p'}} r_i^{\frac{2}{p'} - \frac{3}{\ell}} \|u(\cdot,s)\|_{L^{2\ell}(U_i)}^2 \|u_3(\cdot,s)\|_{L^p} \frac{1}{(-s + r_n^2)} e^{-\frac{r_k^2}{32(-s+r_n^2)}} \, ds \\
&\quad + C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n} \left(\int_{-r_k^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \, ds \right)^{\frac{2p-3}{2p}} r_k^{\frac{1}{p'} - 2} r_i^{\frac{2}{p'} - \frac{3}{\ell}} \|u\|^2_{L^{\frac{4p}{3}}(-r_k^2, 0; L^{2\ell}(U_i))} \\
&\le C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n} \int_{-r_{k+1}^2}^{0} r_k^{\frac{1}{p'}} r_i^{\frac{2}{p'} - \frac{3}{\ell}} \|u(\cdot,s)\|_{L^{2\ell}(U_i)}^2 \|u_3(\cdot,s)\|_{L^p} \frac{1}{(-s + r_n^2)} e^{-\frac{r_k^2}{32(-s+r_n^2)}} \, ds \\
&\quad + C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n} \left(\int_{-r_k^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}} \, ds \right)^{\frac{2p-3}{2p}} \\
&\qquad \cdot r_k^{\frac{1}{p'} - 2} r_i^{\frac{2}{p'} - \frac{3}{\ell}} r_k^{\frac{3}{p} - 3 + \frac{3}{\ell}} \|u\|^2_{L^{\frac{4\ell}{3\ell-3}}(-r_k^2, 0; L^{2\ell}(U_i))} \\
&= \mathrm{II}_1' + \mathrm{II}_2',
\end{aligned}$$

where $1 < \ell < p'$. Note that

$$\begin{aligned}\mathrm{II}_2' &\le C\sum_{i=0}^{n-3}\sum_{k=i+4}^{n}\left(\int_{-r_k^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\tilde q}\,ds\right)^{\frac{1}{\tilde q}} r_k^{2-\frac{3}{p}-\frac{2}{\tilde q}} r_k^{\frac{1}{p'}-2} r_i^{\frac{2}{p'}-\frac{3}{\ell}} r_k^{\frac{3}{p}-3+\frac{3}{\ell}} E_i\\ &\le C\sum_{i=0}^{n-3} r_i^{\frac{2}{p'}-\frac{3}{\ell}}\left(\int_{-r_i^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\tilde q}\,ds\right)^{\frac{1}{\tilde q}}\sum_{k=i+4}^{n} r_k^{2-\frac{3}{p}-\frac{2}{\tilde q}} r_k^{\frac{1}{p'}-2} r_k^{\frac{3}{p}-3+\frac{3}{\ell}} E_i,\end{aligned}$$

where we choose $\tilde q$ close to q and ℓ close to 1 such that

$$2-\frac{3}{p}-\frac{2}{\tilde q}+\frac{1}{p'}-2+\frac{3}{p}-3+\frac{3}{\ell} > 0.$$

Consequently, we have

$$\mathrm{II}_2' \le C\sum_{i=0}^{n-3} r_i^{1-\frac{3}{p}-\frac{2}{\tilde q}-1}\left(\int_{-r_{i+4}^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\tilde q}\,ds\right)^{\frac{1}{\tilde q}} E_i.$$

Thanks to

$$\frac{1}{(-s+r_n^2)}e^{-\frac{r_k^2}{32(-s+r_n^2)}} \le C r_k^{-2},$$

it follows from the Hölder inequality that

$$\begin{aligned}\mathrm{II}_1' &\le C\sum_{i=0}^{n-3}\sum_{k=i+4}^{n}\left(\int_{-r_k^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-3}}\,ds\right)^{\frac{2p-3}{2p}} r_k^{\frac{1}{p'}-2} r_i^{\frac{2}{p'}-\frac{3}{\ell}} r_k^{\frac{3}{p}-3+\frac{3}{\ell}} E_i\\ &\le C\sum_{i=0}^{n-3} r_i^{\frac{2}{p'}-\frac{3}{\ell}}\left(\int_{-r_{i+4}^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\tilde q}\,ds\right)^{\frac{1}{\tilde q}}\sum_{k=i+4}^{n} r_k^{1-\frac{3}{p}-\frac{2}{\tilde q}+1} r_k^{\frac{1}{p'}-2} r_k^{\frac{3}{p}-3+\frac{3}{\ell}} E_i.\end{aligned}$$

Hence, we obtain

$$\mathrm{II}_1' \le C\sum_{i=0}^{n-3} r_i^{1-\frac{3}{p}-\frac{2}{\tilde q}-1}\left(\int_{-r_{i+4}^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\tilde q}\,ds\right)^{\frac{1}{\tilde q}} E_i.$$

Therefore, from the above two estimates, we deduce that

$$\mathrm{II}_{23} \le C\sum_{i=0}^{n-3} r_i^{1-\frac{3}{p}-\frac{2}{\tilde q}-1}\left(\int_{-r_{i+4}^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\tilde q}\,ds\right)^{\frac{1}{\tilde q}} E_i. \tag{3.16}$$

We denote

$$C_i = B_i + r_i^{1-\frac{3}{p}-\frac{2}{\tilde q}}\left(\int_{-r_i^2}^{0}\|u_3(\cdot,s)\|_{L^p}^{\tilde q}\,ds\right)^{\frac{1}{\tilde q}}. \tag{3.17}$$

Then we infer from (3.14), (3.15) and (3.16) that

$$\int_{-1}^{t}\int_{U_0} u_3\pi_0(\partial_3\Phi_n\eta)\,dx\,ds \le C\sum_{i=0}^{n}(r_i^{-1}E_i)C_i.$$

The bound of C_i is guaranteed by B_i and Lemma A.3. ■

Lemma 3.3. *Let* (u, π) *be a suitable weak solution of* (NS) *in* $\mathbb{R}^3 \times (-1, 0)$. *Suppose that* (u, π) *satisfies the assumptions of Theorem* 1.2. *Then we have*

$$\left| \int_{-1}^{t} \int_{U_0} \pi_0 \Phi_n u_3 \partial_3 \eta \, dx \, ds \right| \le C \mathcal{E}^{\frac{3}{2}},$$

$$\left| \int_{-1}^{t} \int_{U_0} \nabla \pi_h \cdot u (\Phi_n \eta) \, dx \, ds \right| \le C \mathcal{E}^{\frac{3}{2}}.$$

The difference between the proof of this lemma and the corresponding pressure estimates in [6] is that the horizontal variable lies in the whole space $\mathbb{R}^2$.

Proof of Lemma 3.3. By the definition of π_0 and the Calderón–Zygmund inequality, we have

$$\begin{aligned} \left| \int_{-1}^{t} \int_{U_0} \pi_0 \Phi_n u_3 \partial_3 \eta \, dx \, ds \right| &\le \int_{-1}^{t} \int_{U_0} |\pi_0 u_3| \, dx \, ds \\ &\le \int_{-1}^{0} \|\pi_0(s)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \|u(s)\|_{L^3(U_0)} \, ds \\ &\le C \int_{-1}^{0} \|u(s)\|_{L^3}^3 \, ds \le C \mathcal{E}^{\frac{3}{2}}. \end{aligned}$$

Now we turn to proving the second inequality of the lemma. We first choose a cut-off function $\zeta(x_3, t) \in C_c^\infty((-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{4}, 0])$ satisfying $\zeta(x_3, t) = 1$ in $(-\frac{1}{4}, \frac{1}{4}) \times (-\frac{1}{16}, 0]$ and $|\partial_3 \zeta| \le C$. Then

$$\begin{aligned} \left| \int_{Q_0} \nabla \pi_h \cdot u (\Phi_n \eta)(1 - \zeta) \, dx \, ds \right| &\le \left| \int_{Q_0} \pi_h u \cdot \nabla (\Phi_n \eta (1 - \zeta)) \, dx \, ds \right| \\ &\le C \|\pi_h\|_{L^{\frac{3}{2}}(Q_0)} \|u\|_{L^3(Q_0)} \le C \mathcal{E}^{\frac{3}{2}}. \end{aligned}$$

Moreover, there exists a sequence of balls centered at $x_j' \in \mathbb{R}^2$ with $j = 1, 2, \ldots$ and radius $\frac{1}{8}$ so that

$$\bigcup_{j=1}^{\infty} \{x' : |x' - x_j'| < \tfrac{1}{8}\} = \mathbb{R}^2,$$

and any point x' is contained within up to 10 balls of $B'(x_j, \frac{1}{2})$. Then we have

$$\begin{aligned} &\left| \int_{-1}^{t} \int_{U_0} \nabla \pi_h \cdot u (\Phi_n \eta) \zeta \, dx \, ds \right| \\ &\quad \le C \sum_{k=1}^{n} \sum_{j=1}^{\infty} r_k^{-1} \int_{Q_k \cap \{|x' - x_j'| < \frac{1}{8}\}} |\nabla \pi_h| \, |u| \, dx \, ds \\ &\quad \le C \sum_{k=1}^{n} \sum_{j=1}^{\infty} r_k^{-1} \|\nabla \pi_h\|_{L^{\frac{3}{2}}(-r_k^2, 0; L^\infty(U_k \cap \{|x' - x_j'| < \frac{1}{8}\}))} \|u\|_{L^3(-r_k^2, 0; L^1(U_k \cap \{|x' - x_j'| < \frac{1}{8}\}))} \end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{k=1}^{n}\sum_{j=1}^{\infty} r_k^{-1/3}\|\nabla\pi_h\|_{L^{\frac32}(-r_k^2,0;L^\infty(U_k\cap\{|x'-x_j'|<\frac18\}))}\|u\|_{L^3(-r_k^2,0;L^3(U_k\cap\{|x'-x_j'|<\frac18\}))}\\
&\le C\sum_{k=1}^{n}\sum_{j=1}^{\infty} r_k^{-1/3}\|\nabla\pi_h\|^{\frac32}_{L^{\frac32}(-r_k^2,0;L^\infty(U_k\cap\{|x'-x_j'|<\frac18\}))}\\
&\quad + C\sum_{k=1}^{n} r_k^{-1/3}\|u\|^3_{L^3(-r_k^2,0;L^3(U_k))}.
\end{aligned}
$$

For any

$$x^*\in U_k\cap\big\{x=(x',x_3);|x'-x_j'|<\tfrac18\big\},$$

we have

$$d(x^*,\partial U_0)>\frac18$$

due to $k\ge 1$. Thus, there exists $x_3^*\in(-\frac12,\frac12)$ such that

$$x^*\in B\big((x_j',x_3^*);\tfrac14\big)\subset U_0\cap\big\{|x'-x_j'|<\tfrac14\big\}.$$

Since π_h is harmonic in U_0, there holds

$$|\nabla\pi_h|(x^*)\le C\int_{B((x_j',x_3^*);\frac14)}|\pi_h|\,dx\le C\|\pi_h\|_{L^{\frac32}(U_0\cap\{|x'-x_j'|<\frac14\})},$$

which implies

$$
\begin{aligned}
\|\nabla\pi_h\|^{\frac32}_{L^{\frac32}(-r_k^2,0;L^\infty(U_k\cap\{|x'-x_j'|<\frac18\}))}&\le\int_{-r_k^2}^{0}\|\nabla\pi_h(\cdot,s)\|^{\frac32}_{L^\infty(U_k\cap\{|x'-x_j'|<\frac18\})}\,ds\\
&\le C\int_{-r_k^2}^{0}\int_{(U_0\cap\{|x'-x_j'|<\frac14\})}|\pi_h|^{\frac32}\,ds.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{-1}^{t}\int_{U_0}\nabla\pi_h\cdot u\Phi_n\eta\zeta\,dx\,ds\\
&\qquad\le C\sum_{k=1}^{n} r_k^{-1/3}\big(\|\pi_h\|^{\frac32}_{L^{\frac32}(-r_k^2,0;L^{\frac32}(U_0))}+\|u\|^3_{L^3(-r_k^2,0;L^3(U_k))}\big)\\
&\qquad\le C\sum_{k=1}^{n} r_k^{1/6}\big(\|\pi_h\|^{\frac32}_{L^2(-r_k^2,0;L^{\frac32}(U_0))}+\|u\|^3_{L^4(-r_k^2,0;L^3(U_k))}\big)\\
&\qquad\le C\big(\|\pi\|^{\frac32}_{L^2(-1,0;L^{\frac32}(U_0))}+\mathcal{E}^{\frac32}\big).
\end{aligned}
$$

Applying Calderón–Zygmund estimates, there holds

$$\left|\int_{-1}^{t}\int_{U_0}\nabla\pi_h\cdot u(\Phi_n\eta)\,dx\,ds\right|\le C\mathcal{E}^{\frac32}.$$

The proof is completed. ■

4. Proofs of Theorem 1.1 and Theorem 1.2

This section is devoted to the proofs of Theorems 1.1 and 1.2. Let us first prove Theorem 1.2.

Proof of Theorem 1.2. By a translation argument, it is enough to consider the point $z_0 = (0,0)$. By Proposition 3.1, we have

$$r_n^{-1}E_n \le C\mathcal{E} + C\mathcal{E}^{\frac{3}{2}} + C\sum_{i=0}^{n-1}(r_i^{-1}E_i)C_i,$$

then, due to $\sum_{i\ge 0} C_i \le C\|u_3\|_{L_t^{q,1}L_x^p}$ and Lemma A.1, we have for any $n\in\mathbb{N}$,

$$r_n^{-1}E_n \le C(\mathcal{E}+\mathcal{E}^{\frac{3}{2}})e^{\sum_{i=0}^{\infty}C_i} \le C(\mathcal{E}+\mathcal{E}^{\frac{3}{2}}),$$

which yields that for any $r\in(0,\frac{1}{2})$,

$$r^{-2}\|u\|_{L^3(Q_r)}^3 \le Cr^{-\frac{3}{2}}\|u\|_{L^4(-r^2,0;L^3(B(r)))}^3 \le C(r_n^{-1}E_n)^{\frac{3}{2}} \le C.$$

Here $0<r_n\le r$. The proof is completed. ∎

Now we prove Theorem 1.1 by applying Theorem 1.2 and the following interior regularity theorem (see [36, Theorem 1.4]).

Theorem 4.1. *Let* (u,π) *be a suitable weak solution of* (NS) *in* Q_1. *If* u *satisfies*

$$\sup_{0<r<1} r^{1-\frac{3}{p}-\frac{2}{q}}\|u\|_{L_t^qL_x^p(Q_r)} \le M < +\infty \tag{4.1}$$

for some (p,q) *with* $1\le\frac{3}{p}+\frac{2}{q}<2$ *and* $1<q\le\infty$, *then there exists a positive constant* ε *depending on* p, q, M *such that* $(0,0)$ *is a regular point if*

$$r_0^{1-\frac{3}{p}-\frac{2}{q}}\|u_3\|_{L_t^qL_x^p(Q_{r_0})} \le \varepsilon \tag{4.2}$$

for some r_0 *with*

$$0<r_0<\min\Big\{\frac{1}{2},\Big(r^{-2}\int_{Q_1}|u(y,s)|^2+|\pi(y,s)|^{\frac{3}{2}}\,dy\,ds\Big)^{-2}\Big\}. \tag{4.3}$$

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Due to $u_0\in L^3(\mathbb{R}^3)$, it follows that (u,π) is regular in $\mathbb{R}^3\times(0,T_0)$ for some $0<T_0\le T$, which implies that it is a suitable weak solution of (NS) in $(0,T_0)$. Assume that T_0 is the first blow-up time. However, we will prove that the point (x,T_0) for any $x\in\mathbb{R}^3$ is a regular point. For this, it is enough to prove that $(0,0)$ is a regular point by a translation argument.

First of all, it follows from the proof of Theorem 1.2 that

$$r^{-2}\|u\|_{L^3(Q_r)}^3 + \sup_{-r^2\le t<0} r^{-1}\int_{B_r(0)} |u(y,t)|^2\,dy + r^{-1}\int_{Q_r} |\nabla u(y,s)|^2\,dy\,ds \le C$$

for any $r \in (0, \frac{1}{2})$ and

$$u_3 \in L^{q,1}(-1,0; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} \le 1,\ 3 < p < \infty.$$

Thus, condition (4.1) holds.

Next we verify condition (4.2). By Hölder inequality, for $q < 3$ and $p > 9$ we have

$$\begin{aligned}
r^{-2}\|u_3\|_{L^3(Q_r)}^3 &\le r^{-2}\int_{-r^2}^0 \Big(\int_{B_r(0)} |u_3|^2\,dx\Big)^{\frac{1}{2}} \Big(\int_{B_r(0)} |u_3|^4\,dx\Big)^{\frac{1}{2}} ds \\
&\le \Big(\sup_{-r^2\le t<0} r^{-1}\int_{B_r(0)} |u_3(y,t)|^2\,dy\Big)^{\frac{1}{2}} r^{-\frac{3}{2}} \int_{-r^2}^0 \Big(\int_{B_r(0)} |u_3|^4\,dx\Big)^{\frac{1}{2}} ds \\
&\le C\|u_3\|_{L_t^q L_x^p(Q_r)}^2,
\end{aligned}$$

which implies

$$\lim_{r\to 0} r^{-\frac{2}{3}}\|u_3\|_{L_t^3 L_x^3(Q_r)} = 0.$$

The remaining case of $q \ge 3$ is obvious, since the local invariant quantity $r^{-2}\|u_3\|_{L^3(Q_r)}^3$ can be controlled by $\|u_3\|_{L_t^q L_x^p(Q_r)}^3$.

Thus, the conditions of Theorem 4.1 are satisfied so that $(0,0)$ is regular point. ■

A. Some basic lemmas

Lemma A.1. *Let $\{b_j\}_{j\in\mathbb{N}}$, $\{y_j\}_{j\in\mathbb{N}}$ be nonnegative series satisfying the inequality*

$$y_n \le C_0 + \sum_{j=0}^{n-1} b_j y_j, \quad n \ge 1 \text{ and } y_0 \le C_0.$$

Then it holds that for any $n \in \mathbb{N}$,

$$y_n \le C_0 e^{\sum_{j=0}^{n-1} b_j}.$$

Proof. We first define the following nonnegative series $\{x_j\}$:

$$x_0 = C_0, \quad x_n = C_0 + \sum_{j=0}^{n-1} b_j x_j, \quad n \ge 1.$$

It is easy to check that for any $j \in \mathbb{N}$, $x_j \geq y_j$. On the other hand, by the definition of $\{x_j\}$, it can be represented as for any $n \geq 1$,

$$x_n = C_0 \prod_{i=0}^{n-1}(1 + b_i) \leq C_0 e^{\sum_{i=0}^{n-1} b_i}.$$

Hence, we deduce that for any $n \geq 1$,

$$y_n \leq x_n \leq C_0 e^{\sum_{i=0}^{n-1} b_i},$$

which along with the condition that $y_0 \leq C_0$ completes the proof of this lemma. ∎

Lemma A.2. *Let $0 < p, q < \infty$. Then for any $f \in L^{p,q}(\mathbb{R})$, there exists a sequence $\{c_n\}_{n\in\mathbb{Z}} \in \ell^q$ and a sequence of functions $\{f_n\}_{n\in\mathbb{Z}}$ with each f_n bounded by $2^{-n/p}$ and supported on a set of measure 2^n such that*

$$f = \sum_{n\in\mathbb{Z}} c_n f_n$$

and

$$c(p,q)\|\{c_n\}\|_{\ell^q} \leq \|f\|_{L^{p,q}} \leq C(p,q)\|\{c_n\}\|_{\ell^q},$$

where the constants $c(p,q)$ and $C(p,q)$ only depend on p, q.

Proof. Let $f \in L^{p,q}(\mathbb{R})$. We denote by f^* the corresponding decreasing rearrangement of f. We let

$$\begin{aligned} c_n &:= 2^{n/p} f^*(2^n), \\ A_n &:= \{x : f^*(2^{n+1}) < |f(x)| \leq f^*(2^n)\}, \\ f_n &:= c_n^{-1} f \mathbf{1}_{A_n}. \end{aligned} \tag{A.1}$$

By a direct calculation, it is easy to check that

$$f = \sum_{n\in\mathbb{Z}} c_n f_n.$$

Now we start to prove the second statement. By the definition of the Lorentz space, we have

$$\begin{aligned} \|f\|_{L^{p,q}}^q &= \sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} (s^{1/p} f^*(s))^q s^{-1}\, ds \\ &\leq \sum_{n\in\mathbb{Z}} (f^*(2^n))^q 2^{nq/p} 2^{-nq/p} \int_{2^n}^{2^{n+1}} s^{\frac{q}{p}-1}\, ds \\ &\leq \frac{p}{q}(2^{q/p} - 1) \sum_{n\in\mathbb{Z}} (f^*(2^n))^q 2^{nq/p} = \frac{p}{q}(2^{q/p} - 1)\|(c_n)\|_{\ell^q}^q \end{aligned}$$

and

$$\begin{aligned}\|f\|_{L^{p,q}}^q &= \sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} (s^{1/p} f^*(s))^q s^{-1}\, ds \\ &\geq \sum_{n\in\mathbb{Z}} 2^{q(n+1)/p} (f^*(2^{n+1}))^q 2^{-q(n+1)/p} \int_{2^n}^{2^{n+1}} s^{q/p-1}\, ds \\ &= \frac{p}{q}(1-2^{-q/p})\|(c_n)\|_{\ell^q}^q.\end{aligned}$$

■

Lemma A.3. *For any*

$$\frac{2p}{2p-3} \leq \tilde{q} < q = \frac{2p}{p-3}, \quad 3 < p < \infty,$$

we have

$$\sum_{k=0}^{\infty} r_k{}^{1-\frac{3}{p}-\frac{2}{\tilde{q}}} \left(\int_{-r_k^2}^0 \|u_3(\cdot,s)\|_{L^p}^{\tilde{q}}\, ds\right)^{\frac{1}{\tilde{q}}} \leq C\|u_3\|_{L_t^{q,1}L_x^p(-r_0,0;\mathbb{R}^3)}.$$

Proof. Let $f(s) = \|u_3(\cdot,s)\|_{L^p}$. By Lemma A.2, we know that

$$f = \sum_{\ell=0}^{+\infty} c_\ell f_\ell, \quad \|f\|_{L^{q,1}} \approx \sum_{\ell=0}^{\infty} |c_\ell|,$$

where

$$|f_\ell| \leq 2^{\frac{\ell}{q}}, \quad |D_\ell| = |\operatorname{supp} f_\ell| \approx 2^{-\ell}.$$

Then we have

$$\begin{aligned}\sum_{k=0}^{\infty} r_k{}^{1-\frac{3}{p}-\frac{2}{\tilde{q}}} \left(\int_{-r_k^2}^0 \|u_3(\cdot,s)\|_{L^p}^{\tilde{q}}\, ds\right)^{\frac{1}{\tilde{q}}} &= \sum_{k=0}^{\infty} r_k{}^{1-\frac{3}{p}-\frac{2}{\tilde{q}}} \left(\int_{I_k} |f|^{\tilde{q}}\, ds\right)^{\frac{1}{\tilde{q}}} \\ \leq \sum_{k=0}^{\infty} r_k{}^{1-\frac{3}{p}-\frac{2}{\tilde{q}}} \sum_\ell |c_\ell| 2^{\frac{\ell}{q}} |D_\ell \cap I_k|^{\frac{1}{\tilde{q}}} &\leq \sum_\ell |c_\ell| 2^{\frac{\ell}{q}} \sum_{k=0}^{\infty} r_k{}^{1-\frac{3}{p}-\frac{2}{\tilde{q}}} |D_\ell \cap I_k|^{\frac{1}{\tilde{q}}} \\ \leq \sum_\ell |c_\ell| 2^{\frac{\ell}{q}} \sum_{k=0}^{\ell/2} r_k{}^{1-\frac{3}{p}-\frac{2}{\tilde{q}}} |D_\ell \cap I_k|^{\frac{1}{\tilde{q}}} &+ \sum_\ell |c_\ell| 2^{\frac{\ell}{q}} \sum_{k=\ell/2}^{\infty} r_k{}^{1-\frac{3}{p}-\frac{2}{\tilde{q}}} |D_\ell \cap I_k|^{\frac{1}{\tilde{q}}},\end{aligned}$$

where $I_k = (-r_k^2, 0)$. On the other hand, we notice that for any $k \leq \ell/2$,

$$r_k{}^{1-\frac{3}{p}-\frac{2}{\tilde{q}}} |D_\ell \cap I_k|^{\frac{1}{\tilde{q}}} \leq C 2^{-\frac{\ell}{\tilde{q}}} 2^{-k(1-\frac{3}{p}-\frac{2}{\tilde{q}})},$$

and for any $\ell/2 \leq k < \infty$,

$$r_k{}^{1-\frac{3}{p}-\frac{2}{\tilde{q}}} |D_\ell \cap I_k|^{\frac{1}{\tilde{q}}} \leq C 2^{-\frac{2k}{\tilde{q}}} 2^{-k(1-\frac{3}{p}-\frac{2}{\tilde{q}})} = C 2^{-k(1-\frac{3}{p})}.$$

Then we have

$$\begin{aligned}&\sum_{k=0}^{\infty} {r_k}^{1-\frac{3}{p}-\frac{2}{q}} \Big(\int_{-r_k^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\tilde{q}}\,ds\Big)^{\frac{1}{\tilde{q}}}\\&\le \sum_{\ell=0}^{\infty} |c_\ell| 2^{\frac{\ell}{q}-\frac{\ell}{\tilde{q}}} \sum_{k=0}^{\ell/2} 2^{-k(1-\frac{3}{p}-\frac{2}{q})} + \sum_{\ell=0}^{\infty} |c_\ell| 2^{\frac{\ell}{q}} \sum_{k=\ell/2}^{\infty} 2^{-k(1-\frac{3}{p})},\end{aligned}$$

which along with the restriction on p, q implies that

$$\sum_{k=0}^{\infty} {r_k}^{1-\frac{3}{p}-\frac{2}{q}} \Big(\int_{-r_k^2}^{0} \|u_3(\cdot,s)\|_{L^p}^{\tilde{q}}\,ds\Big)^{\frac{1}{\tilde{q}}} \le C \sum_{l=0}^{\infty} |c_l| \le C\|u_3\|_{L_t^{q,1}L_x^p(-r_0,0;\mathbb{R}^3)}.$$

The proof is completed. ∎

Finally, let us recall the lemma about the harmonic functions in [6].

Lemma A.4. *Let* $0 < r \le R < \infty$ *and* $h\colon B'(2R) \times (-r,r) \to \mathbb{R}$ *be harmonic. Then for all* $0 < \rho \le \frac{r}{4}$ *and* $1 \le \ell \le p < \infty$*,*

$$\|h\|_{L^p(B'(R)\times(-\rho,\rho))}^p \le c\rho r^{2-3\frac{p}{\ell}} \|h\|_{L^\ell(B'(2R)\times(-r,r))}^p.$$

Funding. W. Wang was supported by NSFC under grant 12071054, 11671067 and the Fundamental Research Funds for the Central Universities. Z. Zhang is partially supported by NSF of China under grant 11425103.

References

[1] D. Albritton, Blow-up criteria for the Navier-Stokes equations in non-endpoint critical Besov spaces. *Anal. PDE* **11** (2018), no. 6, 1415–1456 Zbl 1392.35202 MR 3803715

[2] J. Bourgain and N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D. *J. Funct. Anal.* **255** (2008), no. 9, 2233–2247 Zbl 1161.35037 MR 2473255

[3] L. Caffarelli, R. Kohn, and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.* **35** (1982), no. 6, 771–831 Zbl 0509.35067 MR 673830

[4] M. Cannone, Y. Meyer, and F. Planchon, Solutions auto-similaires des équations de Navier-Stokes. In *Séminaire sur les équations aux dérivées partielles, 1993–1994*, Exp. No. VIII, École Polytechnique, Palaiseau, 1994 Zbl 0882.35090 MR 1300903

[5] C. Cao and E. S. Titi, Regularity criteria for the three-dimensional Navier-Stokes equations. *Indiana Univ. Math. J.* **57** (2008), no. 6, 2643–2661 Zbl 1159.35053 MR 2482994

[6] D. Chae and J. Wolf, On the Serrin-type condition on one velocity component for the Navier-Stokes equations. *Arch. Ration. Mech. Anal.* **240** (2021), no. 3, 1323–1347 Zbl 1472.35268 MR 4264947

[7] J.-Y. Chemin and P. Zhang, On the critical one component regularity for 3-D Navier-Stokes systems. *Ann. Sci. Éc. Norm. Supér. (4)* **49** (2016), no. 1, 131–167 Zbl 1342.35210 MR 3465978

[8] J.-Y. Chemin, P. Zhang, and Z. Zhang, On the critical one component regularity for 3-D Navier-Stokes system: general case. *Arch. Ration. Mech. Anal.* **224** (2017), no. 3, 871–905 Zbl 1371.35193 MR 3621812

[9] B.-Q. Dong and Z. Zhang, On the weak-strong uniqueness of Koch-Tataru's solution for the Navier-Stokes equations. *J. Differential Equations* **256** (2014), no. 7, 2406–2422 Zbl 1285.35064 MR 3160448

[10] L. Escauriaza, G. A. Seregin, and V. Šverák, $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness. *Russ. Math. Surveys* **58** (2003), no. 2, 211–250 *Uspekhi Mat. Nauk* **58** (2003), no. 2(350), 3–44 Zbl 1064.35134 MR 1992563

[11] I. Gallagher, D. Iftimie, and F. Planchon, Asymptotics and stability for global solutions to the Navier-Stokes equations. *Ann. Inst. Fourier (Grenoble)* **53** (2003), no. 5, 1387–1424 Zbl 1038.35054 MR 2032938

[12] I. Gallagher, G. S. Koch, and F. Planchon, A profile decomposition approach to the $L^\infty_t(L^3_x)$ Navier-Stokes regularity criterion. *Math. Ann.* **355** (2013), no. 4, 1527–1559 Zbl 1291.35180 MR 3037023

[13] I. Gallagher, G. S. Koch, and F. Planchon, Blow-up of critical Besov norms at a potential Navier-Stokes singularity. *Comm. Math. Phys.* **343** (2016), no. 1, 39–82 Zbl 1342.35217 MR 3475661

[14] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. *J. Differential Equations* **62** (1986), no. 2, 186–212 Zbl 0577.35058 MR 833416

[15] S. Gustafson, K. Kang, and T.-P. Tsai, Interior regularity criteria for suitable weak solutions of the Navier-Stokes equations. *Comm. Math. Phys.* **273** (2007), no. 1, 161–176 Zbl 1126.35042 MR 2308753

[16] B. Han, Z. Lei, D. Li, and N. Zhao, Sharp one component regularity for Navier-Stokes. *Arch. Ration. Mech. Anal.* **231** (2019), no. 2, 939–970 Zbl 1412.76021 MR 3900817

[17] T. Kato, Strong L^p-solutions of the Navier-Stokes equation in $\mathbf{R}^m$, with applications to weak solutions. *Math. Z.* **187** (1984), no. 4, 471–480 Zbl 0545.35073 MR 760047

[18] C. E. Kenig and G. S. Koch, An alternative approach to regularity for the Navier-Stokes equations in critical spaces. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **28** (2011), no. 2, 159–187 Zbl 1220.35119 MR 2784068

[19] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations. *Adv. Math.* **157** (2001), no. 1, 22–35 Zbl 0972.35084 MR 1808843

[20] I. Kukavica, On partial regularity for the Navier-Stokes equations. *Discrete Contin. Dyn. Syst.* **21** (2008), no. 3, 717–728 Zbl 1147.35071 MR 2399434

[21] I. Kukavica and M. Ziane, One component regularity for the Navier-Stokes equations. *Nonlinearity* **19** (2006), no. 2, 453–469 Zbl 1149.35069 MR 2199398

[22] O. A. Ladyzhenskaya and G. A. Seregin, On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations. *J. Math. Fluid Mech.* **1** (1999), no. 4, 356–387 Zbl 0954.35129 MR 1738171

[23] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* **63** (1934), no. 1, 193–248 Zbl 60.0726.05 MR 1555394

[24] F. Lin, A new proof of the Caffarelli-Kohn-Nirenberg theorem. *Comm. Pure Appl. Math.* **51** (1998), no. 3, 241–257 Zbl 0958.35102 MR 1488514

[25] J. Neustupa, A contribution to the theory of regularity of a weak solution to the Navier-Stokes equations via one component of velocity and other related quantities. *J. Math. Fluid Mech.* **20** (2018), no. 3, 1249–1267 Zbl 1401.35243 MR 3841982

[26] J. Neustupa, A. Novotný, and P. Penel, An interior regularity of a weak solution to the Navier-Stokes equations in dependence on one component of velocity. In *Topics in mathematical fluid mechanics*, pp. 163–183, Quad. Mat. 10, Aracne, Rome 2002 Zbl 1050.35073 MR 2051774

[27] J. Neustupa and P. Penel, Regularity of a suitable weak solution to the Navier-Stokes equations as a consequence of regularity of one velocity component. In *Applied nonlinear analysis*, pp. 391–402, Kluwer/Plenum, New York, 1999 Zbl 0953.35113 MR 1727461

[28] V. Scheffer, Partial regularity of solutions to the Navier-Stokes equations. *Pacific J. Math.* **66** (1976), no. 2, 535–552 Zbl 0325.35064 MR 454426

[29] J. Serrin, The initial value problem for the Navier-Stokes equations. In *Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962)*, pp. 69–98, University of Wisconsin Press, Madison, WI, 1963 Zbl 0115.08502 MR 0150444

[30] M. Struwe, On partial regularity results for the Navier-Stokes equations. *Comm. Pure Appl. Math.* **41** (1988), no. 4, 437–458 Zbl 0632.76034 MR 933230

[31] T. Tao, Quantitative bounds for critically bounded solutions to the Navier-Stokes equations. In *Nine mathematical challenges—an elucidation*, 149–193, Proc. Sympos. Pure Math. 104, American Mathematical Society, Providence, RI, 2021 MR 4337421

[32] G. Tian and Z. Xin, Gradient estimation on Navier-Stokes equations. *Comm. Anal. Geom.* **7** (1999), no. 2, 221–257 Zbl 0939.35139 MR 1685610

[33] A. F. Vasseur, A new proof of partial regularity of solutions to Navier-Stokes equations. *NoDEA Nonlinear Differential Equations Appl.* **14** (2007), no. 5-6, 753–785 Zbl 1142.35066 MR 2374209

[34] W. Wang, L. Zhang, and Z. Zhang, On the interior regularity criteria of the 3-D Navier-Stokes equations involving two velocity components. *Discrete Contin. Dyn. Syst.* **38** (2018), no. 5, 2609–2627 Zbl 1397.35184 MR 3809052

[35] W. Wang and Z. Zhang, On the interior regularity criteria and the number of singular points to the Navier-Stokes equations. *J. Anal. Math.* **123** (2014), 139–170 Zbl 1304.35510 MR 3233577

[36] W. Wang and Z. Zhang, Blow-up of critical norms for the 3-D Navier-Stokes equations. *Sci. China Math.* **60** (2017), no. 4, 637–650 Zbl 1387.35471 MR 3629487

[37] Y. Zhou and M. Pokorný, On the regularity of the solutions of the Navier-Stokes equations via one velocity component. *Nonlinearity* **23** (2010), no. 5, 1097–1107 Zbl 1190.35179 MR 2630092

Received 9 March 2021; revised 21 December 2021; accepted 10 August 2022.

Wendong Wang
School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, P. R. China; wendong@dlut.edu.cn

Di Wu
School of Mathematics, South China University of Technology, Guangzhou, 510640, P. R. China; wudi@scut.edu.cn

Zhifei Zhang
School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China; zfzhang@math.pku.edu.cn