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Solutions to the non-cutoff Boltzmann equation in the grazing limit

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Abstract. It is known that in the parameter range $-2 \le \gamma < -2s$, a spectral gap does not exist for the linearized Boltzmann operator without cutoff, but it does for the linearized Landau operator. This paper is devoted to the understanding of the formation of a spectral gap in this range through the grazing limit. Precisely, we study the Cauchy problems of these two classical collisional kinetic equations around global Maxwellians in a torus and establish the following results which are uniform in the vanishing grazing parameter ε: (i) spectral-gap-type estimates for the collision operators; (ii) global existence of small-amplitude solutions for initial data with low regularity; (iii) propagation of regularity in both space and velocity variables, as well as velocity moments without smallness; (iv) global-in-time asymptotics of the Boltzmann solution toward the Landau solution at the rate $O(\varepsilon)$; (v) continuous transition of decay structure of the Boltzmann operator to the Landau operator. In particular, the result in part (v) captures the uniform-in-ε transition of intrinsic optimal time-decay structures of solutions and reveals how the spectrum of the linearized non-cutoff Boltzmann equation in the mentioned parameter range changes continuously under the grazing limit.

Contents

1. Introduction

In the paper we are concerned with the Cauchy problem on both the Boltzmann and Landau equations in a torus. It is fundamental to study the global existence and large time behavior of solutions in the mathematical theory of these two classical collisional

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kinetic equations and there has been extensive work on it from different frameworks, e.g. [2, 8, 9, 12, 13, 17, 19, 20, 28, 31, 33–37]. The Boltzmann and Landau equations are also closely connected through the so-called grazing collision limit; cf. [3, 5, 10, 11, 16]. Hence, it is interesting to construct uniform global solutions of the Boltzmann equation under the grazing limit so as to develop a unified framework of well-posedness theory for both the Boltzmann and Landau equations. Recently, the second author of this paper, together with his collaborators, has produced a series of works [21–26] on a related topic. In the current work we focus on the uniform grazing limit to the Landau equation from the non-cutoff Boltzmann in a prescribed range of intermolecular interaction potentials $-2 \le \gamma < -2s$. Specifically, our main purpose is to reveal the continuous transition of decay structure of the Boltzmann collision operator in such a range from sub-exponential time decay to exponential time decay in the limit process. We emphasize that this problem is related to the famous spectral gap problem, that is, the linearized non-cutoff Boltzmann operator with $\gamma + 2s < 0$ does not have any spectral gap but the linearized Landau operator with $\gamma + 2 \ge 0$ does.

1.1. Boltzmann and Landau equations in the perturbation framework

For the setting of the study, we first recall the equations. The Cauchy problem on the Boltzmann equation reads

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q^B(F, F), \quad t > 0,\ x \in \mathbb{T}^3,\ v \in \mathbb{R}^3, \\ F|_{t=0} = F_0, \end{cases} \tag{1.1}$$

where $F(t, x, v) \ge 0$ is the density function of particles with velocity $v \in \mathbb{R}^3$ at time $t \ge 0$ and position $x \in \mathbb{T}^3 := [-\pi, \pi]^3$. The Boltzmann collision operator is

$$Q^B(g, h)(v) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)(g'_* h' - g_* h)\, d\sigma\, dv_*. \tag{1.2}$$

Here we have used the standard shorthand notation $h = h(v)$, $g_* = g(v_*)$, $h' = h(v')$ and $g'_* = g(v'_*)$, where v' and v'_* are given by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \quad \sigma \in \mathbb{S}^2.$$

On the other hand, with the same initial data as in (1.1), the Cauchy problem on the Landau equation reads

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q^L(F, F), \quad t > 0,\ x \in \mathbb{T}^3,\ v \in \mathbb{R}^3, \\ F|_{t=0} = F_0, \end{cases} \tag{1.3}$$

where the Landau operator $Q^L(g, h)$ is given by

$$Q^L(g, h)(v) := \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} a(v - v_*)[g(v_*)\nabla_v h(v) - \nabla_{v_*} g(v_*) h(v)]\, dv_* \right\}.$$

Here,

$$a(z) = \Lambda |z|^{\gamma+2}\Big(I_3 - \frac{z \otimes z}{|z|^2}\Big), \tag{1.4}$$

where I_3 is the 3×3 identity matrix and Λ is a positive constant.

In the perturbation framework, that is, by setting $F(t,x,v) = \mu + \mu^{\frac{1}{2}} f(t,x,v)$ with the normalized global Maxwellians $\mu = \mu(v) := (2\pi)^{-3/2} e^{-|v|^2/2}$, the Cauchy problems (1.1) and (1.3) are reduced respectively to

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathcal{L}^B f = \Gamma^B(f,f), & t > 0,\ x \in \mathbb{T}^3,\ v \in \mathbb{R}^3, \\ f|_{t=0} = f_0, \end{cases} \tag{1.5}$$

and

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathcal{L}^L f = \Gamma^L(f,f), & t > 0,\ x \in \mathbb{T}^3,\ v \in \mathbb{R}^3, \\ f|_{t=0} = f_0. \end{cases} \tag{1.6}$$

Here, the linearized Boltzmann operator $\mathcal{L}^B$ and the nonlinear term Γ^B are given respectively by

$$\Gamma^B(g,h) := \mu^{-\frac{1}{2}} Q^B(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), \quad \mathcal{L}^B g := -\Gamma^B(\mu^{\frac{1}{2}}, g) - \Gamma^B(g, \mu^{\frac{1}{2}}).$$

Similarly, the linearized Landau operator $\mathcal{L}^L$ and the nonlinear term Γ^L are

$$\Gamma^L(g,h) := \mu^{-\frac{1}{2}} Q^L(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), \quad \mathcal{L}^L g := -\Gamma^L(\mu^{\frac{1}{2}}, g) - \Gamma^L(g, \mu^{\frac{1}{2}}).$$

In what follows, we impose the following assumptions on the non-cutoff Boltzmann kernel B in (1.2):

(A1) The Boltzmann kernel B takes the form

$$B(v - v_*, \sigma) = C_B |v - v_*|^{\gamma} b(\cos\theta), \quad -3 < \gamma \le 1,\ C_B > 0,$$

where $\cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma$.

(A2) The angular function $b(\cdot)$ is singular in the sense that

$$C_b^{-1} \sin^{-2-2s}\frac{\theta}{2} \le b(\cos\theta) \le C_b \sin^{-2-2s}\frac{\theta}{2}, \quad 0 < s < 1,\ C_b \ge 1.$$

(A3) The parameter γ and s satisfy $\gamma + 2s > -1$.

(A4) Without loss of generality, we assume that $B(v - v_*, \sigma)$ is supported in the interval $0 \le \theta \le \pi/2$, i.e. $\frac{v - v_*}{|v - v_*|} \cdot \sigma \ge 0$, for otherwise B can be replaced by its symmetrized form:

$$\overline{B}(v - v_*, \sigma) = |v - v_*|^{\gamma} \Big\{ b\Big(\frac{v - v_*}{|v - v_*|} \cdot \sigma\Big) + b\Big(\frac{v - v_*}{|v - v_*|} \cdot (-\sigma)\Big)\Big\} \mathbb{1}_{\frac{v - v_*}{|v - v_*|} \cdot \sigma \ge 0},$$

where $\mathbb{1}_A$ is the characteristic function of a set A.

Remark 1.1. The above assumptions on $B(v - v_*, \sigma)$ are motivated by the inverse power law model with potential $U(r) = r^{-p}$, $p > 1$ where $s = \frac{1}{p}$ and $\gamma = \frac{p-4}{p}$ satisfy $\gamma + 4s = 1$.

1.1.1. Mathematical theory on the grazing collisions limit of the Boltzmann equation to the Landau equation. In this subsection we will briefly review existing mathematical work on the grazing limit of the Boltzmann equation to the Landau equation.

Formally, the grazing limit means that when one scales the function of the deviation angle to be concentrated on the collisions that become grazing, the corresponding Boltzmann equation leads to the Landau equation in the limit. Precisely, set

$$\varepsilon = \sin(\theta_{\max}/2), \quad b^\varepsilon(\cos\theta) := (1-s)\varepsilon^{2s-2}\sin^{-2-2s}(\theta/2)\mathbb{1}_{\sin\frac{\theta}{2}\le\varepsilon}, \tag{1.7}$$

where $\theta_{\max}$ is the maximum deviation angle such that collisions happen only when $\theta \le \theta_{\max}$. Then the rescaled Boltzmann kernel $B^\varepsilon(v-v_*,\sigma)$ and the corresponding collision operator Q^ε are given respectively as

$$\begin{aligned} B^\varepsilon(v-v_*,\sigma) &= |v-v_*|^\gamma b^\varepsilon(\cos\theta) \\ &= |v-v_*|^\gamma(1-s)\varepsilon^{2s-2}\sin^{-2-2s}(\theta/2)\mathbb{1}_{\sin\frac{\theta}{2}\le\varepsilon} \end{aligned} \tag{1.8}$$

and

$$Q^\varepsilon(g,h)(v) := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B^\varepsilon(v-v_*,\sigma)(g'_*h' - g_*h)\,d\sigma\,dv_*.$$

At the operator level, it is known that the following asymptotic formula between Q^ε and Q^L holds for suitably smooth functions:

$$\|Q^\varepsilon(f,f) - Q^L(f,f)\|_{L^2} \lesssim \varepsilon\|f\|_{H^3_{\gamma+10}}\|f\|_{H^5_{\gamma+10}}.$$

We refer to [11,21,41] for the details.

Weak convergence of the limit. In the spatially homogeneous case, Arsen'ev–Buryak [5] studied the convergence of weak solutions of the Boltzmann equation to those of the Landau equation under certain assumptions on the Boltzmann kernel. However, the kernel considered in [5] does not include the inverse power law potential. Goudon [16] proved the convergence of weak solutions for the inverse power law potential in the case $\gamma \ge -2$ and $s \le \frac{1}{4}$. Note that this range covers the potential $U(r) = r^{-p}$ for $p \ge \frac{4}{3}$ only. Villani [38] used the symmetry of spherical integrals and introduced a new definition of weak solutions that enables him to show the convergence of weak solutions of the Boltzmann equation to those of the Landau equation by only assuming $\gamma > -4$. Note that the results in [38] hold for the Coulomb potential with $p = 1$.

Based on the renormalized solution theory [38] and the entropy dissipation estimate obtained in [1], an important contribution was made by Alexandre–Villani [3] giving the first study of the problem in the spatially inhomogeneous setting. Thanks to the general setting of weak solutions, the situation in [3] covers a board class of potentials, including the Coulomb interaction.

Classical convergence of the limit. In the spatially homogeneous case, the second author in [21] showed the convergence of (1.9) to (1.3) in weighted Sobolev spaces with an explicit rate. More precisely, supposing that F^ε and F are solutions to (1.9) and (1.3) respectively, it was proved in [21] that

$$\sup_{0\le t\le T} |F^\varepsilon(t) - F(t)|_{H^N} \le \varepsilon C(T, |F_0|_{L^1_{q(N,l)}}, |F_0|_{H_l^{N+3}})$$

for some $T > 0$. Here, T can be extended to ∞ for $\gamma \ge -2s$, whereas $T < \infty$ is required for $-3 < \gamma < -2s$.

In the present work, we are interested in the inverse power law model when the parameters γ and s satisfy $-2 \le \gamma < -2s$, because in this setting the linearized Boltzmann collision operator $\mathscr{L}^B$ does not have a spectral gap while the linearized Landau operator does. Correspondingly, this property induces that for the solutions to the nonlinear equations (1.5) and (1.6), one can derive a sub-exponential time-decay rate for the Boltzmann equation but an exponential decay rate for the Landau equation. As we mentioned above, the grazing collision limit bridges these two equations in the limit process. It is then natural to ask whether one can have a unified framework to show that in the vanishing-in-ε limit process the spectral gap is continuously transferred from nonexistence to existence. Unfortunately, so far we have no idea how to directly answer this question at the level of functional analysis. Thus we resort to finding a continuous transition from sub-exponential structure to exponential structure by studying the time decay of solutions. Mathematically, we are concerned with the rescaled Boltzmann equation:

$$\begin{cases} \partial_t F + v\cdot\nabla_x F = Q^\varepsilon(F,F), \quad t>0,\ x\in\mathbb{T}^3,\ v\in\mathbb{R}^3, \\ F|_{t=0} = F_0, \end{cases} \tag{1.9}$$

as well as the associated Landau equation (1.3) with the same initial data in the limit $\varepsilon \to 0$ in the above perturbation framework.

1.1.2. Mathematical theory of Landau's derivation. In 1936, Landau derived an effective kinetic equation, named the Landau equation (or Fokker–Planck–Landau equation) nowadays, for the charged particles governed by the Coulomb potential in the weak coupling regime. Landau's formal derivation can be found in many books; see [27, 29] for instance. In this situation, it holds that $\gamma = -3$ in (1.4); we refer to [10] for the convergence of the Boltzmann operator to the Landau operator. At the solution level for the limit from (1.9) to (1.3), we refer readers to [3] for convergence of weak solutions, as well as [22] for convergence of classical solutions with an explicit rate $|\ln\varepsilon|^{-1}$. We remark that the Boltzmann kernel in [22] is taken as

$$\widetilde{B}^\varepsilon(v-v_*,\sigma) := |\ln\varepsilon|^{-1}|v-v_*|^{-3}\sin^{-4}\frac{\theta}{2}\mathbb{1}_{\sin\frac{\theta}{2}\ge\varepsilon},$$

and the result holds only locally in time. Very recently, in the near equilibrium framework, in [25] the second and fourth authors proved the global-in-time convergence of solutions of (1.5) with the singular kernel $\widetilde{B}^\varepsilon$ to solutions of (1.6).

1.2. Mathematical setting of the problems

Let us give a detailed mathematical description of the problems to be discussed in this paper. We begin with the function spaces.

Function spaces. We refer to [2, 17] and [18] on global well-posedness theories for the Boltzmann equation without angular cutoff and the Landau equation in weighted Sobolev spaces, respectively. In this paper, we will follow the low regularity function space $L^1_k L^\infty_T L^2$ introduced in [14] to consider the limit, where L^1_k corresponds to the Weiner algebra over a torus. More precisely, the function space is equipped with the norm

$$\|f\|_{L^1_k L^\infty_T L^2} := \sum_{k\in\mathbb{Z}^3} \sup_{0\le t\le T} |\hat{f}(t,k,\cdot)|_{L^2}.$$

Here, $\hat{f}$ is the Fourier transform with respect to x. The $|\cdot|_{L^2}$ is taken with respect to the variable v. Note that in terms of the regularity of the x-variable on a torus, it holds at the formal level that $H^{3/2+\delta}_x \hookrightarrow L^1_k \hookrightarrow L^\infty_x$. To the best of our knowledge, $L^1_k L^\infty_T L^2$ seems to be the largest space in which global well-posedness theory for both the non-cutoff Boltzmann equation and the Landau equation can be established via the direct energy method, in contrast with the recent substantial progress in [4] for constructing the $L^2 \cap L^\infty$ solutions via the De Giorgi argument.

Some well-known facts. Now we list some basic facts on the large time behavior of solutions to both the Boltzmann and the Landau equations in the space $L^1_k L^\infty_T L^2$. Let f^L be the solution to the Cauchy problem (1.6) on the Landau equation. When $\gamma \ge -2$, under a suitable smallness assumption on f_0, it holds (see [14, Theorem 2.1]) that

$$\|f^L(t)\|_{L^1_k L^2} \lesssim e^{-\lambda t}\|f_0\|_{L^1_k L^2} \lesssim e^{-\lambda t}. \tag{1.10}$$

See (1.21) for the precise definition of the norm $\|\cdot\|_{L^1_k L^2}$. The above time-decay property is consistent with the fact that the linearized Landau operator $\mathcal{L}^L$ has a spectral gap if and only if $\gamma \ge -2$. On the other hand, let f^B be the solution to the Cauchy problem (1.5) on the Boltzmann equation. When $-3 < \gamma < -2s$, under a suitable smallness assumption on f_0, it holds (see [14, Theorem 2.1]) that

$$\|f^B(t)\|_{L^1_k L^2} \lesssim e^{-\lambda t^\kappa}\|e^{q\langle v\rangle} f_0\|_{L^1_k L^2} \lesssim e^{-\lambda t^\kappa}, \tag{1.11}$$

where $\kappa = \frac{1}{1+|\gamma+2s|}$, $q > 0$, and $\langle v\rangle = \sqrt{1+|v|^2}$. The time-decay rate in (1.11) is also consistent with the spectrum structure of the linearized Boltzmann operator $\mathcal{L}^B$ in the soft potential regime $\gamma < -2s$ for which there is no spectral gap; cf. [40] and the references therein. To the best of our knowledge, (1.10) and (1.11) provide the optimal decay rate estimates in the existing literatures.

Spectral estimates of the linearized collision operators. The spectral gap estimates for the linearized operators play an important role in the global-in-time well-posedness for the collisional kinetic equation in the perturbation framework. We recall a pioneering work by Wang Chang and Uhlenbeck (see [39]) on the Maxwell molecule model $\gamma = 0$. Let us write $\mathcal{L}^{B,\gamma}$ to address the fact that the linearized collision operator $\mathcal{L}^B$ in fact depends on the parameter γ. In [39], the authors gave explicit formulas for all the eigenvalues and the associated eigenfunctions to $\mathcal{L}^{B,0}$. As a direct consequence, it implies the so-called spectral gap estimate. To be precise, the kernel space of $\mathcal{L}^{B,\gamma}$ and $\mathcal{L}^L$ is defined by

$$\ker := \operatorname{span}\{\mu^{\frac{1}{2}}, \mu^{\frac{1}{2}} v_1, \mu^{\frac{1}{2}} v_2, \mu^{\frac{1}{2}} v_3, \mu^{\frac{1}{2}} |v|^2\}. \tag{1.12}$$

Usually, ker is called the macro-space, and $\ker^\perp$ is called micro-space. Wang Chang–Uhlenbeck proved that for any $f \in \ker^\perp$,

$$\langle \mathcal{L}^{B,0} f, f \rangle \geq \lambda_e |f|^2_{L^2}, \tag{1.13}$$

where λ_e is the first (smallest) nonzero eigenvalue of $\mathcal{L}^{B,0}$ given by

$$\lambda_e := \int_0^{\pi/2} b(\cos\theta) \sin\theta (1 - \cos\theta)\, d\theta. \tag{1.14}$$

Later on, authors in [6,30,32] proved that the spectral gap estimates can be generalized to the other potentials. It was asserted that there exist two constant C_γ and C_b such that for any $f \in \ker^\perp$,

$$\langle \mathcal{L}^{B,\gamma} f, f \rangle \geq C_\gamma C_b |f|^2{}_{L^2_{\gamma/2}}, \tag{1.15}$$

where

$$C_b := \inf_{\sigma_1, \sigma_2 \in \mathbb{S}^2} \int_{\mathbb{S}^2} \min\{b(\sigma_1 \cdot \sigma_3), b(\sigma_2 \cdot \sigma_3)\}\, d\sigma_3.$$

For the angular function b^ε defined in (1.8), one may check that $\lambda_e \sim 1$ while $C_{b^\varepsilon} \to 0$ as ε goes to zero. This shows that estimate (1.13) is robust in the grazing collisions limit process and thus can be thought of as a unified formula for both Boltzmann and Landau collision operators. It also requires us to establish Wang Chang–Uhlenbeck-type estimates for the soft potentials.

Statement of the results. It is obvious that in the regime $-2 \leq \gamma < -2s$ the time asymptotic behaviors of the solutions described in (1.10) and (1.11) are different by noticing that the latter is at the sub-exponential decay rate ($0 < \kappa < 1$) while the former is at the exponential decay rate. Since the grazing collision limit of the Boltzmann equation yields the Landau equation, it is interesting to find out whether the transition from sub-exponential decay structure to exponential decay structure occurs in a continuous way through the limit. Furthermore, one may ask whether one can provide a detailed mathematical description of the time-decay structures in the limit process. To answer the above questions, we first rewrite the rescaled Boltzmann equation (1.9) by letting $F = \mu + \mu^{\frac{1}{2}} f$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathcal{L}^\varepsilon f = \Gamma^\varepsilon(f, f), & t > 0,\ x \in \mathbb{T}^3,\ v \in \mathbb{R}^3, \\ f|_{t=0} = f_0. \end{cases} \tag{1.16}$$

Here, the linearized Boltzmann operator $\mathscr{L}^\varepsilon$ and the nonlinear term Γ^ε are defined by

$$\Gamma^\varepsilon(g,h) := \mu^{-\frac{1}{2}} Q^\varepsilon(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), \quad \mathscr{L}^\varepsilon g := -\Gamma^\varepsilon(\mu^{\frac{1}{2}}, g) - \Gamma^\varepsilon(g, \mu^{\frac{1}{2}}). \tag{1.17}$$

From now on, we assume without of loss of generality the initial perturbation f_0 for (1.16) and (1.6) has zero total mass, momentum, and energy:

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} \mu^{\frac{1}{2}} f_0 \phi \, dx \, dv = 0, \quad \phi(v) = 1, v_1, v_2, v_3, |v|^2.$$

Now the problems to be discussed are the global well-posedness of (1.16), the uniform-in-time asymptotic rate in ε between solutions to (1.16) and (1.6), and the transition from sub-exponential decay (cf. (1.11)) of (1.16) to exponential decay (cf. (1.10)) of (1.6) as ε goes to 0.

1.3. Main results

Before stating the main results in this paper, we first give adequate notation.

- The bracket $\langle\cdot\rangle$ is defined by $\langle v\rangle := (1+|v|^2)^{\frac{1}{2}}$.
- For $N \in \mathbb{N}$, $l \in \mathbb{R}$, and a function $f(v)$ on $\mathbb{R}^3$, set

$$\begin{aligned} |f|_{H^N_l} &:= \sum_{|\beta|\le N} |\langle v\rangle^l \partial_\beta f|_{L^2}, & |f|_{\dot{H}^N_l} &:= \sum_{|\beta|= N} |\langle v\rangle^l \partial_\beta f|_{L^2}, \\ |f|_{L^2_l} &:= |f|_{H^0_l}, & |f|_{L^2} &:= |f|_{L^2_0}. \end{aligned} \tag{1.18}$$

- With the weighted norm $|\cdot|_{\varepsilon,l}$ (from the coercivity estimate for $\mathscr{L}^\varepsilon$ in Theorem 2.1) defined in (1.32), for $N \in \mathbb{N}$, $l \in \mathbb{R}$, and a function $f(v)$ on $\mathbb{R}^3$, set

$$\begin{aligned} |f|_{H^N_{\varepsilon,l}} &:= \sum_{|\beta|\le N} |\partial_\beta f|_{\varepsilon,l}, \quad |f|_{\dot{H}^N_{\varepsilon,l}} := \sum_{|\beta|= N} |\partial_\beta f|_{\varepsilon,l}, \\ |f|_{L^2_{\varepsilon,l}} &:= |f|_{H^0_{\varepsilon,l}} = |f|_{\varepsilon,l}. \end{aligned} \tag{1.19}$$

- For a function $f(t,x,v)$ on $[0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3$, and a norm or seminorm X (defined in (1.18) or (1.19)) on the velocity variable v, we define for $T>0$ and $m\ge 0$ that

$$\begin{aligned} \|f\|_{L^1_{k,m}L^\infty_T X} &:= \sum_{k\in\mathbb{Z}^3} \langle k\rangle^m \sup_{0\le t\le T} |\hat{f}(t,k,\cdot)|_X, \\ \|f\|_{L^1_{k,m}L^2_T X} &:= \sum_{k\in\mathbb{Z}^3} \langle k\rangle^m \Big(\int_0^T |\hat{f}(t,k,\cdot)|_X^2 \, dt\Big)^{\frac{1}{2}}. \end{aligned} \tag{1.20}$$

Here, $\hat{f}$ is the Fourier transform with respect to x. When $m=0$, denote

$$\|f\|_{L^1_k L^\infty_T X} := \|f\|_{L^1_{k,0} L^\infty_T X}, \quad \|f\|_{L^1_k L^2_T X} := \|f\|_{L^1_{k,0} L^2_T X}.$$

Remark 1.2. In (1.20), the notation $L^1_{k,m}$ represents the discrete measure $\langle k\rangle^m$ on the frequency mode $k \in \mathbb{Z}^3$. More precisely, for a function $f(x)$ on $\mathbb{T}^3$, the norm $L^1_{k,m}$ is defined by

$$\|f\|_{L^1_{k,m}} := \sum_{k\in\mathbb{Z}^3} \langle k\rangle^m |\hat{f}(k)|.$$

Note that after taking the summation, the value $\|f\|_{L^1_{k,m}}$ depends on m and f but not on k. Here, the symbol k in the norm is used to emphasize that the norm is taken in the frequency space $k \in \mathbb{Z}^3$ rather than phase space $x \in \mathbb{T}^3$.

- For a function $f(x, v)$ on $\mathbb{T}^3 \times \mathbb{R}^3$, and a norm or seminorm X (defined in (1.18) or (1.19)) on the velocity variable v, define for $m \geq 0$ that

$$\|f\|_{L^1_{k,m}X} := \sum_{k\in\mathbb{Z}^3} \langle k\rangle^m |\hat{f}(k,\cdot)|_X. \tag{1.21}$$

  When $m = 0$, $\|f\|_{L^1_k X} := \|f\|_{L^1_{k,0}X}$.
- Let $n \in \mathbb{N}$ and $m, l \geq 0$. For brevity of notation we denote the energy and dissipation functionals for a function $f(t, x, v)$ on $[0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3$ as

$$\begin{aligned} E_T(f; m, n, l) &:= \sum_{j=0}^{n} \|f\|_{L^1_{k,m+j} L^\infty_T \dot{H}^{n-j}_{l-j(\gamma+2s)}}, \\ D^\varepsilon_T(f; m, n, l) &:= \sum_{j=0}^{n} \|f\|_{L^1_{k,m+j} L^2_T \dot{H}^{n-j}_{\varepsilon,\gamma/2+l-j(\gamma+2s)}}, \end{aligned} \tag{1.22}$$

  respectively, and the norm on the initial data $f_0(x, v)$ as

$$\|f_0\|_{m,n,l} := \sum_{j=0}^{n} \|f_0\|_{L^1_{k,m+j} H^{n-j}_{l-j(\gamma+2s)}}. \tag{1.23}$$

Remark 1.3. Note that in (1.22), the maximum order of regularity for the variable x is $m + n$, while the maximum order of regularity for the variable v is n. The maximum order of the mixed regularity for x, v is $m + n$. The minimum order of the weight for the variable v is l when there is an nth-order derivative on the variable v. The weight increases by $-(\gamma + 2s)$ as the order of the v-derivative decreases by 1.

We begin with Wang Chang–Uhlenbeck-type spectral gap estimates for the linearized Boltzmann collision operator.

Theorem 1.1. *Recall* (1.18). *Let* $-3 < \gamma \leq 0$, $0 < s < 1$. *Let* B *satisfy assumptions* (A1)–(A4). *Suppose that* $\mathcal{L}^B$ *is the linearized collision operator associated with* B. *Then there exists a constant* $C(\gamma, s, \lambda_e)$ *depending only on* γ, s, *and* λ_e *(see* (1.14)) *such that if* $f \in \ker^\perp$,

$$\langle \mathcal{L}^B f, f\rangle \geq C(\gamma, s, \lambda_e)|f|^2_{L^2_{\gamma/2}}.$$

Proof. Let $f \in \ker^{\perp}$ and ε_0 be the universal constant in Theorem 2.2. We denote by $\mathscr{L}^{B}_{\geq\varepsilon_0}$ and $\mathscr{L}^{B}_{\leq\varepsilon_0}$ the linearized operator associated to the Boltzmann kernel $B\mathbb{1}_{\sin\frac{\theta}{2}\geq\varepsilon_0}$ and $B\mathbb{1}_{\sin\frac{\theta}{2}\leq\varepsilon_0}$ respectively.

For $\mathscr{L}^{B}_{\geq\varepsilon_0}$, by (1.15), we have

$$\langle \mathscr{L}^{B}_{\geq\varepsilon_0} f, f\rangle \geq C_\gamma C_{b_{\geq\varepsilon_0}} |f|^2_{L^2_{\gamma/2}},$$

where $b_{\geq\varepsilon_0} := b\mathbb{1}_{\sin\frac{\theta}{2}\geq\varepsilon_0}$.

For $\mathscr{L}^{B}_{\leq\varepsilon_0}$, thanks to Theorem 2.2 and Remark 2.3, we get

$$\langle \mathscr{L}^{B}_{\leq\varepsilon_0} f, f\rangle \geq C\left(\gamma, s, \int_{\mathbb{S}^2} \mathbb{1}_{\sin\frac{\theta}{2}\leq\varepsilon_0} b(\cos\theta)\sin^2\frac{\theta}{2}\,d\sigma\right)|f|^2_{L^2_{\gamma/2}}.$$

Combining these two estimates, we get the desired result and then complete the proof. ∎

Remark 1.4. In comparison with the previous work [6, 30, 32], we have highlighted dependence of the estimate on λ_e in Theorem 1.1. As a direct application, we successfully extend Wang Chang–Uhlenbeck's work to the inverse power law interactions, that is, the kernel B verifies assumptions (A1), (A2), (A4) and the condition $\gamma + 4s = 1$.

With the notation given above, we present the result concerning the global well-posedness, propagation of regularity of solutions, and the asymptotic rate in terms of ε under the grazing limit for the Cauchy problems (1.16) and (1.6) on the Boltzmann and the Landau equation, respectively.

Theorem 1.2. *Let* $-3 < \gamma \leq 0$, $\frac{1}{2} < s < 1$, *and* $\gamma + 2s > -1$. *There exist* $\varepsilon_0, \delta_0 > 0$ *such that if* $0 < \varepsilon \leq \varepsilon_0$, $\mu + \mu^{\frac{1}{2}} f_0 \geq 0$, *and* $\|f_0\|_{L^1_k L^2} \leq \delta_0$, *then the following statements hold:*

(1) (Global well-posedness). *Recall* (1.18), (1.19), (1.20), *and* (1.21). *The Cauchy problem* (1.16) *for the non-cutoff Boltzmann equation admits a unique global solution* f^ε *with* $\mu + \mu^{\frac{1}{2}} f^\varepsilon \geq 0$ *and*

$$\|f^\varepsilon\|_{L^1_k L^\infty_T L^2} + \|f^\varepsilon\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}} \lesssim \|f_0\|_{L^1_k L^2} \tag{1.24}$$

for any $T \geq 0$. *As a result, by passing to the limit* $\varepsilon \to 0$, *the Cauchy problem* (1.6) *with the same initial data* f_0 *for the Landau equation admits a unique global solution* f^L *satisfying* $\mu + \mu^{\frac{1}{2}} f^L \geq 0$ *and*

$$\|f^L\|_{L^1_k L^\infty_T L^2} + \|f^L\|_{L^1_k L^2_T L^2_{0,\gamma/2}} \lesssim \|f_0\|_{L^1_k L^2} \tag{1.25}$$

for any $T \geq 0$.

(2) (Propagation of regularity and velocity moment). *Recall* (1.22) *and* (1.23). *Let* $n \in \mathbb{N}$ *and* $m, l \geq 0$. *There is a constant* $\delta_{m,n,l}$ *with* $0 < \delta_{m,n,l} \leq \delta_0$ *and a polynomial* P_n *with* $P_n(0) = 0$ *such that if the initial data satisfy* $\|f_0\|_{L^1_k L^2} \leq \delta_{m,n,l}$ *and* $\|f_0\|_{m,n,l} < \infty$, *then the following statements are valid. Let* f^ε (f^L) *be the*

solution to the Cauchy problem (1.16) *(problem* (1.6)*) with initial data* f_0*. Then for any* $T \geq 0$*, it holds that*

$$E_T(f^\varepsilon; m, n, l) + D_T^\varepsilon(f^\varepsilon; m, n, l) \lesssim P_n(\|f_0\|_{m,n,l}). \tag{1.26}$$

As a result, by passing to the limit $\varepsilon \to 0$*, for any* $T \geq 0$*, it holds that*

$$E_T(f^L; m, n, l) + D_T^0(f^L; m, n, l) \lesssim P_n(\|f_0\|_{m,n,l}). \tag{1.27}$$

(3) (Asymptotic formula). *Let* f^ε *and* f^L *be the solutions to the Cauchy problems* (1.16) *and* (1.6)*, respectively, with the same initial data* f_0 *satisfying* $\|f_0\|_{0,3,9} < \infty$ *and* $\|f_0\|_{L^1_k L^2} \leq \delta_{0,3,9}$*. Then for any* $T \geq 0$*, it holds that*

$$\|f^\varepsilon - f^L\|_{L^1_k L^\infty_T L^2} + \|f^\varepsilon - f^L\|_{L^1_k L^2_T L^2_{0,\gamma/2}} \lesssim \varepsilon P_3(\|f_0\|_{0,3,9})(1 + P_3(\|f_0\|_{0,3,9})). \tag{1.28}$$

Several remarks on Theorem 1.2 are in order.

Remark 1.5. The restrictions $s > 1/2$ and $\gamma + 2s > -1$ on the parameters s and γ come from Theorem 3.1 for the upper bound for the nonlinear term Γ^ε. By $\gamma + 2s > -1$, the inverse power law potential is covered, because $\gamma + 4s = 1$ is satisfied in this case; cf. also Remark 1.1. Since we aim to investigate the inconsistency of the spectrum in the parameter range $-2 \leq \gamma < -2s$, we only focus on the case of $-3 < \gamma \leq 0$ in Theorem 1.2.

Remark 1.6. Note that all the results in Theorem 1.2 are uniform with respect to the parameter ε and that the smallness assumption is only imposed on $\|f_0\|_{L^1_k L^2}$. In particular, in (1.26) and (1.27), we obtain the propagation of the bounds of solutions in the norm $\|\cdot\|_{L^1_{k,m} H^n_l}$ only under the smallness assumption on $\|f_0\|_{L^1_k L^2}$ and boundedness on $\|f_0\|_{m,n,l}$. In comparison, [14] establishes the propagation in norm $\|\cdot\|_{L^1_{k,m} L^2}$ under the smallness assumption on $\|f_0\|_{L^1_{k,m} L^2}$. Moreover, the asymptotic estimate (1.28) is global in time and has an explicit convergence rate $O(\varepsilon)$.

Remark 1.7. By the weak convergence results in [3] and [38], we can directly use (1.24) and (1.26) to derive (1.25) and (1.27), respectively. This shows that the well-posedness of the Boltzmann and Landau equations can be studied in a unified framework.

Remark 1.8. Theorem 1.2 does not include the Coulomb potential since $\gamma > -3$ is required. However, we can deal with the Coulomb case using the idea in [21]. More precisely, we can take the Boltzmann collision kernel with the mathematical choice of s and γ by

$$s = s_\varepsilon := 1 - \frac{\varepsilon}{4}, \quad \gamma = \gamma_\varepsilon := -3 + \varepsilon,$$

and consider the limit $\varepsilon \to 0$. After all, we need those uniform operator estimates with respect to the parameter s (near 1) and γ (near -3) similar to the situation under consideration. Since in the present paper we are mainly concerned with the spectrum inconsistency in the case $-2 \leq \gamma < -2s$, we leave the Coulomb case for future work.

As the main goal of this work, we state the second result revealing the transition of the decay structure from sub-exponential $e^{-\lambda t^\kappa}$ in (1.11) to exponential $e^{-\lambda t}$ in (1.10) under the grazing limit.

Theorem 1.3 (Transition of decay structure). *Let all the assumptions in Theorem 1.2 be satisfied and further let $-2 \le \gamma < -2s$. If the positive constants λ and q are chosen such that $\lambda \ll \lambda_0$ and $q > 2\lambda$, where $\lambda_0 > 0$ is the constant in Theorem 2.2 given later, then there is a constant $\delta_1 > 0$ such that if $\|e^{q\langle v\rangle} f_0\|_{L^1_k L^2} \le \delta_1$, the solution f^ε to the Cauchy problem (1.16) for the non-cutoff Boltzmann equation satisfies*

$$\|f^\varepsilon(t)\|_{L^1_k L^2} \lesssim \big(\mathbb{1}_{t\le T_\varepsilon}\exp(-\lambda t) + \mathbb{1}_{t>T_\varepsilon}\exp(-\lambda\varepsilon^{-2(1-s)\kappa}t^\kappa)\big)\|e^{q\langle v\rangle} f_0\|_{L^1_k L^2} \quad (1.29)$$

for any $t \ge 0$, where $T_\varepsilon := (\frac{1}{\varepsilon})^{\frac{2(1-s)}{|\gamma+2s|}}$ and $\kappa := \frac{1}{1+|\gamma+2s|}$.

Some remarks on this result are also in order.

Remark 1.9. Theorem 1.3 shows that the transition of the decay structure is continuous in the limit process. Note that the key estimate (1.29) is consistent with the sub-exponential decay rate $e^{-\lambda t^\kappa}$ in (1.11) and the exponential rate $e^{-\lambda t}$ in (1.10) by additionally taking into account the dependence of the rate on the vanishing grazing parameter ε. Moreover, we introduce the time threshold T_ε so as to characterize how the transition occurs as $\varepsilon \to 0$. Note that the estimate of decay rates for the Boltzmann equation with soft potentials from angular cutoff to non-cutoff was studied in [24, 26].

Remark 1.10. Theorem 1.3 provides a detailed picture of the uniform-in-ε and global-in-time dynamics of solutions to the Cauchy problem (1.16) for the non-cutoff Boltzmann equation in the parameter range $-2 \le \gamma < -2s$. More precisely, the perturbation converges to zero with exponential decay rate, i.e. $e^{-\lambda t}$, from initial time to T_ε. When time exactly approaches the critical one T_ε, the convergence rate continuously changes to the sub-exponential decay, i.e. $\exp(-\lambda\varepsilon^{-2(1-s)\kappa}t^\kappa)$, in terms of the definition of T_ε. After the transition time, the solution keeps the sub-exponential decay rate until infinity. Notice $T_\varepsilon \to \infty$ as $\varepsilon \to 0$, which then recovers the exponential decay for the Cauchy problem (1.6) on the Landau equation in case $\gamma + 2 \ge 0$.

Remark 1.11. The exponential velocity weight assumption $\|e^{q\langle v\rangle} f_0\|_{L^1_k L^2} \le \delta_1$ on initial data is used to get the sub-exponential decay for the Cauchy problem (1.16) with $\gamma + 2s < 0$. It is interesting to study what the transition of the decay structure is if only a finite-order polynomial velocity weight is imposed on the initial data.

Remark 1.12. The constructive constant λ in fact gives the lower bound for the first nonzero eigenvalue for the linearized Landau collision operator $\mathcal{L}^L$ in Theorem 1.3. In other words, we have explained the formation of the spectral gap through the large time behavior of the semigroup $e^{t\mathcal{L}^\varepsilon}$ as ε tends to zero. However, understanding the formation of the spectral gap via spectrum theory is still a fundamental and more challenging problem. At the moment we are still far from answering this question.

1.4. Strategy of proof

In this subsection we outline the strategy for proving Theorems 1.2 and 1.3, which will help readers get a better understanding of the key ideas.

1.4.1. Proof of Theorem 1.2. The proofs of both Theorems 1.2 and 1.3 rely on some subtle analysis of the linear operator $\mathcal{L}^\varepsilon$ and the nonlinear operator Γ^ε. Referring to [1], the quantity

$$K^\varepsilon(\xi) := \int_{\mathbb{S}^2} b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big) \min\big\{|\xi|^2\sin^2(\theta/2), 1\big\}\, d\sigma,$$

offers velocity regularity in the frequency space. By Proposition A.1 originating in [21, proof of Theorem 3.1]), we have $K^\varepsilon(\xi) + 1 \gtrsim (W^\varepsilon)^2(\xi)$, where

$$W^\varepsilon(y) := \zeta(\varepsilon|y|)\langle y\rangle + (1-\zeta(\varepsilon|y|))\langle \varepsilon^{-1}\rangle^{1-s}\langle y\rangle^s. \tag{1.30}$$

Here, the function $\zeta\colon [0,\infty) \to [0,1]$ satisfies

$$\zeta \in C^\infty, \quad \zeta(r) = \begin{cases} 1 & \text{if } r\in[0,1/2], \\ 0 & \text{if } r\in[1,\infty), \end{cases} \quad \zeta \text{ is strictly decreasing on } [1/2,1]. \tag{1.31}$$

When $\varepsilon = 0$, we define $W^0(y) := \langle y\rangle$.

As in [26], we call $W^\varepsilon(W^0)$ the characteristic function associated to $\mathcal{L}^\varepsilon(\mathcal{L}^L)$. This is because the function W^ε is the common weight gain in phase space, frequency space, and anisotropic space. More precisely, for $l\in\mathbb{R}$ and $\varepsilon\geq 0$, we define

$$|f|^2_{\varepsilon,l} := |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12})W_l f|^2_{L^2} + |W^\varepsilon(D)W_l f|^2_{L^2} + |W^\varepsilon W_l f|^2_{L^2}. \tag{1.32}$$

Here, $W^\varepsilon(D)$ is the pseudo-differential operator with symbol W^ε, and the operator $W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12})$ is defined in such a way that for $v = r\sigma$ with $r\geq 0$ and $\sigma\in\mathbb{S}^2$,

$$\big(W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12})f\big)(v) := \sum_{l=0}^{\infty}\sum_{m=-l}^{l} W^\varepsilon\big((l(l+1))^{\frac12}\big)Y_l^m(\sigma)f_l^m(r),$$

where $f_l^m(r) = \int_{\mathbb{S}^2} Y_l^m(\sigma)f(r\sigma)\,d\sigma$, and $Y_l^m, -l\leq m\leq l$ are the real spherical harmonics satisfying $(-\Delta_{\mathbb{S}^2})Y_l^m = l(l+1)Y_l^m$.

We use the explicitly defined norm $|\cdot|_{\varepsilon,\gamma/2}$ in (1.32) to characterize the lower bound for the linear operator $\mathcal{L}^\varepsilon$ as well as the upper bound for the nonlinear term Γ^ε.

Step 1: Coercivity estimate. We prove that

$$\langle \mathcal{L}^\varepsilon f, f\rangle + |f|^2_{L^2_{\gamma/2}} \geq \lambda_1 |f|^2_{\varepsilon,\gamma/2} \tag{1.33}$$

for some constant $\lambda_1 > 0$ independent of ε; see Theorem 2.1. That is, the uniform-in-ε coercivity estimate for $\mathcal{L}^\varepsilon$ is obtained by using the norm $|\cdot|_{\varepsilon,\gamma/2}$. Note that by (1.32), the norm $|\cdot|^2_{\varepsilon,\gamma/2}$ has three parts:

$$|W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12})W_{\gamma/2}f|^2_{L^2}, \quad |W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2}, \quad |W^\varepsilon W_{\gamma/2}f|^2_{L^2}.$$

We use some elementary computations to obtain regularity in phase space $|W^\varepsilon W_{\gamma/2} f|^2_{L^2}$ in Proposition 2.1. By the well-known result in [1], we derive frequency space regularity $|W^\varepsilon(D)W_{\gamma/2} f|^2_{L^2}$ in Lemma 2.3 and Proposition 2.4. By referring to [23], we gain the anisotropic norm $|W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})W_{\gamma/2} f|^2_{L^2}$ in Propositions 2.3 and 2.4. Lemma 3.4 shows that $\langle \mathcal{L}^\varepsilon f, f\rangle$ is bounded from above by $|f|^2_{\varepsilon,\gamma/2}$. The lower and upper bounds together demonstrate that $|\cdot|^2_{\varepsilon,\gamma/2}$ is the right norm to characterize the inner product $\langle \mathcal{L}^\varepsilon f, f\rangle$.

Step 2: Spectrum-gap-type estimate. Indeed, we are able to prove that for any $f \in \ker^\perp$, it holds that

$$\langle \mathcal{L}^\varepsilon f, f\rangle \geq \lambda_0 |f|^2_{\varepsilon,\gamma/2} \tag{1.34}$$

for some explicitly computable $\lambda_0 > 0$ independent of ε; see Theorem 2.2. This is motivated by Wang Chang–Uhlenbeck's work [39] on the explicit spectral gap estimate for the Maxwell molecule model $\gamma = 0$. Our main idea is to reduce the desired estimate to the case $\gamma = 0$ and at the same time utilize the coercivity estimate (1.33) to get (1.34) for $-3 < \gamma < 0$; cf. the proof of Theorem 2.2 for more details.

Step 3: Upper bound for nonlinear term. We use the norm $|\cdot|_{\varepsilon,\gamma/2}$ to bound the nonlinear term Γ^ε as

$$|\langle \Gamma^\varepsilon(g,h), f\rangle| \leq C|g|_{L^2}|h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2} \tag{1.35}$$

for some C independent of ε; see Theorem 3.1. When $\gamma < 0$, the relative velocity $v - v_*$ has singularity near 0. For this, we consider $|v - v_*| \lesssim 1$ and $|v - v_*| \gtrsim 1$ separately. When $|v - v_*| \gtrsim 1$, it holds that $|v - v_*|^\gamma \sim \langle v - v_*\rangle^\gamma$ so that there is no singularity. When $|v - v_*| \lesssim 1$, it holds that $\mu^{\frac{1}{2}}(v_*) \lesssim \mu^{\frac{1}{4}}(v)$. Hence, one can make use of $\mu^{\frac{1}{2}}(v_*)$ in the definition of Γ^ε to deal with the weight problem. We call it the weight transferring idea for a general result; cf. Lemma 3.1. We carry out the idea roughly in Proposition 3.3 but fully in Lemma 3.2 and Proposition 3.4.

Step 4: Global well-posedness. With (1.34) and (1.35), we can implement the standard macro–micro decomposition and take advantage of the functional property of the space $L^1_k L^\infty_T L^2$ to prove global well-posedness of the Boltzmann equation (1.16). Since the procedure is well established in [14], we directly conclude the global well-posedness result in the space $L^1_k L^\infty_T L^2$ in Theorem 1.2. We remark that $|g|_{L^2}$ in (1.35) corresponds to the L^2 in $L^1_k L^\infty_T L^2$.

Step 5: Propagation of regularity. To obtain propagation of regularity and velocity moments in Theorem 1.2, we first derive a commutator estimate between $\Gamma^\varepsilon(g,\cdot)$ and the weight function $W_{l,q} := \langle v\rangle^l \exp(q\langle v\rangle)$ in Lemma 3.7. We then derive a commutator estimate between $\mathcal{L}^\varepsilon$ and the weight function $W_{l,q}$ in Lemma 3.10. Note that the case $q = 0$ represents a polynomial weight which is used in Theorem 1.2. Our goal is to prove propagation of the norm $\|\cdot\|_{L^1_{k,m}H^n_l}$ under smallness of $\|f_0\|_{L^1_k L^2}$ and finiteness of $\|f_0\|_{m,n,l}$. We first prove this in Theorem 4.1 for the case $n = 0$ without velocity derivative. Then the case $n \geq 1$ is proved in Theorem 4.2 using the induction argument.

Step 6: Asymptotic formula. To derive the asymptotic formula between the solutions of the Boltzmann and Landau equations in Theorem 1.2, we first show the error estimate for $\Gamma^\varepsilon - \Gamma^L$ in Lemma 4.1. Taking the difference between (1.16) and (1.6), we get an equation for the solution difference $f^\varepsilon - f^L$. We then apply the energy method to the equation to derive (1.28) by using the propagation result (1.26) and error estimate for $\Gamma^\varepsilon - \Gamma^L$.

1.4.2. Proof of Theorem 1.3. We will apply the time-weighted energy method together with the time-velocity splitting technique to establish the transition time-decay structure (1.29) in Theorem 1.3. Such an approach was initiated by Caflisch [7, 8] to treat the spatially homogeneous Boltzmann equation with cutoff soft potentials in a torus and later developed by Strain–Guo [33, 34] in the spatially inhomogeneous setting as well as by Gressman–Strain [17] for the non-cutoff case. The key for obtaining the time decay of solutions is to impose an extra velocity weight on initial data that can be either polynomial or exponential, inducing the polynomial or sub-exponential rate, respectively. In what follows we explain the main points in the proof of Theorem 1.3.

First of all, we apply Caflisch's idea to determine the time threshold T_ε, as it plays the most important role in carrying out the time-velocity splitting technique under the uniform grazing limit. In terms of the energy dissipation norm $|\cdot|_{\varepsilon,\gamma/2}$ in (1.34) and (1.32) and the function W^ε in (1.30), we introduce the following toy model for explanation:

$$\partial_t f + v\cdot\nabla_x f + \lambda a_\varepsilon(v) f = 0,$$

with the constant $\lambda > 0$ suitably small, and

$$a_\varepsilon(v) := \zeta(\varepsilon|v|)\langle v\rangle^{\gamma+2} + [1-\zeta(\varepsilon|v|)]\frac{\langle v\rangle^{\gamma+2s}}{\varepsilon^{2(1-s)}}.$$

The solution is explicitly given by $f(t,x,v) = e^{-\lambda a_\varepsilon(v)t} f_0(x-vt,v)$. Assuming that initial data f_0 decays in velocity at an exponential rate $\exp(-\lambda\langle v\rangle^\vartheta)$ with $0<\vartheta\le 2$, one can formally bound $f(t,x,v)$ as

$$|f(t,x,v)| \lesssim \exp(-\lambda b_\varepsilon(t,v)) \lesssim \exp(-\lambda \inf_v b_\varepsilon(t,v)), \quad b_\varepsilon(t,v) := a_\varepsilon(v)t + \langle v\rangle^\vartheta.$$

To look for a lower bound for $b_\varepsilon(t,v)$ in velocity which should depend only on time, we may compute in the parameter range $-2\le\gamma<-2s$ that

$$\begin{aligned} b_\varepsilon(t,v) &= \zeta(\varepsilon|v|)\{\langle v\rangle^{\gamma+2}t + \langle v\rangle^\vartheta\} + [1-\zeta(\varepsilon|v|)]\Big\{\langle v\rangle^{\gamma+2s}\frac{t}{\varepsilon^{2(1-s)}} + \langle v\rangle^\vartheta\Big\} \\ &\ge \zeta(\varepsilon|v|)t + [1-\zeta(\varepsilon|v|)]\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^\kappa \\ &\ge \min\Big\{t, \Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^\kappa\Big\} = t\mathbb{1}_{t<T_\varepsilon} + \Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^\kappa \mathbb{1}_{t\ge T_\varepsilon}. \end{aligned}$$

Here we have used the inequalities

$$\inf_v\{\langle v\rangle^{\gamma+2}t + \langle v\rangle^\vartheta\} \ge t, \quad \inf_v\Big\{\langle v\rangle^{\gamma+2s}\frac{t}{\varepsilon^{2(1-s)}} + \langle v\rangle^\vartheta\Big\} \ge \Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^\kappa,$$

with $\kappa := \frac{\vartheta}{\vartheta+|\gamma+2s|}$. Moreover, the time threshold $T_\varepsilon > 0$ has to be chosen such that $(\frac{t}{\varepsilon^{2(1-s)}})^\kappa = t$ at $t = T_\varepsilon$, implying that

$$T_\varepsilon = \Big(\frac{1}{\varepsilon^{2(1-s)}}\Big)^{\frac{\kappa}{1-\kappa}} = \Big(\frac{1}{\varepsilon}\Big)^{\frac{2\vartheta(1-s)}{|\gamma+2s|}}.$$

In such a way the solution $f(t,x,v)$ decays in large time as

$$|f(t,x,v)| \lesssim \mathbb{1}_{t\le T_\varepsilon}\exp(-\lambda t) + \mathbb{1}_{t>T_\varepsilon}\exp(-\lambda\varepsilon^{-2(1-s)\kappa}t^\kappa)$$

whenever $\sup_{x,v} e^{\lambda\langle v\rangle^\vartheta}|f_0(x,v)| < \infty$ holds. Therefore, the transition time-decay structure motivates us to define

$$A_\varepsilon(t) := \zeta(T_\varepsilon^{-1}t)t + (1-\zeta(T_\varepsilon^{-1}t))\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^\kappa \tag{1.36}$$

and to obtain the energy estimate on $h(t,x,v) := e^{\lambda A_\varepsilon(t)}f(t,x,v)$ for $\lambda > 0$ suitably small. It turns out that after repeating these known energy estimates, it suffices to obtain the uniform bound on

$$\sqrt{\lambda}\sum_{k\in\mathbb{Z}^3}\Big(\int_0^T A_\varepsilon'(t)\|(1-\zeta)\hat{h}(t,k)\|_{L^2}^2\,dt\Big)^{\frac{1}{2}},$$

where the fact that $0 \le A_\varepsilon'(t) \lesssim 1$ and $1-\zeta$ is supported in $|v| \ge \frac{1}{2\varepsilon}$ has been used. Then, whenever the velocity-weighted norm $\|\exp(q\langle v\rangle^\vartheta)f(t)\|_{L^1_kL^2}$ is bounded uniformly in time, the time-velocity splitting technique can be applied by separating the time interval into three parts as

$$\int_0^T ds = \Big(\int_0^{\frac{1}{2\varepsilon}} + \int_{\frac{1}{2\varepsilon}}^{T_\varepsilon} + \int_{T_\varepsilon}^T\Big)ds$$

for any $T > T_\varepsilon$; cf. Section 5 for more details. Back to propagation of the exponential velocity moments, using the commutator estimate in Lemma 3.7 and the upper bound estimate in Theorem 3.1, we have the weighted upper bound estimate

$$|\langle\Gamma^\varepsilon(g,h), W_{l,q}^2 f\rangle| \lesssim |W_{l,q}g|_{L^2}|W_{l,q}h|_{\varepsilon,\gamma/2}|W_{l,q}f|_{\varepsilon,\gamma/2}, \tag{1.37}$$

as shown in Lemma 3.8. Note that inequality (1.37) is stronger than [14, Lemma 4.1]. Therefore, by using the proof of [14, Theorem 2.1], we get the propagation of the norm $\|\exp(q\langle v\rangle)f(t)\|_{L^1_kL^2}$ under the smallness assumption on $\|\exp(q\langle v\rangle)f_0\|_{L^1_kL^2}$; cf. Theorem 5.1. We remark that for the exponential weight $\exp(q\langle v\rangle^\vartheta)$ we can treat the case $\vartheta = 1$ only (cf. [15] and [14]), due to the specific property of the non-cutoff Boltzmann operator.

1.5. Usual notation and organization of the paper

Denote the multi-index $\beta = (\beta_1,\beta_2,\beta_3)$, with $|\beta| = \beta_1+\beta_2+\beta_3$. Further, $a \lesssim b$ means that there is a generic constant C such that $a \le Cb$. The notation $a \sim b$ implies that

$a \lesssim b$ and $b \lesssim a$. The weight function $W_l(v) := \langle v \rangle^l$. We denote by $C(\lambda_1, \lambda_2, \ldots, \lambda_n)$ or $C_{\lambda_1,\lambda_2,\ldots,\lambda_n}$ a constant depending on the parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. The notations $\langle f, g \rangle := \int_{\mathbb{R}^3} f(v)g(v)\, dv$ and $(f, g) := \int_{\mathbb{R}^3 \times \mathbb{T}^3} fg\, dx\, dv$ are used to denote the two standard inner products for the v variable and for the x, v variables respectively. As usual, $\mathbb{1}_A$ is the characteristic function of the set A. If A, B are two operators, then $[A, B] := AB - BA$. Define $|f|_{L\log L} := \int_{\mathbb{R}^3} |f(v)| \log(1 + |f(v)|)\, dv$.

Finally, the rest of the paper will be organized as follows. In Section 2 we will prove the coercivity estimate given in Theorem 2.1 and the spectral gap estimate given in Theorem 2.2. In Section 3 we will focus on the upper bound estimate given in Theorem 3.1. In addition, with some commutator estimates, we will also prove upper bound estimates with polynomial or exponential weights. The two main theorems will be proved in Sections 4 and 5 respectively. For completeness, in Appendix A we include some known results that are used in Sections 2–5.

2. Coercivity and the spectral gap estimate

In this section, in Theorem 2.1 we will prove a coercivity estimate for the linear operator $\mathcal{L}^\varepsilon$ and in Theorem 2.2 the spectral gap estimate. Unless otherwise specified, the parameter range is $-3 < \gamma \le 0, 0 < s < 1$.

In the rest of the paper, we will omit the range of some frequently used variables in the integrals for brevity. Usually, $\sigma \in \mathbb{S}^2$, $v, v_*, u, \xi \in \mathbb{R}^3$. For example, we set $\int (\cdots)\, d\sigma := \int_{\mathbb{S}^2} (\cdots)\, d\sigma$, $\int (\cdots)\, d\sigma\, dv\, dv_* := \int_{\mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{R}^3} (\cdots)\, d\sigma\, dv\, dv_*$. Integration with respect to other variables should be understood in a similar way. Whenever a new variable appears, we will specify its range once and then omit it thereafter.

2.1. Elementary results

In this subsection we give some preliminaries which will be used frequently in the rest of the paper. We first list some properties of W^ε defined in (1.30). Note that W^ε is a radial function defined on $\mathbb{R}^3$. By the definition (1.30), we have

$$\langle y \rangle \le W^\varepsilon(y) \le \langle \varepsilon^{-1} \rangle^{1-s} \langle y \rangle^s \quad \text{if } \frac{1}{2\varepsilon} \le |y| \le \frac{1}{\varepsilon}. \tag{2.1}$$

$$W^\varepsilon(y) = \langle y \rangle \quad \text{if } |y| \le \frac{1}{2\varepsilon}. \tag{2.2}$$

$$W^\varepsilon(y) = \langle \varepsilon^{-1} \rangle^{1-s} \langle y \rangle^s \quad \text{if } |y| \ge \frac{1}{\varepsilon}. \tag{2.3}$$

$$W^\varepsilon(y) \gtrsim \zeta(\varepsilon|y|) \langle y \rangle. \tag{2.4}$$

$$W^\varepsilon(y) \gtrsim (1 - \zeta(\varepsilon|y|)) \langle \varepsilon^{-1} \rangle^{1-s} \langle y \rangle^s \gtrsim (1 - \zeta(\varepsilon|y|)) \varepsilon^{-1}. \tag{2.5}$$

For any $x, y \in \mathbb{R}^3$, one can check that

$$W^\varepsilon(x) \leq W^\varepsilon(y) \quad \text{if } |x| \leq |y|. \tag{2.6}$$

$$W^\varepsilon(x-y) \lesssim W^\varepsilon(x) W^\varepsilon(y). \tag{2.7}$$

Let us compute an integral regarding the angular function b^ε over the sphere $\mathbb{S}^2$. Recall that $b^\varepsilon(\cos\theta) = (1-s)\varepsilon^{2s-2}\sin^{-2-2s}(\theta/2)\mathbb{1}_{\sin\frac{\theta}{2}\leq\varepsilon}$. Note that $d\sigma = \sin\theta\, d\theta\, d\phi = 4\sin(\theta/2)\, d\sin(\theta/2)\, d\phi$; we have

$$\begin{aligned}
&\int b^\varepsilon(\cos\theta)\sin^2(\theta/2)\, d\sigma \\
&= 4(1-s)\varepsilon^{2s-2}\int_0^\pi \int_0^{2\pi} \mathbb{1}_{\sin\frac{\theta}{2}\leq\varepsilon} \sin^{1-2s}(\theta/2)\, d\sin(\theta/2)\, d\phi \\
&= 8\pi(1-s)\varepsilon^{2s-2}\int_0^\varepsilon u^{1-2s}\, du = 4\pi.
\end{aligned} \tag{2.8}$$

Let us recall the cancellation lemma, which is used when one needs to shift regularity between h and f in the inner product $\langle Q(g,h), f\rangle$. By [1, Lemma 1], we have the following lemma.

Lemma 2.1 (Cancellation lemma, [1]). *Recalling that B^ε in (1.8), then*

$$\int B^\varepsilon(v-v_*,\sigma) g_*(h'-h)\, dv\, dv_*\, d\sigma = C(\varepsilon)\int |v-v_*|^\gamma g_* h\, dv\, dv_*,$$

where $C(\varepsilon)$ is some constant depending on ε and $|C(\varepsilon)| \lesssim 1$ thanks to (2.8).

Next, let us present a result regarding the Riesz potential, whose proof can be found in [26, Lemma 2.7].

Lemma 2.2. *Set $A := \int |v-v_*|^\gamma g_* h f\, dv\, dv_*$. Then*

- *if $-\frac{3}{2} < \gamma \leq 0$, then $|A| \lesssim (|g|_{L^2_{|\gamma|}} + |g|_{L^1_{|\gamma|}})|h|_{L^2_{\gamma/2}}|f|_{L^2_{\gamma/2}}$;*
- *if $-3 < \gamma \leq -\frac{3}{2}$, then for $\eta > 0$, $s_1, s_2 \geq 0$ such that $s_1 + s_2 = -\frac{3}{2} - \gamma + \eta$ there holds*

$$|A| \lesssim C_\eta(|g|_{L^1_{|\gamma|}} + |g|_{H^{s_1}_{|\gamma|}})|h|_{H^{s_2}_{\gamma/2}}|f|_{H^0_{\gamma/2}}.$$

As a result of Lemmas 2.1 and 2.2, we have the following corollary.

Corollary 2.1. *Fix $\eta > 0$ and let $s_1, s_2 \geq 0$ verify $s_1 + s_2 = \max\{-\frac{3}{2} - \gamma + \eta, 0\}$. Then*

$$\left|\int B^\varepsilon(v-v_*,\sigma) g_*((hf)' - hf)\, dv\, dv_*\, d\sigma\right| \lesssim C_\eta(|g|_{L^1_{|\gamma|}} + |g|_{H^{s_1}_{|\gamma|}})|h|_{H^{s_2}_{\gamma/2}}|f|_{H^0_{\gamma/2}}.$$

2.2. Coercivity estimate

We present the coercivity estimate for $\mathcal{L}^\varepsilon$ in the following theorem.

Theorem 2.1. *There exists a constant $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ and any smooth function f, there exists a constant $\lambda_1 > 0$, depending only on γ, such that*

$$\langle \mathcal{L}^\varepsilon f, f\rangle + |f|^2_{L^2_{\gamma/2}} \ge \lambda_1 |f|^2_{\varepsilon,\gamma/2}.$$

In this subsection we will prove Theorem 2.1. Our strategy is based on the following relation in the spirit of the triple norm introduced in [2] (see the proof of Theorem 2.1 in Section 2.2.5):

$$\langle \mathcal{L}^\varepsilon f, f\rangle + |f|^2_{L^2_{\gamma/2}} \gtrsim \mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f) + \mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}}), \tag{2.9}$$

$$\mathcal{N}^{\varepsilon,\gamma}(g,h) := \int b^\varepsilon(\cos\theta)|v-v_*|^\gamma g_*^2 (h'-h)^2 \, d\sigma\, dv\, dv_*. \tag{2.10}$$

Thanks to (2.9), to get the coercivity estimate for $\mathcal{L}^\varepsilon$, it suffices to estimate from below the two functionals $\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f)$ and $\mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}})$. We will study $\mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}})$ in Section 2.2.1 and $\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f)$ in Sections 2.2.2, 2.2.3, and 2.2.4. The coercivity estimate is obtained in Section 2.2.5 by utilizing (2.9).

2.2.1. Gain of weight from $\mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}})$. The functional $\mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}})$ yields weight W^ε in the phase space as shown in the following proposition.

Proposition 2.1. *There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$,*

$$\mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}}) + |f|^2_{L^2_{\gamma/2}} \ge C|W^\varepsilon f|^2_{L^2_{\gamma/2}},$$

where $C > 0$ is a universal constant.

Proof. The proof is divided into four steps.

Step 1: $16/\pi \le |v_*| \le \delta/\varepsilon$. The parameter $0 < \delta \le 1$ will be determined later. We consider the set $A(\varepsilon,\delta) := \{(v_*, v, \sigma) : 16/\pi \le |v_*| \le \delta/\varepsilon,\ |v| \le 8/\pi,\ \sin(\theta/2) \le \varepsilon\}$. When $\varepsilon \le \frac{\pi}{16}\delta$, it is easy to check that $A(\varepsilon,\delta)$ is nonempty. We restrict the integral on the set $A(\varepsilon,\delta)$ to get

$$\mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}}) \ge \int B^\varepsilon \mathbb{1}_{A(\varepsilon,\delta)} f_*^2 ((\mu^{\frac{1}{2}})' - \mu^{\frac{1}{2}})^2 \, d\sigma\, dv\, dv_*. \tag{2.11}$$

Note that $\nabla \mu^{\frac{1}{2}} = -\frac{\mu^{\frac{1}{2}}}{2} v$ and $\nabla^2 \mu^{\frac{1}{2}} = \frac{\mu^{\frac{1}{2}}}{4}(-2I_3 + v\otimes v)$. By Taylor expansion, we have

$$\begin{aligned}\mu^{\frac{1}{2}}(v') - \mu^{\frac{1}{2}}(v) = &-\frac{\mu^{\frac{1}{2}}(v)}{2} v\cdot(v'-v) \\ &+ \int_0^1 (1-\kappa)(\nabla^2\mu^{\frac{1}{2}})(v(\kappa)) : (v'-v)\otimes(v'-v)\, d\kappa,\end{aligned}$$

where $v(\kappa) = v + \kappa(v'-v)$. Thanks to the fact that $(a-b)^2 \ge \frac{a^2}{2} - b^2$, we have

$$(\mu^{\frac{1}{2}}(v') - \mu^{\frac{1}{2}}(v))^2 \ge \frac{\mu(v)}{8}|v\cdot(v'-v)|^2 - \int_0^1 |(\nabla^2\mu^{\frac{1}{2}})(v(\kappa))|^2 |v'-v|^4\, d\kappa.$$

Plugging this into (2.11) we get

$$\begin{aligned}\mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) &\geq \frac{1}{8}\int B^{\varepsilon}\mathbb{1}_{A(\varepsilon,\delta)}\mu(v)|v\cdot(v'-v)|^2 f_*^2\,d\sigma\,dv\,dv_* \\ &\quad -\int B^{\varepsilon}\mathbb{1}_{A(\varepsilon,\delta)}|(\nabla^2\mu^{\frac{1}{2}})(v(\kappa))|^2|v'-v|^4 f_*^2\,d\sigma\,dv\,dv_*\,d\kappa \\ &:= \frac{1}{8}\mathcal{I}_1^{\varepsilon}(\delta)-\mathcal{I}_2^{\varepsilon}(\delta). \end{aligned} \tag{2.12}$$

To estimate $\mathcal{I}_1^{\varepsilon}(\delta)$, for fixed v, v_*, we introduce an orthonormal basis $(h^1_{v,v_*}, h^2_{v,v_*}, \frac{v-v_*}{|v-v_*|})$ such that $d\sigma = \sin\theta\,d\theta\,d\phi$. We express $\frac{v'-v}{|v'-v|}$ and $\frac{v}{|v|}$ using the basis as follows:

$$\begin{aligned}\frac{v'-v}{|v'-v|} &= \cos\frac{\theta}{2}\cos\phi h^1_{v,v_*}+\cos\frac{\theta}{2}\sin\phi h^2_{v,v_*}-\sin\frac{\theta}{2}\frac{v-v_*}{|v-v_*|}, \\ \frac{v}{|v|} &= c_1 h^1_{v,v_*}+c_2 h^2_{v,v_*}+c_3\frac{v-v_*}{|v-v_*|},\end{aligned}$$

where $c_3 = \frac{v}{|v|}\cdot\frac{v-v_*}{|v-v_*|}$ and c_1, c_2 are constants independent of θ and ϕ. Then we have

$$\frac{v}{|v|}\cdot\frac{v'-v}{|v'-v|} = c_1\cos\frac{\theta}{2}\cos\phi + c_2\cos\frac{\theta}{2}\sin\phi - c_3\sin\frac{\theta}{2},$$

and thus

$$\begin{aligned}\left|\frac{v}{|v|}\cdot\frac{v'-v}{|v'-v|}\right|^2 &= c_1^2\cos^2\frac{\theta}{2}\cos^2\phi + c_2^2\cos^2\frac{\theta}{2}\sin^2\phi + c_3^2\sin^2\frac{\theta}{2} \\ &\quad + 2c_1c_2\cos^2\frac{\theta}{2}\cos\phi\sin\phi - 2c_3\cos\frac{\theta}{2}\sin\frac{\theta}{2}(c_1\cos\phi + c_2\sin\phi).\end{aligned}$$

Integrating with respect to σ, we have

$$\begin{aligned}&\int b^{\varepsilon}(\cos\theta)\mathbb{1}_{A(\varepsilon,\delta)}|v\cdot(v'-v)|^2\,d\sigma \\ &= \int_0^{\pi}\int_0^{2\pi} b^{\varepsilon}(\cos\theta)\sin\theta\mathbb{1}_{A(\varepsilon,\delta)}|v\cdot(v'-v)|^2\,d\phi\,d\theta \\ &\geq \pi(c_1^2+c_2^2)|v|^2|v-v_*|^2\mathbb{1}_{B(\varepsilon,\delta)},\end{aligned}$$

where $B(\varepsilon,\delta) = \{(v_*,v) : 16/\pi \leq |v_*| \leq \delta/\varepsilon,\ |v| \leq 8/\pi\}$. Plugging the above estimate into the definition of $\mathcal{I}_1^{\varepsilon}(\delta)$, we get

$$\begin{aligned}\mathcal{I}_1^{\varepsilon}(\delta) &\geq \pi\int (c_1^2+c_2^2)|v-v_*|^{\gamma+2}|v|^2\mathbb{1}_{B(\varepsilon,\delta)}\mu(v)f_*^2\,dv\,dv_* \\ &= \pi\int\left(1-\left(\frac{v}{|v|}\cdot\frac{v_*}{|v_*|}\right)^2\right)|v_*|^2|v-v_*|^{\gamma}|v|^2\mathbb{1}_{B(\varepsilon,\delta)}\mu(v)f_*^2\,dv\,dv_*,\end{aligned}$$

where we have used the facts that $c_1^2+c_2^2+c_3^2 = 1$ and

$$\left(1-\left(\frac{v}{|v|}\cdot\frac{v_*}{|v_*|}\right)^2\right)^{-1}|v-v_*|^2 = (1-c_3^2)^{-1}|v_*|^2.$$

Note that in the region $B(\varepsilon,\delta)$, one has $\frac{1}{2}|v_*| \le |v-v_*| \le \frac{3}{2}|v_*|$. Since $\gamma \le 0$, then

$$\Big(\frac{3}{2}\Big)^{\gamma}|v_*|^{\gamma} \le |v-v_*|^{\gamma} \le \Big(\frac{1}{2}\Big)^{\gamma}|v_*|^{\gamma}. \tag{2.13}$$

We then get

$$\begin{aligned}\mathcal{I}_1^{\varepsilon}(\delta) &\ge \pi\Big(\frac{3}{2}\Big)^{\gamma}\int\Big(1-\Big(\frac{v}{|v|}\cdot\frac{v_*}{|v_*|}\Big)^2\Big)|v_*|^{\gamma+2}|v|^2\mathbb{1}_{B(\varepsilon,\delta)}\mu(v)f_*^2\,dv\,dv_*\\ &= \pi\Big(\frac{3}{2}\Big)^{\gamma}c_1\int|v_*|^{\gamma+2}\mathbb{1}_{\frac{16}{\pi}\le|v_*|\le\frac{\delta}{\varepsilon}}f_*^2\,dv_*,\end{aligned}$$

where

$$c_1 = \int\Big(1-\Big(\frac{v}{|v|}\cdot\frac{v_*}{|v_*|}\Big)^2\Big)|v|^2\mu(v)\mathbb{1}_{|v|\le\frac{8}{\pi}}\,dv$$

is independent of v_*.

We now estimate $\mathcal{I}_2^{\varepsilon}(\delta)$. Recalling that

$$\begin{aligned}B^{\varepsilon} &= (1-s)\varepsilon^{2s-2}\mathbb{1}_{\sin\frac{\theta}{2}\le\varepsilon}|v-v_*|^{\gamma}\sin^{-2-2s}(\theta/2),\\ |v'-v| &= |v-v_*|\sin(\theta/2),\end{aligned}$$

we have

$$\begin{aligned}\mathcal{I}_2^{\varepsilon}(\delta) &= \int B^{\varepsilon}\mathbb{1}_{A(\varepsilon,\delta)}|(\nabla^2\mu^{\frac{1}{2}})(v(\kappa))|^2|v'-v|^4f_*^2\,d\sigma\,dv\,dv_*\,d\kappa\\ &= \varepsilon^{2s-2}\int\sin^{2-2s}(\theta/2)\mathbb{1}_{A(\varepsilon,\delta)}|(\nabla^2\mu^{\frac{1}{2}})(v(\kappa))|^2|v-v_*|^{\gamma+4}f_*^2\,d\sigma\,dv\,dv_*\,d\kappa\\ &\le \Big(\frac{3}{2}\Big)^{\gamma+4}\varepsilon^{2s-2}\int\sin^{2-2s}(\theta/2)\mathbb{1}_{A(\varepsilon,\delta)}|(\nabla^2\mu^{\frac{1}{2}})(v(\kappa))|^2|v_*|^{\gamma+4}f_*^2\,d\sigma\,dv\,dv_*\,d\kappa\\ &= \Big(\frac{3}{2}\Big)^{\gamma+4}\varepsilon^{2s-2}\int_0^{\pi}\int_0^{2\pi}\sin^{2-2s}(\theta/2)\\ &\qquad\times\int\mathbb{1}_{A(\varepsilon,\delta)}|(\nabla^2\mu^{\frac{1}{2}})(v(\kappa))|^2|v_*|^{\gamma+4}f_*^2\sin\theta\,d\theta\,d\phi\,dv\,dv_*\,d\kappa.\end{aligned}$$

Fixing κ, v_*, ϕ, in the change of variable $(v,\theta)\to(v(\kappa),\theta(\kappa))$, where $\theta(\kappa)$ is the angle between σ and $v(\kappa)-v_*$, we have

$$\begin{aligned}\Big|\frac{\partial(v(\kappa),\theta(\kappa))}{\partial(v,\theta)}\Big|^{-1} &\le \Big(1-\frac{\kappa}{2}\Big)^{-5} \le 32 = 2^5,\\ \theta/2 \le \theta(\kappa) \le \theta, &\quad \sin\theta \le 2\sin\theta(\kappa),\\ \sin(\theta(\kappa)/2) \le \sin(\theta/2) &\le 2\sin(\theta(\kappa)/2),\end{aligned} \tag{2.14}$$

$$\sin(\theta/2)\le\varepsilon \Rightarrow \sin(\theta(\kappa)/2)\le\varepsilon. \tag{2.15}$$

Hence, we have

$$\mathbb{1}_{A(\varepsilon,\delta)} \le \mathbb{1}_{\frac{16}{\pi}\le|v_*|\le\frac{\delta}{\varepsilon},\sin\frac{\theta(\kappa)}{2}\le\varepsilon},$$

so that

$$\begin{aligned}\mathcal{I}_2^\varepsilon(\delta) &\leq 2^6 2^{2-2s}\Big(\frac{3}{2}\Big)^{\gamma+4}\varepsilon^{2s-2}\int_0^\pi\int_0^{2\pi}\int \sin^{2-2s}(\theta(\kappa)/2)\mathbb{1}_{\frac{16}{\pi}\leq|v_*|\leq\frac{\delta}{\varepsilon},\sin\frac{\theta(\kappa)}{2}\leq\varepsilon} \\ &\quad\times|(\nabla^2\mu^{\frac{1}{2}})(v(\kappa))|^2|v_*|^{\gamma+4}f_*^2\sin\theta(\kappa)\,d\theta(\kappa)\,d\phi\,dv(\kappa)\,dv_*\,d\kappa \\ &= 2^{9-2s}\pi\Big(\frac{3}{2}\Big)^{\gamma+4}\varepsilon^{2s-2}\Big(\int_0^\pi \sin^{2-2s}(\theta/2)\mathbb{1}_{\sin\frac{\theta}{2}\leq\varepsilon}\sin\theta\,d\theta\Big) \\ &\quad\times\Big(\int \mathbb{1}_{\frac{16}{\pi}\leq|v_*|\leq\frac{\delta}{\varepsilon}}|(\nabla^2\mu^{\frac{1}{2}})(v)|^2|v_*|^{\gamma+4}f_*^2\,dv\,dv_*\Big).\end{aligned}$$

Direct computation gives

$$\int_0^\pi \sin^{2-2s}(\theta/2)\mathbb{1}_{\sin\frac{\theta}{2}\leq\varepsilon}\sin\theta\,d\theta = 4\int_0^\varepsilon u^{3-2s}\,du = \frac{4\varepsilon^{4-2s}}{4-2s}.$$

Let $c_2 = \int|(\nabla^2\mu^{\frac{1}{2}})(v)|^2\,dv$. Then we get

$$\begin{aligned}\mathcal{I}_2^\varepsilon(\delta) &\leq 2^{11-2s}\pi\Big(\frac{3}{2}\Big)^{\gamma+4}(4-2s)^{-1}c_2\varepsilon^2\Big(\int \mathbb{1}_{\frac{16}{\pi}\leq|v_*|\leq\frac{\delta}{\varepsilon}}|v_*|^{\gamma+4}f_*^2\,dv_*\Big) \\ &\leq 2^{11-2s}\pi\Big(\frac{3}{2}\Big)^{\gamma+4}(4-2s)^{-1}c_2\delta^2\int \mathbb{1}_{\frac{16}{\pi}\leq|v_*|\leq\frac{\delta}{\varepsilon}}|v_*|^{\gamma+2}f_*^2\,dv_*,\end{aligned}$$

where we have used $\varepsilon|v_*|\leq\delta$.

Plugging the estimates of $\mathcal{I}_1^\varepsilon(\delta)$ and $\mathcal{I}_2^\varepsilon(\delta)$ into (2.12), we get

$$\mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) \geq (C_1 - C_2\delta^2)\int \mathbb{1}_{\frac{16}{\pi}\leq|v_*|\leq\frac{\delta}{\varepsilon}}|v_*|^{\gamma+2}f_*^2\,dv_*,$$

where $C_1 = 2^{-3}\pi(\frac{3}{2})^\gamma c_1$, $C_2 = 2^{11-2s}\pi(\frac{3}{2})^{\gamma+4}(4-2s)^{-1}c_2$. By choosing δ such that $C_2\delta^2 = C_1/2$, we get

$$\mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) \geq \frac{1}{2}C_1\int \mathbb{1}_{\frac{16}{\pi}\leq|v_*|\leq\frac{\delta}{\varepsilon}}|v_*|^{\gamma+2}f_*^2\,dv_*. \tag{2.16}$$

Step 2: $|v_*|\geq R/\varepsilon$. Here, $R\geq 1$ is a parameter to be determined later. By direct computation, we have

$$\begin{aligned}&\mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) \\ &= \int B^\varepsilon f_*^2((\mu^{\frac{1}{2}})' - \mu^{\frac{1}{2}})^2\,d\sigma\,dv\,dv_* \\ &\geq \int B^\varepsilon \mathbb{1}_{4^{-1}R|v_*|^{-1}\leq\sin\frac{\theta}{2}\leq R|v_*|^{-1}}\mathbb{1}_{|v_*|\geq\frac{R}{\varepsilon}}\mathbb{1}_{|v|\leq 1}f_*^2((\mu^{\frac{1}{2}})' - \mu^{\frac{1}{2}})^2\,d\sigma\,dv\,dv_* \\ &\geq \int b^\varepsilon|v-v_*|^\gamma\mathbb{1}_{4^{-1}R|v_*|^{-1}\leq\sin\frac{\theta}{2}\leq R|v_*|^{-1}}\mathbb{1}_{|v_*|\geq\frac{R}{\varepsilon}}\mathbb{1}_{|v|\leq 1}f_*^2\mu\,d\sigma\,dv\,dv_* \\ &\quad -2\int b^\varepsilon|v-v_*|^\gamma\mathbb{1}_{4^{-1}R|v_*|^{-1}\leq\frac{\theta}{2}\leq R|v_*|^{-1}}\mathbb{1}_{|v_*|\geq\frac{R}{\varepsilon}}\mathbb{1}_{|v|\leq 1}f_*^2(\mu^{\frac{1}{2}})'\mu^{\frac{1}{2}}\,d\sigma\,dv\,dv_* \\ &:= \mathcal{J}_1^\varepsilon(R) - \mathcal{J}_2^\varepsilon(R).\end{aligned}$$

Note that

$$\begin{aligned}\int b^\varepsilon \mathbb{1}_{4^{-1}R|v_*|^{-1}\le \sin\frac{\theta}{2}\le R|v_*|^{-1}}\,d\sigma &= 8\pi(1-s)\varepsilon^{2s-2}\int_{4^{-1}R|v_*|^{-1}}^{R|v_*|^{-1}} u^{-1-2s}\,du\\ &= 4\pi\frac{4^{2s}-1}{s}(1-s)R^{-2s}\varepsilon^{2s-2}|v_*|^{2s}\\ &= C_sR^{-2s}\varepsilon^{2s-2}|v_*|^{2s},\end{aligned} \tag{2.17}$$

where $C_s = 4\pi\frac{4^{2s}-1}{s}(1-s)$. If $\varepsilon \le \frac{1}{2}$, $|v_*| \ge R/\varepsilon \ge 2$, $|v| \le 1$, we have

$$\frac{1}{2}|v_*| \le |v-v_*| \le \frac{3}{2}|v_*|. \tag{2.18}$$

Plugging (2.17) into the definition of $\mathcal{J}_1^\varepsilon(R)$ and using (2.13), we have

$$\begin{aligned}\mathcal{J}_1^\varepsilon(R) &\ge C_sR^{-2s}\varepsilon^{2s-2}\int |v-v_*|^\gamma|v_*|^{2s}\mathbb{1}_{|v_*|\ge\frac{R}{\varepsilon}}\mathbb{1}_{|v|\le 1}f_*^2\mu\,d\sigma\,dv\,dv_*\\ &\ge C_s\Big(\frac{3}{2}\Big)^\gamma c_3R^{-2s}\varepsilon^{2s-2}\int |v_*|^{\gamma+2s}\mathbb{1}_{|v_*|\ge\frac{R}{\varepsilon}}f_*^2\,dv_*,\end{aligned}$$

where $c_3 = \int \mathbb{1}_{|v|\le 1}\mu(v)\,dv$.

Since $\sin(\theta/2) \ge \varepsilon$, there holds $|v'| + |v| \ge |v'-v| = \sin\frac{\theta}{2}|v-v_*| \ge \varepsilon|v-v_*| \ge \varepsilon(|v_*|-|v|)$, and then $|v'| + (1+\varepsilon)|v| \ge \varepsilon|v_*| \ge R$. Thus, $R^2 \le (|v'|+2|v|)^2 \le 8(|v'|^2+|v|^2)$, which implies

$$\mu'^{\frac{1}{2}}\mu^{\frac{1}{2}} = (2\pi)^{-\frac{3}{2}}e^{-\frac{|v'|^2+|v|^2}{4}} \le (2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{8}}e^{-\frac{R^2}{2^6}}. \tag{2.19}$$

Then by (2.19), (2.17), and (2.18), we have

$$\begin{aligned}\mathcal{J}_2^\varepsilon(R) &= 2\int b^\varepsilon|v-v_*|^\gamma \mathbb{1}_{4^{-1}R|v_*|^{-1}\le\sin\frac{\theta}{2}\le R|v_*|^{-1}}\mathbb{1}_{|v_*|\ge\frac{R}{\varepsilon}}\mathbb{1}_{|v|\le 1}f_*^2\mu'^{\frac{1}{2}}\mu^{\frac{1}{2}}\,d\sigma\,dv\,dv_*\\ &\le 2(2\pi)^{-\frac{3}{2}}e^{-\frac{R^2}{2^6}}\int b^\varepsilon|v-v_*|^\gamma\mathbb{1}_{4^{-1}R|v_*|^{-1}\le\sin\frac{\theta}{2}\le R|v_*|^{-1}}\mathbb{1}_{|v_*|\ge\frac{R}{\varepsilon}}\\ &\qquad\times\mathbb{1}_{|v|\le 1}f_*^2e^{-\frac{|v|^2}{8}}\,d\sigma\,dv\,dv_*\\ &\le 2(2\pi)^{-\frac{3}{2}}C_se^{-\frac{R^2}{2^6}}R^{-2s}\varepsilon^{2s-2}\int |v-v_*|^\gamma|v_*|^{2s}\mathbb{1}_{|v_*|\ge\frac{R}{\varepsilon}}\mathbb{1}_{|v|\le 1}f_*^2e^{-\frac{|v|^2}{8}}\,dv\,dv_*\\ &\le 2(2\pi)^{-\frac{3}{2}}C_s\Big(\frac{1}{2}\Big)^\gamma e^{-\frac{R^2}{2^6}}R^{-2s}\varepsilon^{2s-2}\int |v_*|^{\gamma+2s}\mathbb{1}_{|v_*|\ge\frac{R}{\varepsilon}}\mathbb{1}_{|v|\le 1}f_*^2e^{-\frac{|v|^2}{8}}\,dv\,dv_*\\ &= 2(2\pi)^{-\frac{3}{2}}C_s\Big(\frac{1}{2}\Big)^\gamma c_4e^{-\frac{R^2}{2^6}}R^{-2s}\varepsilon^{2s-2}\int |v_*|^{\gamma+2s}\mathbb{1}_{|v_*|\ge\frac{R}{\varepsilon}}f_*^2\,dv_*,\end{aligned}$$

where $c_4 = \int \mathbb{1}_{|v|\le 1}e^{-\frac{|v|^2}{8}}\,dv$. Combining the above estimates for $\mathcal{J}_1^\varepsilon(R)$ and $\mathcal{J}_2^\varepsilon(R)$, we arrive at, for any $\varepsilon \le \frac{1}{2}$, $R \ge 1$,

$$\mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) \ge (C_3 - C_4e^{-\frac{R^2}{2^6}})R^{-2s}\varepsilon^{2s-2}\int |v_*|^{\gamma+2s}\mathbb{1}_{|v_*|\ge\frac{R}{\varepsilon}}f_*^2\,dv_*,$$

where $C_3 = C_s(\frac{3}{2})^\gamma c_3$, $C_4 = 2(2\pi)^{-\frac{3}{2}}C_s(\frac{1}{2})^\gamma c_4$. We choose $R \geq 1$ such that $\frac{1}{2}C_3 \geq C_4 e^{-\frac{R^2}{2^6}}$ and arrive at

$$\mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) \geq \frac{1}{2}C_3 R^{-2s}\varepsilon^{2s-2}\int |v_*|^{\gamma+2s}\mathbb{1}_{|v_*|\geq\frac{R}{\varepsilon}} f_*^2\, dv_*. \tag{2.20}$$

Step 3: $16/\pi \leq |v_*| \leq R/\varepsilon$. Here, R is the fixed constant in Step 2. We also recall the fixed constant δ in Step 1. Since $16/\pi \leq |v_*| \leq R/\varepsilon = \delta/(\delta R^{-1}\varepsilon)$, by (2.16), we have

$$\mathcal{N}^{\delta R^{-1}\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) \geq \frac{1}{2}C_1\int \mathbb{1}_{\frac{16}{\pi}\leq|v_*|\leq\frac{R}{\varepsilon}}|v_*|^{\gamma+2} f_*^2\, dv_*.$$

Observe that

$$\begin{aligned} b^{\delta R^{-1}\varepsilon}(\cos\theta) &= (R/\delta)^{2-2s}(1-s)\varepsilon^{2s-2}\sin^{-2-2s}(\theta/2)\mathbb{1}_{\sin\frac{\theta}{2}\leq\delta R^{-1}\varepsilon} \\ &\leq (R/\delta)^{2-2s} b^{\varepsilon}(\cos\theta), \end{aligned}$$

from which we get

$$\begin{aligned} \mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) &\geq (R/\delta)^{2s-2}\mathcal{N}^{\delta R^{-1}\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) \\ &\geq \frac{1}{2}C_1(R/\delta)^{2s-2}\int \mathbb{1}_{\frac{16}{\pi}\leq|v_*|\leq\frac{R}{\varepsilon}}|v_*|^{\gamma+2} f_*^2\, dv_*. \end{aligned} \tag{2.21}$$

Step 4: To recover weight W^ε. Combining (2.20), (2.21), and

$$|f|^2_{L^2_{\gamma/2}} \geq \int \mathbb{1}_{|v_*|\leq\frac{16}{\pi}}\langle v_*\rangle^\gamma f_*^2\, dv_*,$$

we arrive at

$$\begin{aligned} \mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) + |f|^2_{L^2_{\gamma/2}} \geq &\int \mathbb{1}_{|v_*|\leq\frac{16}{\pi}}\langle v_*\rangle^\gamma f_*^2\, dv_* \\ &+ \frac{1}{4}C_1(R/\delta)^{2s-2}\int |v_*|^{\gamma+2}\mathbb{1}_{\frac{16}{\pi}\leq|v_*|\leq\frac{R}{\varepsilon}} f_*^2\, dv_* \\ &+ \frac{1}{4}C_3 R^{-2s}\varepsilon^{2s-2}\int |v_*|^{\gamma+2s}\mathbb{1}_{|v_*|\geq\frac{R}{\varepsilon}} f_*^2\, dv_*. \end{aligned}$$

Since $|v_*| \geq 16/\pi \geq 4$, we get $|v_*|^2 \leq 1+|v_*|^2 \leq \frac{17}{16}|v_*|^2$, which gives

$$\begin{aligned} |v_*|^{\gamma+2} &\geq \min\{1,(17/16)^{-\gamma/2-1}\}\langle v_*\rangle^{\gamma+2}, \\ |v_*|^{\gamma+2s} &\geq \min\{1,(17/16)^{-\gamma/2-s}\}\langle v_*\rangle^{\gamma+2s}. \end{aligned}$$

Supposing $\varepsilon \leq 1/4$, we have $\varepsilon^{2s-2} \geq (17/16)^{s-1}\langle\varepsilon^{-1}\rangle^{2-2s}$. Therefore, we get

$$\begin{aligned} \mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) + |f|^2_{L^2_{\gamma/2}} \geq &\int \mathbb{1}_{|v_*|\leq\frac{16}{\pi}}\langle v_*\rangle^\gamma f_*^2\, dv_* \\ &+ C_5\int \langle v_*\rangle^{\gamma+2}\mathbb{1}_{\frac{16}{\pi}\leq|v_*|\leq\frac{R}{\varepsilon}} f_*^2\, dv_* \\ &+ C_6\langle\varepsilon^{-1}\rangle^{2-2s}\int \langle v_*\rangle^{\gamma+2s}\mathbb{1}_{|v_*|\geq\frac{R}{\varepsilon}} f_*^2\, dv_*, \end{aligned}$$

where

$$C_5 = \frac{1}{4}C_1(R/\delta)^{2s-2}\min\{1, (17/16)^{-\gamma/2-1}\},$$
$$C_6 = \frac{1}{4}C_3 R^{-2s}(17/16)^{s-1}\min\{1, (17/16)^{-\gamma/2-s}\}.$$

By (2.6) and (2.2), we have

$$\mathbb{1}_{|v_*|\leq\frac{16}{\pi}} W^\varepsilon(v_*) = \langle v_*\rangle \leq W_1(16/\pi) = (1+4(16/\pi)^2)^{\frac{1}{2}}.$$

Then we get

$$\mathbb{1}_{|v_*|\leq\frac{16}{\pi}} \geq (1+4(16/\pi)^2)^{-1}\mathbb{1}_{|v_*|\leq\frac{16}{\pi}}(W^\varepsilon)^2(v_*). \tag{2.22}$$

In the region $16/\pi \leq |v_*| \leq \frac{1}{2}\varepsilon^{-1}$, by (2.2), we have

$$\langle v_*\rangle^2 = (W^\varepsilon)^2(v_*). \tag{2.23}$$

In the region $\frac{1}{2}\varepsilon^{-1} \leq |v_*| \leq \varepsilon^{-1}$, by (2.1) we have

$$\langle v_*\rangle^2 \geq \left\langle\frac{1}{2}\varepsilon^{-1}\right\rangle^{2-2s}\langle v_*\rangle^{2s} \geq \left(\frac{1}{2}\right)^{2-2s}\langle\varepsilon^{-1}\rangle^{2-2s}\langle v_*\rangle^{2s} \geq \left(\frac{1}{2}\right)^{2-2s}(W^\varepsilon)^2(v_*). \tag{2.24}$$

In the region $\varepsilon^{-1} \leq |v_*| \leq R\varepsilon^{-1}$, by (2.3), we have

$$\langle v_*\rangle^2 \geq \langle\varepsilon^{-1}\rangle^{2-2s}\langle v_*\rangle^{2s} = (W^\varepsilon)^2(v_*). \tag{2.25}$$

In the region $|v_*| \geq R\varepsilon^{-1}$, by (2.3), we have

$$\langle\varepsilon^{-1}\rangle^{2-2s}\langle v_*\rangle^{2s} = (W^\varepsilon)^2(v_*). \tag{2.26}$$

Then by (2.22), (2.23), (2.24), (2.25), and (2.26), we get

$$\begin{aligned}\mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) + |f|^2_{L^2_{\gamma/2}} &\geq (1+4(16/\pi)^2)^{-1}\int \mathbb{1}_{|v_*|\leq\frac{16}{\pi}}(W^\varepsilon)^2(v_*)\langle v_*\rangle^\gamma f_*^2\,dv_*\\ &\quad+\left(\frac{1}{2}\right)^{2-2s}C_5\int (W^\varepsilon)^2(v_*)\langle v_*\rangle^\gamma \mathbb{1}_{\frac{16}{\pi}<|v_*|\leq\varepsilon^{-1}} f_*^2\,dv_*\\ &\quad+\min\{C_5, C_6\}\int (W^\varepsilon)^2(v_*)\langle v_*\rangle^\gamma \mathbb{1}_{|v_*|>\varepsilon^{-1}} f_*^2\,dv_*\\ &\geq C(\gamma,s)|W^\varepsilon f|^2_{L^2_{\gamma/2}},\end{aligned}$$

which completes the proof. Here, $C(\gamma,s) := \min\{(1+4(16/\pi)^2)^{-1}, (\frac{1}{2})^{2-2s}C_5, C_6\}$ is a positive constant depending only on γ, s. It is easy to check that $C(\gamma,s) \gtrsim 1$ uniformly when $-3 < \gamma \leq 0, 0 < s < 1$. ■

In the following, we show that Proposition 2.1 is sharp.

Proposition 2.2. *The estimate $\mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) \lesssim |W^\varepsilon f|^2_{L^2_{\gamma/2}}$ holds.*

Proof. First we have

$$
\begin{aligned}
\mathcal{N}^{\varepsilon,\gamma}(f,\mu^{\frac{1}{2}}) &\lesssim \int B^{\varepsilon} f_*^2((\mu^{\frac{1}{4}})'-\mu^{\frac{1}{4}})^2(\mu'^{\frac{1}{2}}+\mu^{\frac{1}{2}})\,d\sigma\,dv\,dv_* \\
&\lesssim \int B^{\varepsilon} f_*^2((\mu^{\frac{1}{4}})'-\mu^{\frac{1}{4}})^2\mu'^{\frac{1}{2}}\,d\sigma\,dv\,dv_* \\
&\quad + \int B^{\varepsilon} f_*^2((\mu^{\frac{1}{4}})'-\mu^{\frac{1}{4}})^2\mu^{\frac{1}{2}}\,d\sigma\,dv\,dv_* \\
&:= \mathcal{K}_1^{\varepsilon,\gamma}(f)+\mathcal{K}_2^{\varepsilon,\gamma}(f).
\end{aligned}
$$

By Taylor expansion, one has

$$
((\mu^{\frac{1}{4}})'-\mu^{\frac{1}{4}})^2 \lesssim \min\{1,|v-v_*|^2\theta^2\} \sim \min\{1,|v'-v_*|^2\theta^2\}.
$$

By Proposition A.1 and (2.7), we have

$$
\int b^{\varepsilon}(\cos\theta)\min\{1,|v-v_*|^2\theta^2\}\,d\sigma \lesssim (W^{\varepsilon})^2(|v-v_*|) \lesssim (W^{\varepsilon})^2(v)(W^{\varepsilon})^2(v_*),
$$

which gives

$$
\begin{aligned}
\mathcal{K}_2^{\varepsilon,\gamma}(f) &\lesssim \int f_*^2|v-v_*|^{\gamma}(W^{\varepsilon})^2(v)(W^{\varepsilon})^2(v_*)\mu^{\frac{1}{2}}\,dv\,dv_* \\
&\lesssim \int f_*^2\langle v_*\rangle^{\gamma}(W^{\varepsilon})^2(v_*)\,dv_* = |W^{\varepsilon}f|^2_{L^2_{\gamma/2}}.
\end{aligned}
$$

Here we have used the fact that $\int |v-v_*|^{\gamma}\mu^{\frac{1}{4}}\,dv \lesssim \langle v_*\rangle^{\gamma}$. By the change of variable $v \to v'$, similarly we have $\mathcal{K}_1^{\varepsilon,\gamma}(f) \lesssim |W^{\varepsilon}f|^2_{L^2_{\gamma/2}}$. The proof of the lemma is completed. ∎

Remark 2.1. By the proof of Proposition 2.2, for $a \geq \frac{1}{8}$, the estimate $\mathcal{N}^{\varepsilon,\gamma}(f,\mu^a) \lesssim |W^{\varepsilon}f|^2_{L^2_{\gamma/2}}$ holds.

2.2.2. Gain of Sobolev regularity from $\mathcal{N}^{\varepsilon,0}(g,f)$. By [1, Corollary 2.1, Lemma 3], and Proposition A.1, we have the following lemma.

Lemma 2.3. *Let g be a function such that $|g^2|_{L^1} \geq \delta > 0$, $|g^2|_{L^1_1}+|g^2|_{L\log L} \leq \lambda < \infty$, then there exists $C(\delta,\lambda)$ such that*

$$
\mathcal{N}^{\varepsilon,0}(g,f)+|f|^2_{L^2} \geq C(\delta,\lambda)|W^{\varepsilon}(D)f|^2_{L^2}.
$$

2.2.3. Gain of anisotropic regularity from $\mathcal{N}^{\varepsilon,0}(g,f)$. In this part, we derive the anisotropic regularity from $\mathcal{N}^{\varepsilon,0}(g,f)$. To this end, we apply a geometric decomposition in the frequency space. More precisely, we will use the following decomposition (see (2.34) in the proof of Proposition 2.3):

$$
\hat{f}(\xi)-\hat{f}(\xi^+) = \underbrace{\hat{f}(\xi)-\hat{f}\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)}_{\text{spherical part}} + \underbrace{\hat{f}\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)-\hat{f}(\xi^+)}_{\text{radial part}}. \tag{2.27}
$$

The "spherical part" gives the anisotropic regularity. Namely, we have the following lemma.

Lemma 2.4. *Set* $\mathcal{A}^\varepsilon(f) := \int b^\varepsilon(\frac{\xi}{|\xi|}\cdot\sigma)|\hat{f}(\xi) - \hat{f}(|\xi|\frac{\xi^+}{|\xi^+|})|^2\, d\xi\, d\sigma$, *where* $\xi^+ = \frac{\xi+|\xi|\sigma}{2}$. *Then*

$$\mathcal{A}^\varepsilon(f) + |f|^2_{L^2} \sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|^2_{L^2} + |f|^2_{L^2}.$$

Proof. Let $r = |\xi|$, $\tau = \xi/|\xi|$, and $\varsigma = \frac{\tau+\sigma}{|\tau+\sigma|}$. Then $\frac{\xi}{|\xi|}\cdot\sigma = 2(\tau\cdot\varsigma)^2 - 1$ and $|\xi|\frac{\xi^+}{|\xi^+|} = r\varsigma$. In the change of variables $(\xi,\sigma) \to (r,\tau,\varsigma)$, one has $d\xi\, d\sigma = 4(\tau\cdot\varsigma)r^2\, dr\, d\tau\, d\varsigma$. Let θ be the angle between τ and σ. Then $2\sin\frac{\theta}{2} = |\tau-\sigma|$ and thus

$$b^\varepsilon(\cos\theta) = (1-s)\varepsilon^{2s-2}2^{2+2s}|\tau-\sigma|^{-2-2s}\mathbb{1}_{|\tau-\sigma|\le 2\varepsilon}. \tag{2.28}$$

Since $\theta/2$ is the angle between the two unit vectors τ and ς, we get $|\tau-\varsigma| = 2(1-\cos\frac{\theta}{2})$. It is easy to check that

$$\frac{1}{2}|\tau-\sigma| \le |\tau-\varsigma| \le |\tau-\sigma|,$$

from which, together with (2.28), we get

$$b^\varepsilon(\cos\theta) \ge (1-s)\varepsilon^{2s-2}|\tau-\varsigma|^{-2-2s}\mathbb{1}_{|\tau-\varsigma|\le\varepsilon},$$
$$b^\varepsilon(\cos\theta) \le (1-s)\varepsilon^{2s-2}2^{2+2s}|\tau-\varsigma|^{-2-2s}\mathbb{1}_{|\tau-\varsigma|\le 2\varepsilon}.$$

By (A.1) in Lemma A.1, we have

$$\begin{aligned}\mathcal{A}^\varepsilon(f) + |f|^2_{L^2} &= 4\int b^\varepsilon(2(\tau\cdot\varsigma)^2-1)|\hat{f}(r\tau)-\hat{f}(r\varsigma)|^2(\tau\cdot\varsigma)r^2\, dr\, d\tau\, d\varsigma + |f|^2_{L^2}\\ &\ge 4(1-s)\varepsilon^{2s-2}\int \frac{|\hat{f}(r\tau)-\hat{f}(r\varsigma)|^2}{|\tau-\varsigma|^{2+2s}}\mathbb{1}_{|\tau-\varsigma|\le\varepsilon}r^2\, dr\, d\tau\, d\varsigma + |f|^2_{L^2}\\ &\sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})\hat{f}|^2_{L^2} + |\hat{f}|^2_{L^2}.\end{aligned}$$

By Remark A.1 we have

$$\begin{aligned}\mathcal{A}^\varepsilon(f) + |f|^2_{L^2} &\le 4(1-s)\varepsilon^{2s-2}2^{2+2s}\int \frac{|\hat{f}(r\tau)-\hat{f}(r\varsigma)|^2}{|\tau-\varsigma|^{2+2s}}\mathbb{1}_{|\tau-\varsigma|\le 2\varepsilon}r^2\, dr\, d\tau\, d\varsigma\\ &\quad + |f|^2_{L^2}\\ &\sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})\hat{f}|^2_{L^2} + |\hat{f}|^2_{L^2}.\end{aligned}$$

With the help of Lemma A.2 and Plancherel's theorem,

$$|W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})\hat{f}|^2_{L^2} = |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|^2_{L^2},$$

which completes the proof. ∎

The "radial part" in (2.27) can be bounded by weight W^ε gain in the phase and frequency space. Namely, we have the following lemma.

Lemma 2.5. *Let*

$$\mathcal{Z}^{\varepsilon,\gamma}(f) := \int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\langle\xi\rangle^\gamma\Big|f\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big) - f(\xi^+)\Big|^2 d\xi\, d\sigma$$

with $\xi^+ = \frac{\xi+|\xi|\sigma}{2}$. *Then*

$$\mathcal{Z}^{\varepsilon,\gamma}(f) \lesssim |W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2} + |W^\varepsilon W_{\gamma/2}f|^2_{L^2}.$$

Proof. We divide the proof into two steps.

Step 1: $\gamma = 0$. As in the proof of Lemma 2.4, with the same change of variables $(\xi,\sigma) \to (r,\tau,\varsigma)$ with $\xi = r\tau$ and $\varsigma = \frac{\sigma+\tau}{|\sigma+\tau|}$, we have

$$\mathcal{Z}^{\varepsilon,0}(f) = 4\int b^\varepsilon(2(\tau\cdot\varsigma)^2 - 1)|f(r\varsigma) - f((\tau\cdot\varsigma)r\varsigma)|^2(\tau\cdot\varsigma)r^2\, dr\, d\tau\, d\varsigma.$$

Recall that

$$b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big) = (1-s)\varepsilon^{2s-2}\Big(\sin\frac{\theta}{2}\Big)^{-2-2s}\mathbb{1}_{\sin\frac{\theta}{2}\le\varepsilon},$$

where θ is the angle between τ and σ. Let $\alpha = \frac{\theta}{2}$ be the angle between τ and σ. Let $u = r\varsigma$. Then we get $r^2\, dr\, d\varsigma\, d\tau = \sin\alpha\, du\, d\alpha\, d\mathbb{S}$. Therefore, we have

$$\begin{aligned}\mathcal{Z}^{\varepsilon,0}(f) &= 8\pi(1-s)\varepsilon^{2s-2}\int_{\mathbb{R}^3}\int_0^\pi(\sin\alpha)^{-1-2s}\mathbb{1}_{\sin\alpha\le\varepsilon}\\ &\qquad\qquad\times|f(u) - f(u\cos\alpha)|^2\cos\alpha\, du\, d\alpha \qquad (2.29)\\ &\lesssim |W^\varepsilon(D)f|^2_{L^2} + |W^\varepsilon f|^2_{L^2}, \qquad (2.30)\end{aligned}$$

where the last inequality is given by Lemma A.5.

Step 2: $\gamma < 0$. We reduce the case when $\gamma < 0$ to the special case $\gamma = 0$. For simplicity, denote $w = |\xi|\frac{\xi^+}{|\xi^+|}$. Then $W_\gamma(\xi) = W_\gamma(w)$. Hence, we have

$$\begin{aligned}&\langle\xi\rangle^\gamma(f(w) - f(\xi^+))^2\\ &\quad= \big\{[(W_{\gamma/2}f)(w) - (W_{\gamma/2}f)(\xi^+)] + (W_{\gamma/2}f)(\xi^+)(1 - W_{\gamma/2}(w)W_{-\gamma/2}(\xi^+))\big\}^2\\ &\quad\le 2[(W_{\gamma/2}f)(\xi^+) - (W_{\gamma/2}f)(w)]^2 + 2|(W_{\gamma/2}f)(\xi^+)|^2|1 - W_{\gamma/2}(w)W_{-\gamma/2}(\xi^+)|^2.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathcal{Z}^{\varepsilon,\gamma}(f) &\le 2\mathcal{Z}^{\varepsilon,0}(W_{\gamma/2}f)\\ &\qquad + 2\int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)|(W_{\gamma/2}f)(\xi^+)|^2|1 - W_{\gamma/2}(w)W_{-\gamma/2}(\xi^+)|^2\, d\xi\, d\sigma\\ &:= \mathcal{Z}^{\varepsilon,0}(W_{\gamma/2}f) + \mathcal{A}.\end{aligned}$$

By noticing that $|W_{\gamma/2}(w)W_{-\gamma/2}(\xi^+) - 1| \lesssim \theta^2$, we have $|\mathcal{A}| \lesssim |W_{\gamma/2}f|^2_{L^2}$, where the change of variable $\xi \to \xi^+$ has been used. The desired result follows from estimate (2.30) in Step 1. ∎

Remark 2.2. With the same notation as in Lemma 2.5, we also have

$$\mathcal{Z}^{\varepsilon,0}(\hat{f}) \lesssim |W^\varepsilon(D)f|_{L^2}^2 + |W^\varepsilon f|_{L^2}^2.$$

Indeed, by (2.29) and Plancherel's theorem, we have

$$\begin{aligned}
&\mathcal{Z}^{\varepsilon,0}(\hat{f}) \\
&= 8\pi(1-s)\varepsilon^{2s-2}\int_{\mathbb{R}^3}\int_0^\pi (\sin\alpha)^{-1-2s}\mathbb{1}_{\sin\alpha\le\varepsilon}|\hat{f}(u)-\hat{f}(u\cos\alpha)|^2\cos\alpha\, du\, d\alpha \\
&= 8\pi(1-s)\varepsilon^{2s-2}\int_{\mathbb{R}^3}\int_0^\pi (\sin\alpha)^{-1-2s}\mathbb{1}_{\sin\alpha\le\varepsilon}|f(u)-f(u/\cos\alpha)|^2\cos\alpha\, du\, d\alpha \\
&\lesssim |W^\varepsilon(D)f|_{L^2}^2 + |W^\varepsilon f|_{L^2}^2,
\end{aligned}$$

where we have used the change of variable $u \to u\cos\alpha$ and the estimate (2.30) in the last inequality.

Now we are in a position to get $|W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|_{L^2_{\gamma/2}}^2$ from $\mathcal{N}^{\varepsilon,0}(g,f)$.

Proposition 2.3. *The following two estimates hold:*

$$\mathcal{N}^{\varepsilon,0}(g,f) + |g|_{L^2_1}^2|W^\varepsilon(D)f|_{L^2}^2 + |g|_{L^2}^2|W^\varepsilon f|_{L^2}^2 \gtrsim |g|_{L^2}^2|W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|_{L^2}^2, \quad (2.31)$$

$$\mathcal{N}^{\varepsilon,0}(g,f) \lesssim |g|_{L^2}^2|W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|_{L^2}^2 + |g|_{L^2_1}^2|W^\varepsilon(D)f|_{L^2}^2 + |g|_{L^2}^2|W^\varepsilon f|_{L^2}^2. \quad (2.32)$$

Proof. By Bobylev's formula, we have

$$\begin{aligned}
\mathcal{N}^{\varepsilon,0}(g,f) &= \frac{1}{(2\pi)^3}\int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big(\widehat{g^2}(0)|\hat{f}(\xi)-\hat{f}(\xi^+)|^2 \\
&\qquad + 2\Re\big((\widehat{g^2}(0)-\widehat{g^2}(\xi^-))\hat{f}(\xi^+)\bar{\hat{f}}(\xi)\big)\Big)\, d\xi\, d\sigma \\
&:= \frac{|g|_{L^2}^2}{(2\pi)^3}\mathcal{I}_1 + \frac{2}{(2\pi)^3}\mathcal{I}_2,
\end{aligned}$$

where $\xi^+ = \frac{\xi+|\xi|\sigma}{2}$ and $\xi^- = \frac{\xi-|\xi|\sigma}{2}$. Thanks to the fact that $\widehat{g^2}(0)-\widehat{g^2}(\xi^-) = \int(1-\cos(v\cdot\xi^-))g^2(v)\,dv$, we have

$$\begin{aligned}
|\mathcal{I}_2| &= \Big|\int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)(1-\cos(v\cdot\xi^-))g^2(v)\Re(\hat{f}(\xi^+)\bar{\hat{f}}(\xi))\, d\sigma\, d\xi\, dv\Big| \\
&\le \Big(\int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)(1-\cos(v\cdot\xi^-))g^2(v)|\hat{f}(\xi^+)|^2\, d\sigma\, d\xi\, dv\Big)^{\frac{1}{2}} \\
&\quad\times\Big(\int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)(1-\cos(v\cdot\xi^-))g^2(v)|\bar{\hat{f}}(\xi)|^2\, d\sigma\, d\xi\, dv\Big)^{\frac{1}{2}}.
\end{aligned}$$

Observe that

$$1-\cos(v\cdot\xi^-) \lesssim |v|^2|\xi^-|^2 = \frac{1}{4}|v|^2|\xi|^2\Big|\frac{\xi}{|\xi|}-\sigma\Big|^2 \sim |v|^2|\xi^+|^2\Big|\frac{\xi^+}{|\xi^+|}-\sigma\Big|^2,$$

thus

$$1-\cos(v\cdot\xi^-)\lesssim \min\Big\{|v|^2|\xi|^2\Big|\frac{\xi}{|\xi|}-\sigma\Big|^2,1\Big\}\sim \min\Big\{|v|^2|\xi^+|^2\Big|\frac{\xi^+}{|\xi^+|}-\sigma\Big|^2,1\Big\}.$$

Note that $\frac{\xi}{|\xi|}\cdot\sigma=2(\frac{\xi^+}{|\xi^+|}\cdot\sigma)^2-1$. By the change of variable from ξ to ξ^+, and the property $W^\varepsilon(|v||\xi|)\lesssim W^\varepsilon(|v|)W^\varepsilon(|\xi|)$, we have

$$\begin{aligned}|\mathcal{I}_2| &\lesssim \int (W^\varepsilon)^2(|v||\xi|)|\hat{f}(\xi)|^2 g^2(v)\,dv\,d\xi\\ &\lesssim |W^\varepsilon g|^2_{L^2}|W^\varepsilon(D)f|^2_{L^2}\lesssim |g|^2_{L^2_1}|W^\varepsilon(D)f|^2_{L^2}.\end{aligned}\tag{2.33}$$

Now we study the lower bound for $\mathcal{I}_1$. By the geometric decomposition

$$\hat{f}(\xi)-\hat{f}(\xi^+)=\hat{f}(\xi)-\hat{f}\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)+\hat{f}\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)-\hat{f}(\xi^+),\tag{2.34}$$

we have

$$\begin{aligned}\mathcal{I}_1 &= \int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)|\hat{f}(\xi)-\hat{f}(\xi^+)|^2\,d\xi\,d\sigma\\ &\geq \frac{1}{2}\int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big|\hat{f}(\xi)-\hat{f}\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)\Big|^2\,d\xi\,d\sigma\\ &\quad-\int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big|\hat{f}\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)-\hat{f}(\xi^+)\Big|^2\,d\xi\,d\sigma\\ &:= \frac{1}{2}\mathcal{I}_{1,1}-\mathcal{I}_{1,2}.\end{aligned}$$

By Lemma 2.4 , we have

$$\mathcal{I}_{1,1}+|f|^2_{L^2}\sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|^2_{L^2}+|f|^2_{L^2}.\tag{2.35}$$

By Remark 2.2, there holds

$$\mathcal{I}_{1,2}\lesssim |W^\varepsilon(D)f|^2_{L^2}+|W^\varepsilon f|^2_{L^2}.\tag{2.36}$$

Combining upper bound estimates (2.33), (2.36) and lower bound estimate (2.35), we get (2.31). On the other hand, by $\mathcal{N}^{\varepsilon,0}(g,f)\lesssim \mathcal{I}_{1,1}+\mathcal{I}_{1,2}+|\mathcal{I}_2|$, one can get (2.32) by the upper bound estimates (2.33), (2.35), (2.36). ∎

2.2.4. Gain of anisotropic regularity from $\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}},f)$. The strategy is to reduce $\mathcal{N}^{\varepsilon,\gamma}$ to $\mathcal{N}^{\varepsilon,0}$ so that the estimates in previous parts can be used.

For technical reasons, we define

$$\widetilde{\mathcal{N}}^{\varepsilon,\gamma}(g,h):=\int_{\mathbb{R}^6\times\mathbb{S}^2} b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma g_*^2(h'-h)^2\,d\sigma\,dv\,dv_*.\tag{2.37}$$

Moreover, we need to consider the "velocity regular" version $\widetilde{\mathcal{N}}^{\varepsilon,\gamma}$ of $\mathcal{N}^{\varepsilon,\gamma}$ to reduce $\widetilde{\mathcal{N}}^{\varepsilon,\gamma}$ to $\mathcal{N}^{\varepsilon,0}$ in the following lemma.

Lemma 2.6. *Let $\gamma \in \mathbb{R}$. Then*

$$\frac{1}{2}C_1\mathcal{N}^{\varepsilon,0}(W_{-|\gamma|/2}g, W_{\gamma/2}f) - C_3|g|^2_{L^2_{|\gamma/2+1|}}|f|^2_{L^2_{\gamma/2}} \\ \leq \widetilde{\mathcal{N}}^{\varepsilon,\gamma}(g,f) \leq 2C_2\mathcal{N}^{\varepsilon,0}(W_{|\gamma|/2}g, W_{\gamma/2}f) + 2C_3|g|^2_{L^2_{|\gamma/2+1|}}|f|^2_{L^2_{\gamma/2}}, \tag{2.38}$$

where C_1, C_2, C_3 are constants depending only on γ that can be chosen as some generic constants if $-3 \leq \gamma \leq 0$.

Proof. Set $F = W_{\gamma/2}f$. By definition, we have

$$\widetilde{\mathcal{N}}^{\varepsilon,\gamma}(g,f) = \int_{\mathbb{R}^6\times\mathbb{S}^2} b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma g_*^2((W_{-\gamma/2}F)' - W_{-\gamma/2}F)^2\,d\sigma\,dv\,dv_*.$$

Make the decomposition

$$(W_{-\gamma/2}F)' - W_{-\gamma/2}F = (W_{-\gamma/2})'(F'-F) + F(W'_{-\gamma/2} - W_{-\gamma/2}) := A + B.$$

From $\frac{1}{2}A^2 - B^2 \leq (A+B)^2 \leq 2A^2 + 2B^2$, we get

$$\frac{1}{2}\mathcal{I}_1 - \mathcal{I}_2 \leq \widetilde{\mathcal{N}}^{\varepsilon,\gamma}(g,f) \leq 2(\mathcal{I}_1 + \mathcal{I}_2), \tag{2.39}$$

where

$$\mathcal{I}_1 := \int_{\mathbb{R}^6\times\mathbb{S}^2} b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma g_*^2 W'_{-\gamma}(F'-F)^2\,d\sigma\,dv\,dv_*,$$
$$\mathcal{I}_2 := \int_{\mathbb{R}^6\times\mathbb{S}^2} b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma g_*^2 F^2(W'_{-\gamma/2} - W_{-\gamma/2})^2\,d\sigma\,dv\,dv_*.$$

Thanks to $|v_* - v| \sim |v_* - v'|$, $\langle v_*\rangle^{-|\gamma|} \lesssim \langle v_* - v'\rangle^\gamma \langle v'\rangle^{-\gamma} \lesssim \langle v_*\rangle^{|\gamma|}$, we get

$$\mathcal{N}^{\varepsilon,0}(W_{-|\gamma|/2}g, W_{\gamma/2}f) \lesssim \mathcal{I}_1 \lesssim \mathcal{N}^{\varepsilon,0}(W_{|\gamma|/2}g, W_{\gamma/2}f). \tag{2.40}$$

By Taylor expansion, one has $(W'_{-\gamma/2} - W_{-\gamma/2})^2 \lesssim \int \langle v(\kappa)\rangle^{-\gamma-2}|v-v_*|^2\sin^2(\theta/2)\,d\kappa$. Note that

$$\langle v-v_*\rangle^\gamma|v-v_*|^2\langle v(\kappa)\rangle^{-\gamma-2} \lesssim \langle v-v_*\rangle^{\gamma+2}\langle v(\kappa)\rangle^{-\gamma-2} \\ \lesssim \langle v(\kappa)-v_*\rangle^{\gamma+2}\langle v(\kappa)\rangle^{-\gamma-2} \lesssim \langle v_*\rangle^{|\gamma+2|}.$$

Then by (2.8), we get

$$\mathcal{I}_2 \lesssim \int g_*^2\langle v_*\rangle^{|\gamma+2|}F^2\,dv\,dv_* \lesssim |g|^2_{L^2_{|\gamma/2+1|}}|F|^2_{L^2}. \tag{2.41}$$

Plugging (2.40) and (2.41) into (2.39), we get (2.38). In addition, one can track the proof for the dependence of C_1, C_2, C_3 on γ. ■

We are now ready to give a lower bound estimate for $\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f)$.

Proposition 2.4. *The following two estimates are valid:*

$$\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f) + |f|^2_{L^2_{\gamma/2}} \gtrsim |W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2}. \tag{2.42}$$

$$\begin{aligned}&\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f) + |W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2} + |W^\varepsilon W_{\gamma/2}f|^2_{L^2}\\ &\quad\gtrsim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})W_{\gamma/2}f|^2_{L^2}.\end{aligned} \tag{2.43}$$

By a suitable combination, we have

$$\begin{aligned}&\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f) + |W^\varepsilon W_{\gamma/2}f|^2_{L^2}\\ &\quad\gtrsim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})W_{\gamma/2}f|^2_{L^2} + |W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2}.\end{aligned} \tag{2.44}$$

Proof. Since $\gamma \leq 0$, then $|v - v_*|^\gamma \geq \langle v - v_*\rangle^\gamma$, and thus $\mathcal{N}^{\varepsilon,\gamma}(g, f) \geq \widetilde{\mathcal{N}}^{\varepsilon,\gamma}(g, f)$. Then as a direct result of Lemma 2.6 with $\eta = 0$, $g = \mu^{\frac{1}{2}}$, we have

$$\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f) + |f|^2_{L^2_{\gamma/2}} \gtrsim \mathcal{N}^{\varepsilon,0}(W_{\gamma/2}\mu^{\frac{1}{2}}, W_{\gamma/2}f) \geq \mathcal{N}^{\varepsilon,0}(W_{-\frac{3}{2}}\mu^{\frac{1}{2}}, W_{\gamma/2}f). \tag{2.45}$$

Note that $|W_{-3}\mu|_{L^1}$ and $|W_{-3}\mu|_{L^1_1} + |W_{-3}\mu|_{L\log L}$ are generic constants. Then according to Lemma 2.3, we have

$$\mathcal{N}^{\varepsilon,0}(W_{-\frac{3}{2}}\mu^{\frac{1}{2}}, W_{\gamma/2}f) + |f|^2_{L^2_{\gamma/2}} \gtrsim |W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2},$$

from which, together with (2.45), we get (2.42).

Taking $g = W_{-\frac{3}{2}}\mu^{\frac{1}{2}}$ in (2.31) of Proposition 2.3, we have

$$\begin{aligned}&\mathcal{N}^{\varepsilon,0}(W_{-\frac{3}{2}}\mu^{\frac{1}{2}}, W_{\gamma/2}f) + |W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2} + |W^\varepsilon W_{\gamma/2}f|^2_{L^2}\\ &\quad\gtrsim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})W_{\gamma/2}f|^2_{L^2},\end{aligned}$$

from which together with (2.45), we get (2.43). The proof is completed. ∎

2.2.5. Coercivity estimate. We are ready to prove the coercivity estimate for $\mathcal{L}^\varepsilon$ in Theorem 2.1.

Proof of Theorem 2.1. By combining Proposition 2.1 and (2.44) in Proposition 2.4, we get

$$\begin{aligned}&\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f) + \mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}}) + |f|^2_{L^2_{\gamma/2}}\\ &\quad\gtrsim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})W_{\gamma/2}f|^2_{L^2} + |W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2} + |W^\varepsilon W_{\gamma/2}f|^2_{L^2}\\ &\quad= |f|^2_{\varepsilon,\gamma/2}.\end{aligned} \tag{2.46}$$

Observe that $\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f) + \mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}})$ corresponds to the anisotropic norm $|||f|||^2$ introduced in [2]. Recalling (1.17), we have

$$\mathcal{L}^\varepsilon = \mathcal{L}^\varepsilon_1 + \mathcal{L}^\varepsilon_2, \quad \mathcal{L}^\varepsilon_1 g := -\Gamma^\varepsilon(\mu^{\frac{1}{2}}, g), \quad \mathcal{L}^\varepsilon_2 g := -\Gamma^\varepsilon(g, \mu^{\frac{1}{2}}). \tag{2.47}$$

By [2, Proposition 2.16] and Corollary 2.1, there holds

$$\langle \mathcal{L}^\varepsilon_1 f, f\rangle \geq \frac{1}{10}(\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f) + \mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}})) - C|f|^2_{L^2_{\gamma/2}}. \tag{2.48}$$

By Lemma 3.3 in the next section, we have

$$|\langle \mathcal{L}^\varepsilon_2 f, f\rangle| \lesssim |\mu^{\frac{1}{8}} f|^2_{L^2} \lesssim |f|^2_{L^2_{\gamma/2}}. \tag{2.49}$$

Combining (2.48), (2.49), and (2.46) completes the proof of the theorem. ■

2.3. Dissipation in microscopic space

In this subsection we consider the dissipative property of $\mathcal{L}^\varepsilon$ in the microscopic space. This is also referred to as the "spectral gap" estimate. Recall the kernel space ker defined in (1.12). An orthonormal basis of ker can be chosen as $\{\mu^{\frac{1}{2}}, \mu^{\frac{1}{2}}v_1, \mu^{\frac{1}{2}}v_2, \mu^{\frac{1}{2}}v_3, \mu^{\frac{1}{2}}(|v|^2-3)/\sqrt{6}\} := \{e_j\}_{1\leq j\leq 5}$. The projection operator $\mathbb{P}$ on the kernel space is defined as

$$\mathbb{P}f := \sum_{j=1}^{5}\langle f, e_j\rangle e_j = (a + b\cdot v + c|v|^2)\mu^{\frac{1}{2}}, \tag{2.50}$$

where for $1 \leq i \leq 3$,

$$a = \int_{\mathbb{R}^3}\Big(\frac{5}{2} - \frac{|v|^2}{2}\Big)\mu^{\frac{1}{2}} f\, dv, \quad b_i = \int_{\mathbb{R}^3} v_i \mu^{\frac{1}{2}} f\, dv, \quad c = \int_{\mathbb{R}^3}\Big(\frac{|v|^2}{6} - \frac{1}{2}\Big)\mu^{\frac{1}{2}} f\, dv.$$

The dissipative property of $\mathcal{L}^\varepsilon$ in the $\ker^\perp$ space is given in the following theorem.

Theorem 2.2. *Let $-3 < \gamma \leq 0$. There are two generic constants $\varepsilon_0, \lambda_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, it holds that*

$$\langle \mathcal{L}^\varepsilon g, g\rangle \geq \lambda_0 |g - \mathbb{P}g|^2_{\varepsilon,\gamma/2}.$$

Remark 2.3. We address that from the proof below, λ_0 depends only on γ and the lower bound for $\int b^\varepsilon(\cos\theta)\sin^2(\theta/2)\, d\sigma$.

To prove Theorem 2.2, we first introduce a special weight function U_δ defined by

$$U_\delta(v) := (1 + \delta^2|v|^2)^{\frac{1}{2}} \geq \max\{\delta|v|, 1\}. \tag{2.51}$$

Here, δ is a sufficiently small parameter. We then introduce a new smooth function χ. Recall the smooth function ζ in (1.31). Let $\chi(\cdot) = \zeta(\cdot/2)$. Then χ is a smooth function verifying $0 \leq \chi \leq 1$, $\chi = 1$ on $[0, 1]$, and $\chi = 0$ on $[2, \infty)$. Let $\chi_R(v) := \chi(|v|/R)$. The following lemma is [25, Lemma 3.2].

Lemma 2.7. *Set*

$$X(\gamma, R, \delta) := \delta^{-\gamma}\big((\chi_R)'(\chi_R)'_*(U_\delta^{\gamma/2})'(U_\delta^{\gamma/2})'_* - \chi_R(\chi_R)_*U_\delta^{\gamma/2}(U_\delta^{\gamma/2})_*\big)^2$$

with $\gamma \le 0 < \delta \le 1 \le R$*. Then*

$$X(\gamma, R, \delta) \lesssim (\delta^2 + R^{-2})\theta^2\langle v\rangle^{\gamma+2}\langle v_*\rangle^2 \mathbb{1}_{|v|\le 4R}.$$

Proof of Theorem 2.2. Supposing $\mathbb{P}g = 0$, then it suffices to prove $\langle \mathscr{L}^\varepsilon g, g\rangle \gtrsim |g|^2_{\varepsilon,\gamma/2}$. In the following, we specify the parameter γ in the operator $\mathscr{L}^\varepsilon$ and denote it by $\mathscr{L}^{\varepsilon,\gamma}$. For brevity, set

$$\begin{aligned}
J^{\varepsilon,\gamma}(g) &:= 4\langle \mathscr{L}^{\varepsilon,\gamma} g, g\rangle,\\
\mathbb{A}(f, g) &:= (f_*g + fg_* - f'_*g' - f'g'_*),\\
\mathbb{F}(f, g) &:= \mathbb{A}^2(f, g).
\end{aligned}$$

With this notation, we have $J^{\varepsilon,\gamma}(g) = \int B^\varepsilon \mathbb{F}(\mu^{\frac12}, g)\, d\sigma\, dv\, dv_*$. The proof is divided into four steps.

Step 1: Localization of $J^{\varepsilon,\gamma}(g)$*.* By (2.51) and the condition $\gamma \le 0$, we get

$$|v - v_*|^{-\gamma} \le C_\gamma\delta^\gamma((\delta|v|)^{-\gamma} + (\delta|v_*|)^{-\gamma}) \le 2C_\gamma\delta^\gamma U_\delta^{-\gamma}(v)U_\delta^{-\gamma}(v_*),$$

which gives $|v - v_*|^\gamma \gtrsim \delta^{-\gamma}U_\delta^\gamma(v)U_\delta^\gamma(v_*)$, so that

$$J^{\varepsilon,\gamma}(g) \gtrsim \delta^{-\gamma}\int b^\varepsilon \chi_R^2(\chi_R^2)_*U_\delta^\gamma(U_\delta^\gamma)_*\mathbb{F}(\mu^{\frac12}, g)\, d\sigma\, dv\, dv_*.$$

We include the function $\chi_R^2(\chi_R^2)_*U_\delta^\gamma(U_\delta^\gamma)_*$ inside $\mathbb{F}(\mu^{\frac12}, g)$, leading to $\mathbb{F}(\chi_R U_\delta^{\gamma/2}\mu^{\frac12}, \chi_R U_\delta^{\gamma/2} g)$ with some correction terms. For simplicity, set $h = \chi_R U_\delta^{\gamma/2}$, $f = \mu^{\frac12}$, then

$$\begin{aligned}
&\chi_R^2(\chi_R^2)_*U_\delta^\gamma(U_\delta^\gamma)_*\mathbb{F}(\mu^{\frac12}, g)\\
&\quad = h_*^2h^2\mathbb{F}(f, g) = \big(hh_*(f_*g + fg_*) - hh_*(f'_*g' + f'g'_*)\big)^2\\
&\quad = \big(hh_*(f_*g + fg_*) - h'h'_*(f'_*g' + f'g'_*) + (h'h'_* - hh_*)(f'_*g' + f'g'_*)\big)^2\\
&\quad \ge \frac12\big(hh_*(f_*g + fg_*) - h'h'_*(f'_*g' + f'g'_*)\big)^2 - (h'h'_* - hh_*)^2(f'_*g' + f'g'_*)^2\\
&\quad = \frac12\mathbb{F}(hf, hg) - (h'h'_* - hh_*)^2(f'_*g' + f'g'_*)^2.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
J^{\varepsilon,\gamma}(g) \gtrsim \frac12\delta^{-\gamma}&\int b^\varepsilon\mathbb{F}(\chi_R U_\delta^{\gamma/2}\mu^{\frac12}, \chi_R U_\delta^{\gamma/2} g)\, d\sigma\, dv\, dv_*\\
&- \delta^{-\gamma}\int b^\varepsilon(h'h'_* - hh_*)^2(f'_*g' + f'g'_*)^2\, d\sigma\, dv\, dv_*. \qquad (2.52)
\end{aligned}$$

We now move $\chi_R U_\delta^{\gamma/2}$ before $\mu^{\frac{1}{2}}$ out of $\mathbb{F}(\chi_R U_\delta^{\gamma/2}\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g)$, which leads to $\mathbb{F}(\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g)$ with some correction terms. That is,

$$\begin{aligned}
&\mathbb{F}(\chi_R U_\delta^{\gamma/2}\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g) \\
&\quad = \mathbb{A}^2(\chi_R U_\delta^{\gamma/2}\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g) \\
&\quad = \big(\mathbb{A}(\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g) - \mathbb{A}\big((1-\chi_R U_\delta^{\gamma/2})\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g\big)\big)^2 \\
&\quad \geq \frac{1}{2}\mathbb{A}^2(\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g) - \mathbb{A}^2\big((1-\chi_R U_\delta^{\gamma/2})\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g\big) \\
&\quad = \frac{1}{2}\mathbb{F}(\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g) - \mathbb{F}((1-\chi_R U_\delta^{\gamma/2})\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g).
\end{aligned} \tag{2.53}$$

By symmetry, we have

$$\begin{aligned}
&\int b^\varepsilon (h' h'_* - h h_*)^2 (f'_* g' + f' g'_*)^2 \, d\sigma \, dv \, dv_* \\
&\qquad \leq 4 \int b^\varepsilon (h' h'_* - h h_*)^2 f_*^2 g^2 \, d\sigma \, dv \, dv_*.
\end{aligned} \tag{2.54}$$

Thanks to (2.52), (2.53), and (2.54), we get

$$\begin{aligned}
J^{\varepsilon,\gamma}(g) \gtrsim{}& \frac{1}{4}\delta^{-\gamma} \int b^\varepsilon \mathbb{F}(\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g) \, d\sigma \, dv \, dv_* \\
&- \frac{1}{2}\delta^{-\gamma} \int b^\varepsilon \mathbb{F}((1-\chi_R U_\delta^{\gamma/2})\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g) \, d\sigma \, dv \, dv_* \\
&- 4\delta^{-\gamma} \int b^\varepsilon (h' h'_* - h h_*)^2 f_*^2 g^2 \, d\sigma \, dv \, dv_* := \frac{1}{4}J_1 - \frac{1}{2}J_2 - 4J_3.
\end{aligned} \tag{2.55}$$

Step 2: Estimates of J_i $(i = 1, 2, 3)$. We will give the estimates term by term.

Lower bound for J_1. We claim that for $\varepsilon \leq 16^{-1} R^{-1}$ and some generic constant C,

$$J_1 \gtrsim \delta^{-\gamma} |g|^2_{L^2_{\gamma/2}} - C(\delta^2 + R^{-2})|g|^2_{\varepsilon,\gamma/2}. \tag{2.56}$$

By Wang Chang–Uhlenbeck [39], for any function F it holds that

$$\langle \mathcal{L}^{\varepsilon,0} F, F\rangle \gtrsim \left(\int b^\varepsilon(\cos\theta)\sin^2(\theta/2) d\sigma\right) |(\mathbb{I}-\mathbb{P})F|^2_{L^2},$$

where $\mathbb{I}$ is the identity operator. Thanks to (2.8), there is a generic constant c_0 such that

$$\langle \mathcal{L}^{\varepsilon,0} F, F\rangle \geq c_0 |(\mathbb{I}-\mathbb{P})F|^2_{L^2}. \tag{2.57}$$

Applying (2.57) with $F = \chi_R U_\delta^{\gamma/2} g$, and using $(a-b)^2 \geq a^2/2 - b^2$, we have

$$\begin{aligned}
J_1 &= \delta^{-\gamma} \int b^\varepsilon \mathbb{F}(\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g) \, d\sigma \, dv \, dv_* = 4\delta^{-\gamma} \langle \mathcal{L}^{\varepsilon,0} \chi_R U_\delta^{\gamma/2} g, \chi_R U_\delta^{\gamma/2} g\rangle \\
&\gtrsim \delta^{-\gamma} |(\mathbb{I}-\mathbb{P})(\chi_R U_\delta^{\gamma/2} g)|^2_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\gtrsim \frac{1}{2}\delta^{-\gamma}|\chi_R U_\delta^{\gamma/2} g|_{L^2}^2 - \delta^{-\gamma}|\mathbb{P}(\chi_R U_\delta^{\gamma/2} g)|_{L^2}^2 \\
&\gtrsim \frac{1}{4}\delta^{-\gamma}|U_\delta^{\gamma/2} g|_{L^2}^2 - \frac{1}{2}\delta^{-\gamma}|(1-\chi_R)U_\delta^{\gamma/2} g|_{L^2}^2 - \delta^{-\gamma}|\mathbb{P}(\chi_R U_\delta^{\gamma/2} g)|_{L^2}^2 \\
&:= J_{1,1} - J_{1,2} - J_{1,3}.
\end{aligned}
$$

- Since $\delta \le 1$ and $\gamma \le 0$, then $U_\delta^{\gamma/2} \ge W_{\gamma/2}$, which yields the leading term

$$
J_{1,1} \gtrsim \delta^{-\gamma}|g|_{L^2_{\gamma/2}}^2. \tag{2.58}
$$

- Thanks to the facts that $\delta^{-\gamma}U_\delta^{\gamma} \le W_\gamma$ and $1-\chi_R(v) = 0$ when $|v| \le R$, we have

$$
\begin{aligned}
J_{1,2} &= \frac{1}{2}\delta^{-\gamma}|(1-\chi_R)U_\delta^{\gamma/2} g|_{L^2}^2 \lesssim |(1-\chi_R)W_{\gamma/2} g|_{L^2}^2 \\
&\lesssim |\mathbb{1}_{|v|\ge R}\zeta(\varepsilon\cdot)W_{\gamma/2} g|_{L^2}^2 + |(1-\zeta(\varepsilon\cdot))W_{\gamma/2} g|_{L^2}^2 \\
&\lesssim R^{-2}|\zeta(\varepsilon\cdot)W_{\gamma/2+1} g|_{L^2}^2 + \varepsilon^2|(1-\zeta(\varepsilon\cdot))\varepsilon^{-2}W_{\gamma/2} g|_{L^2}^2 \\
&\lesssim (R^{-2}+\varepsilon^2)|W^\varepsilon W_{\gamma/2} g|_{L^2}^2,
\end{aligned} \tag{2.59}
$$

  where we have used (2.4) and (2.5) in the last inequality. By the assumption $\varepsilon \le 16^{-1}R^{-1}$, we have

$$
J_{1,2} \lesssim R^{-2}|W^\varepsilon W_{\gamma/2} g|_{L^2}^2. \tag{2.60}
$$

- We now estimate $J_{1,3}$. Recalling (2.50) for the definition of $\mathbb{P}$ and by the condition $\mathbb{P}g = 0$, we have

$$
\mathbb{P}(\chi_R U_\delta^{\gamma/2} g) = \sum_{i=1}^{5} e_i \int e_i \chi_R U_\delta^{\gamma/2} g\, dv = \sum_{i=1}^{5} e_i \int e_i (\chi_R U_\delta^{\gamma/2} - 1) g\, dv.
$$

  Observing that

$$
1 - \chi_R U_\delta^{\gamma/2} \lesssim 1 - \chi_R + \delta|v|\chi_R, \tag{2.61}
$$

  and thus $e_i(1-\chi_R U_\delta^{\gamma/2}) \lesssim (\delta + R^{-1})\mu^{\frac{1}{4}}$, we have $|\int e_i(\chi_R U_\delta^{\gamma/2} - 1) g\, dv| \lesssim (\delta + R^{-1})|\mu^{\frac{1}{8}} g|_{L^2}$, which gives

$$
J_{1,3} = \delta^{-\gamma}|\mathbb{P}(\chi_R U_\delta^{\gamma/2} g)|_{L^2}^2 \lesssim (\delta^2 + R^{-2})|\mu^{\frac{1}{8}} g|_{L^2}^2 \lesssim (\delta^2 + R^{-2})|g|_{L^2_{\gamma/2}}^2. \tag{2.62}
$$

Combining the estimates (2.58), (2.60), and (2.62) gives (2.56).

Upper bound for J_2. For simplicity, by setting $f_\gamma = (1-\chi_R U_\delta^{\gamma/2})\mu^{\frac{1}{2}}$ and $g_\gamma = \chi_R U_\delta^{\gamma/2} g$, we get

$$
\begin{aligned}
J_2 &= \delta^{-\gamma}\int b^\varepsilon \mathbb{F}((1-\chi_R U_\delta^{\gamma/2})\mu^{\frac{1}{2}}, \chi_R U_\delta^{\gamma/2} g)\, d\sigma\, dv\, dv_* \\
&= \delta^{-\gamma}\int b^\varepsilon \mathbb{F}(f_\gamma, g_\gamma)\, d\sigma\, dv\, dv_* \\
&\lesssim \delta^{-\gamma}\int b^\varepsilon (f_\gamma^2)_*(g_\gamma' - g_\gamma)^2\, d\sigma\, dv\, dv_* + \delta^{-\gamma}\int b^\varepsilon (g_\gamma^2)_*(f_\gamma' - f_\gamma)^2\, d\sigma\, dv\, dv_* \\
&:= J_{2,1} + J_{2,2}.
\end{aligned} \tag{2.63}
$$

Thanks to (2.61), we have

$$(f_\gamma^2)_* = ((1-\chi_R U_\delta^{\gamma/2})\mu^{\frac{1}{2}})_*^2 \lesssim (\delta^2 + R^{-2})\mu_*^{\frac{1}{2}}. \tag{2.64}$$

Plugging (2.64) into $J_{2,1}$, we have

$$\begin{aligned} J_{2,1} &\lesssim (\delta^2 + R^{-2})\delta^{-\gamma} \int b^\varepsilon \mu_*^{\frac{1}{2}} (g_\gamma' - g_\gamma)^2 \, d\sigma \, dv \, dv_* \\ &= (\delta^2 + R^{-2})\delta^{-\gamma} \mathcal{N}^{\varepsilon,0}(\mu^{\frac{1}{4}}, g_\gamma) \\ &\lesssim (\delta^2 + R^{-2})\delta^{-\gamma} |\chi_R U_\delta^{\gamma/2} g|_{\varepsilon,\gamma/2}^2 \lesssim (\delta^2 + R^{-2})|g|_{\varepsilon,\gamma/2}^2, \end{aligned} \tag{2.65}$$

where we have used (2.32) and Lemma A.3 with $\Phi = \delta^{-\gamma/2}\chi_R U_\delta^{\gamma/2} \in S_{1,0}^{\gamma/2}$ and $M = W^\varepsilon \in S_{1,0}^1$.

By Taylor expansion up to order 1, $f_\gamma' - f_\gamma = \int_0^1 (\nabla f_\gamma)(v(\kappa)) \cdot (v' - v) \, d\kappa$, from which, together with

$$\begin{aligned} |\nabla f_\gamma| &= |\nabla((1-\chi_R U_\delta^{\gamma/2})\mu^{\frac{1}{2}})| \\ &= |(1-\chi_R U_\delta^{\gamma/2})\nabla\mu^{\frac{1}{2}} - U_\delta^{\gamma/2}\mu^{\frac{1}{2}}\nabla\chi_R - \chi_R\mu^{\frac{1}{2}}\nabla U_\delta^{\gamma/2}| \\ &\lesssim \mu^{\frac{1}{8}}(\delta + R^{-1}), \end{aligned}$$

we get

$$|f_\gamma' - f_\gamma|^2 \lesssim (\delta^2 + R^{-2})\theta^2 \int_0^1 \mu^{\frac{1}{4}}(v(\kappa))|(v(\kappa) - v_*)|^2 \, d\kappa. \tag{2.66}$$

Since $R \le 16^{-1}\varepsilon^{-1}$, by the change $v \to v(\kappa)$, and (2.2), we have

$$\begin{aligned} J_{2,2} &\lesssim (\delta^2 + R^{-2})\delta^{-\gamma} \int b^\varepsilon \theta^2 (\chi_R U_\delta^{\gamma/2} g)_*^2 \mu^{\frac{1}{4}}(v(\kappa))|v(\kappa) - v_*|^2 \, d\sigma \, dv(\kappa) \, dv_* \, d\kappa \\ &\lesssim (\delta^2 + R^{-2})|\chi_R W_{\gamma/2+1} g|_{L^2}^2 \\ &\lesssim (\delta^2 + R^{-2})|W_{\gamma/2} W^\varepsilon g|_{L^2}^2 \lesssim (\delta^2 + R^{-2})|g|_{\varepsilon,\gamma/2}^2. \end{aligned} \tag{2.67}$$

Plugging estimates (2.65) and (2.67) into (2.63), we get

$$J_2 \lesssim (\delta^2 + R^{-2})|g|_{\varepsilon,\gamma/2}^2. \tag{2.68}$$

Upper bound for J_3. By Lemma 2.7, we have

$$\delta^{-\gamma}(h'h_*' - hh_*)^2 \lesssim (\delta^2 + R^{-2})\theta^2 \langle v\rangle^{\gamma+2}\langle v_*\rangle^2 \mathbb{1}_{|v|\le 4R}.$$

Since $8R \le \frac{1}{2}\varepsilon^{-1}$, by (2.2), we have

$$\begin{aligned} J_3 &= \delta^{-\gamma} \int b^\varepsilon (h'h_*' - hh_*)^2 \mu_* g^2 \, d\sigma \, dv \, dv_* \\ &\lesssim (\delta^2 + R^{-2}) \int b^\varepsilon \theta^2 \langle v_*\rangle^2 \langle v\rangle^{\gamma+2} \mu_* \mathbb{1}_{|v|\le 4R} g^2 \, d\sigma \, dv \, dv_* \end{aligned}$$

$$\lesssim (\delta^2 + R^{-2})|\mathbb{1}_{|\cdot|\leq 4R} W_{\gamma/2+1} g|^2_{L^2}$$
$$\lesssim (\delta^2 + R^{-2})|W_{\gamma/2} W^\varepsilon g|^2_{L^2} \leq (\delta^2 + R^{-2})|g|^2_{\varepsilon,\gamma/2}. \tag{2.69}$$

Step 3: Case $-\frac{7}{4} \leq \gamma \leq 0$. Plugging the estimates of J_1 in (2.56), J_2 in (2.68), J_3 in (2.69) into (2.55), for $\varepsilon \leq 16^{-1} R^{-1}, 0 < \delta < 1$, we get

$$J^{\varepsilon,\gamma}(g) \gtrsim \delta^{-\gamma}|g|^2_{L^2_{\gamma/2}} - C(\delta^2 + R^{-2})|g|^2_{\varepsilon,\gamma/2}.$$

Choosing $R = \delta^{-1}$, for some universal constants C_1, C_2, we have

$$J^{\varepsilon,\gamma}(g) \geq C_1 \delta^{-\gamma}|g|^2_{L^2_{\gamma/2}} - C_2\delta^2|g|^2_{\varepsilon,\gamma/2}. \tag{2.70}$$

By the coercivity estimate in Theorem 2.1, for some universal constants C_3, C_4, we have

$$J^{\varepsilon,\gamma}(g) \geq C_3|g|^2_{\varepsilon,\gamma/2} - C_4|g|^2_{L^2_{\gamma/2}}. \tag{2.71}$$

Multiplying (2.71) by $C_5\delta^2$ and adding the resulting inequality to (2.70), we get

$$(1 + C_5\delta^2)J^{\varepsilon,\gamma}(g) \geq (C_1\delta^{-\gamma} - C_4C_5\delta^2)|g|^2_{L^2_{\gamma/2}} + (C_3C_5 - C_2)\delta^2|g|^2_{\varepsilon,\gamma/2}.$$

First, we take C_5 large enough that $C_3C_5 - C_2 \geq C_2$, for example let $C_5 = 2C_2/C_3$. Then we take δ small enough that $C_1\delta^{-\gamma} - C_4C_5\delta^2 \geq 0$, for example, let $\delta = (\frac{C_1}{C_4C_5})^{1/(2+\gamma)} = (\frac{C_1C_3}{2C_4C_2})^{1/(2+\gamma)}$. Then we get

$$J^{\varepsilon,\gamma}(g) \geq C_2\delta^2|g|^2_{\varepsilon,\gamma/2} = C_2\Big(\frac{C_1C_3}{2C_4C_2}\Big)^{2/(2+\gamma)}|g|^2_{\varepsilon,\gamma/2}$$

for any $0 < \varepsilon \leq 16^{-1}R^{-1} = 16^{-1}(\frac{C_1C_3}{2C_4C_2})^{2/(2+\gamma)}$. Without loss of generality, we may assume $\frac{C_1C_3}{2C_4C_2} \leq 1$, which gives, for $-\frac{7}{4} \leq \gamma \leq 0$,

$$J^{\varepsilon,\gamma}(g) \geq C_2\Big(\frac{C_1C_3}{2C_4C_2}\Big)^8|g|^2_{\varepsilon,\gamma/2}. \tag{2.72}$$

Note that $C_2(\frac{C_1C_3}{2C_4C_2})^8$ depends only on the parameter γ and is a universal constant when $-\frac{7}{4} \leq \gamma \leq 0$.

Step 4: Case $-3 < \gamma < -\frac{7}{4}$. In this case, we take $-\frac{7}{4} \leq \alpha, \beta < 0$ such that $\alpha + \beta = \gamma$. Replacing b^ε by $b^\varepsilon|v - v_*|^\alpha$ and γ by β, similarly to (2.55), we get

$$\begin{aligned}
J^{\varepsilon,\gamma}(g) &\gtrsim \frac{1}{4}\delta^{-\beta}\int b^\varepsilon|v - v_*|^\alpha \mathbb{F}(\mu^{\frac{1}{2}}, \chi_R U^{\beta/2}_\delta g)\, d\sigma\, dv\, dv_* \\
&\quad - \frac{1}{2}\delta^{-\beta}\int b^\varepsilon|v - v_*|^\alpha \mathbb{F}((1 - \chi_R U^{\beta/2}_\delta)\mu^{\frac{1}{2}}, \chi_R U^{\beta/2}_\delta g)\, d\sigma\, dv\, dv_* \\
&\quad - 4\delta^{-\beta}\int b^\varepsilon|v - v_*|^\alpha (h'h'_* - hh_*)^2\mu_* g^2\, d\sigma\, dv\, dv_* \\
&:= \frac{1}{4}J^{\alpha,\beta}_1 - \frac{1}{2}J^{\alpha,\beta}_2 - 4J^{\alpha,\beta}_3,
\end{aligned} \tag{2.73}$$

where $h := \chi_R U^{\beta/2}_\delta$.

Lower bound for $J_1^{\alpha,\beta}$. Since $-\frac{7}{4} \le \alpha < 0$, we can use the previous estimate (2.72) to get

$$\begin{aligned} J_1^{\alpha,\beta} &= \delta^{-\beta} J^{\varepsilon,\alpha}(\chi_R U_\delta^{\beta/2} g) \\ &= \delta^{-\beta} J^{\varepsilon,\alpha}((\mathbb{I}-\mathbb{P})\chi_R U_\delta^{\beta/2} g) \gtrsim \delta^{-\beta} |W_{\alpha/2}(\mathbb{I}-\mathbb{P})(\chi_R U_\delta^{\beta/2} g)|_{L^2}^2. \end{aligned}$$

Using $(a-b)^2 \ge a^2/2 - b^2$, we get

$$\begin{aligned} J_1^{\alpha,\beta} &\gtrsim \frac{1}{4}\delta^{-\beta}|W_{\alpha/2}U_\delta^{\beta/2} g|_{L^2}^2 - \frac{1}{2}\delta^{-\beta}|W_{\alpha/2}(1-\chi_R)U_\delta^{\beta/2} g|_{L^2}^2 \\ &\quad - \delta^{-\beta}|W_{\alpha/2}\mathbb{P}(\chi_R U_\delta^{\beta/2} g)|_{L^2}^2 \\ &:= J_{1,1}^{\alpha,\beta} - J_{1,2}^{\alpha,\beta} - J_{1,3}^{\alpha,\beta}. \end{aligned} \tag{2.74}$$

Thanks to $U_\delta \le W$, one has $U_\delta^{\beta/2} \ge W_{\beta/2}$ and

$$J_{1,1}^{\alpha,\beta} \gtrsim \delta^{-\beta}|W_{\alpha/2}W_{\beta/2}g|_{L^2}^2 = \delta^{-\beta}|g|_{L^2_{\gamma/2}}^2. \tag{2.75}$$

Thanks to $\delta^{-\beta}U_\delta^\beta \le W_\beta$, similarly to (2.59) and (2.60), we have

$$J_{1,2}^{\alpha,\beta} \lesssim |W_{\alpha/2}(1-\chi_R)W_{\beta/2}g|_{L^2}^2 \lesssim R^{-2}|W^\varepsilon W_{\gamma/2}g|_{L^2}^2. \tag{2.76}$$

Similarly to (2.62) we get

$$J_{1,3}^{\alpha,\beta} = \delta^{-\beta}|W_{\alpha/2}\mathbb{P}(\chi_R U_\delta^{\beta/2} g)|_{L^2}^2 \lesssim (\delta^2 + R^{-2})|\mu^{\frac{1}{8}} g|_{L^2}^2 \lesssim (\delta^2 + R^{-2})|g|_{L^2_{\gamma/2}}^2. \tag{2.77}$$

Plugging (2.75), (2.76), (2.77) into (2.74), we get

$$J_1^{\alpha,\beta} \gtrsim \delta^{-\beta}|g|_{L^2_{\gamma/2}}^2 - C(\delta^2 + R^{-2})|g|_{\varepsilon,\gamma/2}^2. \tag{2.78}$$

Upper bound for $J_2^{\alpha,\beta}$. Now we estimate

$$J_2^{\alpha,\beta} = \delta^{-\beta}\int b^\varepsilon |v-v_*|^\alpha \mathbb{F}((1-\chi_R U_\delta^{\beta/2})\mu^{\frac{1}{2}}, \chi_R U_\delta^{\beta/2} g)\, d\sigma\, dv\, dv_*.$$

For simplicity, set $f_\beta = (1-\chi_R U_\delta^{\beta/2})\mu^{\frac{1}{2}}$, $g_\beta = \chi_R U_\delta^{\beta/2} g$. Then we get

$$\begin{aligned} J_2^{\alpha,\beta} &\lesssim \delta^{-\beta}\int b^\varepsilon |v-v_*|^\alpha (f_\beta^2)_*(g'_\beta - g_\beta)^2\, d\sigma\, dv\, dv_* \\ &\quad + \delta^{-\beta}\int b^\varepsilon |v-v_*|^\alpha (g_\beta^2)_*(f'_\beta - f_\beta)^2\, d\sigma\, dv\, dv_* \\ &:= J_{2,1}^{\alpha,\beta} + J_{2,2}^{\alpha,\beta}. \end{aligned} \tag{2.79}$$

Similarly to (2.64) we get $f_\beta^2 = ((1-\chi_R U_\delta^{\beta/2})\mu^{\frac{1}{2}})^2 \lesssim (\delta^2 + R^{-2})\mu^{\frac{1}{2}}$, from which, together with $\delta^{-\beta} U_\delta^{\beta/2} \leq W_{\beta/2}$, we get

$$\begin{aligned} J_{2,1}^{\alpha,\beta} &\lesssim (\delta^2 + R^{-2})\delta^{-\beta} \int b^\varepsilon |v-v_*|^\alpha \mu_*^{\frac{1}{2}} (g'_\beta - g_\beta)^2 \, d\sigma \, dv \, dv_* \\ &= (\delta^2 + R^{-2})\delta^{-\beta} \mathcal{N}^{\varepsilon,\alpha}(\mu^{\frac{1}{4}}, g_\beta) \\ &\lesssim (\delta^2 + R^{-2})\delta^{-\beta} |\chi_R U_\delta^{\beta/2} g|^2_{\varepsilon,\alpha/2} \lesssim (\delta^2 + R^{-2})|g|^2_{\varepsilon,\gamma/2}, \end{aligned} \tag{2.80}$$

where we have used Corollary 3.1 and Lemma A.3 with $\Phi = W_{\alpha/2}\delta^{-\beta/2}U_\delta^{\beta/2}\chi_R$, $M = W^\varepsilon$. Similarly to (2.66), we have

$$|f'_\beta - f_\beta|^2 \lesssim (\delta^2 + R^{-2})\theta^2 \int_0^1 \mu^{\frac{1}{4}}(v(\kappa))|v(\kappa) - v_*|^2 \, d\kappa.$$

Thanks to $|v - v_*| \sim |v(\kappa) - v_*|$, since $2R \leq \frac{1}{2}\varepsilon^{-1}$, by the change of variable $v \to v(\kappa)$, we have

$$\begin{aligned} J_{2,2}^{\alpha,\beta} &\lesssim (\delta^2 + R^{-2})\delta^{-\beta} \int b^\varepsilon \theta^2 (\chi_R U_\delta^{\beta/2} g)_*^2 \mu^{\frac{1}{4}}(v(\kappa))|v(\kappa) - v_*|^{2+\alpha} \, d\sigma \, dv(\kappa) \, dv_* \, d\kappa \\ &\lesssim (\delta^2 + R^{-2})|\chi_R W_{\gamma/2+1} g|^2_{L^2} \lesssim (\delta^2 + R^{-2})|g|^2_{\varepsilon,\gamma/2}. \end{aligned} \tag{2.81}$$

Plugging (2.80) and (2.81) into (2.79), we get

$$J_2^{\alpha,\beta} \lesssim (\delta^2 + R^{-2})|g|^2_{\varepsilon,\gamma/2}. \tag{2.82}$$

Upper bound for $J_3^{\alpha,\beta}$. Recall that

$$J_3^{\alpha,\beta} = \delta^{-\beta} \int b^\varepsilon |v - v_*|^\alpha (h'h'_* - hh_*)^2 \mu_* g^2 \, d\sigma \, dv \, dv_*.$$

By Lemma 2.7, we have

$$\delta^{-\beta}(h'h'_* - hh_*)^2 = X(\beta, R, \delta) \lesssim (\delta^2 + R^{-2})\theta^2 \langle v_*\rangle^2 \langle v\rangle^{\beta+2} \mathbb{1}_{|v|\leq 4R}.$$

Thanks to $\int |v - v_*|^\alpha \langle v_*\rangle^2 \mu_* \, dv_* \lesssim \langle v\rangle^\alpha$, since $8R \leq \frac{1}{2}\varepsilon^{-1}$ we get

$$\begin{aligned} J_3^{\alpha,\beta} &\lesssim (\delta^2 + R^{-2}) \int b^\varepsilon |v - v_*|^\alpha \theta^2 \langle v_*\rangle^2 \langle v\rangle^{\beta+2} \mu_* \mathbb{1}_{|v|\leq 4R} g^2 \, d\sigma \, dv \, dv_* \\ &\lesssim (\delta^2 + R^{-2})|\mathbb{1}_{|\cdot|\leq 4R} W_{\gamma/2+1} g|^2_{L^2} \lesssim (\delta^2 + R^{-2})|g|^2_{\varepsilon,\gamma/2}. \end{aligned} \tag{2.83}$$

Plugging estimates (2.78), (2.82), and (2.83) into (2.73), we get

$$J^{\varepsilon,\gamma}(g) \gtrsim \delta^{-\beta}|g|^2_{L^2_{\gamma/2}} - C(\delta^2 + R^{-2})|g|^2_{\varepsilon,\gamma/2}.$$

Choosing $R = \delta^{-1}$, for some universal constants C_6, C_7 we get

$$J^{\varepsilon,\gamma}(g) \geq C_6 \delta^{-\beta} |g|^2_{L^2_{\gamma/2}} - C_7 \delta^2 |g|^2_{\varepsilon,\gamma/2}.$$

Together with the coercivity estimate (2.71), thanks to $-7/4 \leq \beta < 0$, by a similar argument to Step 3, similarly to (2.72) we get for $-3 < \gamma < -7/4$,

$$J^{\varepsilon,\gamma}(g) \geq C_7 \Big(\frac{C_6 C_3}{2C_4 C_7}\Big)^{2/(2+\beta)} |g|^2_{\varepsilon,\gamma/2} \geq C_7 \Big(\frac{C_6 C_3}{2C_4 C_7}\Big)^{8} |g|^2_{\varepsilon,\gamma/2}$$

for any $0 < \varepsilon \leq \min\{16^{-1}(\frac{C_1 C_3}{2C_4 C_2})^8, 16^{-1}(\frac{C_6 C_3}{2C_4 C_7})^8\}$. Note that C_4 depends on γ. And this completes the proof of the theorem. ■

3. Upper bound estimate

Unless otherwise specified, in this section the parameters γ and s satisfy $-3 < \gamma \leq 0$, $\frac{1}{2} < s < 1, \gamma + 2s > -1$. In particular, we specify the parameter γ in Γ^{ε} and use $\Gamma^{\varepsilon,\gamma}$. We derive a uniform upper bound for the nonlinear term $\Gamma^{\varepsilon,\gamma}$ given in the following theorem.

Theorem 3.1. *Let $-3 < \gamma \leq 0, \frac{1}{2} < s < 1, \gamma + 2s > -1$. It holds that*

$$\langle \Gamma^{\varepsilon,\gamma}(g,h), f\rangle \lesssim |g|_{L^2} |h|_{\varepsilon,\gamma/2} |f|_{\varepsilon,\gamma/2}. \tag{3.1}$$

Note that estimate (3.1) matches perfectly with Theorem 2.2, which enables us to establish the well-posedness theory for the Cauchy problem (1.16) near equilibrium. We remark that the parameter constraints $s > 1/2$, $\gamma + 2s > -1$ are a condition of [21, Lemma 1.1].

When $\gamma < 0$, the relative velocity $v - v_*$ has a singularity near 0, which creates some difficulty in obtaining the L^2 norm for the position g in (3.1). To deal with the singularity, we separate the kernel $B^{\varepsilon} = B^{\varepsilon,\gamma,<} + B^{\varepsilon,\gamma,>}$, where $B^{\varepsilon,\gamma,<} := \zeta(|v - v_*|) B^{\varepsilon}$, $B^{\varepsilon,\gamma,>} := (1 - \zeta(|v - v_*|)) B^{\varepsilon}$. We recall that the function ζ is defined in (1.31). We call $|v - v_*| \leq 1$ (support of $\zeta(|v - v_*|)$) the singular region and $|v - v_*| \geq 1/2$ (support of $1 - \zeta(|v - v_*|)$) the regular region.

We associate $Q^{\varepsilon,\gamma,>}$ with kernel $B^{\varepsilon,\gamma,>}$ and denote $\mathcal{L}^{\varepsilon,\gamma,>}$, $\mathcal{L}_1^{\varepsilon,\gamma,>}$, $\mathcal{L}_2^{\varepsilon,\gamma,>}$, $\Gamma^{\varepsilon,\gamma,>}$ correspondingly. Without ambiguity, we explicitly define the Boltzmann operator $Q^{\varepsilon,\gamma,>}$ as

$$Q^{\varepsilon,\gamma,>}(g,h)(v) := \int B^{\varepsilon,\gamma,>}(v - v_*, \sigma)(g'_* h' - g_* h)\, d\sigma\, dv_*. \tag{3.2}$$

Similarly to (1.17) we define

$$\Gamma^{\varepsilon,\gamma,>}(g,h) := \mu^{-\frac{1}{2}} Q^{\varepsilon,\gamma,>}(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), \tag{3.3}$$

$$\mathcal{L}^{\varepsilon,\gamma,>} g := -\Gamma^{\varepsilon,\gamma,>}(\mu^{\frac{1}{2}}, g) - \Gamma^{\varepsilon,\gamma,>}(g, \mu^{\frac{1}{2}}), \tag{3.4}$$

$$\mathcal{L}_1^{\varepsilon,\gamma,>} g := -\Gamma^{\varepsilon,\gamma,>}(\mu^{\frac{1}{2}}, g), \tag{3.5}$$

$$\mathcal{L}_2^{\varepsilon,\gamma,>} g := -\Gamma^{\varepsilon,\gamma,>}(g, \mu^{\frac{1}{2}}). \tag{3.6}$$

Obviously,

$$\begin{aligned}\Gamma^{\varepsilon,\gamma,>}(g,h) &= \mu^{-\frac{1}{2}} Q^{\varepsilon,\gamma,>}(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h)\\ &= \int B^{\varepsilon,\gamma,>}(v-v_*,\sigma)\mu_*^{\frac{1}{2}}(g'_* h' - g_* h)\, d\sigma\, dv_*\\ &= \int B^{\varepsilon,\gamma,>}(v-v_*,\sigma)((\mu^{\frac{1}{2}} g)'_* h' - (\mu^{\frac{1}{2}} g)_* h)\, d\sigma\, dv_*\\ &\quad + \int B^{\varepsilon,\gamma,>}(v-v_*,\sigma)(\mu_*^{\frac{1}{2}} - (\mu^{\frac{1}{2}})'_*) g'_* h'\, d\sigma\, dv_*\\ &= Q^{\varepsilon,\gamma,>}(\mu^{\frac{1}{2}} g, h) + I^{\varepsilon,\gamma,>}(g,h),\end{aligned}$$

where

$$I^{\varepsilon,\gamma,>}(g,h) := \int B^{\varepsilon,\gamma,>}(v-v_*,\sigma)(\mu_*^{\frac{1}{2}} - (\mu^{\frac{1}{2}})'_*) g'_* h'\, d\sigma\, dv_*. \tag{3.7}$$

We use the kernel $B^{\varepsilon,\gamma,<}(v-v_*,\sigma)$ for the Boltzmann operator $Q^{\varepsilon,\gamma,<}$ in the same way as in (3.2). As in (3.3), (3.4), (3.5), (3.6), we define $\Gamma^{\varepsilon,\gamma,<}(g,h)$, $\mathcal{L}^{\varepsilon,\gamma,<} g$, $\mathcal{L}_1^{\varepsilon,\gamma,<} g$, $\mathcal{L}_2^{\varepsilon,\gamma,<} g$ correspondingly. In addition, $I^{\varepsilon,\gamma,<}(g,h)$ is defined using the kernel $B^{\varepsilon,\gamma,<}(v-v_*,\sigma)$ as in (3.7). With this notation in hand, we have

$$\Gamma^{\varepsilon,\gamma}(g,h) = Q^{\varepsilon,\gamma}(\mu^{\frac{1}{2}} g, h) + I^{\varepsilon,\gamma}(g,h), \tag{3.8}$$

$$\Gamma^{\varepsilon,\gamma,>}(g,h) = Q^{\varepsilon,\gamma,>}(\mu^{\frac{1}{2}} g, h) + I^{\varepsilon,\gamma,>}(g,h), \tag{3.9}$$

$$\Gamma^{\varepsilon,\gamma,<}(g,h) = Q^{\varepsilon,\gamma,<}(\mu^{\frac{1}{2}} g, h) + I^{\varepsilon,\gamma,<}(g,h), \tag{3.10}$$

$$Q^{\varepsilon,\gamma}(g,h) = Q^{\varepsilon,\gamma,>}(g,h) + Q^{\varepsilon,\gamma,<}(g,h), \tag{3.11}$$

$$I^{\varepsilon,\gamma}(g,h) = I^{\varepsilon,\gamma,>}(g,h) + I^{\varepsilon,\gamma,<}(g,h). \tag{3.12}$$

We will give the estimates in the following propositions and theorems, as shown in Table 1. The theorems in the table can be derived from the propositions:

- By (3.9), Propositions 3.1 and 3.2 give Theorem 3.2.
- By (3.11), Propositions 3.1 and 3.3 give Theorem 3.3.
- By (3.12), Propositions 3.2 and 3.4 give Theorem 3.4.
- By (3.8), Theorems 3.3 and 3.4 give Theorem 3.1.

Therefore, it remains to prove Propositions 3.1–3.4 in this section.

3.1. Upper bound in the regular region

In this subsection we will give the upper bound for the nonlinear term $\Gamma^{\varepsilon,\gamma,>}(g,h)$. Thanks to (3.9), we have

$$\langle \Gamma^{\varepsilon,\gamma,>}(g,h), f\rangle = \langle Q^{\varepsilon,\gamma,>}(\mu^{\frac{1}{2}} g, h), f\rangle + \langle I^{\varepsilon,\gamma,>}(g,h), f\rangle. \tag{3.13}$$

We first consider $\langle Q^{\varepsilon,\gamma,>}(g,h), f\rangle$ in Section 3.1.1 and then $\langle I^{\varepsilon,\gamma,>}(g,h), f\rangle$ in Section 3.1.2.

Functionals	Proposition or theorem
$\langle Q^{\varepsilon,\gamma,>}(g,h), f\rangle$	Proposition 3.1
$\langle I^{\varepsilon,\gamma,>}(g,h), f\rangle$	Proposition 3.2
$\langle Q^{\varepsilon,\gamma,<}(\mu^{\frac{1}{2}}g,h), f\rangle$	Proposition 3.3
$\langle I^{\varepsilon,\gamma,<}(g,h), f\rangle$	Proposition 3.4
$\langle \Gamma^{\varepsilon,\gamma,>}(g,h), f\rangle$	Theorem 3.2
$\langle Q^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}g,h), f\rangle$	Theorem 3.3
$\langle I^{\varepsilon,\gamma}(g,h), f\rangle$	Theorem 3.4
$\langle \Gamma^{\varepsilon,\gamma}(g,h), f\rangle$	Theorem 3.1

Table 1. Summary of results.

3.1.1. Upper bound for $Q^{\varepsilon,\gamma,>}$. We give the upper bound for $Q^{\varepsilon,\gamma,>}$ in the following proposition.

Proposition 3.1. *The estimate* $|\langle Q^{\varepsilon,\gamma,>}(g,h), f\rangle| \lesssim |g|_{L^1_{|\gamma|+2}}|h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}$ *holds.*

Proof. Define the translation operator T_{v_*} by $(T_{v_*}f)(v) = f(v_* + v)$. By geometric decomposition, we have $\langle Q^{\varepsilon,\gamma,>}(g,h), f\rangle = \mathcal{A}_{\mathrm{r}} + \mathcal{A}_{\mathrm{s}}$, where

$$\mathcal{A}_{\mathrm{r}} := \int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big)|u|^\gamma(1-\zeta(|u|))g_*(T_{v_*}h)(u) \times \Big((T_{v_*}f)(u^+) - (T_{v_*}f)\Big(|u|\frac{u^+}{|u^+|}\Big)\Big)\,d\sigma\,dv_*\,du,$$

$$\mathcal{A}_{\mathrm{s}} := \int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big)|u|^\gamma(1-\zeta(|u|))g_*(T_{v_*}h)(u) \times \Big((T_{v_*}f)\Big(|u|\frac{u^+}{|u^+|}\Big) - (T_{v_*}f)(u)\Big)\,d\sigma\,dv_*\,du.$$

Note that "r" and "s" refer to "radial" and "spherical" respectively. We divide the proof into two steps.

Step 1: Estimate of $\mathcal{A}_{\mathrm{r}}$. By Lemma A.4 and Remark A.2, we have

$$|\mathcal{A}_{\mathrm{r}}| \lesssim \int |g_*|(|W^\varepsilon W_{\gamma/2}T_{v_*}h|_{L^2} + |W^\varepsilon(D)W_{\gamma/2}T_{v_*}h|_{L^2}) \times (|W^\varepsilon W_{\gamma/2}T_{v_*}f|_{L^2} + |W^\varepsilon(D)W_{\gamma/2}T_{v_*}f|_{L^2})\,dv_*. \tag{3.14}$$

Thanks to (2.7), for $u \in \mathbb{R}^3$, we have

$$|W^\varepsilon T_u f|_{L^2} \lesssim W^\varepsilon(u)|W^\varepsilon f|_{L^2}. \tag{3.15}$$

For $u \in \mathbb{R}^3, l \in \mathbb{R}$, we have $(T_u W^l)(v) = \langle v+u\rangle^l \lesssim C(l)\langle u\rangle^{|l|}\langle v\rangle^l$. As a result, we have

$$|T_u f|_{L^2_l} \lesssim \langle u\rangle^{|l|}|f|_{L^2_l}. \tag{3.16}$$

By (3.15) and (3.16), we have

$$
\begin{aligned}
|W^\varepsilon W_{\gamma/2} T_{v_*} h|_{L^2} &\lesssim W^\varepsilon(v_*) W_{|\gamma|/2}(v_*) |W^\varepsilon W_{\gamma/2} h|_{L^2} \\
&\lesssim W_{|\gamma|/2+1}(v_*) |W^\varepsilon W_{\gamma/2} h|_{L^2}.
\end{aligned}
\tag{3.17}
$$

Since $W^\varepsilon \in S^1_{1,0}$, $W_{\gamma/2} \in S^{\gamma/2}_{1,0}$, by Lemma A.3, we have

$$
\begin{aligned}
|W^\varepsilon(D) W_{\gamma/2} T_{v_*} h|_{L^2} &\lesssim |W_{\gamma/2} W^\varepsilon(D) T_{v_*} h|_{L^2} + |T_{v_*} h|_{H^0_{\gamma/2-1}} \\
&= |W_{\gamma/2} T_{v_*} W^\varepsilon(D) h|_{L^2} + |T_{v_*} h|_{H^0_{\gamma/2-1}} \\
&\lesssim W_{|\gamma|/2}(v_*) (|W_{\gamma/2} W^\varepsilon(D) h|_{L^2} + |h|_{L^2_{\gamma/2-1}}) \\
&\lesssim W_{|\gamma|/2}(v_*) |W^\varepsilon(D) W_{\gamma/2} h|_{L^2},
\end{aligned}
\tag{3.18}
$$

where we have used the fact that T_{v_*} and $W^\varepsilon(D)$ are commutable, inequality (3.16), and Lemma A.3. Plugging (3.17) and (3.18) into (3.14), we have

$$
\begin{aligned}
|\mathcal{A}_{\mathrm{r}}| \lesssim\ & |g|_{L^1_{|\gamma|+2}} (|W^\varepsilon(D) W_{\gamma/2} h|_{L^2} + |W^\varepsilon W_{\gamma/2} h|_{L^2}) \\
& \times (|W^\varepsilon(D) W_{\gamma/2} f|_{L^2} + |W^\varepsilon W_{\gamma/2} f|_{L^2}).
\end{aligned}
$$

Step 2: Estimate of $\mathcal{A}_{\mathrm{s}}$. Let $u = r\tau$ and $\varsigma = \frac{\tau+\sigma}{|\tau+\sigma|}$. Then $\frac{u}{|u|} \cdot \sigma = 2(\tau \cdot \varsigma)^2 - 1$ and $|u| \frac{u^+}{|u^+|} = r\varsigma$. In the change of variables $(u, \sigma) \to (r, \tau, \varsigma)$, one has $du\, d\sigma = 4(\tau \cdot \varsigma) r^2 \times dr\, d\tau\, d\varsigma$. Then

$$
\begin{aligned}
\mathcal{A}_{\mathrm{s}} &= 4 \int r^\gamma (1 - \zeta(r)) b^\varepsilon(2(\tau\cdot\varsigma)^2 - 1)(T_{v_*} h)(r\tau) \\
&\qquad \times \big((T_{v_*} f)(r\varsigma) - (T_{v_*} f)(r\tau)\big)(\tau\cdot\varsigma) r^2\, dr\, d\tau\, d\varsigma\, dv_* \\
&= 2 \int r^\gamma (1 - \zeta(r)) b^\varepsilon(2(\tau\cdot\varsigma)^2 - 1)\big((T_{v_*} h)(r\tau) - (T_{v_*} h)(r\varsigma)\big) \\
&\qquad \times \big((T_{v_*} f)(r\varsigma) - (T_{v_*} f)(r\tau)\big)(\tau\cdot\varsigma) r^2\, dr\, d\tau\, d\varsigma\, dv_* \\
&= -\frac{1}{2} \int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big) |u|^\gamma (1 - \zeta(|u|)) g_* \Big((T_{v_*} h)\Big(|u|\frac{u^+}{|u^+|}\Big) - (T_{v_*} h)(u)\Big) \\
&\qquad \times \Big((T_{v_*} f)\Big(|u|\frac{u^+}{|u^+|}\Big) - (T_{v_*} f)(u)\Big)\, d\sigma\, dv_*\, du.
\end{aligned}
$$

Then by the Cauchy–Schwarz inequality and the fact that $|u|^\gamma (1 - \zeta(|u|)) \lesssim \langle u\rangle^\gamma$, we have

$$
\begin{aligned}
|\mathcal{A}_{\mathrm{s}}| \ &\lesssim \Big\{\int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big) \langle u\rangle^\gamma |g_*| \Big((T_{v_*} h)\Big(|u|\frac{u^+}{|u^+|}\Big) - (T_{v_*} h)(u)\Big)^2 d\sigma\, dv_*\, du\Big\}^{\frac{1}{2}} \\
&\quad \times \Big\{\int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big) \langle u\rangle^\gamma |g_*| \Big((T_{v_*} f)\Big(|u|\frac{u^+}{|u^+|}\Big) - (T_{v_*}) f(u)\Big)^2 d\sigma\, dv_*\, du\Big\}^{\frac{1}{2}} \\
&:= (\mathcal{A}_{\mathrm{s}}(h))^{\frac{1}{2}} (\mathcal{A}_{\mathrm{s}}(f))^{\frac{1}{2}}.
\end{aligned}
$$

Note that $\mathcal{A}_{\mathrm{s}}(h)$ and $\mathcal{A}_{\mathrm{s}}(f)$ have exactly the same structure. It suffices to estimate $\mathcal{A}_{\mathrm{s}}(f)$. Since

$$\Big((T_{v_*}f)\Big(|u|\frac{u^+}{|u^+|}\Big)-(T_{v_*}f)(u)\Big)^2 \leq 2\Big((T_{v_*}f)\Big(|u|\frac{u^+}{|u^+|}\Big)-(T_{v_*}f)(u^+)\Big)^2 + 2\big((T_{v_*}f)(u^+)-(T_{v_*}f)(u)\big)^2,$$

we have

$$\begin{aligned}\mathcal{A}_{\mathrm{s}}(f) &\lesssim \int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big)\langle u\rangle^\gamma|g_*|\Big((T_{v_*}f)\Big(|u|\frac{u^+}{|u^+|}\Big)-(T_{v_*}f)(u^+)\Big)^2 d\sigma\, dv_*\, du\\ &\quad+\int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big)\langle u\rangle^\gamma|g_*|\Big((T_{v_*}f)(u^+)-(T_{v_*}f)(u)\Big)^2 d\sigma\, dv_*\, du\\ &:= \mathcal{A}_{\mathrm{s},1}(f)+\mathcal{A}_{\mathrm{s},2}(f).\end{aligned}$$

By Lemma 2.5, and the facts (3.17) and (3.18), we have

$$\mathcal{A}_{\mathrm{s},1}(f)\lesssim\int|g_*|\mathcal{Z}^{\varepsilon,\gamma}(T_{v_*}f)\,dv_*\lesssim|g|_{L^1_{|\gamma|+2}}(|W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2}+|W^\varepsilon W_{\gamma/2}f|^2_{L^2}).$$

Recalling the notation in (2.37), we observe that $\mathcal{A}_{\mathrm{s},2}(f)=\widetilde{\mathcal{N}}^{\varepsilon,\gamma}(\sqrt{|g|},f)$. By Lemma 2.6, we have

$$\widetilde{\mathcal{N}}^{\varepsilon,\gamma}(\sqrt{|g|},f)\lesssim\mathcal{N}^{\varepsilon,0}(W_{|\gamma|/2}\sqrt{|g|},W_{\gamma/2}f)+|g|_{L^1_{|\gamma+2|}}|f|^2_{L^2_{\gamma/2}}.$$

Then by (2.32) in Proposition 2.3, we get

$$\mathcal{N}^{\varepsilon,0}(W_{|\gamma|/2}\sqrt{|g|},W_{\gamma/2}f)\lesssim\big|W_{|\gamma|/2}\sqrt{|g|}\big|^2_{L^2_1}|f|^2_{\varepsilon,\gamma/2}\lesssim|g|_{L^1_{|\gamma|+2}}|f|^2_{\varepsilon,\gamma/2},$$

which gives $\mathcal{A}_{\mathrm{s},2}(f)\lesssim|g|_{L^1_{|\gamma|+2}}|f|^2_{\varepsilon,\gamma/2}$. Combining the estimates for $\mathcal{A}_{\mathrm{s},1}(f)$ and $\mathcal{A}_{\mathrm{s},2}(f)$, we get $\mathcal{A}_{\mathrm{s}}(f)\lesssim|g|_{L^1_{|\gamma|+2}}|f|^2_{\varepsilon,\gamma/2}$. Then we have

$$|\mathcal{A}_{\mathrm{s}}|\lesssim(\mathcal{A}_{\mathrm{s}}(h))^{\frac12}(\mathcal{A}_{\mathrm{s}}(f))^{\frac12}\lesssim|g|_{L^1_{|\gamma|+2}}|h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}.$$

Then the proof of the proposition is completed by the estimates of $\mathcal{A}_{\mathrm{r}}$ and $\mathcal{A}_{\mathrm{s}}$. ∎

3.1.2. Upper bound for $I^{\varepsilon,\gamma,>}$. We now turn to the upper bound estimate for the term $\langle I^{\varepsilon,\gamma,>}(g,h),f\rangle$ that is given in the following proposition.

Proposition 3.2. *The estimate* $|\langle I^{\varepsilon,\gamma,>}(g,h),f\rangle|\lesssim|g|_{L^2}|h|_{\varepsilon,\gamma/2}|W^\varepsilon f|_{L^2_{\gamma/2}}$ *holds.*

Proof. Noticing that $(\mu^{\frac12})'_*-(\mu^{\frac12})_*=((\mu^{\frac14})'_*-\mu^{\frac14}_*)^2+2\mu^{\frac14}_*((\mu^{\frac14})'_*-\mu^{\frac14}_*)$ and $h=(h-h')+h'$, we have

$$\langle I^{\varepsilon,\gamma,>}(g,h),f\rangle=\int B^{\varepsilon,\gamma,>}((\mu^{\frac12})'_*-(\mu^{\frac12})_*)g_*hf'\,d\sigma\,dv_*\,dv$$

$$
\begin{aligned}
&= \int B^{\varepsilon,\gamma,>}((\mu^{\frac{1}{8}})'_* + (\mu^{\frac{1}{8}})_*)^2((\mu^{\frac{1}{8}})'_* - (\mu^{\frac{1}{8}})_*)^2 g_* h f' \, d\sigma \, dv_* \, dv \\
&\quad + 2\int B^{\varepsilon,\gamma,>}((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}})(\mu^{\frac{1}{4}} g)_*(h - h') f' \, d\sigma \, dv_* \, dv \\
&\quad + 2\int B^{\varepsilon,\gamma,>}((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}})(\mu^{\frac{1}{4}} g)_* h' f' \, d\sigma \, dv_* \, dv \\
&:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \qquad (3.19)
\end{aligned}
$$

We divide the proof into three steps for $\mathcal{I}_1$, $\mathcal{I}_2$n, and $\mathcal{I}_3$ defined in (3.19) respectively. In the proof, the following estimate is used often:

$$
\begin{aligned}
((\mu^{\frac{1}{8}})'_* - (\mu^{\frac{1}{8}})_*)^2 &\lesssim \min\{1, |v - v_*|^2 \sin^2(\theta/2)\} \\
&\sim \min\{1, |v' - v_*|^2 \sin^2(\theta/2)\} \\
&\sim \min\{1, |v - v'_*|^2 \sin^2(\theta/2)\}. \qquad (3.20)
\end{aligned}
$$

Step 1: Estimate of $\mathcal{I}_1$. Recall that

$$
\mathcal{I}_1 = \int B^{\varepsilon,\gamma,>}((\mu^{\frac{1}{8}})'_* + (\mu^{\frac{1}{8}})_*)^2((\mu^{\frac{1}{8}})'_* - (\mu^{\frac{1}{8}})_*)^2 g_* h f' \, d\sigma \, dv_* \, dv.
$$

Since $|v - v_*| \geq 1/2$, we have

$$
|v - v_*|^\gamma \sim \langle v - v_* \rangle^\gamma. \qquad (3.21)
$$

By (3.21) and the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
|\mathcal{I}_1| &\lesssim \left\{ \int b^\varepsilon(\cos\theta)\langle v - v_*\rangle^\gamma ((\mu^{\frac{1}{8}})'_* + (\mu^{\frac{1}{8}})_*)^2((\mu^{\frac{1}{8}})'_* - (\mu^{\frac{1}{8}})_*)^2 g_*^2 h^2 \, d\sigma \, dv_* \, dv \right\}^{\frac{1}{2}} \\
&\quad \times \left\{ \int b^\varepsilon(\cos\theta)\langle v - v_*\rangle^\gamma ((\mu^{\frac{1}{8}})'_* + (\mu^{\frac{1}{8}})_*)^2((\mu^{\frac{1}{8}})'_* - (\mu^{\frac{1}{8}})_*)^2 f'^2 \, d\sigma \, dv_* \, dv \right\}^{\frac{1}{2}} \\
&:= (\mathcal{I}_{1,1})^{\frac{1}{2}}(\mathcal{I}_{1,2})^{\frac{1}{2}}.
\end{aligned}
$$

Estimate of $\mathcal{I}_{1,1}$. We claim that

$$
\begin{aligned}
\mathcal{A} &:= \int b^\varepsilon(\cos\theta)\langle v - v_*\rangle^\gamma ((\mu^{\frac{1}{8}})'_* + (\mu^{\frac{1}{8}})_*)^2((\mu^{\frac{1}{8}})'_* - (\mu^{\frac{1}{8}})_*)^2 \, d\sigma \\
&\lesssim (W^\varepsilon)^2(v)\langle v\rangle^\gamma, \qquad (3.22)
\end{aligned}
$$

which yields $\mathcal{I}_{1,1} \lesssim |g|_{L^2}^2 |W^\varepsilon h|_{L^2_{\gamma/2}}^2$. Now we prove (3.22). Since $((\mu^{\frac{1}{8}})'_* + (\mu^{\frac{1}{8}})_*)^2 \leq 2(\mu^{\frac{1}{4}})'_* + 2\mu_*^{\frac{1}{4}}$, we have

$$
\begin{aligned}
\mathcal{A} &\lesssim \int b^\varepsilon(\cos\theta)\langle v - v_*\rangle^\gamma \mu_*^{\frac{1}{4}} ((\mu^{\frac{1}{8}})'_* - (\mu^{\frac{1}{8}})_*)^2 \, d\sigma \\
&\quad + \int b^\varepsilon(\cos\theta)\langle v - v_*\rangle^\gamma (\mu^{\frac{1}{4}})'_* ((\mu^{\frac{1}{8}})'_* - (\mu^{\frac{1}{8}})_*)^2 \, d\sigma := \mathcal{A}_1 + \mathcal{A}_2.
\end{aligned}
$$

By (3.20) and Proposition A.1, one has

$$\mathcal{A}_1 \lesssim \langle v - v_* \rangle^\gamma \mu_*^{\frac{1}{4}} (W^\varepsilon)^2 (v - v_*) \lesssim \langle v \rangle^\gamma \langle v_* \rangle^{|\gamma|} \mu_*^{\frac{1}{4}} (W^\varepsilon)^2 (v) (W^\varepsilon)^2 (v_*) \\ \lesssim (W^\varepsilon)^2 (v) \langle v \rangle^\gamma, \tag{3.23}$$

where we have used (2.7). As for $\mathcal{A}_2$, since $|v - v_*| \sim |v - v'_*|$ so that $\langle v - v_* \rangle^\gamma \lesssim \langle v - v'_* \rangle^\gamma \lesssim \langle v \rangle^\gamma \langle v'_* \rangle^{|\gamma|}$, we have

$$\mathcal{A}_2 \lesssim \langle v \rangle^\gamma \int b^\varepsilon (\cos\theta) (\mu^{\frac{1}{8}})'_* \min\{1, |v - v_*|^2 \sin^2(\theta/2)\}\, d\sigma.$$

If $|v - v_*| \geq 10|v|$, then it holds that

$$|v'_*| = |v'_* - v + v| \geq |v'_* - v| - |v| \geq \Big(\frac{1}{\sqrt{2}} - \frac{1}{10}\Big)|v - v_*| \geq \frac{1}{5}|v - v_*|,$$

and thus $(\mu^{\frac{1}{8}})'_* \lesssim \mu^{\frac{1}{200}}(v - v_*)$, which gives

$$\mathcal{A}_2 \lesssim \langle v \rangle^\gamma \mu^{\frac{1}{200}} (v - v_*) (W^\varepsilon)^2 (v - v_*) \lesssim \langle v \rangle^\gamma.$$

If $|v - v_*| \leq 10|v|$, by Proposition A.1, we have

$$\mathcal{A}_2 \lesssim \langle v \rangle^\gamma \int b^\varepsilon (\cos\theta) \min\{1, |v|^2 \sin^2(\theta/2)\}\, d\sigma \lesssim (W^\varepsilon)^2 (v) \langle v \rangle^\gamma.$$

Combining the estimates of $\mathcal{A}_1$ and $\mathcal{A}_2$ gives (3.22).

Estimate of $\mathcal{I}_{1,2}$. By the changes of variables $(v, v_*) \to (v', v'_*)$ and $(v, v_*, \sigma) \to (v_*, v, -\sigma)$, using (3.21) and $\gamma \leq 0$, we have

$$\mathcal{I}_{1,2} \lesssim \int b^\varepsilon (\cos\theta) |v - v_*|^\gamma ((\mu^{\frac{1}{8}})' - \mu^{\frac{1}{8}})^2 f_*^2 \, d\sigma\, dv_*\, dv \\ = \mathcal{N}^{\varepsilon,\gamma}(f, \mu^{\frac{1}{8}}) \lesssim |W^\varepsilon f|^2_{L^2_{\gamma/2}},$$

where we have used the estimate in Remark 2.1.

Combining the estimates of $\mathcal{I}_{1,1}$ and $\mathcal{I}_{1,2}$, we have

$$\mathcal{I}_1 \lesssim |g|_{L^2} |W^\varepsilon h|_{L^2_{\gamma/2}} |W^\varepsilon f|_{L^2_{\gamma/2}}. \tag{3.24}$$

Step 2: Estimate of $\mathcal{I}_2$. By the Cauchy–Schwarz inequality, we have

$$\mathcal{I}_2 = 2 \int B^{\varepsilon,\gamma,>} ((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}}) (\mu^{\frac{1}{4}} g)_* (h - h') f' \, d\sigma\, dv_*\, dv \\ \lesssim \Big(\int B^{\varepsilon,\gamma,>} |(\mu^{\frac{1}{4}} g)_*| (h - h')^2 \, d\sigma\, dv_*\, dv \Big)^{\frac{1}{2}} \\ \times \Big(\int B^{\varepsilon,\gamma,>} ((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}})^2 |(\mu^{\frac{1}{4}} g)_*| f'^2 \, d\sigma\, dv_*\, dv \Big)^{\frac{1}{2}} := (\mathcal{I}_{2,1})^{\frac{1}{2}} (\mathcal{I}_{2,2})^{\frac{1}{2}}.$$

Estimate of $\mathcal{I}_{2,1}$. Noticing that $(h-h')^2 = (h^2)' - h^2 - 2h(h'-h)$, we have

$$\mathcal{I}_{2,1} = \int B^{\varepsilon,\gamma,>} (\mu^{\frac{1}{4}} g)_* ((h^2)' - h^2)\, d\sigma\, dv_*\, dv - 2\langle Q^{\varepsilon}(|\mu^{\frac{1}{4}} g|, h), h\rangle.$$

By the cancellation lemma and (3.21) we get

$$\begin{aligned}\left| \int B^{\varepsilon,\gamma,>} (\mu^{\frac{1}{4}} g)_* ((h^2)' - h^2)\, d\sigma\, dv_*\, dv \right| &\lesssim \int \langle v - v_*\rangle^{\gamma} |(\mu^{\frac{1}{4}} g)_* h^2|\, dv\, dv_* \\ &\lesssim |\mu^{\frac{1}{8}} g|_{L^2} |h|^2_{L^2_{\gamma/2}}.\end{aligned}$$

By Proposition 3.1, we have

$$|\langle Q^{\varepsilon,\gamma,>}(\mu^{\frac{1}{4}} g, h), h\rangle| \lesssim |\mu^{\frac{1}{4}} g|_{L^1_{|\gamma|+2}} |h|^2_{\varepsilon,\gamma/2} \lesssim |\mu^{\frac{1}{8}} g|_{L^2} |h|^2_{\varepsilon,\gamma/2}.$$

Then the above two estimates give

$$|\mathcal{I}_{2,1}| \lesssim |\mu^{\frac{1}{8}} g|_{L^2} |h|^2_{\varepsilon,\gamma/2}.$$

Estimate of $\mathcal{I}_{2,2}$. Using the change of variable $v \to v'$ and estimate (3.23) of $\mathcal{A}_1$, we have $\mathcal{I}_{2,2} \lesssim |\mu^{\frac{1}{8}} g|_{L^2} |W^{\varepsilon} f|^2_{L^2_{\gamma/2}}$.

Putting together the estimates of $\mathcal{I}_{2,1}$ and $\mathcal{I}_{2,2}$, we get

$$|\mathcal{I}_2| \lesssim |\mu^{\frac{1}{8}} g|_{L^2} |h|_{\varepsilon,\gamma/2} |W^{\varepsilon} f|_{L^2_{\gamma/2}}. \tag{3.25}$$

Step 3: Estimate of $\mathcal{I}_3$. By the changes of variables $(v, v_*) \to (v', v'_*)$ and $(v, v_*, \sigma) \to (v_*, v, -\sigma)$,

$$\mathcal{I}_3 = 2\int B^{\varepsilon,\gamma,>} (\mu^{\frac{1}{4}} - (\mu^{\frac{1}{4}})')(\mu^{\frac{1}{4}} g)' h_* f_*\, d\sigma\, dv_*\, dv.$$

For notational convenience, let

$$\begin{aligned} E_1 &= \{(v, v_*, \sigma) : |v_*| \geq 1/\varepsilon,\ \sin(\theta/2) \leq |v_*|^{-1}\}, \\ E_2 &= \{(v, v_*, \sigma) : |v_*| \geq 1/\varepsilon,\ |v_*|^{-1} \leq \sin(\theta/2) \leq \varepsilon\}, \\ E_3 &= \{(v, v_*, \sigma) : |v_*| \leq 1/\varepsilon\}. \end{aligned}$$

Then $\mathcal{I}_3$ can be decomposed into three parts: $\mathcal{I}_{3,i}$ corresponding to E_i for $i = 1, 2, 3$.

Estimate of $\mathcal{I}_{3,1}$. By Taylor expansion, one has

$$\mu^{\frac{1}{4}} - (\mu^{\frac{1}{4}})' = (\nabla \mu^{\frac{1}{4}})(v') \cdot (v - v') + \int_0^1 (1-\kappa)[(\nabla^2 \mu^{\frac{1}{4}})(v(\kappa)) : (v - v') \otimes (v - v')]\, d\kappa,$$

where $v(\kappa) = v' + \kappa(v - v')$. Observe that for any fixed v_*, it holds that

$$\int B^{\varepsilon,\gamma,>} \mathbb{1}_{|v_*| \geq \frac{1}{\varepsilon}, \sin\frac{\theta}{2} \leq |v_*|^{-1}} (\nabla \mu^{\frac{1}{4}})(v') \cdot (v - v') (\mu^{\frac{1}{4}} g)'\, d\sigma\, dv = 0,$$

which gives

$$|\mathcal{I}_{3,1}| = \bigg|\int_{E_3\times[0,1]} B^{\varepsilon,\gamma,>} \mathbb{1}_{|v_*|\geq\frac{1}{\varepsilon}, \sin\frac{\theta}{2}\leq|v_*|^{-1}} \times (1-\kappa)[(\nabla^2\mu^{\frac14})(v(\kappa)) : (v-v')\otimes(v-v')](\mu^{\frac14}g)'h_* f_*\, d\kappa\, d\sigma\, dv_*\, dv\bigg|$$

$$\lesssim \int b^\varepsilon(\cos\theta)\langle v'-v_*\rangle^{\gamma+2}\sin^2\frac{\theta}{2}\mathbb{1}_{|v_*|\geq\frac{1}{\varepsilon}, \sin\frac{\theta}{2}\leq|v_*|^{-1}}|(\mu^{\frac14}g)'h_* f_*|\, d\sigma\, dv_*\, dv'$$

$$\lesssim \varepsilon^{2s-2}\int \langle v_*\rangle^{\gamma+2s}\mathbb{1}_{|v_*|\geq\frac{1}{\varepsilon}}|(\mu^{\frac18}g)'h_* f_*|\, dv_*\, dv'$$

$$\lesssim |\mu^{\frac{1}{16}}g|_{L^2}|W^\varepsilon h|_{L^2_{\gamma/2}}|W^\varepsilon f|_{L^2_{\gamma/2}},$$

where we have used the fact that $|v'-v_*|\sim|v-v_*|$, the estimate (3.21), the change of variable $v\to v'$, the estimate

$$\int b^\varepsilon(\cos\theta)\sin^2\frac{\theta}{2}\mathbb{1}_{\sin\frac{\theta}{2}\leq|v_*|^{-1}}\, d\sigma \lesssim \varepsilon^{2s-2}|v_*|^{2s-2}\sim\varepsilon^{2s-2}\langle v_*\rangle^{2s-2},$$

$\langle v'-v_*\rangle^{\gamma+2}\lesssim\langle v_*\rangle^{\gamma+2}\langle v'\rangle^{|\gamma+2|}$, and $\langle v'\rangle^{|\gamma+2|}(\mu^{\frac14})'\lesssim(\mu^{\frac18})'$.

Estimate of $\mathcal{I}_{3,2}$. By the estimate

$$\int b^\varepsilon(\cos\theta)\sin^2\frac{\theta}{2}\mathbb{1}_{|v_*|^{-1}\leq\sin\frac{\theta}{2}\leq\varepsilon}\, d\sigma\lesssim\varepsilon^{2s-2}|v_*|^{2s}\sim\varepsilon^{2s-2}\langle v_*\rangle^{2s}$$

and a similar argument to that used for $\mathcal{I}_{3,1}$, we get

$$|\mathcal{I}_{3,2}|\lesssim\int b^\varepsilon(\cos\theta)\mathbb{1}_{|v_*|\geq\frac{1}{\varepsilon}, |v_*|^{-1}\leq\sin\frac{\theta}{2}\leq\varepsilon}\langle v'-v_*\rangle^\gamma|(\mu^{\frac14}g)'h_* f_*|\, d\sigma\, dv_*\, dv'$$

$$\lesssim\varepsilon^{2s-2}\int\langle v_*\rangle^{\gamma+2s}\mathbb{1}_{|v_*|\geq\frac{1}{\varepsilon}}|(\mu^{\frac18}g)'h_* f_*|\, dv_*\, dv'$$

$$\lesssim|\mu^{\frac{1}{16}}g|_{L^2}|W^\varepsilon h|_{L^2_{\gamma/2}}|W^\varepsilon f|_{L^2_{\gamma/2}}.$$

Estimate of $\mathcal{I}_{3,3}$. By estimate (2.8) and the similar argument used for $\mathcal{I}_{3,1}$, we get

$$|\mathcal{I}_{3,3}|\lesssim\int b^\varepsilon(\cos\theta)\langle v'-v_*\rangle^{\gamma+2}\sin^2\frac{\theta}{2}\mathbb{1}_{|v_*|\leq\frac{1}{\varepsilon}}|(\mu^{\frac14}g)'h_* f_*|\, d\sigma\, dv_*\, dv'$$

$$\lesssim\int\langle v_*\rangle^{\gamma+2}\mathbb{1}_{|v_*|\leq\frac{1}{\varepsilon}}|(\mu^{\frac18}g)'h_* f_*|\, dv_*\, dv'$$

$$\lesssim|\mu^{\frac{1}{16}}g|_{L^2}|W^\varepsilon h|_{L^2_{\gamma/2}}|W^\varepsilon f|_{L^2_{\gamma/2}}.$$

The estimates of $\mathcal{I}_{3,1}$, $\mathcal{I}_{3,2}$, and $\mathcal{I}_{3,3}$ give

$$|\mathcal{I}_3|\lesssim|\mu^{\frac18}g|_{L^2}|W^\varepsilon h|_{L^2_{\gamma/2}}|W^\varepsilon f|_{L^2_{\gamma/2}}. \tag{3.26}$$

The proof of the proposition is completed by combining (3.24), (3.25), and (3.26). ■

3.1.3. Upper bound for $\Gamma^{\varepsilon,\gamma,>}(g,h)$. Recalling (3.13), by Propositions 3.1 and 3.2, noting that $|\mu^{\frac{1}{2}}g|_{L^1_{|\gamma|+2}} \lesssim |g|_{L^2}$, we have the following theorem.

Theorem 3.2. *The estimate* $|\langle\Gamma^{\varepsilon,\gamma,>}(g,h), f\rangle| \lesssim |g|_{L^2}|h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}$ *holds.*

3.2. Upper bound in the singular region

By (3.10), we have

$$\langle\Gamma^{\varepsilon,\gamma,<}(g,h), f\rangle = \langle Q^{\varepsilon,\gamma,<}(\mu^{\frac{1}{2}}g,h), f\rangle + \langle I^{\varepsilon,\gamma,<}(g,h), f\rangle.$$

We will give estimates for $\langle Q^{\varepsilon,\gamma,<}(\mu^{\frac{1}{2}}g,h), f\rangle$ and $\langle I^{\varepsilon,\gamma,<}(g,h), f\rangle$ in Sections 3.2.1 and 3.2.2 respectively.

3.2.1. Upper bound for $Q^{\varepsilon,\gamma,<}$. Recall [21, Lemma 1.1],

$$|\langle Q^{\varepsilon,\gamma}(g,h), f\rangle| \lesssim (|g|_{L^1_{N_1}} + |g|_{L^2_{N_1}})|W^\varepsilon(D)W_{N_2}h|_{L^2}|W^\varepsilon(D)W_{N_3}f|_{L^2}, \tag{3.27}$$

where N_1, N_2, N_3 satisfy $N_1 \geq |N_2| + |N_3|$ and $N_2 + N_3 \geq \gamma + 2$. Note that estimate (3.27) requires $(\gamma+2)$-order weight on the latter two functions, while (3.1) allows only γ order. On the other hand, we only need to consider $\langle Q^{\varepsilon,\gamma,<}(\mu^{\frac{1}{2}}g,h), f\rangle$ for which there is a factor $\mu^{\frac{1}{2}}$ for g. In addition, when $|v-v_*| \leq 1$ as in $Q^{\varepsilon,\gamma,<}$, one has $\mu(v_*) \lesssim \mu^{\frac{1}{2}}(v)$, which means weight can be exchanged between v and v_*. By the above observation, we can apply (3.27) to get the following upper bound on $Q^{\varepsilon,\gamma,<}$.

Proposition 3.3. *The estimate*

$$\langle Q^{\varepsilon,\gamma,<}(\mu^{\frac{1}{2}}g,h), f\rangle \lesssim |\mu^{\frac{3}{8}}g|_{L^2}|W^\varepsilon(D)\mu^{\frac{1}{64}}h|_{L^2}|W^\varepsilon(D)\mu^{\frac{1}{64}}f|_{L^2}$$

holds.

Proof. We omit the detail of the proof for brevity because we will use the weight exchange idea in Lemma 3.2 and Proposition 3.4 by using Lemma 3.1. With the weight exchange idea and the proof of (3.27) in [21], the proof for this proposition is straightforward. ∎

As a result of Propositions 3.1 and 3.3, we have the following theorem.

Theorem 3.3. *The estimate* $|\langle Q^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}g,h), f\rangle| \lesssim |\mu^{\frac{3}{8}}g|_{L^2}|h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}$ *holds.*

We turn to derive the upper bound on $\mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f)$ by applying Theorem 3.3.

Corollary 3.1. *The estimate* $\mathcal{N}^{\varepsilon,\gamma}(\mu^a, f) \lesssim |f|^2_{\varepsilon,\gamma/2}$ *holds for any* $a \geq \frac{1}{8}$.

Proof. By (2.10) and the identity $(f'-f)^2 = (f^2)' - f^2 - 2f(f'-f)$, we have

$$\mathcal{N}^{\varepsilon,\gamma}(\mu^a, f) = -2\langle Q^{\varepsilon,\gamma}(\mu^{2a}, f), f\rangle + \int B^\varepsilon \mu_*^{2a}((f^2)' - f^2)\, d\sigma\, dv_*\, dv.$$

Since $2a \geq \frac{1}{4}$, then by Theorem 3.3 and Corollary 2.1, the estimate follows directly. ∎

3.2.2. Upper bound for $I^{\varepsilon,\gamma,<}(g,h)$. The weight exchange idea in Proposition 3.3 is based on the following more general result.

Lemma 3.1. *For $\kappa, \iota \in [0,1]$, let $v(\kappa) := v + \kappa(v' - v)$, $v_*(\iota) := v_* + \iota(v'_* - v_*)$. Suppose $a, b, c, d \in \mathbb{R}$ with $a + b + c + d > 0$. If $|v - v_*| \le 1$, then*

$$\mu^a(v)\mu^b(v_*)\mu^c(v(\kappa))\mu^d(v_*(\iota)) \le (2\pi)^{-\frac{3}{2}(a+b+c+d)} \exp\Big(\frac{1}{2}C(a,b,c,d)\Big),$$

$$C(a,b,c,d) := \Big(a\Big(\frac{4a}{e} - 1\Big)\Big)^+ + \Big(b\Big(\frac{4b}{e} - 1\Big)\Big)^+ + \Big(c\Big(\frac{4c}{e} - 1\Big)\Big)^+ + \Big(d\Big(\frac{4d}{e} - 1\Big)\Big)^+,$$

with $e := a + b + c + d$, $A^+ := \max\{A, 0\}$.

Proof. Since $|v - v_*| \le 1$, we have $|v - v(\kappa)| \le 1$, $|v - v_*(\iota)| \le 1$. We assume without loss of generality $a > 0$. Let $e = a + b + c + d$; without loss of generality, we assume $e = 1$. Otherwise, one may consider $\frac{a}{e}, \frac{b}{e}, \frac{c}{e}, \frac{d}{e}$. Now $a + b + c + d = 1$ and

$$\begin{aligned} a|v|^2 + b|v_*|^2 + c|v(\kappa)|^2 + d|v_*(\iota)|^2 = {} & \frac{1}{4}|v|^2 + \Big(\frac{1}{4} - b\Big)|v|^2 + b|v_*|^2 \\ & + \Big(\frac{1}{4} - c\Big)|v|^2 + c|v(\kappa)|^2 \\ & + \Big(\frac{1}{4} - d\Big)|v|^2 + d|v_*(\iota)|^2. \end{aligned}$$

To estimate $(\frac{1}{4} - b)|v|^2 + b|v_*|^2$, note that for any $0 \le \alpha < 1$, it holds that

$$|x|^2 \ge \alpha|y|^2 - \frac{\alpha}{1-\alpha}|x - y|^2. \tag{3.28}$$

If $b \le 0$, by taking $\alpha = -b(\frac{1}{4} - b)^{-1}$ and (3.28), we have

$$\Big(\frac{1}{4} - b\Big)|v|^2 + b|v_*|^2 \ge 4b\Big(\frac{1}{4} - b\Big)|v - v_*|^2 \ge 4b\Big(\frac{1}{4} - b\Big).$$

If $0 < b < \frac{1}{4}$, it is obvious that

$$\Big(\frac{1}{4} - b\Big)|v|^2 + b|v_*|^2 \ge 0.$$

If $b \ge \frac{1}{4}$, by taking $\alpha = (b - \frac{1}{4})b^{-1}$ and (3.28), we have

$$\Big(\frac{1}{4} - b\Big)|v|^2 + b|v_*|^2 \ge 4b\Big(\frac{1}{4} - b\Big)|v - v_*|^2 \ge -4b\Big(b - \frac{1}{4}\Big).$$

In summary, we get

$$\Big(\frac{1}{4} - b\Big)|v|^2 + b|v_*|^2 \ge -(b(4b - 1))^+.$$

Since $|v - v(\kappa)| \leq 1$, $|v - v_*(\iota)| \leq 1$, similarly, we have

$$\Big(\frac{1}{4}-c\Big)|v|^2+c|v(\kappa)|^2 \geq -(c(4c-1))^+, \quad \Big(\frac{1}{4}-d\Big)|v|^2+d|v_*(\iota)|^2 \geq -(d(4d-1))^+.$$

Therefore,

$$a|v|^2+b|v_*|^2+c|v(\kappa)|^2+d|v_*(\iota)|^2 \geq -(b(4b-1))^+ - (c(4c-1))^+ - (d(4d-1))^+.$$

In the general case when $e \neq 1$ or the case when $a \leq 0$, we have

$$\begin{aligned} a|v|^2 + b|v_*|^2 + c|v(\kappa)|^2 + d|v_*(\iota)|^2 \geq &-\Big(a\Big(4\frac{a}{e}-1\Big)\Big)^+ - \Big(b\Big(4\frac{b}{e}-1\Big)\Big)^+ \\ &-\Big(c\Big(4\frac{c}{e}-1\Big)\Big)^+ - \Big(d\Big(4\frac{d}{e}-1\Big)\Big)^+. \end{aligned}$$

By noting $\mu(v) = (2\pi)^{-\frac{3}{2}} \exp(-|v|^2/2)$, we have the desired estimate. ■

To keep the proof of Proposition 3.4 to a reasonable length, we prepare some estimates in advance in the following Lemma 3.2. First, define

$$\begin{aligned} \mathcal{X}(G, H, F) := \int B^{\varepsilon,\gamma,<}((\mu^{\frac{1}{2}})'_* - (\mu^{\frac{1}{2}})_*)(\mu^{-\frac{1}{16}}G)_*\mu^{-\frac{1}{16}}H \\ \times (\mu^{-\frac{1}{16}}F)'\, d\sigma\, dv_*\, dv. \end{aligned} \tag{3.29}$$

According to (3.56), we have $\langle I^{\varepsilon,\gamma,<}(g, h), f\rangle = \mathcal{X}(G, H, F)$ if we set $G = \mu^{\frac{1}{16}}g$, $H = \mu^{\frac{1}{16}}h$, $F = \mu^{\frac{1}{16}}f$. We have two decompositions on $\mathcal{X}(G, H, F)$. The first is

$$\mathcal{X}(G, H, F) = \mathcal{A}(G, H, F) + \mathcal{B}(G, H, F), \tag{3.30}$$

$$\begin{aligned} \mathcal{A}(G, H, F) := \int B^{\varepsilon,\gamma,<}((\mu^{\frac{1}{2}})'_* - (\mu^{\frac{1}{2}})_*)(\mu^{-\frac{1}{16}}G)_*(\mu^{-\frac{1}{16}}H - (\mu^{-\frac{1}{16}}H)') \\ \times (\mu^{-\frac{1}{16}}F)'\, d\sigma\, dv_*\, dv, \end{aligned}$$

$$\mathcal{B}(G, H, F) := \int B^{\varepsilon,\gamma,<}((\mu^{\frac{1}{2}})'_* - (\mu^{\frac{1}{2}})_*)(\mu^{-\frac{1}{16}}G)_*(\mu^{-1/8}HF)'\, d\sigma\, dv_*\, dv. \tag{3.31}$$

Note that decomposition (3.30) uses regularity of H since $\mathcal{A}(G, H, F)$ contains $\mu^{-\frac{1}{16}}H - (\mu^{-\frac{1}{16}}H)'$. The second decomposition is

$$\mathcal{X}(G, H, F) = \mathcal{C}(G, H, F) + \mathcal{D}(G, H, F), \tag{3.32}$$

$$\begin{aligned} \mathcal{C}(G, H, F) := \int B^{\varepsilon,\gamma,<}((\mu^{\frac{1}{2}})'_* - (\mu^{\frac{1}{2}})_*)(\mu^{-\frac{1}{16}}G)_*\mu^{-\frac{1}{16}}H \\ \times ((\mu^{-\frac{1}{16}}F)' - \mu^{-\frac{1}{16}}F)\, d\sigma\, dv_*\, dv, \end{aligned}$$

$$\begin{aligned} \mathcal{D}(G, H, F) := \int B^{\varepsilon,\gamma,<}((\mu^{\frac{1}{2}})'_* - (\mu^{\frac{1}{2}})_*)(\mu^{-\frac{1}{16}}G)_*\mu^{-\frac{1}{16}}H \\ \times \mu^{-\frac{1}{16}}F\, d\sigma\, dv_*\, dv. \end{aligned} \tag{3.33}$$

Note that decomposition (3.32) uses the regularity of F since $\mathcal{C}(G,H,F)$ contains $(\mu^{-\frac{1}{16}}F)' - \mu^{-\frac{1}{16}}F$.

We now give some rough estimates of $\mathcal{X}(G,H,F)$ in the following lemma. Based on this, a refined estimate will be given in Proposition 3.4.

Lemma 3.2. *Let j be an integer satisfying $2^j \geq \frac{1}{4\varepsilon}$. Then the following estimates hold:*

$$|\mathcal{X}(G,H,F)| \lesssim |G|_{L^2}|H|_{H^1}|F|_{L^2}, \tag{3.34}$$

$$|\mathcal{X}(G,H,F)| \lesssim |G|_{L^2}|H|_{L^2}|F|_{H^1} + |G|_{L^2}|H|_{H^s}|F|_{H^s}, \tag{3.35}$$

$$\begin{aligned} |\mathcal{X}(G,H,F)| \lesssim{}& \varepsilon^{2s-2}2^{(2s-2)j}|G|_{L^2}|H|_{H^1}|F|_{L^2} \\ &+ \varepsilon^{2s-2}2^{(2s-1)j}|G|_{L^2}|H|_{L^2}|F|_{L^2}. \end{aligned} \tag{3.36}$$

Proof. The following is divided into three parts.

Part 1: Proof of (3.34). We first estimate $\mathcal{A}(G,H,F)$. By applying Taylor expansion to $(\mu^{\frac{1}{2}})'_* - (\mu^{\frac{1}{2}})_*$ and $\mu^{-\frac{1}{16}}H - (\mu^{-\frac{1}{16}}H)'$ up to order 1, we get

$$\begin{aligned} |(\mu^{\frac{1}{2}})'_* - \mu_*^{\frac{1}{2}}| &= \left|\int_0^1 (\nabla\mu^{\frac{1}{2}})(v_*(\iota))\cdot(v'-v)\,d\iota\right| \\ &\lesssim \sin(\theta/2)|v-v_*|\int_0^1 |\mu^{\frac{1}{4}}(v_*(\iota))|\,d\iota. \end{aligned} \tag{3.37}$$

$$\begin{aligned} |\mu^{-\frac{1}{16}}H - (\mu^{-\frac{1}{16}}H)'| &= \left|\int_0^1 (\nabla\mu^{-\frac{1}{16}}H)(v(\kappa))\cdot(v'-v)\,d\kappa\right| \\ &\lesssim \sin(\theta/2)|v-v_*|\int_0^1 \mu^{-\frac{1}{8}}(v(\kappa))\big(|H(v(\kappa))| + |\nabla H(v(\kappa))|\big)\,d\kappa. \end{aligned}$$

This implies

$$\begin{aligned} |\mathcal{A}(G,H,F)| &\lesssim \int B^{\varepsilon,\gamma,<}\sin^2(\theta/2)|v-v_*|^2\mu^{\frac{1}{4}}(v_*(\iota))|(\mu^{-\frac{1}{16}}G)_*|\mu^{-\frac{1}{8}}(v(\kappa)) \\ &\qquad \times \big(|H(v(\kappa))| + |\nabla H(v(\kappa))|\big)|(\mu^{-\frac{1}{16}}F)'|\,d\sigma\,dv_*\,dv\,d\iota\,d\kappa \\ &\lesssim \int B^{\varepsilon,\gamma,<}\sin^2(\theta/2)|v-v_*|^2|G_*|\big(|H(v(\kappa))| + |\nabla H(v(\kappa))|\big) \\ &\qquad \times |F'|\,d\sigma\,dv_*\,dv\,d\kappa, \end{aligned} \tag{3.38}$$

where we have used Lemma 3.1 in the last inequality. By the Cauchy–Schwarz inequality and the change of variables $(v,\theta)\to(v(\kappa),\theta(\kappa))$, we get

$$\begin{aligned} |\mathcal{A}(G,H,F)| \lesssim{}& \bigg(\int B^{\varepsilon,\gamma,<}\sin^2(\theta/2)|v-v_*|^2|G_*|\big(|H(v(\kappa))|^2 + |\nabla H(v(\kappa))|^2\big) \\ &\qquad \times d\sigma\,dv_*\,dv\,d\kappa\bigg)^{\frac{1}{2}} \\ &\times \bigg(\int B^{\varepsilon,\gamma,<}\sin^2(\theta/2)|v-v_*|^2|G_*|\,|F'|^2\,d\sigma\,dv_*\,dv\bigg)^{\frac{1}{2}} \end{aligned}$$

$$\lesssim \bigg(\int b^\varepsilon(\cos\theta) \sin^2(\theta/2) |v(\kappa) - v_*|^{\gamma+2} \mathbb{1}_{|v(\kappa)-v_*|\le 1} |G_*| \\ \times \big(|H(v(\kappa))|^2 + |\nabla H(v(\kappa))|^2\big) \sin\theta \, d\theta(\kappa) \, d\phi \, dv_* \, dv(\kappa) \, d\kappa \bigg)^{\frac{1}{2}} \\ \times \bigg(\int b^\varepsilon(\cos\theta) \sin^2(\theta/2) |v' - v_*|^{\gamma+2} \mathbb{1}_{|v'-v_*|\le 1} |G_*| \, |F'|^2 \\ \times \sin\theta \, d\theta' \, d\phi \, dv_* \, dv' \bigg)^{\frac{1}{2}}. \tag{3.39}$$

Here, $\theta(\kappa)$ is the angle between σ and $v(\kappa) - v_*$ and $\theta' = \frac{\theta}{2}$ is the angle between σ and $v' - v_*$. By noting that $b^\varepsilon(\cos\theta) = (1-s)\varepsilon^{2s-2} \sin^{-2-2s}(\theta/2) \mathbb{1}_{\sin\frac{\theta}{2} \le \varepsilon}$, and relations (2.14), (2.15), we have

$$\int_0^\pi b^\varepsilon(\cos\theta) \sin^2(\theta/2) \sin\theta \, d\theta(\kappa) \\ \le 2 \int_0^\pi (1-s)\varepsilon^{2s-2} \sin^{-2s}(\theta/2) \mathbb{1}_{\sin\frac{\theta}{2}\le\varepsilon} \sin\theta(\kappa) \, d\theta(\kappa) \\ \le 8 \int_0^\pi (1-s)\varepsilon^{2s-2} \sin^{1-2s}(\theta(\kappa)/2) \mathbb{1}_{\sin\frac{\theta(\kappa)}{2}\le\varepsilon} \, d\sin(\theta(\kappa)/2) \\ = 8 \int_0^\varepsilon (1-s)\varepsilon^{2s-2} t^{1-2s} \, dt = 4. \tag{3.40}$$

Hence,

$$|\mathcal{A}(G,H,F)| \lesssim \bigg(\int |v - v_*|^{\gamma+2} \mathbb{1}_{|v-v_*|\le 1} |G_*| (|H|^2 + |\nabla H|^2) \, dv_* \, dv \bigg)^{\frac{1}{2}} \\ \times \bigg(\int |v - v_*|^{\gamma+2} \mathbb{1}_{|v-v_*|\le 1} |G_*| \, |F|^2 \, dv_* \, dv \bigg)^{\frac{1}{2}}.$$

Since $\gamma > -3$, for any $v \in \mathbb{R}^3$, we have

$$\int |v - v_*|^{\gamma+2} \mathbb{1}_{|v-v_*|\le 1} |G_*| \, dv_* \le |G|_{L^2} \bigg(\int |v - v_*|^{2\gamma+4} \mathbb{1}_{|v-v_*|\le 1} \, dv_* \bigg)^{\frac{1}{2}} \\ \lesssim |G|_{L^2}, \tag{3.41}$$

which yields

$$|\mathcal{A}(G,H,F)| \lesssim |G|_{L^2} |H|_{H^1} |F|_{L^2}. \tag{3.42}$$

Similarly, we also have

$$|\mathcal{C}(G,H,F)| \lesssim |G|_{L^2} |H|_{L^2} |F|_{H^1}. \tag{3.43}$$

By Taylor expansion and $v'_* - v_* = v - v'$,

$$(\mu^{\frac{1}{2}})'_* - \mu_*^{\frac{1}{2}} = (\nabla \mu^{\frac{1}{2}})_* \cdot (v - v') + \int_0^1 (1-\kappa)(\nabla^2 \mu^{\frac{1}{2}})(v_*(\kappa)) : (v'_* - v_*) \otimes (v'_* - v_*)\, d\kappa. \tag{3.44}$$

Plugging this into (3.31), we have $\mathcal{B}(G,H,F) = \mathcal{B}_1(G,H,F) + \mathcal{B}_2(G,H,F)$, where

$$\mathcal{B}_1(G,H,F) := \int B^{\varepsilon,\gamma,<} (\mu^{-\frac{1}{16}} G)_* (\mu^{-\frac{1}{8}} HF)' (\nabla \mu^{\frac{1}{2}})_* \cdot (v - v')\, d\sigma\, dv_*\, dv,$$

$$\mathcal{B}_2(G,H,F) := \int B^{\varepsilon,\gamma,<} (\mu^{-\frac{1}{16}} G)_* (\mu^{-\frac{1}{8}} HF)' \times \left(\int_0^1 (1-\kappa)(\nabla^2 \mu^{\frac{1}{2}})(v_*(\kappa)) : (v'_* - v_*) \otimes (v'_* - v_*)\, d\kappa \right) d\sigma\, dv_*\, dv.$$

Note that for fixed v_*, one has $\int B^{\varepsilon,\gamma,<} (\mu^{-\frac{1}{8}} HF)'(v - v')\, d\sigma\, dv = 0$, which gives $\mathcal{B}_1(G,H,F) = 0$. By using the arguments in (3.38), (3.39), (3.40), and (3.41), we get

$$\begin{aligned} |\mathcal{B}(G,H,F)| &= |\mathcal{B}_2(G,H,F)| \\ &\lesssim \int B^{\varepsilon,\gamma,<} \sin^2(\theta/2) |v - v_*|^2 |G_* H' F'|\, d\sigma\, dv_*\, dv \\ &\lesssim |G|_{L^2} |H|_{L^2} |F|_{L^2}. \end{aligned} \tag{3.45}$$

Combining (3.42) and (3.45) and noting (3.30), we have (3.34).

Part 2: Proof of (3.35). Plugging (3.44) into (3.33) gives $\mathcal{D}(G,H,F) = \mathcal{D}_1(G,H,F) + \mathcal{D}_2(G,H,F)$, where

$$\mathcal{D}_1(G,H,F) := \int B^{\varepsilon,\gamma,<} (\mu^{-\frac{1}{16}} G)_* \mu^{-\frac{1}{16}} H \mu^{-\frac{1}{16}} F (\nabla \mu^{\frac{1}{2}})_* \cdot (v - v')\, d\sigma\, dv_*\, dv,$$

$$\mathcal{D}_2(G,H,F) := \int B^{\varepsilon,\gamma,<} (\mu^{-\frac{1}{16}} G)_* \mu^{-\frac{1}{16}} H \mu^{-\frac{1}{16}} F \times (1-\kappa)(\nabla^2 \mu^{\frac{1}{2}})(v_*(\kappa)) : (v'_* - v_*) \otimes (v'_* - v_*)\, d\kappa\, d\sigma\, dv_*\, dv.$$

By the symmetry of the σ integral and (2.8), we have

$$\int b^\varepsilon (v - v')\, d\sigma = (v - v_*) \int b^\varepsilon \sin^2(\theta/2)\, d\sigma = 4\pi (v - v_*), \tag{3.46}$$

which gives

$$|\mathcal{D}_1(G,H,F)| = 4\pi \left| \int |v - v_*|^\gamma \zeta(|v - v_*|) (\mu^{-\frac{1}{16}} G)_* \mu^{-\frac{1}{16}} H \mu^{-\frac{1}{16}} F (\nabla \mu^{\frac{1}{2}})_* \cdot (v - v_*)\, dv_*\, dv \right|$$

$$\lesssim \int |v - v_*|^{\gamma+1} \mathbb{1}_{|v-v_*|\le 1} |(\mu^{\frac{1}{16}} G)_* H F| \, dv_* \, dv$$
$$\lesssim \int |v - v_*|^{-2s} \mathbb{1}_{|v-v_*|\le 1} |(\mu^{\frac{1}{16}} G)_* H F| \, dv_* \, dv,$$

where we have used $\gamma + 2s > -1$. By the Hardy inequality, we have $\int |v - v_*|^{-2s} H^2 \, dv \lesssim |H|_{H^s}$ and thus

$$|\mathcal{D}_1(G, H, F)| \lesssim |G|_{L^2} |H|_{H^s} |F|_{H^s}.$$

Similarly to $\mathcal{B}_2(G, H, F)$, we get $|\mathcal{D}_2(G, H, F)| \lesssim |G|_{L^2} |H|_{L^2} |F|_{L^2}$. Therefore,

$$|\mathcal{D}(G, H, F)| \lesssim |G|_{L^2} |H|_{H^s} |F|_{H^s}. \tag{3.47}$$

By combining (3.43) and (3.47) and noting (3.32), we have (3.35).

Part 3: Proof of (3.36). Let $C_j(v) := \min\{2^{-j} |v - v_*|^{-1}, \varepsilon\}$. We decompose as

$$\mathcal{A}(G, H, F) = \mathcal{A}_{\le}(G, H, F) + \mathcal{A}_{\ge}(G, H, F), \tag{3.48}$$

where $\mathcal{A}_{\le}(G, H, F)$ stands for the integration over $\sin(\theta/2) \in [0, C_j(v)]$ and $\mathcal{A}_{\ge}(G, H, F)$ stands for the integration over $\sin(\theta/2) \in [C_j(v), \varepsilon]$. For $\mathcal{A}_{\le}(G, H, F)$, similarly to (3.38), we get

$$|\mathcal{A}_{\le}(G, H, F)| \lesssim \int B^{\varepsilon,\gamma,<} \sin^2(\theta/2) \mathbb{1}_{\sin\frac{\theta}{2} \le C_j(v)} |v - v_*|^2 |G_*| \times \big(|H(v(\kappa))| + |\nabla H(v(\kappa))|\big) |F'| \, d\sigma \, dv_* \, dv \, d\kappa.$$

By the Cauchy–Schwarz inequality and the change of variables $(v, \theta) \to (v(\kappa), \theta(\kappa))$, similarly to (3.39), we get

$$\begin{aligned} &|\mathcal{A}_{\le}(G, H, F)| \\ &\lesssim \bigg(\int b^{\varepsilon}(\cos\theta) \sin^2(\theta/2) \mathbb{1}_{\sin\frac{\theta}{2} \le C_j(v(\kappa))} |v(\kappa) - v_*|^{\gamma+2} \mathbb{1}_{|v(\kappa)-v_*|\le 1} |G_*| \\ &\qquad \times \big(|H(v(\kappa))|^2 + |\nabla H(v(\kappa))|^2\big) \sin\theta \, d\theta(\kappa) \, d\phi \, dv_* \, dv(\kappa) \, d\kappa \bigg)^{\frac{1}{2}} \\ &\quad \times \bigg(\int b^{\varepsilon}(\cos\theta) \sin^2(\theta/2) \mathbb{1}_{\sin\frac{\theta}{2} \le C_j(v')} |v' - v_*|^{\gamma+2} \mathbb{1}_{|v'-v_*|\le 1} \\ &\qquad \times |G_*| \, |F'|^2 \sin\theta \, d\theta' \, d\phi \, dv_* \, dv' \bigg)^{\frac{1}{2}}, \end{aligned} \tag{3.49}$$

where we have used the fact that $C_j(v) \le C_j(v(\kappa))$ since $|v(\kappa) - v_*| \le |v - v_*|$ for $\kappa \in [0, 1]$. Similarly to (3.40), we have

$$\begin{aligned} &\int_0^{\pi} b^{\varepsilon}(\cos\theta) \sin^2(\theta/2) \mathbb{1}_{\sin\frac{\theta}{2} \le C_j(v(\kappa))} \sin\theta \, d\theta(\kappa) \\ &\quad \le 8 \int_0^{2^{-j}|v(\kappa)-v_*|^{-1}} (1 - s)\varepsilon^{2s-2} t^{1-2s} \, dt = 4 \times \varepsilon^{2s-2} 2^{(2s-2)j} |v(\kappa) - v_*|^{2s-2}. \end{aligned}$$

Plugging this into (3.49) gives

$$
\begin{aligned}
&|\mathcal{A}_{\le}(G,H,F)| \\
&\quad\lesssim \varepsilon^{2s-2}2^{(2s-2)j}\Big(\int |v-v_*|^{\gamma+2s}\mathbb{1}_{|v-v_*|\le 1}|G_*|(|H|^2+|\nabla H|^2)\,dv_*\,dv\Big)^{\frac12} \\
&\qquad\times\Big(\int |v-v_*|^{\gamma+2s}\mathbb{1}_{|v-v_*|\le 1}|G_*|\,|F|^2\,dv_*\,dv\Big)^{\frac12}.
\end{aligned}
$$

Since $\gamma+2s>-1$, similarly to (3.41), for any $v\in\mathbb{R}^3$, we have

$$
\int |v-v_*|^{\gamma+2s}\mathbb{1}_{|v-v_*|\le 1}|G_*|\,dv_* \lesssim |G|_{L^2}, \tag{3.50}
$$

which yields

$$
|\mathcal{A}_{\le}(G,H,F)| \lesssim \varepsilon^{2s-2}2^{(2s-2)j}|G|_{L^2}|H|_{H^1}|F|_{L^2}. \tag{3.51}
$$

For $\mathcal{A}_{\ge}(G,H,F)$, by (3.37), we get

$$
\begin{aligned}
|\mathcal{A}_{\ge}(G,H,F)| &\lesssim \int B^{\varepsilon,\gamma,<}\mathbb{1}_{C_j(v)\le\sin\frac{\theta}{2}\le\varepsilon}\sin(\theta/2)|v-v_*|\mu^{\frac14}(v_*(\iota))|(\mu^{-\frac{1}{16}}G)_* \\
&\qquad\times(|\mu^{-\frac{1}{16}}H|+|(\mu^{-\frac{1}{16}}H)'|)|(\mu^{-\frac{1}{16}}F)'| \\
&\lesssim \int B^{\varepsilon,\gamma,<}\mathbb{1}_{C_j(v)\le\sin\frac{\theta}{2}\le\varepsilon}\sin(\theta/2)|v-v_*|\,|G_*|(|H|+|(H)'|)|(F)'|,
\end{aligned}
$$

where we have used Lemma 3.1 in the last inequality. By the Cauchy–Schwarz inequality and the change of variables $(v,\theta)\to(v(\kappa),\theta(\kappa))$, similarly to (3.39), we get

$$
\begin{aligned}
&|\mathcal{A}_{\ge}(G,H,F)| \\
&\quad\lesssim \Big[\Big(\int b^{\varepsilon}(\cos\theta)\sin(\theta/2)\mathbb{1}_{C_j(v)\le\sin\frac{\theta}{2}\le\varepsilon}|v-v_*|^{\gamma+1}\mathbb{1}_{|v-v_*|\le 1}|G_*|\,|H|^2 \\
&\qquad\qquad\times\sin\theta\,d\theta\,d\phi\,dv_*\,dv\Big)^{\frac12} \\
&\qquad+\Big(\int b^{\varepsilon}(\cos\theta)\sin(\theta/2)\mathbb{1}_{C_j(v)\le\sin\frac{\theta}{2}\le\varepsilon}|v'-v_*|^{\gamma+1}\mathbb{1}_{|v'-v_*|\le 1}|G_*|\,|H'|^2 \\
&\qquad\qquad\times\sin\theta\,d\theta'\,d\phi\,dv_*\,dv'\Big)^{\frac12}\Big] \\
&\qquad\times\Big(\int b^{\varepsilon}(\cos\theta)\sin(\theta/2)\mathbb{1}_{C_j(v)\le\sin\frac{\theta}{2}\le\varepsilon}|v'-v_*|^{\gamma+1}\mathbb{1}_{|v'-v_*|\le 1}|G_*|\,|F'|^2 \\
&\qquad\qquad\times\sin\theta\,d\theta'\,d\phi\,dv_*\,dv'\Big)^{\frac12}.
\end{aligned} \tag{3.52}
$$

Similarly to (3.40), we have

$$
\int_0^{\pi} b^{\varepsilon}(\cos\theta)\sin(\theta/2)\mathbb{1}_{C_j(v)\le\sin\frac{\theta}{2}\le\varepsilon}\sin\theta\,d\theta = 4\int_{C_j(v)}^{\varepsilon}(1-s)\varepsilon^{2s-2}t^{-2s}\,dt
$$

$$= 4\frac{1-s}{2s-1}\varepsilon^{2s-2}\big((C_j(v))^{1-2s} - \varepsilon^{1-2s}\big)$$
$$\leq 4\frac{1-s}{2s-1}\varepsilon^{2s-2}2^{(2s-1)j}|v-v_*|^{2s-1}, \tag{3.53}$$

where we have used $2s > 1$ and $(C_j(v))^{1-2s} - \varepsilon^{1-2s} \leq 2^{(2s-1)j}|v-v_*|^{2s-1}$ in the last inequality. Note that

$$C_j(v) = \min\{2^{-j}|v-v_*|^{-1}, \varepsilon\} \geq \{2^{-j}|v'-v_*|^{-1}2^{-1/2}, \varepsilon\} := C_j'(v')$$

and $\theta' = \frac{\theta}{2}$. Thus

$$C_j(v) \leq \sin(\theta/2) \leq \varepsilon \Rightarrow C_j'(v') \leq \sin(\theta') \leq \varepsilon.$$

By this, similarly to (3.40) and (3.53), we have

$$\int_0^\pi b^\varepsilon(\cos\theta)\sin(\theta/2)\mathbb{1}_{C_j(v)\leq \sin\frac{\theta}{2}\leq\varepsilon}\sin\theta\, d\theta'$$
$$\leq 4\int_{C_j'(v')}^{\varepsilon}(1-s)\varepsilon^{2s-2}t^{-2s}\,dt$$
$$\lesssim \frac{1-s}{2s-1}\varepsilon^{2s-2}2^{(2s-1)j}|v'-v_*|^{2s-1}. \tag{3.54}$$

Plugging (3.53) and (3.54) into (3.52) gives

$$|\mathcal{A}_{\geq}(G,H,F)| \lesssim \varepsilon^{2s-2}2^{(2s-1)j}\left(\int |v-v_*|^{\gamma+2s}\mathbb{1}_{|v-v_*|\leq 1}|G_*|\,|H|^2\,dv_*\,dv\right)^{\frac{1}{2}}$$
$$\times\left(\int |v-v_*|^{\gamma+2s}\mathbb{1}_{|v-v_*|\leq 1}|G_*|\,|F|^2\,dv_*\,dv\right)^{\frac{1}{2}}.$$

By (3.50), we obtain

$$|\mathcal{A}_{\geq}(G,H,F)| \lesssim \varepsilon^{2s-2}2^{(2s-1)j}|G|_{L^2}|H|_{L^2}|F|_{L^2}. \tag{3.55}$$

By combining (3.51), (3.55), and (3.45), and by noting (3.48), (3.30), and that

$$\varepsilon^{2s-2}2^{(2s-1)j} \gtrsim \varepsilon^{-1} \geq 1, \quad \text{because } 2^j \gtrsim \varepsilon^{-1} \text{ and } 2s > 1,$$

we obtain (3.36). ∎

We give an estimate for $\langle I^{\varepsilon,\gamma,<}(g,h), f\rangle$ in the following proposition.

Proposition 3.4. *For suitable functions g, h, and f, it holds that*

$$\langle I^{\varepsilon,\gamma,<}(g,h), f\rangle \lesssim |\mu^{\frac{1}{16}}g|_{L^2}|W^\varepsilon(D)\mu^{\frac{1}{16}}h|_{L^2}|W^\varepsilon(D)\mu^{\frac{1}{16}}f|_{L^2}.$$

Proof. Recall that

$$\langle I^{\varepsilon,\gamma,<}(g,h), f\rangle = \int B^{\varepsilon,\gamma,<}((\mu^{\frac{1}{2}})'_* - \mu_*^{\frac{1}{2}}) g_* h f' \, d\sigma \, dv_* \, dv.$$

Let $G = \mu^{\frac{1}{16}} g$, $H = \mu^{\frac{1}{16}} h$, $F = \mu^{\frac{1}{16}} f$. By (3.29), we have

$$\begin{aligned}\langle I^{\varepsilon,\gamma,<}(g,h), f\rangle &= \int B^{\varepsilon,\gamma,<}((\mu^{\frac{1}{2}})'_* - (\mu^{\frac{1}{2}})_*)(\mu^{-\frac{1}{16}} G)_* \mu^{-\frac{1}{16}} H (\mu^{-\frac{1}{16}} F)' \, d\sigma \, dv_* \, dv \\ &= \mathcal{X}(G,H,F).\end{aligned} \tag{3.56}$$

By the function ζ in (1.31), define $\mathfrak{F}_\zeta f := \zeta(\varepsilon|D|) f$ and $\mathfrak{F}^\zeta f := f - \zeta(\varepsilon|D|) f$. We then decompose as

$$\mathcal{X}(G,H,F) = \mathcal{X}(G,\mathfrak{F}_\zeta H, F) + \mathcal{X}(G,\mathfrak{F}^\zeta H, \mathfrak{F}_\zeta F) + \mathcal{X}(G,\mathfrak{F}^\zeta H, \mathfrak{F}^\zeta F).$$

By (3.34) and (2.4), we have

$$\begin{aligned}|\mathcal{X}(G,\mathfrak{F}_\zeta H, F)| &\lesssim |G|_{L^2} |\mathfrak{F}_\zeta H|_{H^1} |F|_{L^2} \\ &\lesssim |G|_{L^2} |W^\varepsilon(D) H|_{L^2} |F|_{L^2}.\end{aligned} \tag{3.57}$$

By (3.35), (2.4), and (2.5) , we have

$$\begin{aligned}|\mathcal{X}(G,\mathfrak{F}^\zeta H, \mathfrak{F}_\zeta F)| &\lesssim |G|_{L^2} |\mathfrak{F}^\zeta H|_{L^2} |\mathfrak{F}_\zeta F|_{H^1} + |G|_{L^2} |\mathfrak{F}^\zeta H|_{H^s} |\mathfrak{F}_\zeta F|_{H^s} \\ &\lesssim |G|_{L^2} |W^\varepsilon(D) H|_{L^2} |W^\varepsilon(D) F|_{L^2}.\end{aligned} \tag{3.58}$$

For $\mathcal{X}(G,\mathfrak{F}^\zeta H, \mathfrak{F}^\zeta F)$, by (3.30), the dyadic decomposition (A.2), (3.36), the Cauchy–Schwarz inequality, and (2.5), we have

$$\begin{aligned}|\mathcal{X}(G,\mathfrak{F}^\zeta H, \mathfrak{F}^\zeta F)| &= \Big| \sum_{j \geq [-\log_2 \varepsilon]-2} \mathcal{X}(G, \varphi_j(D) \mathfrak{F}^\zeta H, \mathfrak{F}^\zeta F) \Big| \\ &\lesssim |G|_{L^2} |\mathfrak{F}^\zeta F|_{L^2} \sum_{j \geq [-\log_2 \varepsilon]-2} \varepsilon^{2s-2} 2^{(2s-1)j} |\varphi_j(D) \mathfrak{F}^\zeta H|_{L^2} \\ &\lesssim |G|_{L^2} |F|_{L^2} \Big(\sum_{j \geq [-\log_2 \varepsilon]-2} \varepsilon^{2s-2} 2^{2sj} |\varphi_j(D) \mathfrak{F}^\zeta H|_{L^2}^2 \Big)^{\frac{1}{2}} \\ &\quad \times \Big(\sum_{j \geq [-\log_2 \varepsilon]-2} \varepsilon^{2s-2} 2^{(2s-2)j} \Big)^{\frac{1}{2}} \\ &\lesssim |G|_{L^2} |W^\varepsilon(D) H|_{L^2} |F|_{L^2}.\end{aligned} \tag{3.59}$$

Combining (3.57), (3.58), and (3.59) completes the proof. ∎

3.2.3. Upper bound for $I^{\varepsilon,\gamma}$. Combining Propositions 3.2 and 3.4 gives the following theorem.

Theorem 3.4. *The estimate $|\langle I^{\varepsilon,\gamma}(g,h), f\rangle| \lesssim |g|_{L^2} |h|_{\varepsilon,\gamma/2} |f|_{\varepsilon,\gamma/2}$ holds.*

3.3. Upper bound for the nonlinear term

Furthermore, by recalling (3.8) and by Theorems 3.3 and 3.4, we conclude the proof for Theorem 3.1.

Taking $g = \mu^{\frac{1}{2}}$ in Theorem 3.1 and recalling that $\mathcal{L}_1^{\varepsilon,\gamma} h = -\Gamma^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, h)$ in (2.47), we have the following corollary.

Corollary 3.2. *The estimate* $|\langle \mathcal{L}_1^{\varepsilon,\gamma} h, f\rangle| \lesssim |h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}$ *holds.*

Recalling that $\mathcal{L}_2^{\varepsilon,\gamma} h = -\Gamma^{\varepsilon,\gamma}(h, \mu^{\frac{1}{2}})$ in (2.47), we have the following lemma.

Lemma 3.3. *The estimate* $|\langle \mathcal{L}_2^{\varepsilon,\gamma} h, f\rangle| \lesssim |\mu^{\frac{1}{8}} h|_{L^2}|\mu^{\frac{1}{8}} f|_{L^2}$ *holds.*

For brevity, we omit the proof of Lemma 3.3. In fact, with Corollary 2.1, one can refer to [2, Lemma 2.15] to prove Lemma 3.3.

Noting that $\mathcal{L}^{\varepsilon,\gamma} h = \mathcal{L}_1^{\varepsilon,\gamma} h + \mathcal{L}_2^{\varepsilon,\gamma} h$, by Corollary 3.2 and Lemma 3.3, we have the following lemma.

Lemma 3.4. *The estimate* $|\langle \mathcal{L}^{\varepsilon,\gamma} h, f\rangle| \lesssim |h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}$ *holds.*

3.4. Weighted upper bound for the nonlinear term

In this subsection we give an upper bound estimate for $\Gamma^{\varepsilon,\gamma}$ with weight.

We will consider both polynomial and exponential weights together. For $l, q \geq 0$, let

$$W_{l,q}(v) := \langle v\rangle^l \exp(q\langle v\rangle).$$

Since $\langle v+u\rangle \leq \langle v\rangle + \langle u\rangle$ and $\langle v+u\rangle \leq \langle v\rangle\langle u\rangle$, we have

$$W_{l,q}(v+u) \leq W_{l,q}(v)W_{l,q}(u). \tag{3.60}$$

In addition, the following estimates hold:

$$\begin{aligned}
\nabla W_{l,q} &= lW_{l,q}\langle v\rangle^{-2}v + qW_{l,q}\langle v\rangle^{-1}v,\\
\nabla^2 W_{l,q} &= lW_{l,q}\langle v\rangle^{-2}I_3 + qW_{l,q}\langle v\rangle^{-1}I_3 - 2lW_{l,q}\langle v\rangle^{-4}v\otimes v - qW_{l,q}\langle v\rangle^{-3}v\otimes v\\
&\quad + l^2W_{l,q}\langle v\rangle^{-4}v\otimes v + q^2W_{l,q}\langle v\rangle^{-2}v\otimes v + 2lqW_{l,q}\langle v\rangle^{-3}v\otimes v.
\end{aligned}$$

Hence

$$|\nabla W_{l,q}| \lesssim (l+q)W_{l,q}, \tag{3.61}$$

$$\nabla^2 W_{l,q} \lesssim (l^2+q^2+l+q)W_{l,q}. \tag{3.62}$$

We first estimate the commutator $[Q^\varepsilon(\mu^{\frac{1}{2}}g, \cdot), W_{l,q}]$.

Lemma 3.5. *Let* $l, q \geq 0$. *It holds that*

$$|\langle Q^\varepsilon(\mu^{\frac{1}{2}}g, W_{l,q}h) - W_{l,q}Q^\varepsilon(\mu^{\frac{1}{2}}g, h), f\rangle| \lesssim |\mu^{\frac{1}{16}}g|_{L^2}|W_{l,q}h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}.$$

Proof. Note that

$$\begin{aligned}
&\langle Q^\varepsilon(\mu^{\frac{1}{2}} g, W_{l,q} h) - W_{l,q} Q^\varepsilon(\mu^{\frac{1}{2}} g, h), f\rangle \\
&\quad = \int B^\varepsilon (W_{l,q} - W'_{l,q}) \mu_*^{\frac{1}{2}} g_* h f' \, d\sigma \, dv_* \, dv \\
&\quad = \int B^\varepsilon (W_{l,q} - W'_{l,q}) \mu_*^{\frac{1}{2}} g_* h (f' - f) \, d\sigma \, dv_* \, dv \\
&\qquad + \int B^\varepsilon (W_{l,q} - W'_{l,q}) \mu_*^{\frac{1}{2}} g_* h f \, d\sigma \, dv_* \, dv \\
&\quad := \mathcal{A}_1 + \mathcal{A}_2.
\end{aligned}$$

We divide the proof into two steps.

Step 1: Estimate of $\mathcal{A}_1$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|\mathcal{A}_1| &\le \left\{ \int B^\varepsilon \mu_*^{\frac{1}{2}} (f' - f)^2 \, d\sigma \, dv_* \, dv \right\}^{\frac{1}{2}} \left\{ \int B^\varepsilon (W_{l,q} - W'_{l,q})^2 \mu_*^{\frac{1}{2}} g_*^2 h^2 \, d\sigma \, dv_* \, dv \right\}^{\frac{1}{2}} \\
&:= (\mathcal{A}_{1,1})^{\frac{1}{2}} (\mathcal{A}_{1,2})^{\frac{1}{2}}.
\end{aligned}$$

Note that $\mathcal{A}_{1,1} = \mathcal{N}^{\varepsilon,\gamma}(\mu^{\frac{1}{4}}, f) \lesssim |f|^2_{\varepsilon,\gamma/2}$ by Corollary 3.1. It remains to estimate $\mathcal{A}_{1,2}$. By Taylor expansion,

$$W'_{l,q} - W_{l,q} = \int_0^1 \nabla W_{l,q}(v(\kappa)) \cdot (v' - v) \, d\kappa.$$

Since $|v(\kappa)| \le |v| + |v_*|$, together with (3.61) and (3.60), we have $|\nabla W_{l,q}(v(\kappa))| \lesssim W_{l,q}(v) W_{l,q}(v_*)$, and thus

$$|W'_{l,q} - W_{l,q}| \lesssim W_{l,q}(v) W_{l,q}(v_*) |v - v_*| \sin \frac{\theta}{2}.$$

By (3.60) and $|v'| \le |v| + |v_*|$, we also have $|W_{l,q} - W'_{l,q}| \lesssim W_{l,q}(v) W_{l,q}(v_*)$. Combining the above two estimates gives

$$|W'_{l,q} - W_{l,q}|^2 \lesssim W^2_{l,q}(v) W^2_{l,q}(v_*) \min\{|v - v_*|^2 \sin^2(\theta/2), 1\}.$$

By this and Proposition A.1, we obtain

$$\begin{aligned}
\int B^\varepsilon (W_{l,q} - W'_{l,q})^2 \mu_*^{\frac{1}{2}} \, d\sigma &\lesssim \mathbb{1}_{|v-v_*| \ge 1} \langle v - v_* \rangle^\gamma W^2_{l,q}(v) W^2_{l,q}(v_*) (W^\varepsilon)^2 (v - v_*) \mu_*^{\frac{1}{2}} \\
&\qquad + \mathbb{1}_{|v-v_*| \le 1} |v - v_*|^{\gamma+2} W^2_{l,q}(v) W^2_{l,q}(v_*) \mu_*^{\frac{1}{2}} \\
&\lesssim \mathbb{1}_{|v-v_*| \ge 1} \langle v \rangle^\gamma W^2_{l,q}(v) (W^\varepsilon)^2 (v) \mu_*^{\frac{1}{8}} \\
&\qquad + \mathbb{1}_{|v-v_*| \le 1} |v - v_*|^{-1} \mu^{\frac{1}{8}} \mu_*^{\frac{1}{8}}. \qquad (3.63)
\end{aligned}$$

Here, when $|v-v_*|\geq 1$, we use $\langle v-v_*\rangle^\gamma \lesssim \langle v\rangle^\gamma \langle v_*\rangle^{|\gamma|}$ and $W^\varepsilon(v-v_*)\lesssim W^\varepsilon(v)W^\varepsilon(v_*)$ by (2.7). When $|v-v_*|\leq 1$, we use Lemma 3.1 to get $\mu_*^{\frac12}\lesssim \mu^{\frac16}\mu_*^{\frac16}$. Then in both cases, the additional weights can be absorbed by the exponential decay in μ. Plugging (3.63) into $\mathcal{A}_{1,2}$ gives

$$
\begin{aligned}
\mathcal{A}_{1,2} &\lesssim \int \langle v\rangle^\gamma W_{l,q}^2(v)(W^\varepsilon)^2(v)\mu_*^{\frac18} g_*^2 h^2\, dv_*\, dv + \int |v-v_*|^{-1}\mu^{\frac18}\mu_*^{\frac18} g_*^2 h^2\, dv_*\, dv \\
&\lesssim |\mu^{\frac{1}{16}} g|_{L^2}^2 |W_{\gamma/2}W^\varepsilon W_{l,q}h|_{L^2}^2 + |\mu^{\frac{1}{16}} g|_{L^2}^2 |\mu^{\frac{1}{16}} h|_{H^{\frac12}}^2 \\
&\lesssim |\mu^{\frac{1}{16}} g|_{L^2}^2 |W_{l,q}h|_{\varepsilon,\gamma/2}^2,
\end{aligned} \tag{3.64}
$$

where we have used the Hardy inequality and $|\cdot|_{\varepsilon,\gamma/2}\geq |\cdot|_{H^{1/2}_{\gamma/2}}$ because $s\geq\frac12$. Combining the estimates for $\mathcal{A}_{1,1}$ and $\mathcal{A}_{1,2}$, we have

$$
|\mathcal{A}_1| \lesssim |\mu^{\frac{1}{16}} g|_{L^2} |W_{l,q}h|_{\varepsilon,\gamma/2} |f|_{\varepsilon,\gamma/2}. \tag{3.65}
$$

Step 2: Estimate of $\mathcal{A}_2$. We want to show that

$$
\begin{aligned}
\left|\int B^\varepsilon (W'_{l,q}-W_{l,q})\mu_*^{\frac12}\, d\sigma\right| &\lesssim \mathbb{1}_{|v-v_*|\geq 1}\langle v\rangle^\gamma (W^\varepsilon)^2(v) W_{l,q}(v)\mu_*^{\frac18} \\
&\quad + \mathbb{1}_{|v-v_*|\leq 1}|v-v_*|^{\gamma+1}\mu^{\frac18}\mu_*^{\frac18}.
\end{aligned} \tag{3.66}
$$

By Taylor expansion, one has

$$
\begin{aligned}
W'_{l,q}-W_{l,q} &= (\nabla W_{l,q})(v)\cdot(v'-v) \\
&\quad + \int_0^1 (1-\kappa)(\nabla^2 W_{l,q})(v(\kappa)) : (v'-v)\otimes(v'-v)\, d\kappa.
\end{aligned} \tag{3.67}
$$

We first consider the case $|v-v_*|\leq 1$. By (3.67), (3.46), (3.61), (3.62), and (3.60) and Lemma 3.1 with $|v-v_*|\leq 1$, we have

$$
\begin{aligned}
&\left|\int B^\varepsilon (W'_{l,q}-W_{l,q})\mu_*^{\frac12}\, d\sigma\right| \\
&\lesssim \left|\int B^\varepsilon (\nabla W_{l,q})(v)\cdot(v'-v)\mu_*^{\frac12}\, d\sigma\right| + \int B^\varepsilon |(\nabla^2 W_{l,q})(v(\kappa))|\,|v'-v|^2\mu_*^{\frac12}\, d\kappa\, d\sigma \\
&\lesssim |v-v_*|^{\gamma+1}\mu_*^{\frac12} W_{l,q}(v)W_{l,q}(v_*) \\
&\lesssim |v-v_*|^{\gamma+1}\mu^{\frac18}\mu_*^{\frac18}.
\end{aligned} \tag{3.68}
$$

We next consider the case $|v-v_*|\geq 1$. Similarly to (3.68), since $|v-v_*|\sim\langle v-v_*\rangle$, we have

$$
\left|\int B^\varepsilon (W'_{l,q}-W_{l,q})\mu_*^{\frac12}\, d\sigma\right| \lesssim \langle v-v_*\rangle^{\gamma+2}\mu_*^{\frac12} W_{l,q}(v)W_{l,q}(v_*). \tag{3.69}
$$

If $|v| \le \varepsilon^{-1}$, then $W^\varepsilon(v) \gtrsim \langle v\rangle$. By (3.69) we have directly

$$\begin{aligned}\left|\int B^\varepsilon (W'_{l,q} - W_{l,q})\mu_*^{\frac12}\, d\sigma\right| &\lesssim \langle v\rangle^{\gamma+2}\mu_*^{\frac18} W_{l,q}(v)\\ &\lesssim \langle v\rangle^{\gamma}(W^\varepsilon)^2(v) W_{l,q}(v)\mu_*^{\frac18}. \end{aligned}\tag{3.70}$$

If $|v| > \varepsilon^{-1}, |v - v_*| \le \varepsilon^{-1}$, then $W^\varepsilon(v) \gtrsim \varepsilon^{s-1}\langle v\rangle^s$. By (3.69), we have

$$\begin{aligned}\left|\int B^\varepsilon (W'_{l,q} - W_{l,q})\mu_*^{\frac12}\, d\sigma\right| &\lesssim \varepsilon^{2s-2}\langle v - v_*\rangle^{\gamma+2s}\mu_*^{\frac14} W_{l,q}(v)\\ &\lesssim \langle v\rangle^{\gamma}(W^\varepsilon)^2(v) W_{l,q}(v)\mu_*^{\frac18}. \end{aligned}\tag{3.71}$$

It remains to consider the last case $|v| > \varepsilon^{-1}$, $|v - v_*| \ge \varepsilon^{-1}$. We divide the angle θ into two parts:

$$\begin{aligned}\int B^\varepsilon (W'_{l,q} - W_{l,q})\mu_*^{\frac12}\, d\sigma &= \mathcal{B}_1 + \mathcal{B}_2,\\ \mathcal{B}_1 := \int B^\varepsilon \mathbb{1}_{\sin\frac{\theta}{2} \le |v-v_*|^{-1}} (W'_{l,q} - W_{l,q})\mu_*^{\frac12}\, d\sigma,&\\ \mathcal{B}_2 := \int B^\varepsilon \mathbb{1}_{\sin\frac{\theta}{2} \ge |v-v_*|^{-1}} (W'_{l,q} - W_{l,q})\mu_*^{\frac12}\, d\sigma.&\end{aligned}$$

For $\mathcal{B}_1$, by using the expansion (3.67), similarly to (3.68), since

$$\int B^\varepsilon \mathbb{1}_{\sin\frac{\theta}{2} \le |v-v_*|^{-1}} \sin^2\frac{\theta}{2}\, d\sigma \lesssim \varepsilon^{2s-2}|v - v_*|^{\gamma+2s-2},$$

we have

$$|\mathcal{B}_1| \lesssim \varepsilon^{2s-2}|v - v_*|^{\gamma+2s}\mu_*^{\frac12} W_{l,q}(v) W_{l,q}(v_*).$$

For $\mathcal{B}_2$, by the fact that $|W_{l,q} - W'_{l,q}| \lesssim W_{l,q}(v)W_{l,q}(v_*)$, since

$$\int B^\varepsilon \mathbb{1}_{\sin\frac{\theta}{2} \ge |v-v_*|^{-1}}\, d\sigma \lesssim \varepsilon^{2s-2}|v - v_*|^{\gamma+2s},$$

we have

$$|\mathcal{B}_2| \lesssim \varepsilon^{2s-2}|v - v_*|^{\gamma+2s}\mu_*^{\frac12} W_{l,q}(v) W_{l,q}(v_*).$$

Similarly to (3.71), the estimates $\mathcal{B}_1$ and $\mathcal{B}_2$ give

$$\begin{aligned}\left|\int B^\varepsilon (W'_{l,q} - W_{l,q})\mu_*^{\frac12}\, d\sigma\right| &\lesssim \varepsilon^{2s-2}|v - v_*|^{\gamma+2s}\mu_*^{\frac12} W_{l,q}(v) W_{l,q}(v_*)\\ &\lesssim \langle v\rangle^{\gamma}(W^\varepsilon)^2(v) W_{l,q}(v)\mu_*^{\frac18}. \end{aligned}\tag{3.72}$$

By combining (3.68), (3.70), (3.71), and (3.72), we have (3.66). Then (3.66) implies

$$\begin{aligned}|\mathcal{A}_2| \lesssim &\int \langle v\rangle^{\gamma}(W^\varepsilon)^2(v) W_{l,q}(v)\mu_*^{\frac18}|g_* h f|\, dv_*\, dv\\ &+ \int \mathbb{1}_{|v-v_*|\le 1}|v - v_*|^{\gamma+1}\mu^{\frac18}\mu_*^{\frac18}|g_* h f|\, dv_*\, dv.\end{aligned}$$

Obviously,

$$\int \langle v\rangle^{\gamma}(W^{\varepsilon})^{2}(v)W_{l,q}(v)\mu_{*}^{\frac{1}{8}}|g_{*}hf|\,dv_{*}\,dv \lesssim |\mu^{\frac{1}{8}}g|_{L^{1}}|W^{\varepsilon}W_{l,q}h|_{L^{2}_{\gamma/2}}|W^{\varepsilon}f|_{L^{2}_{\gamma/2}}$$
$$\lesssim |\mu^{\frac{1}{16}}g|_{L^{2}}|W_{l,q}h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}.$$

By the Cauchy–Schwarz inequality, when $\gamma > -3$, similarly to (3.64), we have

$$\int |v-v_{*}|^{\gamma+1}\mu^{\frac{1}{8}}\mu_{*}^{\frac{1}{8}}|g_{*}hf|\,dv_{*}\,dv \lesssim \left(\int |v-v_{*}|^{-1}\mu^{\frac{1}{8}}\mu_{*}^{\frac{1}{8}}|g_{*}h|^{2}\,dv_{*}\,dv\right)^{\frac{1}{2}}$$
$$\times\left(\int |v-v_{*}|^{\gamma}\mu^{\frac{1}{8}}\mu_{*}^{\frac{1}{8}}f^{2}\,dv_{*}\,dv\right)^{\frac{1}{2}}$$
$$\lesssim |\mu^{\frac{1}{16}}g|_{L^{2}}|W_{l,q}h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}.$$

Then the above two estimates give

$$|\mathcal{A}_{2}| \lesssim |\mu^{\frac{1}{16}}g|_{L^{2}}|W_{l,q}h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}. \tag{3.73}$$

Finally, (3.65) and (3.73) complete the proof. ■

The next lemma gives an estimate for the commutator $[I^{\varepsilon,\gamma}(g,\cdot), W_{l,q}]$.

Lemma 3.6. *Let $l, q \geq 0$. If $-2 \leq \gamma \leq 0$, it holds that*

$$|\langle [I^{\varepsilon,\gamma}(g,\cdot), W_{l,q}]h, f\rangle| \lesssim |\mu^{1/32}g|_{L^{2}}|W_{l,q}h|_{\varepsilon,\gamma/2}|W^{\varepsilon}f|_{L^{2}_{\gamma/2}}$$
$$+ |W_{l,q}g|_{L^{2}}|W_{l,q}h|_{L^{2}_{\gamma/2}}|W^{\varepsilon}f|_{L^{2}_{\gamma/2}}.$$

If $q = 0$, it holds that

$$|\langle [I^{\varepsilon,\gamma}(g,\cdot), W_{l,0}]h, f\rangle| \lesssim |g|_{L^{2}}|W_{l,0}h|_{\varepsilon,\gamma/2}|W^{\varepsilon}f|_{L^{2}_{\gamma/2}}.$$

Proof. By recalling $I^{\varepsilon,\gamma}$ in (3.12), the structure (3.7), and the identity $(\mu^{\frac{1}{2}})'_{*} - \mu_{*}^{\frac{1}{2}} = ((\mu^{\frac{1}{4}})'_{*} - \mu_{*}^{\frac{1}{4}})^{2} + 2\mu_{*}^{\frac{1}{4}}((\mu^{\frac{1}{4}})'_{*} - \mu_{*}^{\frac{1}{4}})$, we have

$$\begin{aligned}
\langle [I^{\varepsilon,\gamma}(g,\cdot), W_{l,q}]h, f\rangle &= \int B^{\varepsilon}((\mu^{\frac{1}{2}})'_{*} - \mu_{*}^{\frac{1}{2}})(W_{l,q} - W'_{l,q})g_{*}hf'\,d\sigma\,dv_{*}\,dv \\
&= \int B^{\varepsilon}((\mu^{\frac{1}{4}})'_{*} - \mu_{*}^{\frac{1}{4}})^{2}(W_{l,q} - W'_{l,q})g_{*}hf'\,d\sigma\,dv_{*}\,dv \\
&\quad + 2\int B^{\varepsilon}\mu_{*}^{\frac{1}{4}}((\mu^{\frac{1}{4}})'_{*} - \mu_{*}^{\frac{1}{4}})(W_{l,q} - W'_{l,q})g_{*}hf'\,d\sigma\,dv_{*}\,dv \\
&:= \mathcal{A}_{1} + 2\mathcal{A}_{2}.
\end{aligned}$$

We divide the proof into two steps.

Step 1: Estimate of $\mathcal{A}_1$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|\mathcal{A}_1| &\leq \left\{\int B^\varepsilon((\mu^{\frac14})'_* - \mu_*^{\frac14})^2 f'^2\, d\sigma\, dv_*\, dv\right\}^{\frac12}\\ &\quad\times\left\{\int B^\varepsilon((\mu^{\frac14})'_* - \mu_*^{\frac14})^2 (W_{l,q} - W'_{l,q})^2 g_*^2 h^2\, d\sigma\, dv_*\, dv\right\}^{\frac12}\\ &:= (\mathcal{A}_{1,1})^{\frac12}(\mathcal{A}_{1,2})^{\frac12}.\end{aligned}$$

By the change of variables $(v, v_*) \to (v'_*, v')$ and Remark 2.1 , we have

$$\mathcal{A}_{1,1} = \int B^\varepsilon((\mu^{\frac14})' - \mu^{\frac14})^2 f_*^2\, d\sigma\, dv_*\, dv \lesssim |W^\varepsilon f|^2_{L^2_{\gamma/2}}. \tag{3.74}$$

Since

$$((\mu^{\frac14})'_* - \mu_*^{\frac14})^2 = ((\mu^{\frac18})'_* + \mu_*^{\frac18})^2((\mu^{\frac18})'_* - \mu_*^{\frac18})^2 \leq 2((\mu^{\frac14})'_* + \mu_*^{\frac14})((\mu^{\frac18})'_* - \mu_*^{\frac18})^2,$$

we have

$$\begin{aligned}\mathcal{A}_{1,2} &\lesssim \int B^\varepsilon \mu_*^{\frac14}((\mu^{\frac18})'_* - \mu_*^{\frac18})^2 (W_{l,q} - W'_{l,q})^2 g_*^2 h^2\, d\sigma\, dv_*\, dv\\ &\quad+ \int B^\varepsilon (\mu^{\frac14})'_*((\mu^{\frac18})'_* - \mu_*^{\frac18})^2 (W_{l,q} - W'_{l,q})^2 g_*^2 h^2\, d\sigma\, dv_*\, dv\\ &:= \mathcal{A}_{1,2,1} + \mathcal{A}_{1,2,2}.\end{aligned}$$

We first estimate $\mathcal{A}_{1,2,2}$. Referring to [15] (more precisely, equation (2.10) on page 170), we get

$$|W_{l,q} - W'_{l,q}| \lesssim |v' - v|\langle v\rangle^{-1}\langle v'_*\rangle^2 W_{l,q}(v) W_{l,q}(v_*). \tag{3.75}$$

This, together with the assumption that $\gamma \geq -2$, gives

$$\begin{aligned}&\int B^\varepsilon(\mu^{\frac14})'_*((\mu^{\frac18})'_* - \mu_*^{\frac18})^2 (W_{l,q} - W'_{l,q})^2\, d\sigma\\ &\quad\lesssim \langle v\rangle^{-2} W^2_{l,q}(v) W^2_{l,q}(v_*) \int |v - v'_*|^{\gamma+2}(\mu^{\frac14})'_*\langle v'_*\rangle^2 b^\varepsilon(\cos\theta)\sin^2\frac{\theta}{2}\, d\sigma\\ &\quad\lesssim \langle v\rangle^\gamma W^2_{l,q}(v) W^2_{l,q}(v_*)\end{aligned}$$

so that $\mathcal{A}_{1,2,2} \lesssim |W_{l,q} g|^2_{L^2}|W_{l,q} h|^2_{L^2_{\gamma/2}}$. If $q = 0$, by [26, proof of (2.84)], we have

$$\int B^\varepsilon(\mu^{\frac14})'_*((\mu^{\frac18})'_* - \mu_*^{\frac18})^2 (W_{l,0} - W'_{l,0})^2\, d\sigma \lesssim \langle v\rangle^{2l+\gamma},$$

which implies $\mathcal{A}_{1,2,2} \lesssim |g|^2_{L^2}|W_{l,0} h|^2_{L^2_{\gamma/2}}$.

Similarly to the estimate for $\mathcal{A}_{1,2}$ in Step 1 of Lemma 3.5, we obtain $\mathcal{A}_{1,2,1} \lesssim |\mu^{\frac{1}{16}} g|^2_{L^2}|W_{l,q} h|^2_{\varepsilon,\gamma/2}$.

In summary, the estimates of $\mathcal{A}_{1,2,1}$ and $\mathcal{A}_{1,2,2}$ imply that $\mathcal{A}_{1,2} \lesssim |W_{l,q} g|^2_{L^2} |W_{l,q} h|^2_{L^2_{\gamma/2}}$ when $q > 0$, and $\mathcal{A}_{1,2} \lesssim |g|^2_{L^2} |W_{l,0} h|^2_{L^2_{\gamma/2}}$ when $q = 0$. Hence, we conclude that when $q > 0$,

$$|\mathcal{A}_1| \lesssim |W_{l,q} g|_{L^2} |W_{l,q} h|_{L^2_{\gamma/2}} |W^\varepsilon f|_{L^2_{\gamma/2}}.$$

When $q = 0$,

$$|\mathcal{A}_1| \lesssim |g|_{L^2} |W_{l,0} h|_{L^2_{\gamma/2}} |W^\varepsilon f|_{L^2_{\gamma/2}}.$$

Step 2: Estimate of $\mathcal{A}_2$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\mathcal{A}_2| &\le \left\{ \int B^\varepsilon ((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}})^2 f'^2 \, d\sigma \, dv_* \, dv \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \int B^\varepsilon \mu_*^{\frac{1}{2}} (W_{l,q} - W'_{l,q})^2 g_*^2 h^2 \, d\sigma \, dv_* \, dv \right\}^{\frac{1}{2}} \\ &:= (\mathcal{A}_{2,1})^{\frac{1}{2}} (\mathcal{A}_{2,2})^{\frac{1}{2}}. \end{aligned}$$

Note that $\mathcal{A}_{2,1} = \mathcal{A}_{1,1}$, then by (3.74), we have $\mathcal{A}_{2,1} \lesssim |W^\varepsilon f|^2_{L^2_{\gamma/2}}$. Similarly to the estimate for $\mathcal{A}_{1,2}$ in Step 1 of Lemma 3.5, we get

$$\mathcal{A}_{2,2} \lesssim |\mu^{\frac{1}{16}} g|^2_{L^2} |W_{l,q} h|^2_{\varepsilon,\gamma/2}.$$

The estimates of $\mathcal{A}_{2,1}$ and $\mathcal{A}_{2,2}$ give

$$|\mathcal{A}_2| \lesssim |\mu^{\frac{1}{16}} g|_{L^2} |W_{l,q} h|_{\varepsilon,\gamma/2} |W^\varepsilon f|_{L^2_{\gamma/2}}.$$

Then the proof of the lemma is completed by combining the estimates of $\mathcal{A}_1$ and $\mathcal{A}_2$. ∎

Recalling (3.8), the following lemma is a direct consequence of Lemmas 3.5 and 3.6.

Lemma 3.7. *Let $l, q \ge 0$. If $-2 \le \gamma \le 0$, then*

$$\begin{aligned} |\langle \Gamma^\varepsilon(g, W_{l,q} h) - W_{l,q} \Gamma^\varepsilon(g, h), f \rangle| &\lesssim |\mu^{1/32} g|_{L^2} |W_{l,q} h|_{\varepsilon,\gamma/2} |f|_{\varepsilon,\gamma/2} \\ &\quad + |W_{l,q} g|_{L^2} |W_{l,q} h|_{L^2_{\gamma/2}} |W^\varepsilon f|_{L^2_{\gamma/2}}. \end{aligned}$$

If $q = 0$, then

$$|\langle \Gamma^\varepsilon(g, W_{l,0} h) - W_{l,0} \Gamma^\varepsilon(g, h), f \rangle| \lesssim |g|_{L^2} |W_{l,0} h|_{\varepsilon,\gamma/2} |f|_{\varepsilon,\gamma/2}.$$

Then Lemma 3.7 and Theorem 3.1 give the following lemma.

Lemma 3.8. *Let $l, q \ge 0$. If $-2 \le \gamma \le 0$, then*

$$|\langle \Gamma^\varepsilon(g, h), W^2_{l,q} f \rangle| \lesssim |W_{l,q} g|_{L^2} |W_{l,q} h|_{\varepsilon,\gamma/2} |W_{l,q} f|_{\varepsilon,\gamma/2}.$$

If $q = 0$, then

$$|\langle \Gamma^\varepsilon(g, h), W^2_{l,0} f \rangle| \lesssim |g|_{L^2} |W_{l,0} h|_{\varepsilon,\gamma/2} |W_{l,0} f|_{\varepsilon,\gamma/2}. \tag{3.76}$$

Recalling that $\mathcal{L}^{\varepsilon,\gamma}h = \mathcal{L}_1^{\varepsilon,\gamma}h + \mathcal{L}_2^{\varepsilon,\gamma}h = -\Gamma^{\varepsilon}(\mu^{\frac{1}{2}}, h) + \mathcal{L}_2^{\varepsilon,\gamma}h$, the next lemma follows from (3.76) and Lemma 3.3.

Lemma 3.9. *Let $l \geq 0$. The estimate $|\langle \mathcal{L}^{\varepsilon,\gamma}h, W_{l,0}^2 f\rangle| \lesssim |W_{l,0}h|_{\varepsilon,\gamma/2}|W_{l,0}f|_{\varepsilon,\gamma/2}$ holds.*

In the following, we give an estimate for the commutator between $\mathcal{L}^{\varepsilon,\gamma}$ and $W_{l,q}$ as a special case.

Lemma 3.10. *Let $l, q \geq 0$, $-2 \leq \gamma \leq 0$. The estimate $|\langle [\mathcal{L}^{\varepsilon,\gamma}, W_{l,q}]f, W_{l,q}f\rangle| \lesssim |W_{l,q}f|^2_{L^2_{\gamma/2}}$ holds.*

Proof. Recall that $\mathcal{L}^{\varepsilon,\gamma} = \mathcal{L}_1^{\varepsilon,\gamma} + \mathcal{L}_2^{\varepsilon,\gamma}$, where $\mathcal{L}_1^{\varepsilon,\gamma}f = -\Gamma^{\varepsilon,\gamma}(\mu^{\frac{1}{2}}, f)$, $\mathcal{L}_2^{\varepsilon,\gamma}f = -\Gamma^{\varepsilon,\gamma}(f, \mu^{\frac{1}{2}})$. Direct computation gives

$$\begin{aligned}\langle [\mathcal{L}_1^{\varepsilon,\gamma}, W_{l,q}]f, W_{l,q}f\rangle &= \int B^{\varepsilon}\mu_*^{\frac{1}{2}}(\mu^{\frac{1}{2}})'_* f f' W_{l,q}(W_{l,q} - W'_{l,q})\, dv\, dv_*\, d\sigma \\ &= \frac{1}{2}\int B^{\varepsilon}\mu_*^{\frac{1}{2}}(\mu^{\frac{1}{2}})'_* f f' (W_{l,q} - W'_{l,q})^2\, dv\, dv_*\, d\sigma.\end{aligned}$$

By using the change of variables $(v, v_*) \to (v', v'_*)$, one has

$$|\langle [\mathcal{L}_1^{\varepsilon,\gamma}, W_{l,q}]f, W_{l,q}f\rangle| \leq \frac{1}{2}\int B^{\varepsilon}\mu_*^{\frac{1}{2}}(\mu^{\frac{1}{2}})'_* f^2 (W_{l,q} - W'_{l,q})^2\, dv\, dv_*\, d\sigma.$$

Then (3.75) gives

$$\begin{aligned}&\int B^{\varepsilon}\mu_*^{\frac{1}{2}}(\mu^{\frac{1}{2}})'_*(W_{l,q} - W'_{l,q})^2\, d\sigma \\ &\qquad \lesssim |v - v_*|^{\gamma+2}\langle v\rangle^{-2} W_{l,q}^2(v)\mu_*^{\frac{1}{2}}\int b^{\varepsilon}(\cos\theta)\sin^2(\theta/2)\, d\sigma,\end{aligned}$$

which implies

$$|\langle [\mathcal{L}_1^{\varepsilon,\gamma}, W_{l,q}]f, W_{l,q}f\rangle| \lesssim \int |v - v_*|^{\gamma+2}\langle v\rangle^{-2} W_{l,q}^2(v)\mu_*^{\frac{1}{2}} f^2\, dv\, dv_* \lesssim |W_{l,q}f|^2_{L^2_{\gamma/2}}.$$

By Lemma 3.3, we have

$$|\langle [\mathcal{L}_2^{\varepsilon,\gamma}, W_{l,q}]f, W_{l,q}f\rangle| \lesssim |\mu^{\frac{1}{16}} f|^2_{L^2}.$$

Then the above two estimates complete the proof. ■

4. Propagation of regularity and the asymptotic formula

In this section we will give the proof of Theorem 1.2. With the coercivity estimate in Theorem 2.1, the spectral gap estimate in Theorem 2.2, and the upper bound estimate in

Theorem 3.1, we can derive the global well-posedness result (1.24) in Theorem 1.2 as in [14].

Then it remains to show the propagation of regularity and the asymptotic formula. We will derive propagation of regularity in Section 4.1 and the asymptotic formula in Section 4.2.

4.1. Propagation of regularity

In this subsection we prove (1.26) as stated in Theorem 4.2. We recall (1.18), (1.19), (1.20), (1.21), (1.22), and (1.23) concerning the norms used.

We first consider propagation of spatial regularity with polynomial weight, i.e. the norm $\|\cdot\|_{L^1_{k,m}L^2_l}$.

Theorem 4.1. *Let $m, l \geq 0$. Suppose f is a solution to the Boltzmann equation (1.16) with initial data f_0 satisfying $\|f_0\|_{L^1_{k,m}L^2_l} < \infty$. There is a constant $\delta > 0$ such that if $\|f_0\|_{L^1_k L^2} < \delta$, then*

$$\|f\|_{L^1_{k,m}L^\infty_T L^2_l} + \|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,l+\gamma/2}} \lesssim \|f_0\|_{L^1_{k,m}L^2_l}(1 + \|f_0\|_{L^1_{k,m}L^2_l}). \tag{4.1}$$

Note that we will show propagation of the norm $\|\cdot\|_{L^1_{k,m}L^2_l}$ only under the smallness assumption on $\|f_0\|_{L^1_k L^2}$ and finiteness on $\|f_0\|_{L^1_{k,m}L^2_l}$.

Proof of Theorem 4.1. Consider

$$\partial_t f + v \cdot \nabla_x f + \mathcal{L}^\varepsilon f = \Gamma^\varepsilon(f, f), \tag{4.2}$$

with the initial condition f_0. For simplicity, denote $\mathcal{H} := \Gamma^\varepsilon(f, f)$.

Recall (1.24) as

$$\|f\|_{L^1_k L^\infty_T L^2} + \|f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}} \lesssim \|f_0\|_{L^1_k L^2}. \tag{4.3}$$

The proof of the theorem is divided into three steps, where $\|\cdot\|_{L^1_{k,m}L^2}$, $\|\cdot\|_{L^1_k L^2_l}$, and $\|\cdot\|_{L^1_{k,m}L^2_l}$ are considered respectively.

Step 1: $\|\cdot\|_{L^1_{k,m}L^2}$. Following the proof of [14, Theorem 5.1], we have

$$\begin{aligned}\|[a, b, c]\|_{L^1_{k,m}L^2_T} \lesssim{}& \|(\mathbb{I} - \mathbb{P})f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}} + \|f\|_{L^1_{k,m}L^\infty_T L^2} + \|f_0\|_{L^1_{k,m}L^2} \\ &+ \sum_j \sum_{k\in\mathbb{Z}^3} \left(\int_0^T |\langle \hat{\mathcal{H}} \langle k\rangle^m, P_j \mu^{\frac{1}{2}} \rangle|^2 \, dt \right)^{\frac{1}{2}},\end{aligned} \tag{4.4}$$

where $\{P_j\}_j$ is a set of polynomials with degree ≤ 4.

Taking the Fourier transform of (4.2) with respect to x, at mode $k \in \mathbb{Z}^3$, we have

$$\partial_t \hat{f}(k) + \mathrm{i} v \cdot k \hat{f}(k) + \mathcal{L}^\varepsilon \hat{f}(k) = \hat{\mathcal{H}}(k). \tag{4.5}$$

Taking the inner product with $\langle k\rangle^{2m}\hat{f}$, similarly to [14, equation (3.7)], we have

$$\begin{aligned}\|f\|_{L^1_{k,m}L^\infty_T L^2}&+\|(\mathbb{I}-\mathbb{P})f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}}\\&\lesssim \|f_0\|_{L^1_{k,m}L^2}+\sum_{k\in\mathbb{Z}^3}\left(\int_0^T\left|\left\langle\widehat{\mathscr{H}}(k)\langle k\rangle^m,\langle k\rangle^m\hat{f}\right\rangle\right|dt\right)^{\frac12}.\end{aligned}\tag{4.6}$$

By suitably combining (4.4) and (4.6), and noting that

$$\|[a,b,c]\|_{L^1_{k,m}L^2_T}\sim\|\mathbb{P}f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}},$$

we have

$$\begin{aligned}\|f\|_{L^1_{k,m}L^\infty_T L^2}+\|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}}\lesssim \|f_0\|_{L^1_{k,m}L^2}&\\+\sum_j\sum_{k\in\mathbb{Z}^3}\left(\int_0^T|\langle\widehat{\mathscr{H}}\langle k\rangle^m,P_j\mu^{\frac12}\rangle|^2\,dt\right)^{\frac12}&\\+\sum_{k\in\mathbb{Z}^3}\left(\int_0^T\left|\left\langle\widehat{\mathscr{H}}(k)\langle k\rangle^m,\langle k\rangle^m\hat{f}\right\rangle\right|dt\right)^{\frac12}&.\end{aligned}\tag{4.7}$$

Recalling that $\widehat{\mathscr{H}}(k)=\sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\hat{f}(k-p),\hat{f}(p))$, by Theorem 3.1 and

$$\langle k\rangle^m\lesssim\langle k-p\rangle^m+\langle p\rangle^m\tag{4.8}$$

we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3}\left(\int_0^T\left|\left\langle\langle k\rangle^m\sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\hat{f}(k-p),\hat{f}(p)),P_j\mu^{\frac12}\right\rangle\right|^2dt\right)^{\frac12}\\&\quad\lesssim\|f\|_{L^1_{k,m}L^\infty_T L^2}\|f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}}+\|f\|_{L^1_k L^\infty_T L^2}\|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}},\end{aligned}$$

and (similarly to the estimate for the upper bound for [14, equation (3.8)])

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3}\left(\int_0^T\left|\left\langle\langle k\rangle^m\sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\hat{f}(k-p),\hat{f}(p)),\langle k\rangle^m\hat{f}\right\rangle\right|dt\right)^{\frac12}\\&\quad\lesssim\eta\|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}}+\frac{1}{4\eta}\|f\|_{L^1_{k,m}L^\infty_T L^2}\|f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}}\\&\qquad+\frac{1}{4\eta}\|f\|_{L^1_k L^\infty_T L^2}\|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}}.\end{aligned}$$

Plugging the above two inequalities into (4.7), for $0<\eta\le1$, we get

$$\begin{aligned}&\|f\|_{L^1_{k,m}L^\infty_T L^2}+\|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}}\\&\quad\lesssim\|f_0\|_{L^1_{k,m}L^2}+\eta\|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}}+\frac{1}{\eta}\|f\|_{L^1_{k,m}L^\infty_T L^2}\|f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}}\\&\qquad+\frac{1}{\eta}\|f\|_{L^1_k L^\infty_T L^2}\|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}}.\end{aligned}$$

By choosing η small, recalling (4.3), and under the smallness assumption on $\|f_0\|_{L^1_k L^2}$, we arrive at

$$\|f\|_{L^1_{k,m} L^\infty_T L^2} + \|f\|_{L^1_{k,m} L^2_T L^2_{\varepsilon,\gamma/2}} \lesssim \|f_0\|_{L^1_{k,m} L^2}. \tag{4.9}$$

Step 2: $\|\cdot\|_{L^1_k L^2_l}$. We now consider propagation of polynomial moments. Starting from (4.5), taking the inner product with $W_{2l}\hat{f}$, similarly to [14, equation (3.7)], we have

$$\begin{aligned}&\|f\|_{L^1_k L^\infty_T L^2_l} + \|(\mathbb{I}-\mathbb{P})W_l f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}} \\ &\quad \lesssim \|f_0\|_{L^1_k L^2_l} + \sum_{k\in\mathbb{Z}^3}\left(\int_0^T |\langle \hat{\mathcal{H}}, W_l \hat{f}\rangle|\, dt\right)^{\frac{1}{2}}.\end{aligned} \tag{4.10}$$

In this step, $\hat{\mathcal{H}}(k) = [\mathcal{L}^\varepsilon, W_l]\hat{f}(k) + W_l \sum_{p\in\mathbb{Z}^3} \Gamma^\varepsilon(\hat{f}(k-p), \hat{f}(p))$. By (2.50) and (4.3), we have

$$\|\mathbb{P} W_l f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}} \lesssim \|f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}} \lesssim \|f_0\|_{L^1_k L^2}. \tag{4.11}$$

A suitable combination of (4.10) and (4.11) gives

$$\|f\|_{L^1_k L^\infty_T L^2_l} + \|f\|_{L^1_k L^2_T L^2_{\varepsilon,l+\gamma/2}} \lesssim \|f_0\|_{L^1_k L^2_l} + \sum_{k\in\mathbb{Z}^3}\left(\int_0^T |\langle \hat{\mathcal{H}}, W_l \hat{f}\rangle|\, dt\right)^{\frac{1}{2}}. \tag{4.12}$$

For the term involving $[\mathcal{L}^\varepsilon, W_l]\hat{f}(k)$, by Lemma 3.10 we get

$$\sum_{k\in\mathbb{Z}^3}\left(\int_0^T |\langle [\mathcal{L}^\varepsilon, W_l]\hat{f}, W_l \hat{f}\rangle|\, dt\right)^{\frac{1}{2}} \lesssim \|f\|_{L^1_k L^2_T L^2_{l+\gamma/2}}.$$

Since $|\cdot|_{\varepsilon,l} \geq |\cdot|_{L^2_{l+s}} \geq |\cdot|_{L^2_{l+1/2}}$, we have $|f|_{L^2_{l+\gamma/2}} \leq \eta |f|_{L^2_{\varepsilon,l+\gamma/2}} + C(\eta,l)|f|_{L^2_{\varepsilon,\gamma/2}}$ which gives

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3}\left(\int_0^T |\langle [\mathcal{L}^\varepsilon, W_l]\hat{f}, W_l \hat{f}\rangle|\, dt\right)^{\frac{1}{2}} \\ &\quad \lesssim \eta\|f\|_{L^1_k L^2_T L^2_{\varepsilon,l+\gamma/2}} + C(\eta,l)\|f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}}.\end{aligned} \tag{4.13}$$

For the term involving $W_l \sum_{p\in\mathbb{Z}^3} \Gamma^\varepsilon(\hat{f}(k-p), \hat{f}(p))$, by (3.76) in Lemma 3.8 we get

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3}\left(\int_0^T \left|\left\langle W_l \sum_{p\in\mathbb{Z}^3} \Gamma^\varepsilon(\hat{f}(k-p), \hat{f}(p)), W_l \hat{f}\right\rangle\right| dt\right)^{\frac{1}{2}} \\ &\quad \lesssim \eta\|f\|_{L^1_k L^2_T L^2_{\varepsilon,l+\gamma/2}} + \frac{1}{4\eta}\|f\|_{L^1_k L^\infty_T L^2}\|f\|_{L^1_k L^2_T L^2_{\varepsilon,l+\gamma/2}}.\end{aligned} \tag{4.14}$$

Plugging (4.13) and (4.14) into (4.12), by choosing η small and using (4.3), under the smallness assumption on $\|f_0\|_{L^1_k L^2}$ we have

$$\|f\|_{L^1_k L^\infty_T L^2_l} + \|f\|_{L^1_k L^2_T L^2_{\varepsilon,l+\gamma/2}} \lesssim \|f_0\|_{L^1_k L^2_l} + \|f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}} \lesssim \|f_0\|_{L^1_k L^2_l}, \tag{4.15}$$

where we have used (4.3) in the last inequality.

Step 3: $\|\cdot\|_{L^1_{k,m} L^2_l}$. We now show the propagation of spatial regularity with polynomial moment. Starting from (4.5), taking the inner product with $\langle k\rangle^{2m} W_{2l}\hat{f}$, similarly to [14, equation (3.7)], we have

$$\begin{aligned}&\|f\|_{L^1_{k,m} L^\infty_T L^2_l} + \|(\mathbb{I}-\mathbb{P})W_l f\|_{L^1_{k,m} L^2_T L^2_{\varepsilon,\gamma/2}}\\ &\quad\lesssim \|f_0\|_{L^1_{k,m} L^2_l} + \sum_{k\in\mathbb{Z}^3}\left(\int_0^T |\langle \widehat{\mathcal{H}}, \langle k\rangle^{2m} W_l \hat{f}\rangle|\,dt\right)^{\frac12}.\end{aligned} \tag{4.16}$$

In this step, $\widehat{\mathcal{H}}(k) = [\mathcal{L}^\varepsilon, W_l]\hat{f}(k) + W_l \sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\hat{f}(k-p), \hat{f}(p))$. By (2.50) and (4.9), we have

$$\|\mathbb{P}W_l f\|_{L^1_{k,m} L^2_T L^2_{\varepsilon,\gamma/2}} \lesssim \|f\|_{L^1_{k,m} L^2_T L^2_{\varepsilon,\gamma/2}} \lesssim \|f_0\|_{L^1_{k,m} L^2}. \tag{4.17}$$

A suitable combination of (4.16) and (4.17) gives

$$\begin{aligned}&\|f\|_{L^1_{k,m} L^\infty_T L^2_l} + \|f\|_{L^1_{k,m} L^2_T L^2_{\varepsilon,l+\gamma/2}}\\ &\quad\lesssim \|f_0\|_{L^1_{k,m} L^2_l} + \sum_{k\in\mathbb{Z}^3}\left(\int_0^T |\langle \widehat{\mathcal{H}}, \langle k\rangle^{2m} W_{2l} \hat{f}\rangle|\,dt\right)^{\frac12}.\end{aligned} \tag{4.18}$$

Similarly to (4.13), we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3}\left(\int_0^T |\langle [\mathcal{L}^\varepsilon, W_l]\hat{f}, \langle k\rangle^{2m} W_l \hat{f}\rangle|\,dt\right)^{\frac12}\\ &\quad\lesssim \eta\|f\|_{L^1_{k,m} L^2_T L^2_{\varepsilon,l+\gamma/2}} + C(\eta,l)\|f\|_{L^1_{k,m} L^2_T L^2_{\varepsilon,\gamma/2}}.\end{aligned} \tag{4.19}$$

For the term involving $W_l \sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\hat{f}(k-p), \hat{f}(p))$, by (3.76) in Lemma 3.8 with (4.8), we get

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3}\left(\int_0^T \Big|\Big\langle \langle k\rangle^m W_l \sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\hat{f}(k-p), \hat{f}(p)), \langle k\rangle^m W_l \hat{f}\Big\rangle\Big|\,dt\right)^{\frac12}\\ &\quad\lesssim \eta\|f\|_{L^1_{k,m} L^2_T L^2_{\varepsilon,l+\gamma/2}} + \frac{1}{4\eta}\|f\|_{L^1_k L^\infty_T L^2}\|f\|_{L^1_{k,m} L^2_T L^2_{\varepsilon,l+\gamma/2}}\\ &\qquad + \frac{1}{4\eta}\|f\|_{L^1_{k,m} L^\infty_T L^2}\|f\|_{L^1_k L^2_T L^2_{\varepsilon,l+\gamma/2}}.\end{aligned} \tag{4.20}$$

Plugging (4.19) and (4.20) into (4.18), by choosing η small, recalling (4.3), under the smallness assumption on $\|f_0\|_{L^1_k L^2}$, we have

$$\begin{aligned}&\|f\|_{L^1_{k,m}L^\infty_T L^2_l} + \|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,l+\gamma/2}}\\&\quad\lesssim \|f_0\|_{L^1_{k,m}L^2_l} + \|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}} + \|f\|_{L^1_{k,m}L^\infty_T L^2}\|f\|_{L^1_k L^2_T L^2_{\varepsilon,l+\gamma/2}}.\end{aligned}$$

By (4.9), (4.15), and (4.1) we complete the proof of the theorem. ■

We now turn to the propagation of spatial and velocity regularity with polynomial moment. Taking the v-derivative of ∂_β of (4.5), we get

$$\begin{aligned}&\partial_t\partial_\beta \hat{f}(k) + \mathrm{i}v\cdot k\partial_\beta \hat{f}(k) + \mathscr{L}^\varepsilon\partial_\beta \hat{f}(k)\\&\quad= \mathrm{i}[v\cdot k,\partial_\beta]\hat{f}(k) + [\mathscr{L}^\varepsilon,\partial_\beta]\hat{f}(k) + \partial_\beta\widehat{\Gamma^\varepsilon(f,f)}(k).\end{aligned}$$

Taking the inner product with $\langle k\rangle^{2m}\langle v\rangle^{2l}\partial_\beta\hat{f}$, where $\beta\in\mathbb{Z}^3$, $m,l\geq 0$, similarly to [14, equation (3.7)], we have

$$\begin{aligned}&\|\partial_\beta f\|_{L^1_{k,m}L^\infty_T L^2_l} + \|(\mathbb{I}-\mathbb{P})W_l\partial_\beta f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}}\\&\quad\lesssim \|\partial_\beta f_0\|_{L^1_{k,m}L^2_l} + \sum_{k\in\mathbb{Z}^3}\left(\int_0^T \left|\left\langle \hat{\mathscr{H}}, \langle k\rangle^{2m}\langle v\rangle^{l}\partial_\beta \hat{f}\right\rangle\right| dt\right)^{\frac{1}{2}},\end{aligned}\tag{4.21}$$

where

$$\begin{aligned}\hat{\mathscr{H}}(k) &= \mathrm{i}W_l[v\cdot k,\partial_\beta]\hat{f}(k) + [\mathscr{L}^\varepsilon, W_l]\partial_\beta\hat{f}(k) + W_l[\mathscr{L}^\varepsilon,\partial_\beta]\hat{f}(k)\\&\quad + W_l\partial_\beta\sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\hat{f}(k-p),\hat{f}(p)).\end{aligned}\tag{4.22}$$

By (2.50) and integrating by parts, it holds that

$$\|\mathbb{P}W_l\partial_\beta f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}} \lesssim \|f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,\gamma/2}} \lesssim \|f_0\|_{L^1_{k,m}L^2},\tag{4.23}$$

where we have used (4.9) in the last inequality.

A suitable combination of (4.21) and (4.23) gives

$$\begin{aligned}&\|\partial_\beta f\|_{L^1_{k,m}L^\infty_T L^2_l} + \|\partial_\beta f\|_{L^1_{k,m}L^2_T L^2_{\varepsilon,l+\gamma/2}}\\&\quad\lesssim \|f_0\|_{L^1_{k,m}H^{|\beta|}_l} + \sum_{k\in\mathbb{Z}^3}\left(\int_0^T \left|\left\langle \hat{\mathscr{H}}, \langle k\rangle^{2m}\langle v\rangle^{l}\partial_\beta \hat{f}\right\rangle\right| dt\right)^{\frac{1}{2}}.\end{aligned}\tag{4.24}$$

We now estimate the last term in (4.24). Recalling (4.22), we will estimate term by term.

For the term involving $\mathrm{i}[v\cdot k,\partial_\beta]\hat{f}(k)$, if $\beta=(\beta_1,\beta_2,\beta_3)$, then

$$[v\cdot k,\partial_\beta]\hat{f}(k) = -\sum_{j=1}^3 \beta_j k_j \partial_{\beta-e_j}\hat{f}(k).$$

Hence,

$$\sum_{k\in\mathbb{Z}^3}\left(\int_0^T \left|\langle [v\cdot k,\partial_\beta]\hat f(k),\langle k\rangle^{2m}\langle v\rangle^{2l}\partial_\beta \hat f\rangle\right| dt\right)^{\frac12}$$
$$\lesssim \sum_{k\in\mathbb{Z}^3}\left(\int_0^T \sum_{j=1}^3 \beta_j |\langle k\rangle^{m+1}\partial_{\beta-e_j}\hat f(k)|_{L^2_{l-(\gamma/2+s)}}|\langle k\rangle^{m}\partial_{\beta}\hat f(k)|_{L^2_{l+\gamma/2+s}}\,dt\right)^{\frac12}$$
$$\lesssim \eta\|\partial_\beta f\|_{L^1_{k,m}L^2_TL^2_{\varepsilon,l+\gamma/2}}+\frac{1}{\eta}\|f\|_{L^1_{k,m+1}L^2_T\dot H^N_{\varepsilon,\gamma/2+l-(\gamma+2s)}}. \tag{4.25}$$

Similarly to (4.19), we have

$$\sum_{k\in\mathbb{Z}^3}\left(\int_0^T \left|\langle [\mathcal{L}^\varepsilon,W_l]\partial_\beta \hat f,\langle k\rangle^{2m}W_l\partial_\beta \hat f\rangle\right| dt\right)^{\frac12}$$
$$\lesssim \eta\|\partial_\beta f\|_{L^1_{k,m}L^2_TL^2_{\varepsilon,l+\gamma/2}}+C(\eta,l)\|\partial_\beta f\|_{L^1_{k,m}L^2_TL^2_{\varepsilon,\gamma/2}}. \tag{4.26}$$

By Lemma 3.9 we get

$$\sum_{k\in\mathbb{Z}^3}\left(\int_0^T \left|\langle W_l[\mathcal{L}^\varepsilon,\partial_\beta]\hat f(k),\langle k\rangle^{2m}W_l\partial_\beta \hat f\rangle\right| dt\right)^{\frac12}$$
$$\lesssim \sum_{k\in\mathbb{Z}^3}\left(\int_0^T \sum_{\beta_1<\beta}|\langle k\rangle^{m}\partial_{\beta_1}\hat f|_{L^2_{\varepsilon,l+\gamma/2}}|\langle k\rangle^{m}\partial_{\beta}\hat f|_{L^2_{\varepsilon,l+\gamma/2}}\,dt\right)^{\frac12}$$
$$\lesssim \eta\|\partial_\beta f\|_{L^1_{k,m}L^2_TL^2_{\varepsilon,l+\gamma/2}}+\frac{1}{\eta}\|f\|_{L^1_{k,m}L^2_TH^N_{\varepsilon,l+\gamma/2}}. \tag{4.27}$$

For the term involving $\partial_\beta\sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\hat f(k-p),\hat f(p))$, by (3.76) in Lemma 3.8, with (4.8), we get

$$\sum_{k\in\mathbb{Z}^3}\left(\int_0^T \left|\left\langle \partial_\beta\sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\hat f(k-p),\hat f(p)),\langle k\rangle^{2m}\langle v\rangle^{2l}\partial_\beta \hat f\right\rangle\right| dt\right)^{\frac12}$$
$$\lesssim \eta\|\partial_\beta f\|_{L^1_{k,m}L^2_TL^2_{\varepsilon,l+\gamma/2}}$$
$$+\frac{1}{4\eta}\|f\|_{L^1_kL^\infty_TL^2}\|\partial_\beta f\|_{L^1_{k,m}L^2_TL^2_{\varepsilon,l+\gamma/2}}+\frac{1}{4\eta}\|f\|_{L^1_{k,m}L^\infty_TL^2}\|\partial_\beta f\|_{L^1_kL^2_TL^2_{\varepsilon,l+\gamma/2}}$$
$$+\frac{1}{4\eta}\|\partial_\beta f\|_{L^1_kL^\infty_TL^2}\|f\|_{L^1_{k,m}L^2_TL^2_{\varepsilon,l+\gamma/2}}+\frac{1}{4\eta}\|\partial_\beta f\|_{L^1_{k,m}L^\infty_TL^2}\|f\|_{L^1_kL^2_TL^2_{\varepsilon,l+\gamma/2}}$$
$$+\frac{1}{4\eta}\|f\|_{L^1_kL^\infty_TH^N}\|f\|_{L^1_{k,m}L^2_TH^N_{\varepsilon,l+\gamma/2}}+\frac{1}{4\eta}\|f\|_{L^1_{k,m}L^\infty_TH^N}\|f\|_{L^1_kL^2_TH^N_{\varepsilon,l+\gamma/2}}. \tag{4.28}$$

With the above preparation, we are ready to prove the propagation of both spatial and velocity regularity with polynomial weight in the following theorem.

Theorem 4.2. *Let $n \in \mathbb{N}$, $m, l \geq 0$. Suppose f is a solution to the Boltzmann equation (1.16) with initial data f_0 verifying $\|f_0\|_{m,n,l} < \infty$. There is a constant $\delta > 0$ and a polynomial P_n with $P_n(0) = 0$ such that if $\|f_0\|_{L^1_k L^2} < \delta$, then*

$$E_T(f; m, n, l) + D^{\varepsilon}_T(f; m, n, l) \lesssim P_n(\|f_0\|_{m,n,l}). \tag{4.29}$$

Proof. Recall the notation $E_T(f; m, n, l)$, $D^{\varepsilon}_T(f; m, n, l)$ in (1.22) and $\|f_0\|_{m,n,l}$ in (1.23). We will prove (4.29) by induction. First, by (4.1), we see that (4.29) is valid for $n = 0$ with $P_0(x) = x(1 + x)$.

Let $N \geq 0$ be an integer. Let us assume that (4.29) is valid for any $0 \leq n \leq N$ and $m, l \geq 0$. We will prove the above statement (4.29) is also valid for $n = N + 1$, $m, l \geq 0$. To be clear, we fix two parameters $m_*, l_* \geq 0$ and prove (4.29) for $n = N + 1$, $m = m_*$, $l = l_*$.

We concentrate on $\|\cdot\|_{L^1_{k,m_*} \dot{H}^{N+1}_{l_*}}$. We divide the proof into four steps for the estimation on $\|\cdot\|_{L^1_k \dot{H}^{N+1}}$, $\|\cdot\|_{L^1_{k,m_*} \dot{H}^{N+1}}$, $\|\cdot\|_{L^1_k \dot{H}^{N+1}_{l_*}}$, and $\|\cdot\|_{L^1_{k,m_*} \dot{H}^{N+1}_{l_*}}$ respectively.

Step 1. $\|\cdot\|_{L^1_k \dot{H}^{N+1}}$. We start from (4.24) by taking $|\beta| = N + 1$, $m = l = 0$. In this case, $[\mathcal{L}^{\varepsilon}, W_l] = 0$ in (4.22).

Plugging (4.25), (4.27), and (4.28) for the case $m = l = 0$ into (4.24), taking the sum over $|\beta| = N + 1$, by choosing η small and under the smallness assumption on $\|f_0\|_{L^1_k L^2}$ which implies $\|f\|_{L^1_k L^\infty_T L^2}$ and $\|f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}}$ are small, and by using (4.29) with $n = N$, $m = 1$, $l = -(\gamma + 2s)$, we have

$$\begin{aligned} &\|f\|_{L^1_k L^\infty_T \dot{H}^{N+1}} + \|f\|_{L^1_k L^2_T \dot{H}^{N+1}_{\varepsilon,\gamma/2}} \\ &\quad\lesssim \|f_0\|_{L^1_k H^{N+1}} + P_N(\|f_0\|_{1,N,-(\gamma+2s)})\big(1 + P_N(\|f_0\|_{1,N,-(\gamma+2s)})\big). \end{aligned} \tag{4.30}$$

Let us denote $P^{0,N+1,0}(x) := P_N(x)(1 + P_N(x))$. Adding (4.30) and (4.29) with $n = N$, $m = 1$, $l = -(\gamma + 2s)$, we have

$$E_T(f; 0, N + 1, 0) + D^{\varepsilon}_T(f; 0, N + 1, 0) \lesssim P^{0,N+1,0}(\|f_0\|_{0,N+1,0}). \tag{4.31}$$

Step 2. $\|\cdot\|_{L^1_{k,m_*} \dot{H}^{N+1}}$. We start from (4.24) by taking $|\beta| = N + 1$, $m = m_*$, $l = 0$. In this case, $[\mathcal{L}^{\varepsilon}, W_l] = 0$ in (4.22).

Plugging (4.25), (4.27), and (4.28) for the case $m = m_*$, $l = 0$ into (4.24), taking the sum over $|\beta| = N + 1$, by choosing η small and under the smallness assumption on $\|f_0\|_{L^1_k L^2}$ which implies $\|f\|_{L^1_k L^\infty_T L^2}$ and $\|f\|_{L^1_k L^2_T L^2_{\varepsilon,\gamma/2}}$ are small, using (4.29) with $n = N$, $m = m_* + 1$, $l = -(\gamma + 2s)$ and (4.31), we have

$$\begin{aligned} &\|f\|_{L^1_{k,m_*} L^\infty_T \dot{H}^{N+1}} + \|f\|_{L^1_{k,m_*} L^2_T \dot{H}^{N+1}_{\varepsilon,\gamma/2}} \\ &\quad\lesssim \|f_0\|_{L^1_{k,m_*} H^{N+1}} + P_0(\|f_0\|_{m_*,0,0}) P^{0,N+1,0}(\|f_0\|_{0,N+1,0}) \\ &\qquad + P_N(\|f_0\|_{m_*+1,N,-(\gamma+2s)})\big(1 + P_N(\|f_0\|_{m_*+1,N,-(\gamma+2s)})\big). \end{aligned} \tag{4.32}$$

Let us define $P^{m_*,N+1,0}(x) := P_0(x)P^{0,N+1,0}(x) + P_N(x)(1 + P_N(x))$. Adding (4.32) and (4.29) with $n = N$, $m = m_* + 1$, $l = -(\gamma + 2s)$, we have

$$E_T(f; m_*, N+1, 0) + D_T^\varepsilon(f; m_*, N+1, 0) \lesssim P^{m_*,N+1,0}(\|f_0\|_{m_*,N+1,0}). \quad (4.33)$$

Step 3. $\|\cdot\|_{L^1_k \dot{H}^{N+1}_{l_*}}$. We start from (4.24) by taking $|\beta| = N+1$, $m = 0$, $l = l_*$.

Plugging (4.25), (4.26), (4.27), and (4.28) for the case $m = 0$, $l = l_*$ into (4.24), taking the sum over $|\beta| = N+1$, by choosing η small and under the smallness assumption on $\|f_0\|_{L^1_k L^2}$ which implies $\|f\|_{L^1_k L^\infty_T L^2}$ is small, using (4.29) with $n = N$, $m = 1$, $l = l_* - (\gamma + 2s)$ and (4.31), we have

$$\begin{aligned}
&\|f\|_{L^1_k L^\infty_T \dot{H}^{N+1}_{l_*}} + \|f\|_{L^1_k L^2_T \dot{H}^{N+1}_{\varepsilon, l_*+\gamma/2}} \\
&\quad \lesssim \|f_0\|_{L^1_k H^{N+1}_{l_*}} + P_0(\|f_0\|_{0,0,l_*})P^{0,N+1,0}(\|f_0\|_{0,N+1,0}) \\
&\qquad + P_N(\|f_0\|_{1,N,l_*-(\gamma+2s)})\big(1 + P_N(\|f_0\|_{1,N,l_*-(\gamma+2s)})\big). \quad (4.34)
\end{aligned}$$

Let us define $P^{0,N+1,l_*}(x) := P_0(x)P^{0,N+1,0}(x) + P_N(x)(1 + P_N(x))$. Adding (4.34) and (4.29) with $n = N$, $m = 1$, $l = l_* - (\gamma + 2s)$, we have

$$E_T(f; 0, N+1, l_*) + D_T^\varepsilon(f; 0, N+1, l_*) \lesssim P^{0,N+1,l_*}(x)(\|f_0\|_{0,N+1,l_*}). \quad (4.35)$$

Step 4. $\|\cdot\|_{L^1_{k,m_*} \dot{H}^{N+1}_{l_*}}$. We start from (4.24) by taking $|\beta| = N+1$, $m = m_*$, $l = l_*$.

Plugging (4.25), (4.26), (4.27), and (4.28) for the case $m = m_*$, $l = l_*$ into (4.24), taking the sum over $|\beta| = N+1$, by choosing η small and under the smallness assumption on $\|f_0\|_{L^1_k L^2}$ which implies $\|f\|_{L^1_k L^\infty_T L^2}$ is small, and by using (4.31), (4.33), (4.35), and (4.29) with $n = N$, $m = m_* + 1$, $l = l_* - (\gamma + 2s)$, we have

$$\begin{aligned}
&\|f\|_{L^1_{k,m_*} L^\infty_T \dot{H}^{N+1}_{l_*}} + \|f\|_{L^1_{k,m_*} L^2_T \dot{H}^{N+1}_{\varepsilon, l_*+\gamma/2}} \\
&\quad \lesssim \|f_0\|_{L^1_{k,m_*} H^{N+1}_{l_*}} + P_0(\|f_0\|_{m_*,0,l_*})P^{0,N+1,0}(\|f_0\|_{0,N+1,0}) \\
&\qquad + P_0(\|f_0\|_{m_*,0,0})P^{0,N+1,l_*}(\|f_0\|_{0,N+1,l_*}) \\
&\qquad + P_0(\|f_0\|_{0,0,l_*})P^{m_*,N+1,0}(\|f_0\|_{m_*,N+1,0}) \\
&\qquad + P_N(\|f_0\|_{m_*+1,N,l_*-(\gamma+2s)})\big(1 + P_N(\|f_0\|_{m_*+1,N,l_*-(\gamma+2s)})\big). \quad (4.36)
\end{aligned}$$

Define

$$\begin{aligned}
P_{N+1}(x) := {}& P_0(x)P^{0,N+1,l_*}(x) + P_0(x)P^{0,N+1,0}(x) \\
&+ P_0(x)P^{m_*,N+1,0}(x) + P_N(x)(1 + P_N(x)).
\end{aligned}$$

Note that P_{N+1} is independent of m_*, l_*. Summing (4.36) and (4.29) with $n = N$, $m = m_* + 1$, $l = l_* - (\gamma + 2s)$ gives

$$E_T(f; m_*, N+1, l_*) + D_T^\varepsilon(f; m_*, N+1, l_*) \lesssim P_{N+1}(\|f_0\|_{m_*,N+1,l_*}).$$

And this completes the proof of the theorem. ∎

4.2. Global asymptotics

We will prove (1.28) in this subsection. We first give an estimate of the operator $\Gamma^\varepsilon - \Gamma^L$.

Lemma 4.1. *For suitable functions g, h, f, there holds*

$$|\langle(\Gamma^\varepsilon - \Gamma^L)(g,h), f\rangle| \lesssim \varepsilon |g|_{H^3} |h|_{H^3_{9+\gamma/2}} |f|_{L^2_{\gamma/2}}.$$

Proof. Note that

$$\langle(\Gamma^\varepsilon - \Gamma^L)(g,h), f\rangle = \langle Q^\varepsilon(\mu^{\frac{1}{2}}g, \mu^{\frac{1}{2}}h) - Q^L(\mu^{\frac{1}{2}}g, \mu^{\frac{1}{2}}h), \mu^{-\frac{1}{2}}f\rangle.$$

By the proof in [41], it holds that

$$|\langle Q^\varepsilon(G,H) - Q^L(G,H), F\rangle| \lesssim \varepsilon |G|_{H^3_{8+\gamma}} |H|_{H^3_{6+\gamma/2}} |F|_{L^2_{\gamma/2}}.$$

By using the fact that

$$|\nabla^3(\mu^{\frac{1}{2}}h)| \lesssim \langle v\rangle^3 \mu^{\frac{1}{2}}(|h| + |\nabla h| + |\nabla^2 h| + |\nabla^3 h|), \tag{4.37}$$

and the proof in [41], the estimate in the lemma follows. Note that the additional 3 weight (from $6+\gamma/2$ to $9+\gamma/2$) for the function h comes from $\langle v\rangle^3$ in (4.37). On the other hand, the factor $\mu^{\frac{1}{2}}$ before g absorbs any polynomial weight. ∎

We are ready to prove (1.28) in Theorem 1.2.

Proof of (1.28). Let f^ε and f^L be the solutions to (1.16) and (1.6) respectively with the initial data f_0. Set $F^\varepsilon_R := \varepsilon^{-1}(f^\varepsilon - f^L)$. Then it solves

$$\begin{aligned}\partial_t F^\varepsilon_R + v\cdot\nabla_x F^\varepsilon_R + \mathcal{L}^L F^\varepsilon_R = \varepsilon^{-1}[(\mathcal{L}^L - \mathcal{L}^\varepsilon)f^\varepsilon + (\Gamma^\varepsilon - \Gamma^L)(f^\varepsilon, f^L)] \\ + \Gamma^\varepsilon(f^\varepsilon, F^\varepsilon_R) + \Gamma^L(F^\varepsilon_R, f^L).\end{aligned}$$

For simplicity, we denote the right-hand side by $\mathcal{H}$,

$$\begin{aligned}\mathcal{H} := \varepsilon^{-1}[(\mathcal{L}^L - \mathcal{L}^\varepsilon)f^\varepsilon + (\Gamma^\varepsilon - \Gamma^L)(f^\varepsilon, f^L)] \\ + \Gamma^\varepsilon(f^\varepsilon, F^\varepsilon_R) + \Gamma^L(F^\varepsilon_R, f^L).\end{aligned}$$

Taking the Fourier transform with respect to x, for the mode $k \in \mathbb{Z}^3$, we have

$$\partial_t \widehat{F^\varepsilon_R}(k) + \mathrm{i}v\cdot k\widehat{F^\varepsilon_R}(k) + \mathcal{L}^L\widehat{F^\varepsilon_R}(k) = \widehat{\mathcal{H}}(k), \tag{4.38}$$

where

$$\begin{aligned}\widehat{\mathcal{H}}(k) := {}& \varepsilon^{-1}(\mathcal{L}^L - \mathcal{L}^\varepsilon)\widehat{f^\varepsilon}(k) \\ & + \sum_{p\in\mathbb{Z}^3} \varepsilon^{-1}(\Gamma^\varepsilon - \Gamma^L)(\widehat{f^\varepsilon}(k-p), \widehat{f^L}(p)) \\ & + \sum_{p\in\mathbb{Z}^3} \Gamma^\varepsilon(\widehat{f^\varepsilon}(k-p), \widehat{F^\varepsilon_R}(p)) \\ & + \sum_{p\in\mathbb{Z}^3} \Gamma^L(\widehat{F^\varepsilon_R}(k-p), \widehat{f^L}(p)).\end{aligned} \tag{4.39}$$

We divide the proof into three steps.

Step 1: Macroscopic part. For the estimate of $[a,b,c]$ of F_R^ε defined by (2.50), by [14, Theorem 5.1], since $F_R^\varepsilon(0)=0$, it holds that

$$\|[a,b,c]\|_{L^1_kL^2_T} \lesssim \|(\mathbb{I}-\mathbb{P})F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}} + \|F_R^\varepsilon\|_{L^1_kL^\infty_TL^2} + \sum_j \sum_{k\in\mathbb{Z}^3}\Big(\int_0^T |\langle \widehat{\mathscr{H}}(k), P_j\mu^{\frac12}\rangle|^2\,dt\Big)^{\frac12},$$

where $\{P_j\}_j$ is a set of polynomials with degree ≤ 4.

Note that $\widehat{\mathscr{H}}(k)$ has four terms in (4.39). For the term $\sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\widehat{f^\varepsilon}(k-p),\widehat{F_R^\varepsilon}(p))$, by Theorem 3.1, we have

$$\sum_{k\in\mathbb{Z}^3}\Big(\int_0^T\Big|\Big\langle \sum_{p\in\mathbb{Z}^3}\Gamma^\varepsilon(\widehat{f^\varepsilon}(k-p),\widehat{F_R^\varepsilon}(p)), P_j\mu^{\frac12}\Big\rangle\Big|^2\,dt\Big)^{\frac12} \lesssim \|f^\varepsilon\|_{L^1_kL^\infty_TL^2}\|F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}}.$$

Similarly, for the term $\sum_{p\in\mathbb{Z}^3}\Gamma^L(\widehat{F_R^\varepsilon}(k-p),\widehat{f^L}(p))$, we have

$$\sum_{k\in\mathbb{Z}^3}\Big(\int_0^T\Big|\Big\langle \sum_{p\in\mathbb{Z}^3}\Gamma^L(\widehat{F_R^\varepsilon}(k-p),\widehat{f^L}(p)), P_j\mu^{\frac12}\Big\rangle\Big|^2\,dt\Big)^{\frac12} \lesssim \|F_R^\varepsilon\|_{L^1_kL^\infty_TL^2}\|f^L\|_{L^1_kL^2_TL^2_{0,\gamma/2}}.$$

For the term $\sum_{p\in\mathbb{Z}^3}\varepsilon^{-1}(\Gamma^\varepsilon-\Gamma^L)(\widehat{f^\varepsilon}(k-p),\widehat{f^L}(p))$, by Lemma 4.1 we have

$$\sum_{k\in\mathbb{Z}^3}\Big(\int_0^T\Big|\Big\langle \sum_{p\in\mathbb{Z}^3}\varepsilon^{-1}(\Gamma^\varepsilon-\Gamma^L)(\widehat{f^\varepsilon}(k-p),\widehat{f^L}(p)), P_m\mu^{\frac12}\Big\rangle\Big|^2\,dt\Big)^{\frac12} \lesssim \|f^\varepsilon\|_{L^1_kL^\infty_TH^3}\|f^L\|_{L^1_kL^2_TH^3_{9+\gamma/2}}.$$

Similarly, for the term $\varepsilon^{-1}(\mathscr{L}^L-\mathscr{L}^\varepsilon)\widehat{f^\varepsilon}(k)$, by Lemma 4.1 we have

$$\sum_{k\in\mathbb{Z}^3}\Big(\int_0^T|\langle\varepsilon^{-1}(\mathscr{L}^L-\mathscr{L}^\varepsilon)\widehat{f^\varepsilon}(k), P_m\mu^{\frac12}\rangle|^2\,dt\Big)^{\frac12} \lesssim \|f^\varepsilon\|_{L^1_kL^2_TH^3_{9+\gamma/2}}.$$

In summary, we have

$$\begin{aligned}\|[a,b,c]\|_{L^1_kL^2_T} \le{}& C_1\|(\mathbb{I}-\mathbb{P})F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}} + C_1\|F_R^\varepsilon\|_{L^1_kL^\infty_TL^2}\\ &+ C_1\|f^\varepsilon\|_{L^1_kL^\infty_TL^2}\|F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}}\\ &+ C_1\|F_R^\varepsilon\|_{L^1_kL^\infty_TL^2}\|f^L\|_{L^1_kL^2_TL^2_{0,\gamma/2}}\\ &+ C_1\|f^\varepsilon\|_{L^1_kL^\infty_TH^3}\|f^L\|_{L^1_kL^2_TH^3_{9+\gamma/2}} + C_1\|f^\varepsilon\|_{L^1_kL^2_TH^3_{9+\gamma/2}}. \end{aligned} \tag{4.40}$$

Step 2: Microscopic part. From (4.38), taking the inner product with $\widehat{F_R^\varepsilon}(k)$, similarly to [14, equation (3.7)], with $F_R^\varepsilon(0) = 0$, we have

$$\|F_R^\varepsilon\|_{L^1_k L^\infty_T L^2} + \|(\mathbb{I} - \mathbb{P})F_R^\varepsilon\|_{L^1_k L^2_T L^2_{0,\gamma/2}} \lesssim \sum_{k\in\mathbb{Z}^3} \left(\int_0^T |\langle \hat{\mathscr{H}}(k), \widehat{F_R^\varepsilon}(k)\rangle| \, dt \right)^{\frac{1}{2}}.$$

We estimate $\hat{\mathscr{H}}(k)$ term by term as follows. For the term $\sum_{p\in\mathbb{Z}^3} \Gamma^\varepsilon(\widehat{f^\varepsilon}(k-p), \widehat{F_R^\varepsilon}(p))$, by Theorem 3.1, we have

$$\sum_{k\in\mathbb{Z}^3} \left(\int_0^T \left| \left\langle \sum_{p\in\mathbb{Z}^3} \Gamma^\varepsilon(\widehat{f^\varepsilon}(k-p), \widehat{F_R^\varepsilon}(p)), \widehat{F_R^\varepsilon}(k) \right\rangle \right| dt \right)^{\frac{1}{2}}$$
$$\lesssim \eta \|F_R^\varepsilon\|_{L^1_k L^2_T L^2_{0,\gamma/2}} + \frac{1}{4\eta} \|f^\varepsilon\|_{L^1_k L^\infty_T L^2} \|F_R^\varepsilon\|_{L^1_k L^2_T L^2_{0,\gamma/2}}.$$

Similarly, for the term $\sum_{p\in\mathbb{Z}^3} \Gamma^L(\widehat{F_R^\varepsilon}(k-p), \widehat{f^L}(p))$, we have

$$\sum_{k\in\mathbb{Z}^3} \left(\int_0^T \left| \left\langle \sum_{p\in\mathbb{Z}^3} \Gamma^L(\widehat{F_R^\varepsilon}(k-p), \widehat{f^L}(p)), \widehat{F_R^\varepsilon}(k) \right\rangle \right| dt \right)^{\frac{1}{2}}$$
$$\lesssim \eta \|F_R^\varepsilon\|_{L^1_k L^2_T L^2_{0,\gamma/2}} + \frac{1}{4\eta} \|F_R^\varepsilon\|_{L^1_k L^\infty_T L^2} \|f^L\|_{L^1_k L^2_T L^2_{0,\gamma/2}}.$$

For the term $\sum_{p\in\mathbb{Z}^3} \varepsilon^{-1}(\Gamma^\varepsilon - \Gamma^L)(\widehat{f^\varepsilon}(k-p), \widehat{f^L}(p))$, by Lemma 4.1 we have

$$\sum_{k\in\mathbb{Z}^3} \left(\int_0^T \left| \left\langle \sum_{p\in\mathbb{Z}^3} \varepsilon^{-1}(\Gamma^\varepsilon - \Gamma^L)(\widehat{f^\varepsilon}(k-p), \widehat{f^L}(p)), \widehat{F_R^\varepsilon}(k) \right\rangle \right| dt \right)^{\frac{1}{2}}$$
$$\lesssim \eta \|F_R^\varepsilon\|_{L^1_k L^2_T L^2_{0,\gamma/2}} + \frac{1}{4\eta} \|f^\varepsilon\|_{L^1_k L^\infty_T H^3} \|f^L\|_{L^1_k L^2_T H^3_{9+\gamma/2}}.$$

Finally, for the term $\varepsilon^{-1}(\mathscr{L}^L - \mathscr{L}^\varepsilon)\widehat{f^\varepsilon}(k)$, by Lemma 4.1 we have

$$\sum_{k\in\mathbb{Z}^3} \left(\int_0^T |\langle \varepsilon^{-1}(\mathscr{L}^L - \mathscr{L}^\varepsilon)\widehat{f^\varepsilon}(k), \widehat{F_R^\varepsilon}(k)\rangle| \, dt \right)^{\frac{1}{2}}$$
$$\lesssim \eta \|F_R^\varepsilon\|_{L^1_k L^2_T L^2_{0,\gamma/2}} + \frac{1}{4\eta} \|f^\varepsilon\|_{L^1_k L^2_T H^3_{9+\gamma/2}}.$$

Combining the above estimates gives

$$\begin{aligned}
&\|F_R^\varepsilon\|_{L^1_k L^\infty_T L^2} + \|(\mathbb{I} - \mathbb{P})F_R^\varepsilon\|_{L^1_k L^2_T L^2_{0,\gamma/2}} \\
&\quad \le \eta \|F_R^\varepsilon\|_{L^1_k L^2_T L^2_{0,\gamma/2}} + \frac{C_2}{\eta} \|f^\varepsilon\|_{L^1_k L^\infty_T L^2} \|F_R^\varepsilon\|_{L^1_k L^2_T L^2_{0,\gamma/2}} \\
&\qquad + \frac{C_2}{\eta} \|F_R^\varepsilon\|_{L^1_k L^\infty_T L^2} \|f^L\|_{L^1_k L^2_T L^2_{0,\gamma/2}} + \frac{C_2}{\eta} \|f^\varepsilon\|_{L^1_k L^\infty_T H^3} \|f^L\|_{L^1_k L^2_T H^3_{9+\gamma/2}} \\
&\qquad + \frac{C_2}{\eta} \|f^\varepsilon\|_{L^1_k L^2_T H^3_{9+\gamma/2}}.
\end{aligned} \tag{4.41}$$

Step 3: Micro–macro components. The combination (4.40) $\times \frac{1}{2C_1}$ + (4.41) gives

$$\begin{aligned}&\frac{1}{2}\|F_R^\varepsilon\|_{L^1_kL^\infty_TL^2}+\frac{1}{2}\|(\mathbb{I}-\mathbb{P})F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}}+\frac{1}{2C_1}\|[a,b,c]\|_{L^1_kL^2_T}\\&\quad\le\eta\|F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}}+\Big(\frac{C_2}{\eta}+\frac{1}{2}\Big)\|f^\varepsilon\|_{L^1_kL^\infty_TL^2}\|F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}}\\&\qquad+\Big(\frac{C_2}{\eta}+\frac{1}{2}\Big)\|F_R^\varepsilon\|_{L^1_kL^\infty_TL^2}\|f^L\|_{L^1_kL^2_TL^2_{0,\gamma/2}}\\&\qquad+\Big(\frac{C_2}{\eta}+\frac{1}{2}\Big)\|f^\varepsilon\|_{L^1_kL^\infty_TH^3}\|f^L\|_{L^1_kL^2_TH^3_{9+\gamma/2}}+\Big(\frac{C_2}{\eta}+\frac{1}{2}\Big)\|f^\varepsilon\|_{L^1_kL^2_TH^3_{9+\gamma/2}}.\end{aligned}$$

Note that

$$\frac{1}{2}\|(\mathbb{I}-\mathbb{P})F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}}+\frac{1}{2C_1}\|[a,b,c]\|_{L^1_kL^2_T}\ge c_1\|F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}}.$$

Then by choosing $\eta=\frac{c_1}{2}$, we have

$$\begin{aligned}\frac{1}{2}\|F_R^\varepsilon\|_{L^1_kL^\infty_TL^2}+\frac{c_1}{2}\|F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}}&\le\Big(\frac{2C_2}{c_1}+\frac{1}{2}\Big)\|f^\varepsilon\|_{L^1_kL^\infty_TL^2}\|F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}}\\&\quad+\Big(\frac{2C_2}{c_1}+\frac{1}{2}\Big)\|F_R^\varepsilon\|_{L^1_kL^\infty_TL^2}\|f^L\|_{L^1_kL^2_TL^2_{0,\gamma/2}}\\&\quad+\Big(\frac{2C_2}{c_1}+\frac{1}{2}\Big)\|f^\varepsilon\|_{L^1_kL^\infty_TH^3}\|f^L\|_{L^1_kL^2_TH^3_{9+\gamma/2}}\\&\quad+\Big(\frac{2C_2}{c_1}+\frac{1}{2}\Big)\|f^\varepsilon\|_{L^1_kL^2_TH^3_{9+\gamma/2}}.\end{aligned}$$

Under the smallness assumption on $\|f_0\|_{L^1_kL^2}$, by (1.24) we have

$$\begin{aligned}&\|F_R^\varepsilon\|_{L^1_kL^\infty_TL^2}+\|F_R^\varepsilon\|_{L^1_kL^2_TL^2_{0,\gamma/2}}\\&\quad\lesssim\|f^\varepsilon\|_{L^1_kL^2_TH^3_{9+\gamma/2}}+\|f^\varepsilon\|_{L^1_kL^\infty_TH^3}\|f^L\|_{L^1_kL^2_TH^3_{9+\gamma/2}}\\&\quad\lesssim P_3(\|f_0\|_{0,3,9})(1+P_3(\|f_0\|_{0,3,9})),\end{aligned}$$

where we have used the propagation estimates (1.26) and (1.27) with $m=0, n=3, l=9$. Note that $F_R^\varepsilon=\varepsilon^{-1}(f^\varepsilon-f^L)$ and this gives (1.28). ∎

5. Propagation of moment and decay transition

With Lemma 3.8 and [14, proof of Theorem 2.1], we have the propagation of moment stated in the following theorem.

Theorem 5.1. *Under the assumptions in Theorem 1.2, let $l,q\ge 0$, $-2\le\gamma\le 0$. There is a constant $\delta_1>0$, such that if $\|W_{l,q}f_0\|_{L^1_kL^2}\le\delta_1$, then the solution f^ε to the Boltzmann equation (1.16) satisfies*

$$\|W_{l,q}f^\varepsilon\|_{L^1_kL^\infty_TL^2}\lesssim\|W_{l,q}f_0\|_{L^1_kL^2}.$$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Consider

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \mathscr{L}^\varepsilon f^\varepsilon = \Gamma^\varepsilon(f^\varepsilon, f^\varepsilon). \tag{5.1}$$

Recall that $T_\varepsilon = (\frac{1}{\varepsilon})^{\frac{2(1-s)}{|\gamma+2s|}}$, $\kappa = \frac{1}{1+|\gamma+2s|}$, and the function $A_\varepsilon(t)$ defined in (1.36) is

$$A_\varepsilon(t) = \zeta(T_\varepsilon^{-1}t)t + (1-\zeta(T_\varepsilon^{-1}t))\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^\kappa.$$

Here, the function ζ is defined in (1.31). Multiplying (5.1) by $g(t) := \exp(\lambda A_\varepsilon(t))$, with $h^\varepsilon(t) := g(t) f^\varepsilon(t)$, we have

$$\partial_t h^\varepsilon + v \cdot \nabla_x h^\varepsilon + \mathscr{L}^\varepsilon h^\varepsilon = \Gamma^\varepsilon(f^\varepsilon, h^\varepsilon) + \lambda A_\varepsilon'(t) h.$$

Similarly to [14, equation (6.3)], we get

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^3} \sup_{0\le t\le T} \|\hat{h}(t,k)\|_{L^2} + \sqrt{\lambda_0} \sum_{k\in\mathbb{Z}^3} \Big(\int_0^T |\hat{h}(t,k)|^2_{\varepsilon,\gamma/2}\, dt\Big)^{\frac12} \\
&\quad\lesssim \sum_{k\in\mathbb{Z}^3} \|\hat{f}_0(k)\|_{L^2} + \sqrt{\lambda} \sum_{k\in\mathbb{Z}^3} \Big(\int_0^T A_\varepsilon'(t) \|\hat{h}(t,k)\|^2_{L^2}\, dt\Big)^{\frac12}.
\end{aligned}$$

Now we use dissipation and propagation of moment to cancel the last term. Note $0 \le A_\varepsilon'(t) \lesssim 1$ and

$$|\hat{h}(t,k)|^2_{\varepsilon,\gamma/2} \ge |\langle v\rangle^{1+\gamma/2} \zeta \hat{h}(t,k)|^2_{L^2} \ge |\zeta \hat{h}(t,k)|^2_{L^2},$$

which gives, for $\lambda/\lambda_0 \ll 1$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^3} \sup_{0\le t\le T} \|\hat{h}(t,k)\|_{L^2} + \sqrt{\lambda_0} \sum_{k\in\mathbb{Z}^3} \Big(\int_0^T |\hat{h}(t,k)|^2_{\varepsilon,\gamma/2}\, dt\Big)^{\frac12} \\
&\quad\lesssim \sum_{k\in\mathbb{Z}^3} \|\hat{f}_0(k)\|_{L^2} + \sqrt{\lambda} \sum_{k\in\mathbb{Z}^3} \Big(\int_0^T A_\varepsilon'(t) \|(1-\zeta)\hat{h}(t,k)\|^2_{L^2}\, dt\Big)^{\frac12}.
\end{aligned} \tag{5.2}$$

We claim that for any $T > 0$,

$$\sum_{k\in\mathbb{Z}^3} \sup_{0\le t\le T} \|\hat{h}(t,k)\|_{L^2} + \sqrt{\lambda_0} \sum_{k\in\mathbb{Z}^3} \Big(\int_0^T |\hat{h}(t,k)|^2_{\varepsilon,\gamma/2}\, dt\Big)^{\frac12} \lesssim \|W_{0,q} f_0\|_{L^1_k L^2}. \tag{5.3}$$

Note that the support of $(1-\zeta)$ is $|v| \ge \frac{1}{2\varepsilon}$. We prove the claim by considering three cases.

Case 1: $T \le \frac{1}{2\varepsilon}$. In this case, $t \le T \le \frac{1}{2\varepsilon} \le |v|$. With the fact that $A_\varepsilon(t) \le 2t$, by the propagation of moment in Theorem 5.1, we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^3}\left(\int_0^T A_\varepsilon'(t)\|(1-\zeta)\hat{h}(t,k)\|_{L^2}^2\,dt\right)^{\frac12}\\
&\quad\lesssim \sum_{k\in\mathbb{Z}^3}\left(\int_0^T \exp(4\lambda t-2qt)\|(1-\zeta)\exp(q\langle v\rangle)\hat{f}(t,k)\|_{L^2}^2\,dt\right)^{\frac12}\\
&\quad\lesssim C(q,\lambda)\sum_{k\in\mathbb{Z}^3}\|(1-\zeta)\exp(q\langle v\rangle)\hat{f}(t,k)\|_{L^\infty_T L^2}^2\\
&\quad\lesssim C(q,\lambda)\|\exp(q\langle v\rangle)f_0\|_{L^1_k L^2},
\end{aligned}
\tag{5.4}
$$

because $q > 2\lambda$. Plugging (5.4) into (5.2), we have (5.3) for $T \le \frac{1}{2\varepsilon}$.

Case 2: $\frac{1}{2\varepsilon} \le T \le T_\varepsilon$. In view of (5.4), we only need to consider

$$
\sum_{k\in\mathbb{Z}^3}\left(\int_{\frac{1}{2\varepsilon}}^T A_\varepsilon'(t)\|(1-\zeta)\hat{h}(t,k)\|_{L^2}^2\,dt\right)^{\frac12}.
$$

The domain of $\frac{1}{2\varepsilon} \le t \le T$, $v \in \mathbb{R}^3$ can be divided into two parts:

$$
\mathrm{D}_1 := \{\langle v\rangle \le T_\varepsilon\},\quad \mathrm{D}_2 := \{\langle v\rangle > T_\varepsilon\}.
$$

In D_1, recalling that $T_\varepsilon = (\frac{1}{\varepsilon})^{\frac{2(1-s)}{|\gamma+2s|}}$, we have $\langle v\rangle^{\gamma+2s}\varepsilon^{2(1-s)} \ge 1$ so that

$$
\begin{aligned}
&\sqrt{\lambda}\sum_{k\in\mathbb{Z}^3}\left(\int_{\frac{1}{2\varepsilon}}^T A_\varepsilon'(t)\|\mathbb{1}_{\mathrm{D}_1}(1-\zeta)\hat{h}(t,k)\|_{L^2}^2\,dt\right)^{\frac12}\\
&\quad\lesssim \sqrt{\lambda}\sum_{k\in\mathbb{Z}^3}\left(\int_{\frac{1}{2\varepsilon}}^T \|\hat{h}(t,k)\|_{\varepsilon,\gamma/2}^2\,dt\right)^{\frac12}.
\end{aligned}
\tag{5.5}
$$

Note that by taking λ small enough such that $\lambda/\lambda_0 \ll 1$, this can be absorbed by the dissipation. In D_2, since $\langle v\rangle > T_\varepsilon \ge T \ge t$, similarly to (5.4), by using Theorem 5.1 we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^3}\left(\int_{\frac{1}{2\varepsilon}}^T A_\varepsilon'(t)\|\mathbb{1}_{\mathrm{D}_2}(1-\zeta)\hat{h}(t,k)\|_{L^2}^2\,dt\right)^{\frac12}\\
&\quad\lesssim \sum_{k\in\mathbb{Z}^3}\left(\int_{\frac{1}{2\varepsilon}}^T \exp(4\lambda t-2qt)\|(1-\zeta)\exp(q\langle v\rangle)\hat{f}(t,k)\|_{L^2}^2\,dt\right)^{\frac12}\\
&\quad\lesssim C(q,\lambda)\|\exp(q\langle v\rangle)f_0\|_{L^1_k L^2}.
\end{aligned}
\tag{5.6}
$$

Plugging (5.5) and (5.6) into (5.2), we have (5.3) for $\frac{1}{2\varepsilon} \le T \le T_\varepsilon$.

Case 3: $T > T_\varepsilon$. In view of (5.4), (5.5), and (5.6), we only need to consider

$$\sum_{k\in\mathbb{Z}^3}\Big(\int_{T_\varepsilon}^{T} A'_\varepsilon(t)\|(1-\zeta)\hat{h}(t,k)\|_{L^2}^2\,dt\Big)^{\frac{1}{2}},$$

where $A'_\varepsilon(t)$ gives some decay since $0<\kappa<1$,

$$A'_\varepsilon(t)=\kappa\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^{\kappa-1}\frac{1}{\varepsilon^{2(1-s)}}.$$

Note that when $t\ge T_\varepsilon$, we have $A_\varepsilon(t)=(\frac{t}{\varepsilon^{2(1-s)}})^{\kappa}$.

Divide the domain $T_\varepsilon\le t\le T$, $v\in\mathbb{R}^3$ into two parts,

$$\mathrm{D}_3:=\Big\{\langle v\rangle^{\gamma+2s}\ge\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^{\kappa-1}\Big\},$$
$$\mathrm{D}_4:=\Big\{\langle v\rangle^{\gamma+2s}<\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^{\kappa-1}\Big\}.$$

In D_3, we have

$$\langle v\rangle^{\gamma+2s}\frac{1}{\varepsilon^{2(1-s)}}\ge\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^{\kappa-1}\frac{1}{\varepsilon^{2(1-s)}}\ge A'_\varepsilon(t)$$

so that

$$\begin{aligned}&\sqrt{\lambda}\sum_{k\in\mathbb{Z}^3}\Big(\int_{T_\varepsilon}^{T} A'_\varepsilon(t)\|\mathbb{1}_{\mathrm{D}_3}(1-\zeta)\hat{h}(t,k)\|_{L^2}^2\,dt\Big)^{\frac{1}{2}}\\&\quad\lesssim\sqrt{\lambda}\sum_{k\in\mathbb{Z}^3}\Big(\int_{T_\varepsilon}^{T}\|\hat{h}(t,k)\|_{\varepsilon,\gamma/2}^2\,dt\Big)^{\frac{1}{2}}.\end{aligned}\tag{5.7}$$

Note that by taking λ small enough such that $\lambda/\lambda_0\ll 1$, this can be absorbed by the dissipation. In D_4, since $\langle v\rangle>(\frac{t}{\varepsilon^{2(1-s)}})^{\kappa}$, similarly to (5.4), by using Theorem 5.1 we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3}\Big(\int_{T_\varepsilon}^{T} A'_\varepsilon(t)\|\mathbb{1}_{\mathrm{D}_4}(1-\zeta)\hat{h}(t,k)\|_{L^2}^2\,dt\Big)^{\frac{1}{2}}\\&\quad=\sum_{k\in\mathbb{Z}^3}\Big(\int_{T_\varepsilon}^{T} A'_\varepsilon(t)\exp\Big(2\lambda\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^{\kappa}\Big)\|\mathbb{1}_{\mathrm{D}_4}(1-\zeta)\hat{f}(t,k)\|_{L^2}^2\,dt\Big)^{\frac{1}{2}}\\&\quad\lesssim\sum_{k\in\mathbb{Z}^3}\Big(\int_{T_\varepsilon}^{T} A'_\varepsilon(t)\exp\Big(2(\lambda-q)\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^{\kappa}\Big)\|(1-\zeta)\exp(q\langle v\rangle)\hat{f}(t,k)\|_{L^2}^2\,dt\Big)^{\frac{1}{2}}\\&\quad\lesssim C(q,\lambda,\kappa)\sum_{k\in\mathbb{Z}^3}\|(1-\zeta)\exp(q\langle v\rangle)\hat{f}(t,k)\|_{L^\infty_T L^2}^2\\&\quad\lesssim C(q,\lambda,\kappa)\|\exp(q\langle v\rangle)f_0\|_{L^1_k L^2},\end{aligned}\tag{5.8}$$

where for $0 < \kappa < 1$, $\lambda < q$, we have used the estimate

$$\begin{aligned}\int_{T_\varepsilon}^{T} & A'_\varepsilon(t)\exp\Big(2(\lambda-q)\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^{\kappa}\Big)\,dt \\ &= \int_{T_\varepsilon}^{T} \kappa\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^{\kappa-1}\frac{1}{\varepsilon^{2(1-s)}}\exp\Big(2(\lambda-q)\Big(\frac{t}{\varepsilon^{2(1-s)}}\Big)^{\kappa}\Big)\,dt \\ &\le \int_0^{\infty} z^{\kappa-1}\exp(2(\lambda-q)z^{\kappa})\,dt \lesssim C(q,\lambda,\kappa).\end{aligned}$$

Plugging (5.7) and (5.8) into (5.2), we have (5.3) for $T > T_\varepsilon$.

Since $h^\varepsilon(t) := \exp(\lambda A_\varepsilon(t)) f^\varepsilon(t)$, by (5.3) we get

$$\|f^\varepsilon(t)\|_{L^1_k L^2} \lesssim \exp(-\lambda A_\varepsilon(t))\|\exp(q\langle v\rangle) f_0\|_{L^1_k L^2},$$

By the definition of A_ε, we obtain (1.29). ∎

A. Supplementary formulas and estimates

Lemma A.1 ([23, Lemma 4.1]). *For any function f defined on $\mathbb{S}^2$, it holds that*

$$\begin{aligned}(1-s)\varepsilon^{2s-2}\int_{\mathbb{S}^2\times\mathbb{S}^2} & \frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}\mathbb{1}_{|\sigma-\tau|\le\varepsilon}\,d\sigma\,d\tau + |f|^2_{L^2(\mathbb{S}^2)} \\ &\sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12})f|^2_{L^2(\mathbb{S}^2)} + |f|^2_{L^2(\mathbb{S}^2)}.\end{aligned}$$

As consequence, for any function f defined on $\mathbb{R}^3$, it holds that

$$\begin{aligned}(1-s)\varepsilon^{2s-2}\int_{\mathbb{R}_+\times\mathbb{S}^2\times\mathbb{S}^2} & \frac{|f(r\sigma)-f(r\tau)|^2}{|\sigma-\tau|^{2+2s}}\mathbb{1}_{|\sigma-\tau|\le\varepsilon} r^2\,d\sigma\,d\tau\,dr + |f|^2_{L^2} \\ &\sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12})f|^2_{L^2} + |f|^2_{L^2}. \end{aligned}\tag{A.1}$$

Remark A.1. Lemma A.1 also holds if we replace $\mathbb{1}_{|\sigma-\tau|\le\varepsilon}$ by $\mathbb{1}_{|\sigma-\tau|\le 2\varepsilon}$.

Similarly to [23, Lemma 5.8], we have the following lemma.

Lemma A.2. *Let $\mathcal{F}$ be Fourier transform operator. Then*

$$\mathcal{F} W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12}) = W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12})\mathcal{F}.$$

Proposition A.1. *Suppose*

$$E^\varepsilon(\xi) := \frac{1}{4\pi}\int_{\mathbb{S}^2} b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\min\big\{|\xi|^2\sin^2(\theta/2),1\big\}\,d\sigma.$$

Then we have

$$E^\varepsilon(\xi) = \mathbb{1}_{|\xi|\le\varepsilon^{-1}}|\xi|^2 + \mathbb{1}_{|\xi|>\varepsilon^{-1}}\varepsilon^{2s-2}\Big[\frac{1}{s}(|\xi|^{2s}-\varepsilon^{-2s}) + \varepsilon^{-2s}\Big].$$

As a result, we have

$$E^\varepsilon(\xi) + 1 \sim (W^\varepsilon)^2(\xi),$$

where W^ε is defined in (1.30). Here, the constants in $\sim$ may depend on s.

Proof. Recalling (1.7) and $d\sigma = 4\sin(\theta/2)\, d\mathbb{S}\, d\sin(\theta/2)$, we have

$$E^\varepsilon(\xi) = 2(1-s)\varepsilon^{2s-2}\int_0^\pi \sin^{-1-2s}(\theta/2)\sin\theta \mathbb{1}_{\sin\frac{\theta}{2}\le\varepsilon} \min\{|\xi|^2\sin^2(\theta/2), 1\}\, d\sin(\theta/2).$$

By the change of variable $t = \sin(\theta/2)$, we have

$$E^\varepsilon(\xi) = 2(1-s)\varepsilon^{2s-2}\int_0^\varepsilon t^{-1-2s}\min\{|\xi|^2 t^2, 1\}\, dt.$$

When $|\xi| \le \varepsilon^{-1}$, we have

$$E^\varepsilon(\xi) = 2(1-s)\varepsilon^{2s-2}|\xi|^2\int_0^\varepsilon t^{1-2s}\, dt = |\xi|^2.$$

When $|\xi| > \varepsilon^{-1}$, we have

$$\begin{aligned} E^\varepsilon(\xi) &= 2(1-s)\varepsilon^{2s-2}|\xi|^2\int_0^{|\xi|^{-1}} t^{1-2s}\, dt + 2(1-s)\varepsilon^{2s-2}\int_{|\xi|^{-1}}^\varepsilon t^{-1-2s}\, dt \\ &= \varepsilon^{2s-2}\Big[\frac{1}{s}(|\xi|^{2s} - \varepsilon^{-2s}) + \varepsilon^{-2s}\Big]. \end{aligned}$$

The proof is completed by combining the above two cases. ∎

We now recall the definition of the symbol class $S^m_{1,0}$.

Definition A.1. A smooth function $a(v,\xi)$ is a symbol of type $S^m_{1,0}$ if $a(v,\xi)$ satisfies, for any multi-indices α and β,

$$|(\partial_\xi^\alpha \partial_v^\beta a)(v,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|},$$

where $C_{\alpha,\beta}$ is a constant depending only on α and β.

Lemma A.3 ([23, Lemma 5.3]). *Let $l, s, r \in \mathbb{R}$, $M \in S^r_{1,0}$, $\Phi \in S^l_{1,0}$. The estimate $|[M(D), \Phi]f|_{H^s} \lesssim |f|_{H^{r+s-1}_{l-1}}$ holds.*

We now recall the dyadic decomposition. Let $B_{4/3} := \{x \in \mathbb{R}^3 : |x| \le 4/3\}$ and $C := \{x \in \mathbb{R}^3 : 3/4 \le |x| \le 8/3\}$. Denote two radial functions $\phi \in C_0^\infty(B_{4/3})$ and $\psi \in C_0^\infty(C)$ which satisfy

$$0 \le \phi, \psi \le 1 \quad \text{and} \quad \phi(x) + \sum_{j\ge 0}\psi(2^{-j}x) = 1 \quad \text{for all } x \in \mathbb{R}^3. \tag{A.2}$$

Set $\varphi_{-1}(x) := \phi(x)$ and $\varphi_j(x) := \psi(2^{-j}x)$ for any $x \in \mathbb{R}^3$ and $j \ge 0$. Then the dyadic decomposition $f = \sum_{j=-1}^\infty \varphi_j f$ holds for any function defined on $\mathbb{R}^3$.

Proposition A.2. *(*[26, Proposition 5.2]*) It holds that*

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{S}^2} b\Big(\frac{u}{|u|}\cdot\sigma\Big)h(u)\Big(f(u^+)-f\Big(\frac{|u|}{|u^+|}u^+\Big)\Big)\,d\sigma\,du\\&\quad=\int_{\mathbb{R}^3\times\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big(\hat h(\xi^+)-\hat h\Big(\frac{|\xi|}{|\xi^+|}\xi^+\Big)\Big)\bar{\hat f}(\xi)\,d\sigma\,d\xi.\end{aligned}$$

Lemma A.4. *Let* $\mathcal{Y}^{\varepsilon,\gamma}(h,f):=\int b^\varepsilon(\frac{u}{|u|}\cdot\sigma)\langle u\rangle^\gamma h(u)[f(u^+)-f(|u|\frac{u^+}{|u^+|})]\,du\,d\sigma$, *where* $u^+=\frac{u+|u|\sigma}{2}$. *Then*

$$\begin{aligned}|\mathcal{Y}^{\varepsilon,\gamma}(h,f)|\lesssim{}&(|W^\varepsilon W_{\gamma/2}h|_{L^2}+|W^\varepsilon(D)W_{\gamma/2}h|_{L^2})\\&\times(|W^\varepsilon W_{\gamma/2}f|_{L^2}+|W^\varepsilon(D)W_{\gamma/2}f|_{L^2}).\end{aligned}$$

Proof. We divide the proof into two steps.

Step 1: $\gamma=0$. Since the support of b^ε is in $\frac{u}{|u|}\cdot\sigma\ge 0$, we get $|u|/\sqrt{2}\le|u^+|\le|u|$. Recalling the function ζ in (1.31), we define $\zeta_4(\cdot):=\zeta(\frac{\cdot}{4})$. We apply the decomposition

$$\begin{aligned}\mathcal{Y}^{\varepsilon,0}(h,f)&=\mathcal{Y}^{\varepsilon,0}(h,\zeta(\varepsilon v)f)+\mathcal{Y}^{\varepsilon,0}\big(h,(1-\zeta(\varepsilon v))f\big)\\&=\mathcal{Y}^{\varepsilon,0}(\zeta_4(\varepsilon v)h,\zeta(\varepsilon v)f)+\mathcal{Y}^{\varepsilon,0}\big(h,(1-\zeta(\varepsilon v))f\big).\end{aligned}\tag{A.3}$$

Note that the second equality is ensured by the definition of ζ and the fact that $|u|/\sqrt{2}\le|u^+|\le|u|$. The first term in (A.3) can be decomposed further as

$$\begin{aligned}\mathcal{Y}^{\varepsilon,0}(\zeta_4(\varepsilon v)h,\zeta(\varepsilon v)f)={}&\mathcal{Y}^{\varepsilon,0}(\zeta(\varepsilon D)\zeta_4(\varepsilon v)h,\zeta(\varepsilon v)f)\\&+\mathcal{Y}^{\varepsilon,0}\big((1-\zeta(\varepsilon D))\zeta_4(\varepsilon v)h,\zeta(\varepsilon v)f\big).\end{aligned}\tag{A.4}$$

Step 1.1: Estimate of $\mathcal{Y}^{\varepsilon,0}(\zeta(\varepsilon D)\zeta_4(\varepsilon v)h,\zeta(\varepsilon v)f)$. By Proposition A.2 and the fact that $|\xi|/\sqrt{2}\le|\xi^+|\le|\xi|$, we have

$$\begin{aligned}&\mathcal{Y}^{\varepsilon,0}(\zeta(\varepsilon D)\zeta_4(\varepsilon v)h,\zeta(\varepsilon v)f)\\&\quad=\mathcal{Y}^{\varepsilon,0}(\zeta(\varepsilon D)\zeta_4(\varepsilon v)h,\zeta_4(\varepsilon D)\zeta(\varepsilon v)f)\\&\quad=\int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big)(\zeta(\varepsilon D)\zeta_4(\varepsilon v)h)(u)[(\zeta_4(\varepsilon D)\zeta(\varepsilon v)f)(u^+)\\&\qquad-(\zeta_4(\varepsilon D)\zeta(\varepsilon v)f)\Big(|u|\frac{u^+}{|u^+|}\Big)]\,du\,d\sigma.\end{aligned}$$

By Taylor expansion,

$$\begin{aligned}&(\zeta_4(\varepsilon D)\zeta(\varepsilon v)f)(u^+)-(\zeta_4(\varepsilon D)\zeta(\varepsilon v)f)\Big(|u|\frac{u^+}{|u^+|}\Big)\\&\quad=\Big(1-\frac{1}{\cos\theta}\Big)\int_0^1\big(\nabla(\zeta_4(\varepsilon D)\zeta(\varepsilon v)f)\big)(u^+(\kappa))\cdot u^+\,d\kappa,\end{aligned}$$

where $u^+(\kappa) = (1-\kappa)|u|\frac{u^+}{|u^+|} + \kappa u^+$. Then by the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\mathcal{Y}^{\varepsilon,0}(\zeta(\varepsilon D)\zeta_4(\varepsilon v)h, \zeta(\varepsilon v) f)\\
&\quad = \int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big)\Big(1-\frac{1}{\cos\theta}\Big)(\zeta(\varepsilon D)\zeta(\varepsilon v)h)(u)\\
&\qquad \times \int_0^1 \big(\nabla(\zeta_4(\varepsilon D)\zeta(\varepsilon v) f)\big)(u^+(\kappa))\cdot u^+ \, d\kappa\, du\, d\sigma.\\
&\quad \lesssim \Big(\int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big)\sin^2\frac{\theta}{2}|(\zeta(\varepsilon D)\zeta_4(\varepsilon v)h)(u)|^2|u^+|^2\, du\, d\sigma\Big)^{\frac12}\\
&\qquad \times \Big(\int b^\varepsilon\Big(\frac{u}{|u|}\cdot\sigma\Big)\sin^2\frac{\theta}{2}\big|\big(\nabla(\zeta_4(\varepsilon D)\zeta(\varepsilon v) f)\big)(u^+(\kappa))\big|^2\, du\, d\sigma\Big)^{\frac12}\\
&\quad \lesssim |\zeta(\varepsilon D)\zeta_4(\varepsilon v)h|_{L^2_1}|\zeta_4(\varepsilon D)\zeta(\varepsilon v) f|_{H^1}\\
&\quad \lesssim |W^\varepsilon h|_{L^2}|W^\varepsilon(D) f|_{L^2},
\end{aligned} \tag{A.5}$$

where we have used the fact that $|u^+| \sim |u|$, the change of variable $u \to u^+(\kappa)$, and the estimate (2.8).

Step 1.2: Estimate of $\mathcal{Y}^{\varepsilon,0}((1-\zeta(\varepsilon D))\zeta_4(\varepsilon v)h, \zeta(\varepsilon v) f)$. By Proposition A.2 and the dyadic decomposition in the frequency space, we have

$$\begin{aligned}
&\mathcal{Y}^{\varepsilon,0}\big((1-\zeta(\varepsilon D))\zeta_4(\varepsilon v)h, \zeta(\varepsilon v) f\big)\\
&\quad = \int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big[\big((1-\zeta(\varepsilon\xi))\widehat{\zeta_4(\varepsilon\cdot)h}\big)(\xi^+)\\
&\qquad\qquad - \big((1-\zeta(\varepsilon\xi))\widehat{\zeta_4(\varepsilon\cdot)h}\big)\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)\Big]\overline{\widehat{\zeta(\varepsilon\cdot) f}}(\xi)\, d\xi\, d\sigma\\
&\quad = \sum_{l\ge[-\log_2\varepsilon]-4} \mathcal{Y}_l,
\end{aligned} \tag{A.6}$$

$$\mathcal{Y}_l := \int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big[(\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h})(\xi^+) - (\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h})\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)\Big](\tilde\varphi_l\overline{\widehat{\zeta(\varepsilon\cdot) f}})(\xi)\, d\xi\, d\sigma,$$

where $\tilde\varphi_l := \sum_{|j-l|\le 2, j\ge -1}\varphi_j$. Decompose as $\mathcal{Y}_l = \mathcal{Y}_{l,\le} + \mathcal{Y}_{l,\ge}$ according to $\sin(\theta/2) \le 2^{-l}$ and $\sin(\theta/2) \ge 2^{-l}$. By Taylor expansion,

$$(\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h})(\xi^+) - (\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h})\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big) = \Big(1-\frac{1}{\cos\theta}\Big)\int_0^1 (\nabla\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h})(\xi^+(\kappa))\cdot\xi^+\, d\kappa,$$

where $\xi^+(\kappa) = (1-\kappa)|\xi|\frac{\xi^+}{|\xi^+|} + \kappa\xi^+$. Then by the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\mathcal{Y}_{l,\le} &= \int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\mathbb{1}_{\sin\frac{\theta}{2}\le 2^{-l}}\Big(1-\frac{1}{\cos\theta}\Big)\Big(\int_0^1 (\nabla\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h})(\xi^+(\kappa))\cdot\xi^+\, d\kappa\Big)\\
&\qquad \times (\tilde\varphi_l\overline{\widehat{\zeta(\varepsilon\cdot) f}})(\xi)\, d\xi\, d\sigma
\end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{l}\Big(\int b^{\varepsilon}\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\mathbb{1}_{\sin\frac{\theta}{2}\le 2^{-l}}\sin^2\frac{\theta}{2}|(\nabla\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h})(\xi^+(\kappa))|^2\,d\xi\,d\sigma\Big)^{\frac12}\\
&\quad\times\Big(\int b^{\varepsilon}\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\mathbb{1}_{\sin\frac{\theta}{2}\le 2^{-l}}\sin^2\frac{\theta}{2}|(\tilde\varphi_l\widehat{\zeta(\varepsilon\cdot)f})(\xi)|^2\,d\xi\,d\sigma\Big)^{\frac12}\\
&\lesssim \varepsilon^{2s-2}2^{l(2s-1)}|\nabla\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h}|_{L^2}|\tilde\varphi_l\widehat{\zeta(\varepsilon\cdot)f}|_{L^2}\\
&\lesssim \varepsilon^{s-1}2^{ls}|\nabla\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h}|_{L^2}|\tilde\varphi_l\widehat{\zeta(\varepsilon\cdot)f}|_{L^2},
\end{aligned}
\tag{A.7}
$$

where we have used the fact that $|\xi^+|\sim|\xi|\sim 2^l$, $2^l\gtrsim\varepsilon^{-1}$, the change of variable $\xi\to\xi^+(\kappa)$, and the estimate

$$
\int b^{\varepsilon}\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\sin^2\frac{\theta}{2}\mathbb{1}_{\sin\frac{\theta}{2}\le 2^{-l}}\,d\sigma\lesssim(1-s)\varepsilon^{2s-2}\int_0^{2^{-l}}t^{1-2s}\,dt\lesssim\varepsilon^{2s-2}2^{l(2s-2)}.
$$

Since

$$
|\nabla\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h}|=|\varphi_l\widehat{\zeta_4(\varepsilon\cdot)vh}+(\nabla\varphi_l)\widehat{\zeta_4(\varepsilon\cdot)h}|\lesssim|\varphi_l\widehat{\zeta_4(\varepsilon\cdot)vh}|+2^{-l}|\tilde\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h}|,
$$

we have

$$
\begin{aligned}
&\sum_{l\ge[-\log_2\varepsilon]-4}\mathcal{Y}_{l,\le}\\
&\quad\lesssim\sum_{l\ge[-\log_2\varepsilon]-4}\varepsilon^{s-1}2^{ls}(|\varphi_l\widehat{\zeta_4(\varepsilon\cdot)vh}|_{L^2}+2^{-2l}|\tilde\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h}|_{L^2})|\tilde\varphi_l\widehat{\zeta(\varepsilon\cdot)f}|_{L^2}\\
&\quad\lesssim\Big(\sum_{l\ge[-\log_2\varepsilon]-4}(|\varphi_l\widehat{\zeta_4(\varepsilon\cdot)vh}|^2_{L^2}+2^{-l}|\tilde\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h}|^2_{L^2})\Big)^{\frac12}\\
&\quad\quad\times\Big(\sum_{l\ge[-\log_2\varepsilon]-4}\varepsilon^{2s-2}2^{2ls}|\tilde\varphi_l\widehat{\zeta(\varepsilon\cdot)f}|^2_{L^2}\Big)^{\frac12}\\
&\quad\lesssim|W^{\varepsilon}h|_{L^2}|W^{\varepsilon}(D)f|_{L^2}.
\end{aligned}
\tag{A.8}
$$

By the Cauchy–Schwarz inequality, the change of variables $\xi\to\xi^+$ and $\xi\to|\xi|\frac{\xi^+}{|\xi^+|}$, and the estimate

$$
\int b^{\varepsilon}\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\mathbb{1}_{\sin\frac{\theta}{2}\ge 2^{-l}}\,d\sigma\lesssim(1-s)\varepsilon^{2s-2}\int_{2^{-l}}^{\varepsilon}t^{-1-2s}\,dt\lesssim\varepsilon^{2s-2}2^{2sl},
$$

we have similarly

$$
\begin{aligned}
\mathcal{Y}_{l,\ge}&\le\int b^{\varepsilon}\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\mathbb{1}_{\sin\frac{\theta}{2}\ge 2^{-l}}\Big[|(\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h})(\xi^+)|+\Big|(\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h})\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)\Big|\Big]\\
&\quad\times|(\tilde\varphi_l\overline{\widehat{\zeta(\varepsilon\cdot)f}})(\xi)|\,d\xi\,d\sigma\\
&\lesssim\varepsilon^{2s-2}2^{2ls}\|\varphi_l\widehat{\zeta_4(\varepsilon\cdot)h}|_{L^2}|\tilde\varphi_l\widehat{\zeta(\varepsilon\cdot)f}|_{L^2}.
\end{aligned}
\tag{A.9}
$$

By taking the sum, we get

$$\sum_{l\geq[-\log_2 \varepsilon]-4} \mathcal{Y}_{l,\geq} \lesssim \Big(\sum_{l\geq[-\log_2 \varepsilon]-4} \varepsilon^{2s-2} 2^{2ls} |\varphi_l \widehat{\zeta_4(\varepsilon\cdot) h}|^2_{L^2} \Big)^{\frac{1}{2}}$$

$$\times \Big(\sum_{l\geq[-\log_2 \varepsilon]-4} \varepsilon^{2s-2} 2^{2ls} |\tilde{\varphi}_l \widehat{\zeta(\varepsilon\cdot) f}|_{L^2} \Big)^{\frac{1}{2}}$$

$$\lesssim |W^\varepsilon(D)h|_{L^2} |W^\varepsilon(D) f|_{L^2}. \tag{A.10}$$

By combining (A.8) and (A.10), (A.6) gives

$$\mathcal{Y}^{\varepsilon,0}\big((1-\zeta(\varepsilon D))\zeta_4(\varepsilon v)h, \zeta(\varepsilon v) f\big)$$
$$\lesssim (|W^\varepsilon h|_{L^2} + |W^\varepsilon(D)h|_{L^2}) |W^\varepsilon(D) f|_{L^2}. \tag{A.11}$$

Step 1.3: Estimate of $\mathcal{Y}^{\varepsilon,0}(h, (1-\zeta(\varepsilon v)) f)$. Note that

$$\mathcal{Y}^{\varepsilon,0}\big(h, (1-\zeta(\varepsilon v)) f\big) = \sum_{k\geq[-\log_2 \varepsilon]-4} \mathcal{Y}^{\varepsilon,0}(\tilde{\varphi}_k h, \varphi_k f)$$
$$= \sum_{k\geq[-\log_2 \varepsilon]-4} \mathcal{Y}^{\varepsilon,0}(\tilde{\varphi}_k h, \zeta(\varepsilon D)\varphi_k f)$$
$$+ \sum_{k\geq[-\log_2 \varepsilon]-4} \mathcal{Y}^{\varepsilon,0}\big(\tilde{\varphi}_k h, (1-\zeta(\varepsilon D))\varphi_k f\big). \tag{A.12}$$

We first consider $\sum_{k\geq[-\log_2 \varepsilon]-4} \mathcal{Y}^{\varepsilon,0}(\tilde{\varphi}_k h, \zeta(\varepsilon D)\varphi_k f)$. Decompose as

$$\mathcal{Y}^{\varepsilon,0}(\tilde{\varphi}_k h, \zeta(\varepsilon D)\varphi_k f) = \mathcal{Y}_{k,\leq} + \mathcal{Y}_{k,\geq}$$

according to $\sin(\theta/2) \leq 2^{-k}$ and $\sin(\theta/2) \geq 2^{-k}$.

For $\mathcal{Y}_{k,\leq}$, by Taylor expansion of $\zeta(\varepsilon D)\varphi_k f$, similarly to (A.7), we have

$$\mathcal{Y}_{k,\leq} = \int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big) \mathbb{1}_{\sin\frac{\theta}{2}\leq 2^{-k}} \Big(1-\frac{1}{\cos\theta}\Big)(\tilde{\varphi}_k h)(u)$$
$$\times \int_0^1 (\nabla\zeta(\varepsilon D)\varphi_k f)(u^+(\kappa))\cdot u^+ \, d\kappa \, du \, d\sigma$$
$$\lesssim \varepsilon^{s-1} 2^{ks} |\tilde{\varphi}_k h|_{L^2} |\nabla\zeta(\varepsilon D)\varphi_k f|_{L^2}.$$

By taking the sum, since $|\nabla\zeta(\varepsilon D)\varphi_k f|_{L^2} \lesssim |W^\varepsilon(D)\varphi_k f|_{L^2}$, we have

$$\sum_{k\geq[-\log_2 \varepsilon]-4} \mathcal{Y}_{k,\leq} \lesssim \sum_{k\geq[-\log_2 \varepsilon]-4} \varepsilon^{s-1} 2^{ks} |\tilde{\varphi}_k h|_{L^2} |W^\varepsilon(D)\varphi_k f|_{L^2}$$
$$\lesssim |W^\varepsilon h|_{L^2} |W^\varepsilon(D) f|_{L^2}, \tag{A.13}$$

where we have used

$$\begin{aligned}\sum_{k\geq -1}^{\infty} |W^\varepsilon(D)\varphi_k f|_{L^2}^2 &= \sum_{k\geq -1}^{\infty} 2^{-2k}|W^\varepsilon(D)2^k\varphi_k f|_{L^2}^2 \\ &\lesssim \sum_{k\geq -1}^{\infty} 2^{-2k}(|2^k\varphi_k W^\varepsilon(D) f|_{L^2}^2 + |f|_{H^0}^2) \\ &\lesssim |W^\varepsilon(D) f|_{L^2}^2, \end{aligned} \tag{A.14}$$

because $W^\varepsilon \in S^1_{1,0}$, $2^k\varphi_k \in S^1_{1,0}$, and Lemma A.3.

For $\mathcal{Y}_{k,\geq}$, similarly to (A.9), we get $\mathcal{Y}_{k,\geq} \lesssim \varepsilon^{2s-2}2^{2ks}|\tilde{\varphi}_k h|_{L^2}|\zeta(\varepsilon D)\varphi_k f|_{L^2}$, and

$$\begin{aligned}\sum_{k\geq[-\log_2\varepsilon]-4} \mathcal{Y}_{k,\geq} &\lesssim \sum_{k\geq[-\log_2\varepsilon]-4} \varepsilon^{2s-2}2^{2ks}|\tilde{\varphi}_k h|_{L^2}|\varphi_k f|_{L^2} \\ &\lesssim |W^\varepsilon h|_{L^2}|W^\varepsilon f|_{L^2}. \end{aligned} \tag{A.15}$$

Combining (A.13) and (A.15), we get

$$\sum_{k\geq[-\log_2\varepsilon]-4} \mathcal{Y}^{\varepsilon,0}(\tilde{\varphi}_k h, \zeta(\varepsilon D)\varphi_k f) \lesssim |W^\varepsilon h|_{L^2}|W^\varepsilon(D) f|_{L^2} + |W^\varepsilon h|_{L^2}|W^\varepsilon f|_{L^2}. \tag{A.16}$$

Now we consider $\sum_{k\geq[-\log_2\varepsilon]-4} \mathcal{Y}^{\varepsilon,0}(\tilde{\varphi}_k h, (1-\zeta(\varepsilon D))\varphi_k f)$. By Proposition A.2 and the dyadic decomposition in the frequency space, we have

$$\begin{aligned}&\mathcal{Y}^{\varepsilon,0}\big(\tilde{\varphi}_k h, (1-\zeta(\varepsilon D))\varphi_k f\big) \\ &\quad= \int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big[(\widehat{\tilde{\varphi}_k h})(\xi^+) - (\widehat{\tilde{\varphi}_k h})\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)\Big](1-\zeta(\varepsilon\xi))\overline{\widehat{\varphi_k f}}(\xi)\,d\xi\,d\sigma \\ &\quad= \sum_{l\geq[-\log_2\varepsilon]-4} \int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big[(\tilde{\varphi}_l\widehat{\tilde{\varphi}_k h})(\xi^+) - (\tilde{\varphi}_l\widehat{\tilde{\varphi}_k h})\Big(|\xi|\frac{\xi^+}{|\xi^+|}\Big)\Big]\varphi_l\overline{\widehat{\varphi_k f}}(\xi)\,d\xi\,d\sigma \\ &\quad:= \sum_{l\geq[-\log_2\varepsilon]-4} \mathcal{Y}_{k,l}. \end{aligned}$$

Decompose as $\mathcal{Y}_{k,l} = \mathcal{Y}_{k,l,\leq} + \mathcal{Y}_{k,l,\geq}$ according to $\sin(\theta/2) \leq 2^{-(k+l)/2}$ and $\sin(\theta/2) \geq 2^{-(k+l)/2}$.

For $\mathcal{Y}_{k,l,\leq}$, by Taylor expansion for $\tilde{\varphi}_l\widehat{\tilde{\varphi}_k h}$, similarly to (A.7), we have

$$\begin{aligned}\mathcal{Y}_{k,l,\leq} &= \int b^\varepsilon\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\mathbb{1}_{\sin\frac{\theta}{2}\leq 2^{-(k+l)/2}}\Big(1-\frac{1}{\cos\theta}\Big) \\ &\quad\times\Big(\int_0^1 (\nabla\tilde{\varphi}_l\widehat{\tilde{\varphi}_k h})(\xi^+(\kappa))\cdot\xi^+\,d\kappa\Big)\varphi_l\overline{\widehat{\varphi_k f}}(\xi)\,d\xi\,d\sigma \\ &\lesssim 2^l\varepsilon^{2s-2}2^{(s-1)(k+l)}|\nabla\tilde{\varphi}_l\widehat{\tilde{\varphi}_k h}|_{L^2}|\varphi_l\widehat{\varphi_k f}|_{L^2} \\ &= \varepsilon^{2s-2}2^{s(k+l)}2^{-k}|\nabla\tilde{\varphi}_l\widehat{\tilde{\varphi}_k h}|_{L^2}|\varphi_l\widehat{\varphi_k f}|_{L^2}. \end{aligned}$$

Since

$$|\nabla\tilde{\varphi}_l\widehat{\tilde{\varphi}_k h}| = |\tilde{\varphi}_l\widehat{v\tilde{\varphi}_k h} + (\nabla\tilde{\varphi}_l)\widehat{\tilde{\varphi}_k h}| \lesssim |\tilde{\varphi}_l\widehat{v\tilde{\varphi}_k h}| + 2^{-l}|\tilde{\tilde{\varphi}}_l\widehat{\tilde{\varphi}_k h}|,$$

where $\tilde{\tilde{\varphi}}_l := \sum_{|j-l|\le 4, j\ge -1}\varphi_j$, we have

$$\begin{aligned}
&\sum_{k,l\ge[-\log_2\varepsilon]-4}\mathcal{Y}_{k,l,\le}\\
&\lesssim \sum_{k,l\ge[-\log_2\varepsilon]-4}\varepsilon^{2s-2}2^{s(k+l)}2^{-k}(|\tilde{\varphi}_l\widehat{v\tilde{\varphi}_k h}|_{L^2} + 2^{-l}|\tilde{\tilde{\varphi}}_l\widehat{\tilde{\varphi}_k h}|_{L^2})|\varphi_l\widehat{\varphi_k f}|_{L^2}\\
&\lesssim \Big(\sum_{k,l\ge[-\log_2\varepsilon]-4}(\varepsilon^{2s-2}2^{2sk}2^{-2k}|\tilde{\varphi}_l\widehat{v\tilde{\varphi}_k h}|^2_{L^2} + \varepsilon^{2s-2}2^{2sk}2^{-2k}2^{-2l}|\tilde{\tilde{\varphi}}_l\widehat{\tilde{\varphi}_k h}|^2_{L^2})\Big)^{\frac{1}{2}}\\
&\quad\times\Big(\sum_{k,l\ge[-\log_2\varepsilon]-4}\varepsilon^{2s-2}2^{2ls}|\varphi_l\widehat{\varphi_k f}|^2_{L^2}\Big)^{\frac{1}{2}}\\
&\lesssim |W^\varepsilon h|_{L^2}|W^\varepsilon(D)f|_{L^2}.
\end{aligned}\tag{A.17}$$

For $\mathcal{Y}_{k,l,\ge}$, we have

$$\mathcal{Y}_{k,l,\ge} \lesssim \varepsilon^{2s-2}2^{(k+l)s}|\tilde{\varphi}_l\widehat{\tilde{\varphi}_k h}|_{L^2}|\varphi_l\widehat{\varphi_k f}|_{L^2}.$$

Thus by (A.14),

$$\begin{aligned}
\sum_{k,l\ge[-\log_2\varepsilon]-4}\mathcal{Y}_{k,l,\ge} &\lesssim \sum_{k,l\ge[-\log_2\varepsilon]-4}\varepsilon^{2s-2}2^{(k+l)s}|\tilde{\varphi}_l\widehat{\tilde{\varphi}_k h}|_{L^2}|\varphi_l\widehat{\varphi_k f}|_{L^2}\\
&\lesssim |W^\varepsilon h|_{L^2}|W^\varepsilon(D)f|_{L^2}.
\end{aligned}\tag{A.18}$$

Combining (A.17) and (A.18) gives

$$\sum_{k\ge[-\log_2\varepsilon]-4}\mathcal{Y}^{\varepsilon,0}\big(\tilde{\varphi}_k h, (1-\zeta(\varepsilon D))\varphi_k f\big) \lesssim |W^\varepsilon h|_{L^2}|W^\varepsilon(D)f|_{L^2}.\tag{A.19}$$

With (A.16) and (A.19), (A.12) gives

$$\mathcal{Y}^{\varepsilon,0}\big(h, (1-\zeta(\varepsilon v))f\big) \lesssim |W^\varepsilon h|_{L^2}|W^\varepsilon(D)f|_{L^2} + |W^\varepsilon h|_{L^2}|W^\varepsilon f|_{L^2}.\tag{A.20}$$

Back to (A.3) and (A.4), by combining (A.5), (A.11), and (A.20), we get

$$|\mathcal{Y}^{\varepsilon,0}(h,f)| \lesssim (|W^\varepsilon h|_{L^2} + |W^\varepsilon(D)h|_{L^2})^{\frac{1}{2}}(|W^\varepsilon f|_{L^2} + |W^\varepsilon(D)f|_{L^2})^{\frac{1}{2}}.\tag{A.21}$$

Step 2: $\gamma \neq 0$. For simplicity, denote $w = |u|\frac{u^+}{|u^+|}$. Then $W_{\gamma/2}(u) = W_{\gamma/2}(w)$. Note that

$$\begin{aligned}
\langle u\rangle^\gamma h(u)[f(u^+) - f(w)] = &(W_{\gamma/2}h)(u)[(W_{\gamma/2}f)(u^+) - (W_{\gamma/2}f)(w)]\\
&+ (W_{\gamma/2}h)(u)(W_{\gamma/2}f)(u^+)(W_{\gamma/2}(w)W_{-\gamma/2}(u^+) - 1),
\end{aligned}$$

which yields

$$\mathcal{Y}^{\varepsilon,\gamma}(h,f) = \mathcal{Y}^{\varepsilon,0}(W_{\gamma/2}h, W_{\gamma/2}f) + \mathcal{A},$$
$$\mathcal{A} := \int b^{\varepsilon}\Big(\frac{u}{|u|}\cdot\sigma\Big)(W_{\gamma/2}h)(u)(W_{\gamma/2}f)(u^+)(W_{\gamma/2}(w)W_{-\gamma/2}(u^+)-1)\,du\,d\sigma.$$

By the Cauchy–Schwarz inequality, $|W_{\gamma/2}(u)W_{-\gamma/2}(u^+)-1| \lesssim \sin^2\frac{\theta}{2}$, and the estimate (2.8), we have

$$\begin{aligned}|\mathcal{A}| &\lesssim \Big(\int b^{\varepsilon}\Big(\frac{u}{|u|}\cdot\sigma\Big)|(W_{\gamma/2}h)(u)|^2|W_{\gamma/2}(w)W_{-\gamma/2}(u^+)-1|\,du\,d\sigma\Big)^{\frac12}\\ &\quad\times\Big(\int b^{\varepsilon}\Big(\frac{u}{|u|}\cdot\sigma\Big)|(W_{\gamma/2}f)(u^+)|^2|W_{\gamma/2}(w)W_{-\gamma/2}(u^+)-1|\,du\,d\sigma\Big)^{\frac12}\\ &\lesssim |W_{\gamma/2}h|_{L^2}|W_{\gamma/2}f|_{L^2},\end{aligned}$$

where the change of variable $u \to u^+$ has been used in the estimate for f. This together with (A.21) completes the proof of the lemma. ∎

Remark A.2. Set

$$\mathcal{X}^{\varepsilon,\gamma}(h,f) := \int b^{\varepsilon}\Big(\frac{u}{|u|}\cdot\sigma\Big)|u|^{\gamma}(1-\zeta(|u|))h(u)\Big[f(u^+) - f\Big(|u|\frac{u^+}{|u^+|}\Big)\Big]\,du\,d\sigma.$$

Then

$$\begin{aligned}|\mathcal{X}^{\varepsilon,\gamma}(h,f)| \lesssim{}& (|W^{\varepsilon}W_{\gamma/2}h|_{L^2} + |W^{\varepsilon}(D)W_{\gamma/2}h|_{L^2})\\ &\times(|W^{\varepsilon}W_{\gamma/2}f|_{L^2} + |W^{\varepsilon}(D)W_{\gamma/2}f|_{L^2}).\end{aligned}$$

Indeed, by the identity

$$|u|^{\gamma}(1-\zeta(|u|)) = \langle u\rangle^{\gamma}(|u|^{\gamma}\langle u\rangle^{-\gamma}-1)(1-\zeta(|u|)) + \langle u\rangle^{\gamma}(1-\zeta(|u|)),$$

we have

$$\mathcal{X}^{\varepsilon,\gamma}(h,f) = \mathcal{Y}^{\varepsilon,\gamma}((|\cdot|^{\gamma}\langle\cdot\rangle^{-\gamma}-1)(1-\zeta)h, f) + \mathcal{Y}^{\varepsilon,\gamma}((1-\zeta)h, f).$$

Then the estimate follows from Lemmas A.4 and A.3 because $(|\cdot|^{\gamma}\langle\cdot\rangle^{-\gamma}-1)(1-\zeta)$, $1-\zeta \in S^0_{1,0}$.

Lemma A.5. *It holds that*

$$(1-s)\varepsilon^{2s-2}\int_{\mathbb{R}^3}\int_0^{2\varepsilon}\theta^{-1-2s}|f(v)-f(v\cos\theta)|^2\,dv\,d\theta \lesssim |W^{\varepsilon}(D)f|^2_{L^2} + |W^{\varepsilon}f|^2_{L^2}.$$

We omit the proof for brevity because the localization techniques used in Lemma A.4 can be applied similarly by considering $f(v)(f(v)-f(v\cos\theta))$ and $f(v\cos\theta)(f(v)-f(v\cos\theta))$ separately.

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