Prismatic and q-crystalline sites of higher level

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Abstract – In this article, we define the m-prismatic site and the m-q-crystalline site, which are higher-level analogs of the prismatic site and the q-crystalline site respectively. We prove a certain equivalence between the category of crystals on the m-prismatic site (resp. the m-q-crystalline site) and that on the prismatic site (resp. the q-crystalline site), which can be regarded as the prismatic (resp. the q-crystalline) analog of the Frobenius descent due to Berthelot and the Cartier transform due to Ogus–Vologodsky, Oyama and Xu. We also prove equivalence between the category of crystals on the m-prismatic site and that on the $(m-1)$-q-crystalline site.

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1. Introduction

Fix a prime p. In [5], Bhatt and Scholze defined two new p-adic cohomology theories generalizing crystalline cohomology, called prismatic cohomology and q-crystalline cohomology. They are defined as the cohomologies of the corresponding sites called the prismatic site and the q-crystalline site respectively. The notion of a prismatic site $(X/A)_{\mathbb{\Delta}}$ is defined for a p-adic formal scheme X over A/I, where (A, I) is a bounded prism. Also, the q-crystalline site $(X/D)_{q\text{-crys}}$ is defined for a p-completely smooth affine formal scheme X over D/I, where (D, I) is a q-PD pair. (The assumptions imply that A, D are endowed with a lift ϕ of Frobenius on A/pA, D/pD respectively. Also, D is a $\mathbf{Z}_p[\![q - 1]\!]$-algebra such that $(D, [p]_q D)$, where $[p]_q = \frac{q^p - 1}{q - 1}$, is a bounded prism and $I \subseteq \phi^{-1}([p]_q D)$. Moreover, when $q = 1$ in D, the q-PD pair (D, I) is a PD ring.) Among other things, they proved the following comparison theorems:

(1) If (D, I) is a q-PD pair and X is a p-completely smooth affine formal scheme over D/I, then the q-crystalline cohomology of X over D is isomorphic to the prismatic cohomology of its Frobenius pullback $X' := X \,\hat{\times}_{\mathrm{Spf}(D/I), \phi^*} \mathrm{Spf}(D/[p]_q D)$ over D ([5, Theorem 16.18]).

(2) If (D, I) is a q-PD pair with $q = 1$ in D and X is an affine p-completely smooth formal scheme over D/I, then the q-crystalline cohomology of X over D is isomorphic to the crystalline cohomology of X over D (a special case of [5, Theorem 16.14]).

On the other hand, for an integer $m \geq 0$, Berthelot ([2]; see also the works of Le Stum–Quirós [13] and Miyatani [14]) defined the notion of a crystalline cohomology of level m as the cohomology of the m-crystalline site. The notion of an m-crystalline site $(X/D)_{m\text{-crys}}$ can be defined for a smooth scheme X over D/I, where (D, J, I, γ) is a p-torsion-free p-complete m-PD ring or an m-PD ring in which p is nilpotent such that the ideal I contains p. When $m = 0$, it coincides with the usual crystalline site. He proved the following results, which are called the Frobenius descent:

(3) The category of crystals on the m-crystalline site of X over D is equivalent to that on the crystalline site of X' over D, where $X' := X \times_{\mathrm{Spec}(D/I), (\phi^m)^*} \mathrm{Spec}(D/I)$ is the pullback of X by the m-fold iteration ϕ^m of the Frobenius lift ([3, Corollaire 2.3.7, Théorème 4.1.3] in local situation). (See also the equivalence $\tilde{\sigma}^*$ in Section 5.)

(4) The crystalline cohomology of level m of X over D is isomorphic to the crystalline cohomology of X' over D, where X' is as above ([14, Proposition 5.4], [13, Proposition 5.5]).

The purpose of the present article is to introduce the notion of an m-prismatic site and an m-q-crystalline site, which are the level-m versions of the prismatic site and the q-crystalline site respectively, and prove a prismatic and a q-crystalline version of the equivalence of (3) in Theorems 2.25, 3.18. The q-crystalline version of (3) is actually compatible with the equivalence of (3) when $q = 1$ and $p \in I$. We also prove the equivalences of categories of crystals corresponding to the higher-level versions of (1) and (2) in Theorem 4.2 and Proposition 5.28 respectively.

Let us explain the content of each section. In Section 2, for a bounded prism (A, I) and a smooth and separated formal scheme X over A/J with $J := (\phi^m)^{-1}(I)$, we first define the m-prismatic site $(X/A)_{m\text{-}\mathbb{\Delta}}$ (Definition 2.1). Next we prove that the category of crystals on the m-prismatic site $(X/A)_{m\text{-}\mathbb{\Delta}}$ is equivalent to that on the prismatic site $(X'/A)_{\mathbb{\Delta}}$, where $X' := X \hat{\times}_{\mathrm{Spf}(A/J),(\phi^m)^*} \mathrm{Spf}(A/I)$ (Theorem 2.25). We work with the category of crystals with respect to certain categories of modules with some technical conditions (Definition 2.16) so that our argument works. Our proof of equivalence is based on the idea developed by Oyama [17] and Xu [20] in their study of the Cartier transform in positive characteristic or the case modulo p^n: we define a functor $\rho \colon (X/A)_{m\text{-}\mathbb{\Delta}} \to (X'/A)_{\mathbb{\Delta}}$ of sites and prove that it induces an equivalence of topoi $\widetilde{(X/A)}_{m\text{-}\mathbb{\Delta}} \to \widetilde{(X'/A)}_{\mathbb{\Delta}}$ (Theorem 2.15) and, moreover, that this equivalence preserves the categories of crystals on both sides.

In Section 3, for a q-PD pair (D, I) and a smooth and separated formal scheme X over D/J with $J := (\phi^m)^{-1}(I)$, we first define the m-q-crystalline site $(X/D)_{m\text{-}q\text{-crys}}$ (Definition 3.3). Next we prove that the category of crystals on $(X/D)_{m\text{-}q\text{-crys}}$ is equivalent to that on the q-crystalline site $(X'/D)_{q\text{-crys}}$, where $X' := X \hat{\times}_{\mathrm{Spf}(D/J),(\phi^m)^*} \mathrm{Spf}(D/I)$ (Theorem 3.18). The proof of equivalence is parallel to that in Section 2. We also introduce the category of stratifications, which is equivalent to the category of crystals on $(X/D)_{m\text{-}q\text{-crys}}$ (Definition 3.21, Proposition 3.22).

In Section 4 we compare the category of crystals on the m-prismatic site with that on the $(m-1)$-q-crystalline site. By the comparison theorems in (1) and (4) above, it is natural to regard the prismatic cohomology as a kind of "level-(-1) q-crystalline cohomology". So it would be natural to compare the m-prismatic site with the $(m-1)$-q-crystalline site. Based on this observation, for a q-PD pair (D, I) with $J_q := (\phi^{m-1})^{-1}(I)$, $J_{\mathbb{\Delta}} := (\phi^m)^{-1}([p]_q D)$ and a smooth and separated formal scheme X over D/J_q, we prove that the category of crystals on the m-prismatic site $(\widetilde{X}/D)_{m\text{-}\mathbb{\Delta}}$ is equivalent to that on the $(m-1)$-q-crystalline site $(X/D)_{(m-1)\text{-}q\text{-crys}}$, where $\widetilde{X} := X \hat{\times}_{\mathrm{Spf}(D/J_q)} \mathrm{Spf}(D/J_{\mathbb{\Delta}})$ (Theorem 4.2). The category of stratifications we introduced in Section 3 plays a central role in the proof.

In Section 5 we relate our equivalence in Section 3 to the Frobenius descent in (3). Let (D, J, I, γ) be a p-torsion-free p-complete m-PD ring with $p \in I$, and let $\widetilde{X}$ be a

smooth and separated scheme over D/I. First we give an alternative site-theoretic proof (a proof based on the idea of Oyama [17] and Xu [20]) of the equivalence in (3), namely, equivalence between the category of crystals on the m-crystalline site $(\widetilde{X}/D)_{m\text{-crys}}$ and that on the crystalline site $(X'/D)_{\text{crys}}$, where $X' := \widetilde{X} \times_{\text{Spec}(D/I),(\phi^m)^*} \text{Spec}(D/I)$ (Corollary 5.21). Note that the original definition of the $(m$-$)$crystalline site is not suitable to apply the site-theoretic argument of Oyama and Xu. Our strategy is to introduce the variants $(X/D)_{m\text{-crys,new}}$ (where $X := \widetilde{X} \times_{\text{Spec}(D/I)} \text{Spec}(D/J)$) and $(X'/D)_{\text{crys,new}}$ of the $(m$-$)$crystalline site (Definition 5.12) which do not change the categories of crystals and for which the site-theoretic argument works. Next, assuming that (D, I) is a q-PD pair with $q = 1$ in D and $p \in I$, we prove that the equivalence of categories of crystals in Section 3 is compatible with the Frobenius descent by comparing the categories of crystals on the $(m$-$)q$-crystalline site and those on the variants of the $(m$-$)$crystalline site.

In Section 6 we establish relationships between our result and results in the works of Xu [20], Gros–Le Stum–Quirós [7] and Morrow–Tsuji [15]. We will see that our equivalences between the category of crystals on the prismatic site, that on the 1-prismatic site and that on the q-crystalline site are compatible with the equivalence of the Cartier transform in the case modulo p^n by Xu, and the equivalence between the category of twisted hyper-stratified modules of level (-1) and that of level 0 by Gros–Le Stum–Quirós. The equivalence of categories of crystals in Section 2 also fits naturally into the diagram involving the category of generalized representations, the category of modules with flat q-Higgs field and the category of modules with flat q-connection that appeared in the work of Morrow–Tsuji.

The original motivation for introducing m-crystalline cohomology would be to develop a p-adic cohomology theory over a ramified base, for example, when the base is a complete discrete valuation ring V of mixed characteristic $(0, p)$ in which p is not a uniformizer. However, there exists no δ-ring structure on the ring V above. So our definition of the m-prismatic site and the m-q-crystalline site is not enough for this purpose. We hope to generalize our results to the case of a possibly ramified base in the future.

2. The m-prismatic site

In this section we first define the m-prismatic site, which is a higher-level analog of the prismatic site defined in [5] as well as a prismatic analog of the level-m crystalline site. Next we prove that there exists an equivalence between the category of crystals on the m-prismatic site of a smooth formal scheme X and that on the usual prismatic site

of X', where X' is the pullback of X by the m-fold iteration ϕ^m of the Frobenius lift ϕ on the base prism.

We fix a non-negative integer m and a bounded prism (A, I). Then I is closed for the p-adic topology of A. (Indeed, I is p-adically complete because I is invertible as an A-module, and the boundedness of the prism (A, I) implies that the p-adic topology of I coincides with the topology on I induced by the p-adic topology of A by the argument in the proof of [6, Proposition 4.3].) Also, the map $\phi \colon A \to A$ is continuous for the p-adic topology by [5, Lemma 2.17]. Thus $(\phi^m)^{-1}(I)$ is closed for the p-adic topology of A.

DEFINITION 2.1. Let $J = (\phi^m)^{-1}(I)$, and let X be a p-adic formal scheme smooth and separated over A/J. We define the *m-prismatic site* $(X/A)_{m\text{-}\mathbb{\Delta}}$ of X over A as follows. Objects are maps $(A, I) \to (E, I_E)$ of bounded prisms together with a map $\mathrm{Spf}(E/J_E) \to X$ over A/J satisfying the following condition, where $J_E = (\phi^m)^{-1}(I_E)$:

(∗) $\mathrm{Spf}(E/J_E) \to X$ factors through some affine open $\mathrm{Spf}(R) \subseteq X$.

We will often denote such an object by

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X) \in (X/A)_{m\text{-}\mathbb{\Delta}}$$

or (E, I_E) if no confusion arises. A morphism

$$(\mathrm{Spf}(E') \leftarrow \mathrm{Spf}(E'/J_{E'}) \to X) \to (\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X)$$

is a map of bounded prisms $(E, I_E) \to (E', I_{E'})$ over (A, I) such that the induced morphism

$$\mathrm{Spf}(E'/J_{E'}) \to \mathrm{Spf}(E/J_E)$$

is compatible with the maps $\mathrm{Spf}(E'/J_{E'}) \to X$, $\mathrm{Spf}(E/J_E) \to X$. When we denote such an object by (E, I_E), we will write $(E, I_E) \to (E', I_{E'})$ (not $(E', I_{E'}) \to (E, I_E)$) for a morphism from $(E', I_{E'})$ to (E, I_E). A map $(E, I_E) \to (E', I_{E'})$ in $(X/A)_{m\text{-}\mathbb{\Delta}}$ is a cover if it is a faithfully flat map of prisms, i.e., E' is (p, I_E)-completely faithfully flat over E.

REMARK 2.2. By [5, Lemma 3.5], for an object (E, I_E) of $(X/A)_{m\text{-}\mathbb{\Delta}}$, the ideal I_E is always equal to the ideal IE.

We need to check that the category $(X/A)_{m\text{-}\mathbb{\Delta}}$ endowed with the topology as defined above forms a site. Thanks to [15, Lemma 3.8], we have the following lemma:

LEMMA 2.3. *Let $(E_1, I_{E_1}) \xleftarrow{f} (E, I_E) \xrightarrow{g} (E_2, I_{E_2})$ be maps in $(X/A)_{m\text{-}\Delta}$ such that f is a cover. Let $E_3 := E_1 \widehat{\otimes}_E E_2$, where the completion is the classical (p, I)-completion. Then (E_3, I_{E_3}) is the object that represents the coproduct $(E_1, I_{E_1}) \sqcup_{(E, I_E)} (E_2, I_{E_2})$ in $(X/A)_{m\text{-}\Delta}$, and the canonical map $(E_2, I_{E_2}) \to (E_3, I_{E_3})$ is a cover.*

By Lemma 2.3, the set of covers in Definition 2.1 actually forms a pretopology on the category $(X/A)_{m\text{-}\Delta}$.

REMARK 2.4. When $m = 0$, we denote the site $(X/A)_{m\text{-}\Delta}$ simply by $(X/A)_\Delta$ and call it the prismatic site. This is equal to the prismatic site defined in [5, Definition 4.1], except that the technical condition $(*)$ in Definition 2.1, which we imposed to facilitate some arguments below, is not assumed in [5]. One easily checks that the topos is unchanged even if we do not impose condition $(*)$.

In order to establish equivalence between the category of crystals on the m-prismatic site and that on the usual prismatic site, we first construct a functor between these sites.

CONSTRUCTION 2.5. Under the notation and assumption in Definition 2.1, let $X' = X \, \widehat{\times}_{\mathrm{Spf}(A/J), (\phi^m)^*} \mathrm{Spf}(A/I)$. Then we have a diagram:

$$
\begin{array}{ccccc}
\mathrm{Spf}(A) & \longleftarrow & \mathrm{Spf}(A/J) & \longleftarrow & X \\
{\scriptstyle (\phi^m)^*}\uparrow & & {\scriptstyle (\phi^m)^*}\uparrow & \square & \uparrow \\
\mathrm{Spf}(A) & \longleftarrow & \mathrm{Spf}(A/I) & \longleftarrow & X'.
\end{array}
$$

We define a functor ρ from the m-prismatic site of X over A to the usual prismatic site of X' over A in the following way: for an object $(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X)$ of $(X/A)_{m\text{-}\Delta}$, we define the object $\rho(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X)$ of $(X'/A)_\Delta$ by

$$
(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/I_E) \xrightarrow{f} X'),
$$

where the right map f is defined as

$$
\mathrm{Spf}(E/I_E) \xrightarrow{g} \mathrm{Spf}(E/J_E) \underset{\mathrm{Spf}(A/J), (\phi^m)^*}{\widehat{\times}} \mathrm{Spf}(A/I)
$$
$$
\to X \underset{\mathrm{Spf}(A/J), (\phi^m)^*}{\widehat{\times}} \mathrm{Spf}(A/I) = X'.
$$

Here the first map g is induced by the map of rings

$$
E/J_E \underset{A/J, \phi^m}{\widehat{\otimes}} A/I \to E/I_E, \quad e \otimes a \mapsto \phi^m(e)a.
$$

This defines the functor $\rho: (X/A)_{m\text{-}\Delta} \to (X'/A)_\Delta$.

Next we want to show that ρ induces an equivalence of topoi. We follow the proof of [20, Theorem 9.2], where the author proved an analogous result for the Oyama topoi. The main properties of the functor ρ that we need are summarized in the following propositions.

PROPOSITION 2.6 (Cf. [20, Proposition 9.3 (i)]). *The functor ρ is fully faithful.*

PROOF. The functor ρ is clearly faithful. To prove fullness, suppose that

$$\alpha\colon \rho(\mathrm{Spf}(E') \leftarrow \mathrm{Spf}(E'/J_{E'}) \to X) \to \rho(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X)$$

is a map in $(X'/A)_{\mathbb{\Delta}}$. We consider the diagram

$$
\begin{array}{ccc}
\mathrm{Spf}(E'/I_{E'}) & \xrightarrow{\ (\phi^m)^*\ } & \mathrm{Spf}(E'/J_{E'}) \\
\bar{\alpha}\downarrow \searrow & X' \longrightarrow X & \nearrow \downarrow \bar{\alpha} \\
\mathrm{Spf}(E/I_E) & \xrightarrow{\ (\phi^m)^*\ } & \mathrm{Spf}(E/J_E),
\end{array}
$$

where the morphisms $\bar{\alpha}$ are induced by α and the morphisms $(\phi^m)^*$ are induced by ϕ^m on E' and E. It is enough to check that the right triangle is commutative. Since the left triangle, the outer square and the two trapezoids are commutative, it suffices to prove the following claim:

CLAIM. *Let $f_i\colon \mathrm{Spf}(E'/J_{E'}) \to X$ ($i = 1, 2$) be maps which factor through some affine opens $\mathrm{Spf}(R_i) \subseteq X$ ($i = 1, 2$) respectively, and satisfies*

$$f_1 \circ (\phi^m)^* = f_2 \circ (\phi^m)^*\colon \mathrm{Spf}(E'/I_{E'}) \to X.$$

Then $f_1 = f_2$.

We prove the claim. Let $\mathrm{Spf}(R_3) = \mathrm{Spf}(R_1) \cap \mathrm{Spf}(R_2)$ (note that, since X is separated by assumption, the intersection on the right-hand side is affine). Then the map $f_1 \circ (\phi^m)^* = f_2 \circ (\phi^m)^*$ factors through $\mathrm{Spf}(R_3)$. Since $(\phi^m)^*$ is homeomorphic as a map of topological spaces, this implies that both f_1 and f_2 factor through $\mathrm{Spf}(R_3)$. If f_i^* denote the maps $R_3 \to E'/J_{E'}$ corresponding to f_i ($i = 1, 2$), then we have equality of the map of rings $\phi^m \circ f_1^* = \phi^m \circ f_2^*\colon R_3 \to E'/I_{E'}$. Since the map $\phi^m\colon E'/J_{E'} \to E'/I_{E'}$ is injective, we conclude that $f_1^* = f_2^*$, and hence $f_1 = f_2$. $\blacksquare$

PROPOSITION 2.7 (Cf. [20, Proposition 9.3 (ii)]). *The functor ρ is continuous.*

PROOF. By definition, $\alpha\colon (E, I_E) \to (E', I_{E'})$ is a cover in $(X/A)_{m\text{-}\mathbb{\Delta}}$ if and only if $\rho(\alpha)\colon \rho(E, I_E) \to \rho(E', I_{E'})$ is a cover in $(X'/A)_{\mathbb{\Delta}}$ (note that the definition of covers in the prismatic site is the $m = 0$ case of Definition 2.1). On the other hand, if

$$(E_1, IE_1) \xleftarrow{f} (E, IE) \xrightarrow{g} (E_2, IE_2)$$

are maps in $(X/A)_{m\text{-}\mathbb{\Delta}}$ such that the map f is a cover, then by Lemma 2.3, we see that $\rho(E_1 \sqcup_E E_2) \to \rho(E_1) \sqcup_{\rho(E)} \rho(E_2)$ is an isomorphism. These imply that ρ is continuous, as desired. ∎

PROPOSITION 2.8 (Cf. [20, Proposition 9.3 (ii)]). *The functor ρ is cocontinuous.*

PROOF. Suppose that

$$\alpha'\colon (\mathrm{Spf}(E') \leftarrow \mathrm{Spf}(E'/I_{E'}) \to X') \to \rho(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X)$$

is a cover in $(X'/A)_{\mathbb{\Delta}}$. Condition $(*)$ in Definition 2.1 ensures that the map $\mathrm{Spf}(E/J_E)$ $\to X$ factors through some affine open $\mathrm{Spf}(R) \subseteq X$. Then α' induces the following commutative diagram:

$$R' = R \underset{A/J,\phi^m}{\hat{\otimes}} A/I \xrightarrow{g \otimes \mathrm{id}} E/J_E \underset{A/J,\phi^m}{\hat{\otimes}} A/I \longrightarrow E/I_E \twoheadleftarrow E$$

where g (resp. g') is the map induced by $\mathrm{Spf}(E/J_E) \to X$ (resp. $\mathrm{Spf}(E'/I_{E'}) \to X'$). If we define g'' as the composite $R \to R' \xrightarrow{g'} E'/I_{E'}$, the commutativity of the above diagram implies that the solid arrows in

$$R \xrightarrow{g} E/J_E \xhookrightarrow{\phi^m} E/I_E \twoheadleftarrow E$$

commute. Then we can define the dotted arrow as the composite $\bar{\alpha}' \circ g$ so that the diagram commutes and that it defines an object $(\mathrm{Spf}(E') \leftarrow \mathrm{Spf}(E'/J_{E'}) \to X)$ and a morphism

$$(\mathrm{Spf}(E') \leftarrow \mathrm{Spf}(E'/J_{E'}) \to X) \to (\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X)$$

in $(X/A)_{m\text{-}\mathbb{\Delta}}$. If we denote it by α, then $\rho(\alpha) = \alpha'$. The map α is also a cover by assumption. These imply that ρ is cocontinuous, as desired. ∎

In order to prove another important proposition to prove the equivalence of topoi, we prepare several lemmas.

LEMMA 2.9. *Let A be an I-adically complete ring where I is a finitely generated ideal. Let $J \subseteq A$ be a closed ideal such that the image of J in A/I is a nil ideal, and let $\bar{f} \colon \overline{U} \to \mathrm{Spf}(A/J)$ be a smooth morphism (resp. an open immersion) with $\overline{U}$ affine. Then there is a unique smooth morphism (resp. open immersion) $f \colon U \to \mathrm{Spf}(A)$ which lifts $\bar{f}$. Moreover, U is affine.*

PROOF. We follow the proof of [16, Lemma 2.3.14], where the authors proved an analogous results for log schemes. We may replace A by A/I^n and Spf by Spec to prove the lemma. Then J is a nil ideal of A and the morphism $\bar{f} \colon \overline{U} \to \mathrm{Spec}(A/J)$ is a smooth morphism (resp. an open immersion) of schemes.

We first treat the case that $\bar{f}$ is smooth. Express J as the inductive limit of the inductive system $(J_\lambda)_\lambda$ of finitely generated nilpotent ideals of A: $J = \varinjlim_\lambda J_\lambda$. Since

$$\mathrm{Spec}(A/J) = \varprojlim_\lambda \mathrm{Spec}(A/J_\lambda),$$

by [9, Théorème 8.8.2 (ii) and Proposition 17.7.8], there exists a smooth scheme $\overline{U_\lambda}$ over $\mathrm{Spec}(A/J_\lambda)$ for some λ such that $\overline{U} = \overline{U_\lambda} \times_{\mathrm{Spec}(A/J_\lambda)} \mathrm{Spec}(A/J)$. By [9, Théorème 8.10.5 (viii)], we may assume $\overline{U_\lambda}$ is affine. By replacing $\overline{U} \to \mathrm{Spec}(A/J)$ by $\overline{U_\lambda} \to \mathrm{Spec}(A/J_\lambda)$, we may assume that J is a finitely generated nilpotent ideal. Next, fix a generator $x_1, \ldots, x_n$ of J and write A as the inductive limit of the inductive system $(A_\lambda)_\lambda$ of Noetherian subrings containing $x_1, \ldots, x_n$. If we write J_λ for the ideal of A_λ generated by $x_1, \ldots, x_n$, we have

$$\mathrm{Spec}(A/J) = \varprojlim_\lambda \mathrm{Spec}(A_\lambda/J_\lambda).$$

Then, by a similar argument to above, we may replace $J \subseteq A$ by $J_\lambda \subseteq A_\lambda$ for some λ and so we may assume furthermore that A is Noetherian. Then the existence and the uniqueness of U follows from [10, Exposé III, Corollaire 6.8].

It remains to check that the scheme U is affine. We can reduce to the case where J is nilpotent by the argument above, and then we may assume that $J^2 = 0$. Let $\mathcal{I}$ be a coherent ideal sheaf of $\mathcal{O}_U$. By the proof of [11, Theorem 3.7], for a scheme X, the following conditions on X are equivalent:

(1) X is affine.

(2) $H^1(X, \mathcal{I}) = 0$ for all coherent ideal sheaves $\mathcal{I}$.

So we need to prove that $H^1(U, \mathcal{I}) = 0$. We consider the exact sequence

$$0 \to J\mathcal{I} \to \mathcal{I} \to \mathcal{I}/J\mathcal{I} \to 0.$$

As $\overline{U}$ is affine, we see that $H^1(U, \mathfrak{I}/J\mathfrak{I}) = H^1(\overline{U}, \mathfrak{I}/J\mathfrak{I}) = 0$. Similarly, one can show that $H^1(U, J\mathfrak{I}) = 0$. Hence $H^1(X, \mathfrak{I}) = 0$.

Next we consider the case that $\bar{f}$ is an open immersion. Then the existence of f follows as $\operatorname{Spec}(A/J)$ is homeomorphic to $\operatorname{Spec}(A)$. The uniqueness and the affinity of U follow from the previous case because an open immersion is smooth. So we are done. ∎

LEMMA 2.10. *Let* (E, I_E) *be a bounded prism. Let* $\bar{f}$ *be an open immersion*

$$\bar{f} : \operatorname{Spf}(\overline{E_i}) \to \operatorname{Spf}(E/I_E)$$

or an open immersion

$$\bar{f} : \operatorname{Spf}(\overline{E_i}) \to \operatorname{Spf}(E/J_E).$$

Then there is a unique open immersion $f : \operatorname{Spf}(E_i) \to \operatorname{Spf}(E)$ *which lifts* $\bar{f}$. *Moreover, the corresponding map of rings induces a map* $(E, I_E) \to (E_i, I_E E_i)$ *of bounded prisms.*

PROOF. We first check that the images of I_E, J_E in $E/(p, I_E)$ are nil ideals. The assertion is trivial for I_E. For J_E, it follows immediately from the definitions of J_E and the Frobenius lift ϕ. Then the existence, the uniqueness and the affinity of the lifting follow from Lemma 2.9. We denote the lifting by $f : \operatorname{Spf}(E_i) \to \operatorname{Spf}(E)$.

As f is an open immersion, there exists an open cover

$$\operatorname{Spf}(E_i) = \bigcup_{j=1}^{N} \operatorname{Spf}(\widehat{E}_{g_j}),$$

where $\widehat{E}_{g_j}$ is the classical (p, I_E)-completion of the localization E_{g_j} of E by $g_j \in E$. As E_{g_j} is an étale E-algebra, the derived (p, I_E)-completion $E_{g_j}^D$ of E_{g_j} is (p, I_E)-completely étale and it admits a unique δ-structure compatible with the one on E by [5, Lemma 2.18]. Then $(E_{g_j}^D, I_E E_{g_j}^D)$ is a bounded prism by [5, Lemma 3.7 (2), (3)], and so $\widehat{E}_{g_j}$ is equal to $E_{g_j}^D$ by [5, Lemma 3.7 (1)]. We see also that $\widehat{E}_{g_{j_1} \cdots g_{j_n}}$ is equal to $E_{g_{j_1} \cdots g_{j_n}}^D$ for the same reason.

By the sheaf property and the vanishing of higher cohomologies of the structure sheaf of $\operatorname{Spf}(E_i)$, we have a quasi-isomorphism

$$E_i \xrightarrow{\simeq} \left[\prod_j \widehat{E}_{g_j} \to \prod_{j_1 < j_2} \widehat{E}_{g_{j_1} g_{j_2}} \to \cdots \to \widehat{E}_{g_1 \cdots g_N} \right].$$

We write $K^\bullet$ for the complex on the right-hand side. As each $\widehat{E}_{g_{j_1} \cdots g_{j_n}} = E_{g_{j_1} \cdots g_{j_n}}^D$ is (p, I_E)-completely flat over E, $K^\bullet \otimes_E^L N$ is concentrated in degree ≥ 0 for any

(p, I_E)-torsion E-module N. On the other hand, $E_i \otimes_E^L N$ is concentrated in degree ≤ 0 by definition. Hence $E_i \otimes_E^L N$ is concentrated in degree 0. Then, since

$$E_i \otimes_E^L E/(p, I_E) = E_i/(p, I_E)$$

is étale over $E/(p, I_E)$, E_i is (p, I_E)-completely étale over E. Thus it admits a unique δ-structure compatible with the one on E by [5, Lemma 2.18] and then $(E_i, I_E E_i)$ is a bounded prism by [5, Lemma 3.7 (2), (3)]. ∎

The next result will also be used to prove the equivalence of topoi.

PROPOSITION 2.11 (Cf. [20, Lemma 9.8 (i)]). *Let $(E', I_{E'})$ be an object in $(X'/A)_{\mathbb{\Delta}}$. Then there exists an object (E, I_E) in $(X/A)_{m\text{-}\mathbb{\Delta}}$ and a cover of the form*

$$(E', I_{E'}) \to \rho(E, I_E).$$

PROOF. Let $X = \bigcup_i \mathrm{Spf}(R_i)$ be a finite affine open cover of X. By the definition of X' in Construction 2.5, we see that

$$\mathrm{Spf}(R_i) \mathbin{\hat{\times}_X} X' = \mathrm{Spf}(R_i \mathbin{\underset{A/J,\phi^m}{\hat{\otimes}}} A/I).$$

We denote this formal scheme by $\mathrm{Spf}(R_i')$. On the other hand, condition $(*)$ in Definition 2.1 ensures that the map $\mathrm{Spf}(E'/I_{E'}) \to X'$ factors through some affine open $\mathrm{Spf}(R'') \subseteq X'$. As X' is separated by assumption, we see that $\mathrm{Spf}(R_i') \cap \mathrm{Spf}(R'') = \mathrm{Spf}(R_i^*)$ for some R_i^*. Then the formal scheme $\mathrm{Spf}(R_i') \mathbin{\hat{\times}_{X'}} \mathrm{Spf}(E'/I_{E'})$ is identified with $\mathrm{Spf}(R_i' \mathbin{\hat{\otimes}_{R_i^*}} E'/I_{E'})$. In particular, it is affine. We denote this affine open subscheme of $\mathrm{Spf}(E'/I_{E'})$ by $\mathrm{Spf}(\overline{E_i'})$. By Lemma 2.10, there is a unique bounded prism $(E_i', I_{E_i'}) \in (X'/A)_{\mathbb{\Delta}}$ for which the corresponding formal scheme $\mathrm{Spf}(E_i')$ is an affine open formal subscheme of $\mathrm{Spf}(E')$ and lifts $\mathrm{Spf}(\overline{E_i'})$, namely $E_i'/I_{E_i'} = \overline{E_i'}$. By construction, we see that

$$(E', I_{E'}) \to \prod_i (E_i', I_{E_i'})$$

is a cover in $(X'/A)_{\mathbb{\Delta}}$. To prove the proposition, we may replace $(E', I_{E'})$ by $(E_i', I_{E_i'})$ to assume that the structure morphism $\mathrm{Spf}(E'/I_{E'}) \to X'$ factors through an affine open $\mathrm{Spf}(R') \subseteq X'$ such that R' is of the form $R \mathbin{\hat{\otimes}_{A/J,\phi^m}} A/I$ for some affine open $\mathrm{Spf}(R) \subseteq X$.

As the map $A/J \to R$ is smooth, [19, Tag 00TA] tells us that there exists an open cover of $\mathrm{Spf}(R)$ by standard opens $\mathrm{Spf}(\widehat{R_g})$ such that each $\widehat{R_g}/p$ is standard smooth over $A/(p, J)$. After refining the given cover of X in the previous paragraph we may

assume that R/p is standard smooth over $A/(p, J)$. By [19, Tag 00T7], there exists a surjection $A[x_1, \ldots, x_n]^\wedge \to R$ whose kernel is the ideal $(J, y_1, \ldots, y_r)$ such that

$$\overline{y_1}, \ldots, \overline{y_r} \in A/(p, J)[x_1, \ldots, x_n]$$

form a regular sequence. On the other hand, there is a natural map

$$A[x_1, \ldots, x_n]^\wedge \to A\{x_1, \ldots, x_n\}^\wedge,$$

where the symbol {} denotes the adjoining of elements in the theory of δ-rings (see [5, Notation 2.8]). By Corollary 2.13 below about the regular sequence, we see that the sequence

$$\phi^m(y_1), \ldots, \phi^m(y_r) \in A\{x_1, \ldots, x_n\}^\wedge$$

is (p, I)-completely regular relative to A. Then we can construct a map

$$A\{x_1, \ldots, x_n\}^\wedge \to S \stackrel{\text{def}}{=} \big((A\{x_1, \ldots, x_n\})^\wedge\{\tfrac{K}{I}\}\big)^\wedge$$

to the prismatic envelope by [5, Proposition 3.13], where K denotes the ideal $(I, \phi^m(y_1), \ldots, \phi^m(y_r)) \subseteq A\{x_1, \ldots, x_n\}^\wedge$. By the construction of the prismatic envelope, $\phi^m(y_i) \in I_S$ and so $y_i \in J_S = (\phi^m)^{-1}(I_S)$ for all i. This gives a map $R \to S/J_S$.

Next, let $A[x_1, \ldots, x_n]^\wedge \to R \widehat{\otimes}_{A, \phi^m} A$ be the base change of $A[x_1, \ldots, x_n]^\wedge \to R$ in the previous paragraph along $\phi^m \colon A \to A$. Passing to the quotient then induces $A[x_1, \ldots, x_n]^\wedge \to R'$. The kernel of this map is the ideal $(I, y_1', \ldots, y_r')$, where the y_i' are the images of the y_i under the map $A[x_1, \ldots, x_n]^\wedge \to A[x_1, \ldots, x_n]^\wedge$ sending $\sum_{\underline{j}} \alpha_{\underline{j}} x^{\underline{j}}$ to $\sum_{\underline{j}} \phi^m(\alpha_{\underline{j}}) x^{\underline{j}}$ (here we write $\underline{j} := (j_1, \ldots, j_n)$ for the multi-index). By the definition of the y_i, the sequence $\overline{y_1'}, \ldots, \overline{y_r'} \in A/(p, I)[x_1, \ldots, x_n]$ is a regular sequence. By Corollary 2.13 below, the sequence $y_1', \ldots, y_r' \in A\{x_1, \ldots, x_n\}^\wedge$ is (p, I)-completely regular relative to A. Then we can construct a map

$$A\{x_1, \ldots, x_n\}^\wedge \to S' \stackrel{\text{def}}{=} \big((A\{x_1, \ldots, x_n\})^\wedge\{\tfrac{K'}{I}\}\big)^\wedge$$

to the prismatic envelope, where K' denotes the ideal $(I, y_1', \ldots, y_r') \subseteq A\{x_1, \ldots, x_n\}^\wedge$. Then we have a diagram:

$$
\begin{array}{ccc}
A/I & \longrightarrow & R' \\
\big\uparrow & & \big\uparrow \\
\big\uparrow & & \big\uparrow \\
A \longrightarrow A[x_1, \ldots, x_n]^\wedge & \to A\{x_1, \ldots, x_n\}^\wedge \to & S' \stackrel{\text{def}}{=} \big((A\{x_1, \ldots, x_n\})^\wedge\{\tfrac{K'}{I}\}\big)^\wedge.
\end{array}
$$

By the definitions of y_i and y_i', we have a map of δ-A-algebras $(\phi^m)' \colon S' \to S$ sending x_i to $\phi^m(x_i)$. By Proposition 2.14 below, we see that $(\phi^m)'$ is (p, I)-completely faithfully flat.

Let $(E', I_{E'}) \in (X'/A)_{\mathbb{\Delta}}$ be as above. We have a map $f_1 \colon A[x_1, \ldots, x_n]^\wedge \to E'/I_{E'}$ determined as the composition of the map $A[x_1, \ldots, x_n]^\wedge \to R'$ in the above diagram with $R' \to E'/I_{E'}$. As $A[x_1, \ldots, x_n]^\wedge$ is the completion of a polynomial ring, one can choose a map $f_2 \colon A[x_1, \ldots, x_n]^\wedge \to E'$ lifting f_1. As E' is a δ-A-algebra, f_2 extends uniquely to a δ-A-algebra map $f_3 \colon A\{x_1, \ldots, x_n\}^\wedge \to E'$. By construction, this extension carries K' into $I_{E'}$. By the universal property of S', f_3 extends uniquely to a δ-A-algebra map $g \colon S' \to E'$. If we set $h \colon E' \to E' \,\widehat{\otimes}_{S'}\, S$ to be the base change of $(\phi^m)' \colon S' \to S$ along g, then by (p, I)-complete faithful flatness of $(\phi^m)'$, we see that the same holds true for h.

It remains to check that the map h defines a morphism

$$(E', I_{E'}) \to \rho\Big(E' \,\widehat{\underset{S'}{\otimes}}\, S, I_{E' \widehat{\underset{S'}{\otimes}} S} \Big)$$

in $(X'/A)_{\mathbb{\Delta}}$. To see this, it is enough to check that the bottom-right square in the following diagram is commutative, in which all the other squares are commutative:

Here, i_2 is the map to the second component of the coproduct in Lemma 2.3. We can check the commutativity of the bottom-right square by tracing the elements x_i:

so the proposition follows. ∎

We prove the claims which were used in the proof above. We first review several notions that appeared in [5, Section 2.6].

A map $A \to B$ of simplicial commutative rings is called flat if $\pi_0(A) \to \pi_0(B)$ is flat and $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$ is an isomorphism for all i. This can be

checked after the derived base change along $A \to \pi_0(A)$. For a commutative ring A, a finitely generated ideal $I = (f_1, \ldots, f_n) \subseteq A$ and an I-completely flat commutative A-algebra B, a sequence $x_1, \ldots, x_r \in B$ is I-completely regular relative to A if the map of simplicial commutative rings

$$A \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n]} \mathbb{Z} \to B \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n, x_1, \ldots, x_r]} \mathbb{Z}$$

is flat. Here, the maps $\mathbb{Z}[f_1, \ldots, f_n] \to \mathbb{Z}$, $\mathbb{Z}[f_1, \ldots, f_n, x_1, \ldots, x_r] \to \mathbb{Z}$ are defined by sending each f_i, x_i to 0.

LEMMA 2.12. *Let $A, I = (f_1, \ldots, f_n) \subseteq A$ and B be as above and let $x_1, \ldots, x_r \in B$ be a sequence of elements satisfying the following conditions:*

(1) *The images $\overline{x_1}, \ldots, \overline{x_r}$ of $x_1, \ldots, x_r$ in B/IB form a regular sequence.*

(2) *$B/(I, x_1, \ldots, x_r)$ is flat over A/I.*

Then $x_1, \ldots, x_r$ is I-completely regular relative to A.

PROOF. We prove the flatness of the map

$$A \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n]} \mathbb{Z} \to B \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n, x_1, \ldots, x_r]} \mathbb{Z}.$$

This can be checked after the derived base change along

$$A \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n]} \mathbb{Z} \to \pi_0(A \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n]} \mathbb{Z}) = A/I,$$

which is the composition of the maps

$$A \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n]} \mathbb{Z} \xrightarrow{f_1} A/I \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n]} \mathbb{Z} \xrightarrow{f_2} A/I,$$

where f_1 is the map induced by the projection $A \to A/I$ and f_2 is the map taking the degree 0 part of the Koszul complex. By taking the base change along f_1, we obtain the map

$$
\begin{aligned}
A/I \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n]} \mathbb{Z} &\to (A/I \otimes^L_A B) \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n, x_1, \ldots, x_r]} \mathbb{Z} \\
&\cong B/IB \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n, x_1, \ldots, x_r]} \mathbb{Z} \\
&\cong (B/IB \otimes^L_{\mathbb{Z}[f_1, \ldots, f_n, x_1, \ldots, x_r]} \mathbb{Z}[x_1, \ldots, x_r]) \otimes^L_{\mathbb{Z}[x_1, \ldots, x_r]} \mathbb{Z},
\end{aligned}
$$

where the first isomorphism follows from the I-complete flatness of B over A. Then, taking the base change along f_2, we obtain the map

$$A/I \to B/IB \otimes^L_{\mathbb{Z}[x_1, \ldots, x_r]} \mathbb{Z}.$$

Note that the complex $B/IB \otimes^{L}_{\mathbb{Z}[x_1,\ldots,x_r]} \mathbb{Z}$ is the Koszul complex corresponding to B/IB and $\overline{x_1},\ldots,\overline{x_r}$. As $\overline{x_1},\ldots,\overline{x_r}$ form a regular sequence in B/IB, we see that the complex $B/IB \otimes^{L}_{\mathbb{Z}[x_1,\ldots,x_r]} \mathbb{Z}$ is identified with $B/(I, x_1,\ldots,x_r)$, which is flat over A/I by assumption, as desired. ∎

COROLLARY 2.13. *With notation as in the proof of Proposition* 2.11, *the sequence* $\phi^m(y_1),\ldots,\phi^m(y_r) \in A\{x_1,\ldots,x_n\}^{\wedge}$ *and the sequence* $y'_1,\ldots,y'_r \in A\{x_1,\ldots,x_n\}^{\wedge}$ *are* (p, I)-*completely regular relative to* A.

PROOF. We first treat the case of the sequence $y'_1,\ldots,y'_r$. Note that the map

$$A/(p, I)[x_1,\ldots,x_n] \to A/(p, I)\{x_1,\ldots,x_n\}$$

is flat. Recall that $\overline{y'_1},\ldots,\overline{y'_r} \in A/(p, I)[x_1,\ldots,x_n]$ form a regular sequence and the ring $A[x_1,\ldots,x_n]/(p, I, y'_1,\ldots,y'_r) = R'/pR'$ is smooth (and hence flat) over $A/(p, I)$, so the claim follows from Lemma 2.12.

Next we prove the claim for the sequence $\phi^m(y_1),\ldots,\phi^m(y_r)$. Recall that the sequence

$$\overline{y_1},\ldots,\overline{y_r} \in A/(p, J)[x_1,\ldots,x_n]$$

is a regular sequence and $A[x_1,\ldots,x_n]/(p, J, y_1,\ldots,y_r) = R/p$ is smooth over $A/(p, J)$. If $J^{(p^m)}$ denotes the ideal generated by x^{p^m} for all elements x of J, then we have $(p, J^{(p^m)}) \subseteq (p, I)$. Since (p, J) is a nil ideal in $A/(p, J^{(p^m)})[x_1,\ldots,x_n]$, the Jacobian criterion for smoothness implies that

$$\overline{y_1},\ldots,\overline{y_r} \in A/(p, J^{(p^m)})[x_1,\ldots,x_n]$$

form a regular sequence and that the quotient $A[x_1,\ldots,x_n]/(p, J^{(p^m)}, y_1,\ldots,y_r)$ is smooth over $A/(p, J^{(p^m)})$. Then it implies that $\overline{y_1},\ldots,\overline{y_r} \in A/(p, I)[x_1,\ldots,x_n]$ form a regular sequence and that the quotient $A[x_1,\ldots,x_n]/(p, I, y_1,\ldots,y_r)$ is smooth over $A/(p, I)$. As $\overline{\phi^m(y_i)} = \overline{y_i}^{p^m}$ in $A/(p, I)\{x_1,\ldots,x_n\}$, we see that

$$\overline{\phi^m(y_1)},\ldots,\overline{\phi^m(y_r)} \in A/(p, I)\{x_1,\ldots,x_n\}$$

form a regular sequence and that $A\{x_1,\ldots,x_n\}/(p, I, \phi^m(y_1),\ldots,\phi^m(y_r))$ is flat over $A/(p, I)$. By Lemma 2.12, we conclude that the sequence

$$\phi^m(y_1),\ldots,\phi^m(y_r) \in A\{x_1,\ldots,x_n\}^{\wedge}$$

is (p, I)-completely regular relative to A. ∎

PROPOSITION 2.14. *With notation as in the proof of Proposition* 2.11, *the map*

$$(\phi^m)' \colon S' \to S$$

is (p, I)-*completely faithfully flat.*

PROOF. Since the map $(\phi^m)'$ is the derived (p, I)-completion of the base change of the δ-A-algebra map

$$(\phi^m)'' \colon A\{x_1, \ldots, x_n\}^\wedge \to A\{x_1, \ldots, x_n\}^\wedge, \quad x_i \mapsto \phi^m(x_i),$$

it suffices to prove that $(\phi^m)''$ is (p, I)-completely faithfully flat. By [5, Lemma 2.11], the δ-A-algebra map

$$(\phi^m)''' \colon A\{x_1, \ldots, x_n\} \to A\{x_1, \ldots, x_n\}, \quad x_i \mapsto \phi^m(x_i)$$

is faithfully flat. As the map $(\phi^m)''$ is the derived (p, I)-completion of the base change of $(\phi^m)'''$ along $A\{x_1, \ldots, x_n\} \to A\{x_1, \ldots, x_n\}^\wedge$, we conclude that $(\phi^m)''$ is (p, I)-completely faithfully flat. ■

We are now prepared to prove the equivalence of topoi. First, we note that the functor of sites $\rho \colon (X/A)_{m\text{-}\mathbb{\Delta}} \to (X'/A)_{\mathbb{\Delta}}$ in Construction 2.5 induces a functor between the categories of presheaves of sets

$$\hat{\rho}^* \colon \widehat{(X'/A)}_{\mathbb{\Delta}} \to \widehat{(X/A)}_{m\text{-}\mathbb{\Delta}}, \quad \mathcal{G} \mapsto \mathcal{G} \circ \rho,$$

and it admits a right adjoint

$$\hat{\rho}_* \colon \widehat{(X/A)}_{m\text{-}\mathbb{\Delta}} \to \widehat{(X'/A)}_{\mathbb{\Delta}}.$$

By Propositions 2.7 and 2.8, we obtain a morphism of topoi

$$\mathrm{C} \colon \widehat{(X/A)}_{m\text{-}\mathbb{\Delta}} \to \widehat{(X'/A)}_{\mathbb{\Delta}}, \quad \mathrm{C}^* = \hat{\rho}^*, \ \mathrm{C}_* = \hat{\rho}_*.$$

THEOREM 2.15. *The morphism* $\mathrm{C} \colon \widehat{(X/A)}_{m\text{-}\mathbb{\Delta}} \to \widehat{(X'/A)}_{\mathbb{\Delta}}$ *is an equivalence of topoi.*

PROOF. By [17, Proposition 4.2.1] (see also [20, Proposition 9.10]), we are reduced to checking the conditions which were proved in Propositions 2.6, 2.7, 2.8 and 2.11. ■

We want to prove that C induces an equivalence between the categories of crystals. First, we define a suitable category of crystals with some technical conditions so that our argument works.

DEFINITION 2.16. Let (E, I_E) be a bounded prism.

(1) Let $\widetilde{\mathcal{M}}_{\mathbb{\Delta}}(E, I_E)$ be the category of E-modules M such that, for any map of bounded prisms $(E, I_E) \to (E_0, I_{E_0})$ and any faithfully flat map of bounded prisms $(E_0, I_{E_0}) \to (E_0', I_{E_0'})$, the sequence

$$0 \to M \,\widehat{\otimes}_E\, E_0 \to M \,\widehat{\otimes}_E\, E_0' \to M \,\widehat{\otimes}_E\, (E_0' \,\widehat{\otimes}_{E_0}\, E_0')$$

is exact, where the completion is the classical (p, I_{E_0})-completion.

(2) Let $\{\mathcal{M}_{\mathbb{\Delta}}(E, I_E) \subseteq \widetilde{\mathcal{M}}_{\mathbb{\Delta}}(E, I_E)\}_{(E, I_E)}$ be the largest family of full subcategories such that, for any $M \in \mathcal{M}_{\mathbb{\Delta}}(E, I_E)$, any map $(E, I_E) \to (E_0', I_{E_0'})$ of bounded prisms and any faithfully flat map of bounded prisms $(E_0, I_{E_0}) \to (E_0', I_{E_0'})$, any descent datum ε on $M \,\widehat{\otimes}_E\, E_0'$ (i.e., an isomorphism

$$(E_0' \,\widehat{\otimes}_{E_0}\, E_0') \,\widehat{\otimes}_{E_0'}\, (M \,\widehat{\otimes}_E\, E_0') \cong (M \,\widehat{\otimes}_E\, E_0') \,\widehat{\otimes}_{E_0'}\, (E_0' \,\widehat{\otimes}_{E_0}\, E_0')$$

satisfying the cocycle condition on $E_0' \,\widehat{\otimes}_{E_0}\, E_0' \,\widehat{\otimes}_{E_0}\, E_0'$) descends uniquely to an E_0-module $M_0 \in \mathcal{M}_{\mathbb{\Delta}}(E_0, I_{E_0})$.

We will need the following variant of Definition 2.16; in this variant, the ring E does not necessarily admit a δ-structure, and we assume that the ideal I_E above is equal to (p).

DEFINITION 2.17. Let E be a p-torsion-free p-complete ring.

(1) Let $\widetilde{\mathcal{M}}(E)$ be the category of E-modules M such that, for any map $E \to E_0$ of p-torsion-free p-complete rings and any p-completely faithfully flat map $E_0 \to E_0'$ of p-torsion-free p-complete rings, the sequence

$$0 \to M \,\widehat{\otimes}_E\, E_0 \to M \,\widehat{\otimes}_E\, E_0' \to M \,\widehat{\otimes}_E\, (E_0' \,\widehat{\otimes}_{E_0}\, E_0')$$

is exact, where the completion is the classical p-completion.

(2) Let $\{\mathcal{M}(E) \subseteq \widetilde{\mathcal{M}}(E)\}_E$ be the largest family of full subcategories such that, for any $M \in \mathcal{M}(E)$, any map $E \to E_0'$ of p-torsion-free p-complete rings and any p-completely faithfully flat map $E_0 \to E_0'$ of p-torsion-free p-complete rings, any descent datum ε on $M \,\widehat{\otimes}_E\, E_0'$ (i.e., an isomorphism

$$(E_0' \,\widehat{\otimes}_{E_0}\, E_0') \,\widehat{\otimes}_{E_0'}\, (M \,\widehat{\otimes}_E\, E_0') \cong (M \,\widehat{\otimes}_E\, E_0') \,\widehat{\otimes}_{E_0'}\, (E_0' \,\widehat{\otimes}_{E_0}\, E_0')$$

satisfying the cocycle condition on $E_0' \,\widehat{\otimes}_{E_0}\, E_0' \,\widehat{\otimes}_{E_0}\, E_0'$) descends uniquely to an E_0-module $M_0 \in \mathcal{M}(E_0)$.

The categories introduced above have the following properties.

PROPOSITION 2.18. *We keep the notation above:*

(1) *For any $M \in \widetilde{\mathcal{M}}_{\mathbb{\Delta}}(E, I_E)$ and any map $(E, I_E) \to (E', I_{E'})$ of bounded prisms, $M \, \widehat{\otimes}_E \, E'$ belongs to $\widetilde{\mathcal{M}}_{\mathbb{\Delta}}(E', I_{E'})$. An analogous property holds true for the other categories $\mathcal{M}_{\mathbb{\Delta}}(E, I_E)$, $\widetilde{\mathcal{M}}(E)$ and $\mathcal{M}(E)$.*

(2) *Let $f \colon M \to M'$ be a morphism in $\widetilde{\mathcal{M}}_{\mathbb{\Delta}}(E, I_E)$ and let $(E, I_E) \to (E', I_{E'})$ be a faithfully flat map of bounded prisms. Assume that f induces an isomorphism*

$$M \, \widehat{\otimes}_E \, E' \simeq M' \, \widehat{\otimes}_E \, E'.$$

Then f is also an isomorphism. An analogous property holds true for $\widetilde{\mathcal{M}}(E)$.

PROOF. Part (1) follows immediately from the definition of $\widetilde{\mathcal{M}}_{\mathbb{\Delta}}(E, I_E)$. For part (2), we consider the map of exact sequences:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & M & \longrightarrow & M \, \widehat{\otimes}_E \, E' & \rightrightarrows & M \, \widehat{\otimes}_E \, (E' \, \widehat{\otimes}_E \, E') \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \widehat{\otimes} \mathrm{id}} & & \downarrow{\scriptstyle f \widehat{\otimes} (\mathrm{id} \widehat{\otimes} \mathrm{id})} \\
0 & \longrightarrow & M' & \longrightarrow & M' \, \widehat{\otimes}_E \, E' & \rightrightarrows & M' \, \widehat{\otimes}_E \, (E' \, \widehat{\otimes}_E \, E').
\end{array}
$$

The map $f \, \widehat{\otimes} \, \mathrm{id}$ is an isomorphism by assumption. It follows that $f \, \widehat{\otimes} \, (\mathrm{id} \, \widehat{\otimes} \, \mathrm{id})$ is also an isomorphism. These imply that f is an isomorphism. ∎

REMARK 2.19. Let us make some remarks on the categories introduced above:

(1) Let $\mathcal{M}^{\mathrm{fp}}(E)$ be the category of finite projective E-modules. Also, let $\mathcal{M}^{p^n\text{-tors}}(E)$ be the category of

$$
\begin{cases}
(p, I_E)^n\text{-torsion } E\text{-modules} & \text{(when } (E, I_E) \text{ is a bounded prism)}, \\
p^n\text{-torsion } E\text{-modules} & \text{(when } E \text{ is a } p\text{-torsion-free } p\text{-complete ring)},
\end{cases}
$$

and let $\mathcal{M}^{\mathrm{tors}}(E)$ be $\bigcup_n \mathcal{M}^{p^n\text{-tors}}(E)$. Then we have an inclusion

$$\mathcal{M}^{\mathrm{fp}}(E) \subseteq \mathcal{M}_{\mathbb{\Delta}}(E, I_E), \quad \mathcal{M}(E)$$

by [1, Proposition A.12]. We also have an inclusion

$$\mathcal{M}^{\mathrm{tors}}(E) \subseteq \mathcal{M}_{\mathbb{\Delta}}(E, I_E), \quad \mathcal{M}(E)$$

by the usual descent argument.

(2) If E is a p-torsion-free p-complete δ-ring, then we have an inclusion

$$\widetilde{\mathcal{M}}(E) \subseteq \widetilde{\mathcal{M}}_{\mathbb{\Delta}}(E, pE)$$

by the definition. The same holds true for $\mathcal{M}(E)$ and $\mathcal{M}_{\mathbb{\Delta}}(E, pE)$.

We will simply write $\widetilde{\mathcal{M}}_{\mathbb{\Delta}}(E)$ (resp. $\mathcal{M}_{\mathbb{\Delta}}(E)$) for $\widetilde{\mathcal{M}}_{\mathbb{\Delta}}(E, I_E)$ (resp. $\mathcal{M}_{\mathbb{\Delta}}(E, I_E)$).

Next we define the category of crystals with respect to the categories of modules $\mathcal{M}_{\mathbb{\Delta}}(E)$ in Definition 2.16.

DEFINITION 2.20. Let $\mathcal{C}_{\mathbb{\Delta}}((X/A)_{m\text{-}\mathbb{\Delta}})$ (resp. $\mathcal{C}^{\mathrm{fp}}((X/A)_{m\text{-}\mathbb{\Delta}})$, $\mathcal{C}^{\mathrm{tors}}((X/A)_{m\text{-}\mathbb{\Delta}})$) be the category of abelian presheaves $\mathcal{F}$ on $(X/A)_{m\text{-}\mathbb{\Delta}}$ such that, for any object (E, I_E) in $(X/A)_{m\text{-}\mathbb{\Delta}}$, $\mathcal{F}(E, I_E) \in \mathcal{M}_{\mathbb{\Delta}}(E)$ (resp. $\mathcal{M}^{\mathrm{fp}}(E)$, $\mathcal{M}^{\mathrm{tors}}(E)$), and for any morphism $(E, I_E) \to (E_1, I_{E_1})$ in $(X/A)_{m\text{-}\mathbb{\Delta}}$, the map $\mathcal{F}(E, I_E) \to \mathcal{F}(E_1, I_{E_1})$ is compatible with the module structures in the usual sense and the canonical map $\mathcal{F}(E, I_E) \,\widehat{\otimes}_E\, E_1 \to \mathcal{F}(E_1, I_{E_1})$ is an isomorphism of E_1-modules. This condition means that $\mathcal{C}_{\mathbb{\Delta}}((X/A)_{m\text{-}\mathbb{\Delta}})$ (resp. $\mathcal{C}^{\mathrm{fp}}((X/A)_{m\text{-}\mathbb{\Delta}})$, $\mathcal{C}^{\mathrm{tors}}((X/A)_{m\text{-}\mathbb{\Delta}})$) is the category of crystals with respect to $\mathcal{M}_{\mathbb{\Delta}}$ (resp. $\mathcal{M}^{\mathrm{fp}}$, $\mathcal{M}^{\mathrm{tors}}$).

REMARK 2.21. Let us make some remarks on Definition 2.20:

(1) Presheaves $\mathcal{F}$ in the above definition are automatically sheaves by Definition 2.16 (1).

(2) One can prove that the category $\mathcal{C}_{\mathbb{\Delta}}((X/A)_{m\text{-}\mathbb{\Delta}})$ is unchanged even if we do not impose condition $(*)$ in Definition 2.1.

In order to prove that the morphism C induces an equivalence between the categories of crystals, we will use the following propositions. We follow the proof of [20, Theorem 9.12].

PROPOSITION 2.22 (Cf. [20, Lemma 9.5]). *Let (E, I_E) be an object of $(X/A)_{m\text{-}\mathbb{\Delta}}$ and let $g\colon \rho(E, I_E) \to (E', I_{E'})$ be a morphism in $(X'/A)_{\mathbb{\Delta}}$. Then there exist an object (E_1, I_{E_1}) of $(X/A)_{m\text{-}\mathbb{\Delta}}$ and a morphism $f\colon (E, I_E) \to (E_1, I_{E_1})$ in $(X/A)_{m\text{-}\mathbb{\Delta}}$ such that $g = \rho(f)$ (so $(E_1, I_{E_1}) = (E', I_{E'})$ as a bounded prism).*
If g is a cover, then so is f.

PROOF. We have shown this result in the proof of Proposition 2.8. ∎

PROPOSITION 2.23 (Cf. [20, Lemma 9.8 (ii)]). *Let $g\colon (E', I_{E'}) \to (E'_1, I_{E'_1})$ be a morphism in $(X'/A)_{\mathbb{\Delta}}$. Then there exist a morphism $h\colon (E, I_E) \to (E_1, I_{E_1})$ in $(X/A)_{m\text{-}\mathbb{\Delta}}$ and covers $f\colon (E', I_{E'}) \to \rho(E, I_E)$, $f_1\colon (E'_1, I_{E'_1}) \to \rho(E_1, I_{E_1})$ in $(X'/A)_{\mathbb{\Delta}}$ such that the following diagram is a pushout diagram:*

$$
\begin{array}{ccc}
(E'_1, I_{E'_1}) & \xrightarrow{\ f_1\ } & \rho(E_1, I_{E_1}) \\[4pt]
{\scriptstyle g}\big\uparrow & & {\scriptstyle \rho(h)}\big\uparrow \\[4pt]
(E', I_{E'}) & \xrightarrow{\ f\ } & \rho(E, I_E).
\end{array}
$$

PROOF. Let f be the cover constructed in Proposition 2.11, and let $(E'', I_{E''})$ be the pushout of the diagram $(E'_1, I_{E'_1}) \xleftarrow{g} (E', I_{E'}) \xrightarrow{f} \rho(E, I_E)$. Applying Proposition 2.22 to the map $\rho(E, I_E) \to (E'', I_{E''})$, we obtain the desired diagram. ∎

PROPOSITION 2.24 (Cf. [20, Lemma 9.9]). *Let $(E', I_{E'})$ be an object of $(X'/A)_\triangle$, let (E, I_E) be an object of $(X/A)_{m\text{-}\triangle}$ and let $(E', I_{E'}) \to \rho(E, I_E)$ be a cover. Then there exist an object (E_2, I_{E_2}) of $(X/A)_{m\text{-}\triangle}$ and two morphisms $p_1, p_2 \colon (E, I_E) \to (E_2, I_{E_2})$ in $(X/A)_{m\text{-}\triangle}$ such that*

$$\rho(E_2, I_{E_2}) = \rho(E, I_E) \underset{(E', I_{E'})}{\widehat{\otimes}} \rho(E, I_E),$$

and $\rho(p_1)$ (resp. $\rho(p_2)$) is the map

$$\rho(E, I_E) \to \rho(E, I_E) \underset{(E', I_{E'})}{\widehat{\otimes}} \rho(E, I_E)$$

to the first (resp. second) component.

PROOF. Applying Proposition 2.22 to the coprojection

$$\rho(E, I_E) \to \rho(E, I_E) \underset{(E', I_{E'})}{\widehat{\otimes}} \rho(E, I_E)$$

into the first component, we obtain the map $p_1 \colon (E, I_E) \to (E_2, I_{E_2})$ satisfying the conditions. The existence of p_2 follows from the fullness of ρ. ∎

We are now prepared to prove the equivalence of categories of crystals.

THEOREM 2.25. *The functors C_*, C^* induce equivalences of categories*

$$\mathcal{C}_\triangle((X/A)_{m\text{-}\triangle}) \rightleftarrows \mathcal{C}_\triangle((X'/A)_\triangle)$$

quasi-inverse to each other.

PROOF. We want to prove that the functors C_*, C^* preserve crystals. For C^*, it follows easily from the equality

$$C^*(\mathcal{F})(E, I_E) = \mathcal{F}(\rho(E, I_E)).$$

We show the claim for C_*. Let $\mathcal{F}$ be an object in $\mathcal{C}_\triangle((X/A)_{m\text{-}\triangle})$ and let $g \colon (E', I_{E'}) \to (E'_1, I_{E'_1})$ be a morphism in $(X'/A)_\triangle$. We want to prove that $C_*(\mathcal{F})(E', I_{E'}) \in \mathcal{M}_\triangle(E')$, $C_*(\mathcal{F})(E'_1, I_{E'_1}) \in \mathcal{M}_\triangle(E'_1)$ and that the map

$$C_*(\mathcal{F})(E', I_{E'}) \underset{E'}{\widehat{\otimes}} E'_1 \to C_*(\mathcal{F})(E'_1, I_{E'_1})$$

is an isomorphism. By Proposition 2.23, we have a pushout diagram

$$
\begin{array}{ccc}
(E_1', I_{E_1'}) & \xrightarrow{\;f_1\;} & \rho(E_1, I_{E_1}) \\
{\scriptstyle g}\big\uparrow & & \big\uparrow{\scriptstyle \rho(h)} \\
(E', I_{E'}) & \xrightarrow{\;f\;} & \rho(E, I_E)
\end{array}
$$

where f and f_1 are covers. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
C_*(\mathcal{F})(E', I_{E'}) \mathbin{\hat{\otimes}_{E'}} E_1' \mathbin{\hat{\otimes}_{E_1'}} E_1 \xrightarrow{\;\sim\;} C_*(\mathcal{F})(E', I_{E'}) \mathbin{\hat{\otimes}_{E'}} E \mathbin{\hat{\otimes}_{E}} E_1 \rightarrow C_*(\mathcal{F})(\rho(E, I_E)) \mathbin{\hat{\otimes}_{E}} E_1 \\
\big\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big\downarrow \\
C_*(\mathcal{F})(E_1', I_{E_1'}) \mathbin{\hat{\otimes}_{E_1'}} E_1 \xrightarrow{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} C_*(\mathcal{F})(\rho(E_1, I_{E_1})).
\end{array}
$$

Using Proposition 2.18, it is enough to show that $C_*(\mathcal{F})(E', I_{E'}) \in \mathcal{M}_\triangle(E')$, $C_*(\mathcal{F})(E_1', I_{E_1'}) \in \mathcal{M}_\triangle(E_1')$ and that the left vertical arrow is an isomorphism. To show that the left vertical arrow is an isomorphism, it suffices to check that the other arrows are all isomorphisms. As $C^*C_* \simeq \mathrm{id}$, we see that for each $(E, I_E) \in (X/A)_{m\text{-}\triangle}$,

$$
(2.1) \qquad \mathcal{F}(E, I_E) = C^*C_*(\mathcal{F})(E, I_E) = C_*(\mathcal{F})(\rho(E, I_E)).
$$

As $\mathcal{F}$ is a crystal, equality (2.1) implies that the right vertical arrow is an isomorphism. So it is enough to show that for any cover $f \colon (E', I_{E'}) \to \rho(E, I_E)$, one has $C_*(\mathcal{F})(E', I_{E'}) \in \mathcal{M}_\triangle(E')$ and the map

$$
C_*(\mathcal{F})(E', I_{E'}) \mathbin{\hat{\otimes}_{E'}} E \to C_*(\mathcal{F})(\rho(E, I_E))
$$

is an isomorphism.

In the following, we simply write E, E' for (E, I_E), $(E', I_{E'})$ respectively in order to lighten notation. We consider the diagram

$$
E' \xrightarrow{\;f\;} \rho(E) \rightrightarrows \rho(E) \mathbin{\hat{\otimes}_{E'}} \rho(E) \underset{\longrightarrow}{\overset{\longrightarrow}{\rightarrow}} \rho(E) \mathbin{\hat{\otimes}_{E'}} \rho(E) \mathbin{\hat{\otimes}_{E'}} \rho(E),
$$

where all the arrows except f are the maps constructed by the coprojections. By Proposition 2.24 and the definition of ρ, the above diagram may be rewritten as

$$
E' \xrightarrow{\;f\;} \rho(E) \rightrightarrows \rho(E_2) \underset{\longrightarrow}{\overset{\longrightarrow}{\rightarrow}} \rho(E_3),
$$

where $E_2, E_3 \in (X/A)_{m\text{-}\triangle}$ such that

$$
E_2 = E \mathbin{\hat{\otimes}_{E'}} E, \qquad E_3 = E \mathbin{\hat{\otimes}_{E'}} E \mathbin{\hat{\otimes}_{E'}} E
$$

as prisms and all the arrows except f come from the morphisms in $(X/A)_{m\text{-}\underline{\mathbb{\Delta}}}$. Then we have a diagram

$$C_*(\mathcal{F})(\rho(E)) \rightrightarrows C_*(\mathcal{F})(\rho(E_2)) \Rrightarrow C_*(\mathcal{F})(\rho(E_3))$$

which is identified with the following diagram by using equality (2.1):

$$\mathcal{F}(E) \rightrightarrows \mathcal{F}(E_2) \Rrightarrow \mathcal{F}(E_3).$$

As $\mathcal{F}$ is a crystal, the above diagram defines a descent datum on $\mathcal{F}(E)$ relative to the faithfully flat map of bounded prisms $(E', I_{E'}) \to (E, I_E)$. So it descends uniquely to an object M in $\mathcal{M}_{\underline{\mathbb{\Delta}}}(E')$. In particular, we have an isomorphism of E-modules $M \,\widehat{\otimes}_{E'}\, E \cong \mathcal{F}(E)$ and an exact sequence

$$0 \longrightarrow M \longrightarrow \mathcal{F}(E) \rightrightarrows \mathcal{F}(E_2).$$

On the other hand, since $C_*(\mathcal{F})$ is a sheaf, we have an exact sequence

$$0 \longrightarrow C_*(\mathcal{F})(E') \longrightarrow C_*(\mathcal{F})(\rho(E)) \rightrightarrows C_*(\mathcal{F})(\rho(E_2)).$$

By these two exact sequences and equality (2.1), we see that

$$C_*(\mathcal{F})(E') \cong M \in \mathcal{M}_{\underline{\mathbb{\Delta}}}(E').$$

Thus,

$$C_*(\mathcal{F})(E', I_{E'}) \,\widehat{\underset{E'}{\otimes}}\, E \to C_*(\mathcal{F})(\rho(E, I_E))$$

is an isomorphism, as desired. ∎

The following is proved in exactly the same way as Theorem 2.25.

COROLLARY 2.26. *The functors* C_*, C^* *induce equivalences of categories*

$$\mathcal{C}^{\mathrm{fp}}((X/A)_{m\text{-}\underline{\mathbb{\Delta}}}) \rightleftarrows \mathcal{C}^{\mathrm{fp}}((X'/A)_{\underline{\mathbb{\Delta}}}),$$
$$\mathcal{C}^{\mathrm{tors}}((X/A)_{m\text{-}\underline{\mathbb{\Delta}}}) \rightleftarrows \mathcal{C}^{\mathrm{tors}}((X'/A)_{\underline{\mathbb{\Delta}}})$$

quasi-inverse to each other.

3. The m-q-crystalline site

In this section we prove a q-crystalline version of the theorems in the previous section, namely, we define the m-q-crystalline site which is a higher-level analog of

the q-crystalline site defined in [5, Section 16], and prove equivalence between the category of crystals on the m-q-crystalline site of a smooth formal scheme X and that on the usual q-crystalline site of X', where X' is the pullback of X by the m-fold iteration ϕ^m of the Frobenius lift ϕ on the base q-PD pair.

First, we give the definition of q-PD pairs we consider. Set $A = \mathbf{Z}_p[\![q-1]\!]$ with δ-structure given by $\delta(q) = 0$.

DEFINITION 3.1. A q-PD pair is given by a derived $(p, [p]_q)$-complete δ-pair (D, I) over $(A, (q-1))$ satisfying the following conditions:

(1) For any $f \in I$, we have $\phi(f) - [p]_q \delta(f) \in [p]_q I$.

(2) The pair $(D, ([p]_q))$ is a bounded prism over $(A, ([p]_q))$, i.e., D is $[p]_q$-torsion-free and $D/([p]_q)$ has bounded p^∞-torsion.

(3) The ring $D/(q-1)$ is p-torsion-free with finite $(p, [p]_q)$-complete Tor-amplitude over D.

(4) D/I is classically p-complete.

REMARK 3.2. Let us make some remarks on Definition 3.1:

(1) This definition of q-PD pairs follows that in [12, Section 7.1]. Condition (4) enables us to consider the affine p-adic formal scheme $\mathrm{Spf}(D/I)$, but this is not imposed in [5, Definition 16.2]. Our notion of a q-PD pair differs from that in [6, Definition 3.1]; conditions (3), (4) in Definition 3.1, the condition that $D/([p]_q)$ has bounded p^∞-torsion and the condition of $(p, [p]_q)$-completeness of D are not imposed in [6, Definition 3.1] (but the latter two conditions are imposed in the definition of the q-crystalline site in [6, Section 7]).

(2) By conditions (1) and (2), we see that for any $x \in I$, the element

$$\gamma(x) \stackrel{\mathrm{def}}{=} \phi(x)/[p]_q - \delta(x) \in I$$

is well defined.

(3) Let $J = (\phi^m)^{-1}(I)$. As ϕ is continuous, the ideal $J \subseteq D$ is closed for the p-adic topology of D. So D/J is also classically p-complete.

(4) Condition (4) in Definition 3.1 implies that I is closed for the $(p, [p]_q)$-adic topology of D because

$$I + (p, [p]_q)^n \subseteq I + (p, q-1)^n \subseteq I + (p)^n.$$

Next we define the m-q-crystalline site, which is a higher-level analog of the q-crystalline site as well as a q-analog of the level–m crystalline site. We fix a non-negative integer m and a q-PD pair (D, I).

DEFINITION 3.3. Let $J = (\phi^m)^{-1}(I)$ and let X be a p-adic formal scheme smooth and separated over D/J. We define the *m-q-crystalline site* $(X/D)_{m\text{-}q\text{-crys}}$ of X over D as follows. Objects are maps $(D, I) \to (E, I_E)$ of q-PD pairs together with a map $\mathrm{Spf}(E/J_E) \to X$ over D/J satisfying the following condition, where $J_E = (\phi^m)^{-1}(I_E)$:

($*$) $\mathrm{Spf}(E/J_E) \to X$ factors through some affine open $\mathrm{Spf}(R) \subseteq X$.

We will often denote such an object by

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X) \in (X/D)_{m\text{-}q\text{-crys}}$$

or (E, I_E) if no confusion arises. A morphism

$$(\mathrm{Spf}(E') \leftarrow \mathrm{Spf}(E'/J_{E'}) \to X) \to (\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X)$$

is a map of q-PD pairs $(E, I_E) \to (E', I_{E'})$ over (D, I) such that the induced morphism

$$\mathrm{Spf}(E'/J_{E'}) \to \mathrm{Spf}(E/J_E)$$

is compatible with the maps $\mathrm{Spf}(E'/J_{E'}) \to X$, $\mathrm{Spf}(E/J_E) \to X$. When we denote such an object by (E, I_E), we will write $(E, I_E) \to (E', I_{E'})$ (not $(E', I_{E'}) \to (E, I_E)$) for a morphism from $(E', I_{E'})$ to (E, I_E). A map $(E, I_E) \to (E', I_{E'})$ in $(X/D)_{m\text{-}q\text{-crys}}$ is a cover if it is a $(p, [p]_q)$-completely faithfully flat map and satisfies

$$(3.1) \qquad\qquad\qquad I_{E'} = \overline{I_E E'},$$

where the right-hand side means the $(p, [p]_q)$-complete ideal of E' generated by $I_E E'$.

REMARK 3.4. Note that equality (3.1) does not imply the equality $J_{E'} = \overline{J_E E'}$ in general. Indeed, in the case of the inclusion

$$(E, I_E) := (\mathbf{Z}_p[\![q - 1]\!], (q - 1)) \hookrightarrow (\mathbf{Z}_p[\![q^{1/p} - 1]\!], (q - 1)) =: (E', I_{E'}),$$

where the δ-ring structure on E' is defined by $\delta(q^{1/p}) = 0$, we have $I_{E'} = \overline{I_E E'}$ but $\overline{J_E E'} = (q - 1) \neq (q^{1/p} - 1) = J_{E'}$.

We need to check that the category $(X/D)_{m\text{-}q\text{-crys}}$ endowed with the topology as defined above forms a site. We have the following lemma as in the case of the m-prismatic site:

LEMMA 3.5. *Let* $(E_1, I_{E_1}) \xleftarrow{f} (E, I_E) \xrightarrow{g} (E_2, I_{E_2})$ *be maps in* $(X/D)_{m\text{-}q\text{-crys}}$ *such that f is a cover. Let* $E_3 := E_1 \widehat{\otimes}_E E_2$, *where the completion is the classical* $(p, [p]_q)$-*completion. Then* $(E_3, \overline{I_{E_2} E_3})$ *is the object that represents the coproduct*

$$(E_1, I_{E_1}) \sqcup_{(E, I_E)} (E_2, I_{E_2})$$

in $(X/D)_{m\text{-}q\text{-crys}}$, *and the canonical map* $(E_2, I_{E_2}) \to (E_3, I_{E_3})$ *is a cover.*

Proof. By Lemma 2.3, $(E_3, ([p]_q))$ represents the object $(E_1, ([p]_q)) \sqcup_{(E, ([p]_q))}$ $(E_2, ([p]_q))$ in the category of bounded prisms. Also, by [5, Lemma 16.5 (5)], $(E_3, \overline{I_{E_2} E_3})$ is a q-PD pair. We can check that it represents the object

$$(E_1, I_{E_1}) \sqcup_{(E, I_E)} (E_2, I_{E_2})$$

in $(X/D)_{m\text{-}q\text{-crys}}$ by using Remark 3.2 (4).   ∎

By Lemma 3.5, the set of covers in Definition 3.3 actually forms a pretopology on the category $(X/D)_{m\text{-}q\text{-crys}}$.

Remark 3.6. When $m = 0$, we denote the site $(X/D)_{m\text{-}q\text{-crys}}$ simply by $(X/D)_{q\text{-crys}}$ and call it the q-crystalline site. This site is similar to the one in [5] and that in [12] but there are slight differences: First, the topology considered here (flat topology) is different from the one in [5] (indiscrete topology) and the one in [12] (étale topology). Second, we imposed the technical condition $(*)$ which is not assumed in [5] and [12]. (However, we note that our topos is unchanged if we do not impose condition $(*)$.) Third, X is assumed to be affine in [5] but we do not assume that X is affine.

In order to establish equivalence between the category of crystals on the m-q-crystalline site and that on the usual q-crystalline site, we first construct a functor between these sites, as in the prismatic case.

Construction 3.7. Under the notation and assumption in Definition 3.3, let X' be $X \mathbin{\hat{\times}}_{\mathrm{Spf}(D/J), (\phi^m)^*} \mathrm{Spf}(D/I)$. Then we have a diagram:

$$
\begin{array}{ccccc}
\mathrm{Spf}(D) & \longleftarrow & \mathrm{Spf}(D/J) & \longleftarrow & X \\
{\scriptstyle (\phi^m)^*} \uparrow & & {\scriptstyle (\phi^m)^*} \uparrow & \square & \uparrow \\
\mathrm{Spf}(D) & \longleftarrow & \mathrm{Spf}(D/I) & \longleftarrow & X'.
\end{array}
$$

We define a functor ρ from the m-q-crystalline site of X over D to the usual q-crystalline site of X' over D as in the case of the m-prismatic site: for an object

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X)$$

of $(X/D)_{m\text{-}q\text{-crys}}$, we define the object $\rho(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X)$ of $(X'/D)_{q\text{-crys}}$ by

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/I_E) \xrightarrow{f} X'),$$

where the right map f is defined as

$$\mathrm{Spf}(E/I_E) \overset{g}{\to} \mathrm{Spf}(E/J_E) \underset{\mathrm{Spf}(D/J),(\phi^m)^*}{\widehat{\times}} \mathrm{Spf}(D/I)$$

$$\to X \underset{\mathrm{Spf}(D/J),(\phi^m)^*}{\widehat{\times}} \mathrm{Spf}(D/I) = X'.$$

Here, the first map g is induced by the map of rings

$$E/J_E \underset{D/J,\phi^m}{\widehat{\otimes}} D/I \to E/I_E, \quad e \otimes d \mapsto \phi^m(e)d.$$

This defines the functor $\rho\colon (X/D)_{m\text{-}q\text{-crys}} \to (X'/D)_{q\text{-crys}}$.

Next we want to show that ρ induces an equivalence of topoi. Since the m-q-crystalline site is defined in a very similar way to the m-prismatic site, almost all propositions can be proved in exactly the same way as in the case of the m-prismatic site by replacing the bounded prism (A, I) with the q-PD pair (D, I), and so we omit the details. In particular, we have the following proposition.

PROPOSITION 3.8. *The functor ρ is fully faithful, continuous and cocontinuous.*

As the construction of the q-PD envelope is different from that of the prismatic envelope that appeared in the proof of Proposition 2.11, we give a proof of the q-analog of Proposition 2.11. We will need the following q-analog of Lemma 2.10:

LEMMA 3.9. *Let (E, I_E) be a q-PD pair. Let $\bar{f}$ be an open immersion*

$$\bar{f}\colon \mathrm{Spf}(\overline{E_i}) \to \mathrm{Spf}(E/I_E)$$

or an open immersion

$$\bar{f}\colon \mathrm{Spf}(\overline{E_i}) \to \mathrm{Spf}(E/J_E).$$

Then there is a unique open immersion $f\colon \mathrm{Spf}(E_i) \to \mathrm{Spf}(E)$ which lifts $\bar{f}$. Moreover, the corresponding map of rings induces a map $(E, I_E) \to (E_i, \overline{I_E E_i})$ of q-PD pairs.

PROOF. We first check that the images of I_E, J_E in $E/(p, [p]_q)$ are nil ideals. For I_E, it follows immediately from Definition 3.1 (1) and the definition of the Frobenius lift ϕ. For J_E, we note that the image of J_E in E/I_E is a nil ideal by the definitions of J_E and ϕ, so the claim follows. Then the existence, the uniqueness and the affinity of the lifting follow from Lemma 2.9. We denote the lifting by $f\colon \mathrm{Spf}(E_i) \to \mathrm{Spf}(E)$. We can give a δ-structure on E_i such that the corresponding map $E \to E_i$ is a map of δ-rings as in the proof of Lemma 2.10 by replacing the ideal I_E with the ideal $([p]_q)$.

It remains to show that $(E_i, \overline{I_E E_i})$ is a q-PD pair. Condition (4) in Definition 3.1 follows as the ideal $\overline{I_E E_i}$ is closed. To prove the other conditions, by [5, Lemma 16.5 (5)],

it is enough to prove that the map of δ-rings $E \to E_i$ is $(p, [p]_q)$-completely flat. Applying the same argument as in the proof of Lemma 2.10 by replacing the ideal I_E with the ideal $([p]_q)$, we see that $E \to E_i$ is $(p, [p]_q)$-completely flat (actually $(p, [p]_q)$-completely étale), as desired.   ∎

The next result will be used to prove the equivalence of topoi.

PROPOSITION 3.10. *Let $(E', I_{E'})$ be an object in $(X'/D)_{q\text{-crys}}$. Then there exists an object (E, I_E) in $(X/D)_{m\text{-}q\text{-crys}}$ and a cover of the form $(E', I_{E'}) \to \rho(E, I_E)$.*

PROOF. Let $X = \bigcup_i \mathrm{Spf}(R_i)$ be a finite affine open cover of X. By the definition of X' in Construction 3.7, we see that $\mathrm{Spf}(R_i) \hat{\times}_X X' = \mathrm{Spf}(R_i \hat{\otimes}_{D/J, \phi^m} D/I)$. We denote this formal scheme by $\mathrm{Spf}(R_i')$. On the other hand, condition $(\ast)$ in Definition 3.1 ensures that the map $\mathrm{Spf}(E'/I_{E'}) \to X'$ factors through some affine open $\mathrm{Spf}(R'') \subseteq X'$. As X' is separated by assumption, we see that $\mathrm{Spf}(R_i') \cap \mathrm{Spf}(R'') = \mathrm{Spf}(R_i^*)$ for some R_i^*. Then the formal scheme $\mathrm{Spf}(R_i') \hat{\times}_{X'} \mathrm{Spf}(E'/I_{E'})$ is identified with $\mathrm{Spf}(R_i' \hat{\otimes}_{R_i^*} E'/I_{E'})$. In particular, it is affine. We denote this affine open subscheme of $\mathrm{Spf}(E'/I_{E'})$ by $\mathrm{Spf}(\overline{E_i'})$. By Lemma 3.9, there is a unique q-PD pair $(E_i', \overline{I_{E'} E_i'}) \in (X'/D)_{q\text{-crys}}$ for which the corresponding formal scheme $\mathrm{Spf}(E_i')$ is an affine open formal subscheme of $\mathrm{Spf}(E')$ and lifts $\mathrm{Spf}(\overline{E_i'})$, namely $E_i'/\overline{I_{E'} E_i'} = \overline{E_i'}$. By construction, we see that

$$(E', I_{E'}) \to \prod_i (E_i', \overline{I_{E'} E_i'})$$

is a cover in $(X'/D)_{q\text{-crys}}$. To prove the proposition, we may replace $(E', I_{E'})$ by $(E_i', \overline{I_{E'} E_i'})$ to assume that the structure morphism $\mathrm{Spf}(E'/I_{E'}) \to X'$ factors through an affine open $\mathrm{Spf}(R') \subseteq X'$ such that R' is of the form $R \hat{\otimes}_{D/J, \phi^m} D/I$ for some affine open $\mathrm{Spf}(R) \subseteq X$.

As the map $D/J \to R$ is smooth, [19, Tag 00TA] tells us that there exists an open cover of $\mathrm{Spf}(R)$ by standard opens $\mathrm{Spf}(\widehat{R_g})$ such that each $\widehat{R_g}/p$ is standard smooth over $D/(p, J)$. After refining the given cover of X in the previous paragraph, we may assume that R/p is standard smooth over $D/(p, J)$. By [19, Tag 00T7], there exists a surjection $D[x_1, \ldots, x_n]^\wedge \to R$ whose kernel is the ideal $(J, y_1, \ldots, y_r)$ such that

$$\overline{y_1}, \ldots, \overline{y_r} \in D/(p, J)[x_1, \ldots, x_n]$$

form a regular sequence. On the other hand, there is a natural map

$$D[x_1, \ldots, x_n]^\wedge \to D\{x_1, \ldots, x_n\}^\wedge,$$

where the symbol $\{\}$ denotes the adjoining of elements in the theory of δ-rings. By a similar argument to that in the proof of Corollary 2.13, we see that the sequence

$$\phi^m(y_1), \ldots, \phi^m(y_r) \in D\{x_1, \ldots, x_n\}^\wedge$$

is $(p, [p]_q)$-completely regular relative to D. Then we can construct a map

$$D\{x_1, \ldots, x_n\}^\wedge \to S \stackrel{\text{def}}{=} \left(D\left\{ x_1, \ldots, x_n, \frac{\phi^{m+1}(y_1)}{[p]_q}, \ldots, \frac{\phi^{m+1}(y_r)}{[p]_q} \right\} \right)^\wedge$$

to the q-PD envelope of $(D\{x_1, \ldots, x_n\}^\wedge, K)$ by [5, Lemma 16.10], where K denotes the ideal $(I, \phi^m(y_1), \ldots, \phi^m(y_r)) \subseteq D\{x_1, \ldots, x_n\}^\wedge$. By the construction of the q-PD envelope, $\phi^m(y_i) \in I_S$ and so $y_i \in J_S = (\phi^m)^{-1}(I_S)$ for all i. This gives a map $R \to S/J_S$.

Next, let $D[x_1, \ldots, x_n]^\wedge \to R \,\widehat{\otimes}_{D,\phi^m}\, D$ be the base change of $D[x_1, \ldots, x_n]^\wedge \to R$ in the previous paragraph along $\phi^m \colon D \to D$. Passing to the quotient then induces $D[x_1, \ldots, x_n]^\wedge \to R'$. The kernel of this map is the ideal $(I, y_1', \ldots, y_r')$, where the y_i' are the images of the y_i under the map $D[x_1, \ldots, x_n]^\wedge \to D[x_1, \ldots, x_n]^\wedge$ sending $\sum_j \alpha_j x^{\underline{j}}$ to $\sum_j \phi^m(\alpha_j) x^{\underline{j}}$ (here we write $\underline{j} := (j_1, \ldots, j_n)$ for the multi-index). By the definition of the y_i, the sequence $\overline{y_1'}, \ldots, \overline{y_r'} \in D/(p, I)[x_1, \ldots, x_n]$ is a regular sequence. By a similar argument to that in the proof of Corollary 2.13, the sequence $y_1', \ldots, y_r' \in D\{x_1, \ldots, x_n\}^\wedge$ is $(p, [p]_q)$-completely regular relative to D. Then we can construct a map

$$D\{x_1, \ldots, x_n\}^\wedge \to S' \stackrel{\text{def}}{=} \left(D\left\{ x_1, \ldots, x_n, \frac{\phi(y_1')}{[p]_q}, \ldots, \frac{\phi(y_r')}{[p]_q} \right\} \right)^\wedge$$

to the q-PD envelope of $(D\{x_1, \ldots, x_n\}^\wedge, K')$, where K' denotes the ideal

$$(I, y_1', \ldots, y_r') \subseteq D\{x_1, \ldots, x_n\}^\wedge.$$

Then we have a diagram:

$$
\begin{array}{ccc}
D/I & \longrightarrow & R' \\
\big\uparrow & & \big\uparrow \\
D & \longrightarrow D[x_1, \ldots, x_n]^\wedge \to D\{x_1, \ldots, x_n\}^\wedge \to & S' \stackrel{\text{def}}{=} (D\{x_1, \ldots, x_n, \frac{\phi(y_1')}{[p]_q}, \ldots, \frac{\phi(y_r')}{[p]_q}\})^\wedge.
\end{array}
$$

By the definitions of y_i and y_i', we have a map of δ-D-algebras $(\phi^m)' \colon S' \to S$ sending x_i to $\phi^m(x_i)$. By Proposition 3.11 below, we see that $(\phi^m)'$ is $(p, [p]_q)$-completely faithfully flat.

Let $(E', I_{E'}) \in (X'/D)_{q\text{-crys}}$ be as above. We have a map

$$f_1 \colon D[x_1, \ldots, x_n]^\wedge \to E'/I_{E'}$$

determined as the composition of the map $D[x_1, \ldots, x_n]^\wedge \to R'$ in the above diagram with $R' \to E'/I_{E'}$. As $D[x_1, \ldots, x_n]^\wedge$ is the completion of a polynomial ring, one can choose a map $f_2 \colon D[x_1, \ldots, x_n]^\wedge \to E'$ lifting f_1. As E' is a δ-D-algebra, f_2 extends uniquely to a δ-D-algebra map $f_3 \colon D\{x_1, \ldots, x_n\}^\wedge \to E'$. By construction, this extension carries K' into $I_{E'}$. By the universal property of S', f_3 extends uniquely to a δ-D-algebra map $g \colon S' \to E'$. If we set $h \colon E' \to E' \, \widehat{\otimes}_{S'} \, S$ to be the base change of $(\phi^m)' \colon S' \to S$ along g, then by $(p, [p]_q)$-complete faithful flatness of $(\phi^m)'$, we see that the same holds true for h.

It remains to check that the map h defines a morphism

$$(E', I_{E'}) \to \rho\left(E' \, \widehat{\underset{S'}{\otimes}} \, S, \, I_{E' \widehat{\underset{S'}{\otimes}} S}\right)$$

in $(X'/D)_{q\text{-crys}}$. To see this, it is enough to check that the bottom-right square in the following diagram is commutative, in which all the other squares are commutative:

$$
\begin{array}{ccccc}
D[x_1, \ldots, x_n]^\wedge & \longrightarrow & S & \xrightarrow{\ i_2\ } & E' \, \widehat{\underset{S'}{\otimes}} \, S \\
\downarrow & & \downarrow & & \downarrow \\
D[x_1, \ldots, x_n]^\wedge \twoheadrightarrow R' = R \, \widehat{\underset{D/J, \phi^m}{\otimes}} \, D/I & \to & S/J_S \, \widehat{\underset{D/J, \phi^m}{\otimes}} \, D/I & \xrightarrow{\ \overline{i_2}\ } & E' \, \widehat{\underset{S'}{\otimes}} \, S/J_{E' \widehat{\otimes}_{S'} S} \, \widehat{\underset{D/J, \phi^m}{\otimes}} \, D/I \\
\downarrow{\scriptstyle f_2} & & \downarrow & & \downarrow \\
E' & \twoheadrightarrow & E'/I_{E'} & \xrightarrow{\quad \bar{h} \quad} & E' \, \widehat{\underset{S'}{\otimes}} \, S/I_{E' \widehat{\otimes}_{S'} S}.
\end{array}
$$

Here i_2 is the map to the second component of the coproduct in Lemma 3.5. We can check the commutativity of the bottom-right square by tracing the elements x_i:

$$
\begin{array}{ccccc}
x_i & \longmapsto & x_i & \longmapsto & 1 \otimes x_i \\
\downarrow & & & & \downarrow \\
x_i & \longmapsto \overline{x_i} \otimes 1 & & \longmapsto & \overline{1 \otimes x_i} \otimes 1 \\
\downarrow & & & & \downarrow \\
f_2(x_i) & \longmapsto \overline{f_2(x_i)} & & \longmapsto & \overline{1 \otimes \phi^m(x_i)} = \overline{f_2(x_i) \otimes 1},
\end{array}
$$

so the proposition follows. ∎

We prove the claim used in the proof above.

PROPOSITION 3.11. *With notation as in the proof of Proposition 3.10, the map*

$$(\phi^m)' \colon S' \to S$$

is $(p, [p]_q)$-completely faithfully flat.

Proof. Since the map $(\phi^m)'$ is the derived $(p, [p]_q)$-completion of the base change of the δ-D-algebra map

$$(\phi^m)'': D\{x_1, \ldots, x_n\}^\wedge \to D\{x_1, \ldots, x_n\}^\wedge, \quad x_i \mapsto \phi^m(x_i),$$

it suffices to prove that $(\phi^m)''$ is $(p, [p]_q)$-completely faithfully flat. By [5, Lemma 2.11], the δ-D-algebra map

$$(\phi^m)''': D\{x_1, \ldots, x_n\} \to D\{x_1, \ldots, x_n\}, \quad x_i \mapsto \phi^m(x_i)$$

is faithfully flat. As the map $(\phi^m)''$ is the derived $(p, [p]_q)$-completion of the base change of $(\phi^m)'''$ along $D\{x_1, \ldots, x_n\} \to D\{x_1, \ldots, x_n\}^\wedge$, we conclude that $(\phi^m)''$ is $(p, [p]_q)$-completely faithfully flat. ∎

We are now prepared to prove the equivalence of topoi. First, we note that the functor of sites $\rho: (X/D)_{m\text{-}q\text{-crys}} \to (X'/D)_{q\text{-crys}}$ in Construction 3.7 induces a functor between the categories of presheaves of sets

$$\hat{\rho}^*: \widetilde{(X'/D)}_{q\text{-crys}} \to \widetilde{(X/D)}_{m\text{-}q\text{-crys}}, \quad \mathcal{G} \mapsto \mathcal{G} \circ \rho,$$

and it admits a right adjoint

$$\hat{\rho}_*: \widetilde{(X/D)}_{m\text{-}q\text{-crys}} \to \widetilde{(X'/D)}_{q\text{-crys}}.$$

By Proposition 3.8, we obtain a morphism of topoi

$$\mathrm{C}: \widetilde{(X/D)}_{m\text{-}q\text{-crys}} \to \widetilde{(X'/D)}_{q\text{-crys}}, \quad \mathrm{C}^* = \hat{\rho}^*, \ \mathrm{C}_* = \hat{\rho}_*.$$

THEOREM 3.12. *The morphism* $\mathrm{C}: \widetilde{(X/D)}_{m\text{-}q\text{-crys}} \to \widetilde{(X'/D)}_{q\text{-crys}}$ *is an equivalence of topoi.*

Proof. By [17, Proposition 4.2.1] (see also [20, Proposition 9.10]), we are reduced to checking the conditions which were proved in Propositions 3.8 and 3.10. ∎

Next we define the category of crystals with respect to the categories of modules $\mathcal{M}_\mathbb{\Delta}(E)$ and $\mathcal{M}(E)$ in Definition 2.16.

DEFINITION 3.13. Let $\mathcal{M}_\mathbb{\Delta}(E)$ and $\mathcal{M}(E)$ be the categories of modules as in Definitions 2.16 and 2.17:

(1) Let

$$\mathcal{C}_\mathbb{\Delta}((X/D)_{m\text{-}q\text{-crys}}) \quad (\text{resp. } \mathcal{C}^{\mathrm{fp}}((X/D)_{m\text{-}q\text{-crys}}), \ \mathcal{C}^{\mathrm{tors}}((X/D)_{m\text{-}q\text{-crys}}))$$

be the category of abelian presheaves $\mathcal{F}$ on $(X/D)_{m\text{-}q\text{-crys}}$ such that, for any object (E, I_E) in $(X/D)_{m\text{-}q\text{-crys}}$, $\mathcal{F}(E, I_E) \in \mathcal{M}_\triangle(E)$ (resp. $\mathcal{M}^{\text{fp}}(E)$, $\mathcal{M}^{\text{tors}}(E)$), and for any morphism $(E, I_E) \to (E_1, I_{E_1})$ in $(X/D)_{m\text{-}q\text{-crys}}$, the map

$$\mathcal{F}(E, I_E) \to \mathcal{F}(E_1, I_{E_1})$$

is compatible with the module structures in the usual sense and the canonical map $\mathcal{F}(E, I_E) \,\widehat{\otimes}_E\, E_1 \to \mathcal{F}(E_1, I_{E_1})$ is an isomorphism of E_1-modules.

(2) Assume that (D, I) is a q-PD pair with $q = 1$ in D. Then we define the category $\mathcal{C}((X/D)_{m\text{-}q\text{-crys}})$ as the category of abelian presheaves $\mathcal{F}$ on $(X/D)_{m\text{-}q\text{-crys}}$ such that, for any object (E, I_E) in $(X/D)_{m\text{-}q\text{-crys}}$, $\mathcal{F}(E, I_E) \in \mathcal{M}(E)$, and for any morphism $(E, I_E) \to (E_1, I_{E_1})$ in $(X/D)_{m\text{-}q\text{-crys}}$, the map

$$\mathcal{F}(E, I_E) \to \mathcal{F}(E_1, I_{E_1})$$

is compatible with the module structures in the usual sense and the canonical map $\mathcal{F}(E, I_E) \,\widehat{\otimes}_E\, E_1 \to \mathcal{F}(E_1, I_{E_1})$ is an isomorphism of E_1-modules.

REMARK 3.14. As in Remark 2.21, one can prove that presheaves $\mathcal{F}$ in the above definition are automatically sheaves and the category $\mathcal{C}_\triangle((X/D)_{m\text{-}q\text{-crys}})$ is unchanged even if we do not impose condition $(*)$ in Definition 3.3.

We are now prepared to prove equivalence of categories of crystals. The following results can be proved in exactly the same way as in the case of an m-prismatic site, so we omit the details.

PROPOSITION 3.15. *Let (E, I_E) be an object of $(X/D)_{m\text{-}q\text{-crys}}$ and let*

$$g\colon \rho(E, I_E) \to (E', I_{E'})$$

be a morphism in $(X'/D)_{q\text{-crys}}$. Then there exist an object (E_1, I_{E_1}) of $(X/D)_{m\text{-}q\text{-crys}}$ and a morphism $f\colon (E, I_E) \to (E_1, I_{E_1})$ in $(X/D)_{m\text{-}q\text{-crys}}$ such that $g = \rho(f)$. If g is a cover, then so is f.

PROPOSITION 3.16. *Let $g\colon (E', I_{E'}) \to (E'_1, I_{E'_1})$ be a morphism in $(X'/D)_{q\text{-crys}}$. Then there exist a morphism $h\colon (E, I_E) \to (E_1, I_{E_1})$ in $(X/D)_{m\text{-}q\text{-crys}}$ and covers $f\colon (E', I_{E'}) \to \rho(E, I_E)$, $f_1\colon (E'_1, I_{E'_1}) \to \rho(E_1, I_{E_1})$ in $(X'/D)_{q\text{-crys}}$ such that the following diagram is a pushout diagram:*

$$
\begin{array}{ccc}
(E'_1, I_{E'_1}) & \xrightarrow{f_1} & \rho(E_1, I_{E_1}) \\
{\scriptstyle g}\big\uparrow & & {\scriptstyle \rho(h)}\big\uparrow \\
(E', I_{E'}) & \xrightarrow{f} & \rho(E, I_E).
\end{array}
$$

PROPOSITION 3.17. *Let $(E', I_{E'})$ be an object of $(X'/D)_{q\text{-crys}}$, let (E, I_E) be an object of $(X/D)_{m\text{-}q\text{-crys}}$ and let $(E', I_{E'}) \to \rho(E, I_E)$ be a cover. Then there exist an object (E_2, I_{E_2}) of $(X/D)_{m\text{-}q\text{-crys}}$ and two morphisms $p_1, p_2 \colon (E, I_E) \to (E_2, I_{E_2})$ in $(X/D)_{m\text{-}q\text{-crys}}$ such that $\rho(E_2, I_{E_2}) = \rho(E, I_E) \hat{\otimes}_{(E', I_{E'})} \rho(E, I_E)$, and $\rho(p_1)$ (resp. $\rho(p_2)$) is the map $\rho(E, I_E) \to \rho(E, I_E) \hat{\otimes}_{(E', I_{E'})} \rho(E, I_E)$ to the first (resp. second) component.*

THEOREM 3.18. *The functors C_*, C^* induce equivalences of categories*

$$\mathcal{C}_{\triangle}((X/D)_{m\text{-}q\text{-crys}}) \rightleftarrows \mathcal{C}_{\triangle}((X'/D)_{q\text{-crys}})$$

quasi-inverse to each other.

COROLLARY 3.19. *The functors C_*, C^* induce equivalences of categories*

$$\mathcal{C}^{\text{fp}}((X/D)_{m\text{-}q\text{-crys}}) \rightleftarrows \mathcal{C}^{\text{fp}}((X'/D)_{q\text{-crys}}),$$
$$\mathcal{C}^{\text{tors}}((X/D)_{m\text{-}q\text{-crys}}) \rightleftarrows \mathcal{C}^{\text{tors}}((X'/D)_{q\text{-crys}})$$

quasi-inverse to each other.

COROLLARY 3.20. *Assume that (D, I) is a q-PD pair with $q = 1$ in D. Then the functors C_*, C^* induce equivalences of categories*

$$\mathcal{C}((X/D)_{m\text{-}q\text{-crys}}) \rightleftarrows \mathcal{C}((X'/D)_{q\text{-crys}})$$

quasi-inverse to each other.

Finally, we establish a relationship between the category of crystals on the m-q-crystalline site and the category of certain stratifications.

DEFINITION 3.21. Let (D, I) be a q-PD pair, let $J = (\phi^m)^{-1}(I)$ and let X be a p-adic formal scheme smooth and separated over D/J.

(1) We define the category $\mathcal{S}_{m,(D,I)}$ as follows. Objects are maps $\sigma \colon T \to R$, where T is a finite set and $\mathrm{Spf}(R) \subseteq X$ is an affine open formal subscheme such that the induced map $D/(p, J) \to R/p$ is standard smooth in the sense of [19, Tag 00T6]; moreover, σ is a map satisfying the following conditions:

(a) $\begin{aligned} f_\sigma \colon D_\sigma &:= D[x_t]_{t \in T}^{\wedge} \to R, \\ x_t &\mapsto \sigma(t) \end{aligned}$ is surjective.

(b) There exists a sequence $y_1, \ldots, y_r \in D[x_t]_{t \in T}^{\wedge}$ such that the kernel of f_σ can be described as the ideal $(J, (y_w)_{w \in W})$ (where $W = \{1, \ldots, r\}$), and that $\overline{y_1}, \ldots, \overline{y_r} \in D/(p, J)[x_t]_{t \in T}$ form a regular sequence.

A morphism from $\sigma\colon T \to R$ to $\sigma'\colon T' \to R'$ in $\mathcal{S}_{m,(D,I)}$ is a pair (f,g) where f is a map of sets $f\colon T \to T'$ and g is a map of p-complete rings $g\colon R \to R'$, such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{\sigma} & R \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ T' & \xrightarrow{\sigma'} & R' \end{array}$$

commutes and that the map $g^*\colon \mathrm{Spf}(R') \to \mathrm{Spf}(R)$ corresponding to g is compatible with the open immersions $\mathrm{Spf}(R') \subseteq X$, $\mathrm{Spf}(R) \subseteq X$.

Let $\sigma\colon T \to R$ and $\sigma'\colon T' \to R'$ be objects of $\mathcal{S}_{m,(D,I)}$. Their sum $\sigma \sqcup \sigma'$ is given by the map

$$T \sqcup T' \to R \mathbin{\widehat{\underset{D/J}{\otimes}}} R' \to R'',$$

where the first map is given by $t \mapsto \sigma(t) \otimes 1\,(t \in T)$, $t' \mapsto 1 \otimes \sigma'(t')\,(t' \in T')$, R'' is defined by the equality $\mathrm{Spf}(R'') = \mathrm{Spf}(R) \cap \mathrm{Spf}(R') \subseteq X$ (it is well defined as X is separated) and the second map is induced by the open immersions $\mathrm{Spf}(R'') \subseteq \mathrm{Spf}(R)$, $\mathrm{Spf}(R'') \subseteq \mathrm{Spf}(R')$. We will simply write $\mathcal{S}_{m,D}$ instead of $\mathcal{S}_{m,(D,I)}$ if no confusion arises.

(2) For $\sigma\colon T \to R \in \mathcal{S}_{m,D}$, we define the ring S_σ in the same way as in the proof of Proposition 3.10: namely, we define

$$S_\sigma = D\left\{ x_t, \frac{\phi^{m+1}(y_w)}{[p]_q} \right\}^{\wedge}_{t \in T, w \in W}$$

as the q-PD envelope of $(D\{x_t\}^{\wedge}_{t \in T}, K)$, where the ideal $K = (I, (\phi^m(y_w))_{w \in W})$. Note that it is independent of the choice of the elements $y_w\,(w \in W)$. Indeed, the ideal $(J, (y_w)_{w \in W})$ is independent of the choice when regarded as an ideal of $D\{x_t\}^{\wedge}_{t \in T}$, and so is the ideal $(I, (\phi^m(y_w))_{w \in W}) = (I, \phi^m((J, (y_w)_{w \in W})))$. Since the q-PD envelope only depends on the ideal $(I, (\phi^m(y_w))_{w \in W})$ by [5, Lemma 16.10], this gives the desired independence. In particular, we see that the construction of S_σ is functorial in $\sigma \in \mathcal{S}_{m,D}$.

(3) A *stratification* with respect to $\mathcal{S}_{m,D}$ and $\mathcal{M}_{\mathbb{\Delta}}$ is a pair

$$((M_\sigma)_{\sigma \in \mathcal{S}_{m,D}}, (\varphi_{\sigma\sigma'})_{\sigma \to \sigma'}),$$

where $M_\sigma \in \mathcal{M}_{\mathbb{\Delta}}(S_\sigma)$ and $\varphi_{\sigma\sigma'}\colon M_\sigma \mathbin{\widehat{\otimes}_{S_\sigma}} S_{\sigma'} \xrightarrow{\ \simeq\ } M_{\sigma'}$ is an isomorphism of $S_{\sigma'}$-modules satisfying the cocycle condition. If (D, I) is a q-PD pair with $q = 1$ in D, then we can also define a *stratification* with respect to $\mathcal{S}_{m,D}$ and $\mathcal{M}$ as above by replacing $\mathcal{M}_{\mathbb{\Delta}}$ with $\mathcal{M}$. We denote the category of stratifications with respect to $\mathcal{S}_{m,D}$ and $\mathcal{M}_{\mathbb{\Delta}}$ (resp. $\mathcal{M}$) by $\mathrm{Str}_{\mathbb{\Delta}}(\mathcal{S}_{m,D})$ (resp. $\mathrm{Str}(\mathcal{S}_{m,D})$).

PROPOSITION 3.22. *There is an equivalence of categories*

$$\mathcal{C}_{\underline{\Delta}}((X/D)_{m\text{-}q\text{-crys}}) \xrightarrow{\simeq} \text{Str}_{\underline{\Delta}}(\mathcal{S}_{m,D}).$$

PROOF. The functor $\mathcal{C}_{\underline{\Delta}}((X/D)_{m\text{-}q\text{-crys}}) \to \text{Str}_{\underline{\Delta}}(\mathcal{S}_{m,D})$ is given by

$$\mathcal{F} \mapsto \left((\mathcal{F}(S_\sigma))_{\sigma \in \mathcal{S}_{m,D}}, \left(\mathcal{F}(S_\sigma) \underset{S_\sigma}{\widehat{\otimes}} S_{\sigma'} \xrightarrow{\simeq} \mathcal{F}(S_{\sigma'})\right)_{\sigma \to \sigma'}\right).$$

We can define the functor $\text{Str}_{\underline{\Delta}}(\mathcal{S}_{m,D}) \to \mathcal{C}_{\underline{\Delta}}((X/D)_{m\text{-}q\text{-crys}})$ as follows. Given $((M_\sigma)_{\sigma \in \mathcal{S}_{m,D}}, (\varphi_{\sigma\sigma'})_{\sigma \to \sigma'}) \in \text{Str}_{\underline{\Delta}}(\mathcal{S}_{m,D})$ and $(E, I_E) \in (X/D)_{m\text{-}q\text{-crys}}$, we choose an affine open $\text{Spf}(R) \subseteq X$ such that $\text{Spf}(E/J_E) \to X$ factors through $\text{Spf}(R)$. There exists an open cover of $\text{Spf}(R)$ by standard opens $\text{Spf}(\widehat{R_{g_i}})$ such that each $\widehat{R_{g_i}}/p$ is standard smooth over $D/(p, J)$ as in the proof of Proposition 3.10. Then the fiber product $\text{Spf}(\widehat{R_{g_i}} \widehat{\otimes}_R E/J_E)$ is an affine open of $\text{Spf}(E/J_E)$. We denote the formal scheme $\text{Spf}(\widehat{R_{g_i}} \widehat{\otimes}_R E/J_E)$ by $\text{Spf}(\overline{E_i})$. By Lemma 3.9, there is a unique q-PD pair $(E_i, \widehat{I_E E_i}) \in (X/D)_{m\text{-}q\text{-crys}}$ for which the corresponding formal scheme $\text{Spf}(E_i)$ is an affine open formal subscheme of $\text{Spf}(E)$, and lifts $\text{Spf}(\overline{E_i})$.

First, we define $\mathcal{F}(E_i)$ for such E_i. By construction, $\text{Spf}(E_i/J_{E_i}) \to X$ factors through some affine open $\text{Spf}(\widetilde{R})$ which satisfies the condition in Definition 3.21 (1), so there exists an object in $\mathcal{S}_{m,D}$ of the form $\sigma : T \to \widetilde{R}$. Then we have a map $f_1 : D[x_t]_{t \in T}^{\wedge} \to E_i/J_{E_i}$ determined as the composition of the map $f_\sigma : D[x_t]_{t \in T}^{\wedge} \to \widetilde{R}$ and $\widetilde{R} \to E_i/J_{E_i}$. As $D[x_t]_{t \in T}^{\wedge}$ is the completion of a polynomial ring, one can choose a map $f_2 : D[x_t]_{t \in T}^{\wedge} \to E_i$ lifting f_1. As E_i is a δ-D-algebra, f_2 extends uniquely to a δ-D-algebra map $f_3 : D\{x_t\}_{t \in T}^{\wedge} \to E_i$. Since the image of y_w ($w \in W$) under f_3 belongs to J_{E_i}, the image of $\phi^m(y_w)$ belongs to I_{E_i}. As f_3 carries $K := (I, (\phi^m(y_w))_{w \in W})$ into I_{E_i} and S_σ is the q-PD envelope of $(D\{x_t\}_{t \in T}^{\wedge}, K)$, f_3 extends uniquely to a δ-D-algebra map $g : S_\sigma \to E_i$ in $(X/D)_{m\text{-}q\text{-crys}}$. We define

$$\mathcal{F}(E_i) := M_\sigma \underset{S_\sigma}{\widehat{\otimes}} E_i \in \mathcal{M}_{\underline{\Delta}}(E_i).$$

We must show that this is well defined. If we choose another affine open $\text{Spf}(\widetilde{R}')$, $\sigma' : T' \to \widetilde{R}'$ which is an object in $\mathcal{S}_{m,D}$ and a morphism $S_{\sigma'} \to E_i$, then we have a diagram

$$\begin{array}{ccc} S_\sigma & & \\ \downarrow & \searrow & \\ S_{\sigma \sqcup \sigma'} & \longrightarrow & E_i. \\ \uparrow & \nearrow & \\ S_{\sigma'} & & \end{array}$$

So we have the isomorphism

$$
\begin{aligned}
M_\sigma \underset{S_\sigma}{\widehat{\otimes}} E_i \quad &= \quad \left(M_\sigma \underset{S_\sigma}{\widehat{\otimes}} S_{\sigma \sqcup \sigma'} \right) \underset{S_{\sigma \sqcup \sigma'}}{\widehat{\otimes}} E_i \\
&\xrightarrow[\simeq]{\varphi_{\sigma, \sigma \sqcup \sigma'}} \quad M_{\sigma \sqcup \sigma'} \underset{S_{\sigma \sqcup \sigma'}}{\widehat{\otimes}} E_i \\
&\xleftarrow[\simeq]{\varphi_{\sigma', \sigma \sqcup \sigma'}} \quad \left(M_{\sigma'} \underset{S_{\sigma'}}{\widehat{\otimes}} S_{\sigma \sqcup \sigma'} \right) \underset{S_{\sigma \sqcup \sigma'}}{\widehat{\otimes}} E_i \\
&= \quad M_{\sigma'} \underset{S_{\sigma'}}{\widehat{\otimes}} E_i .
\end{aligned}
$$

Hence, $\mathcal{F}(E_i)$ is independent of the choice. Moreover, for a morphism

$$(E_i, I_{E_i}) \to (E', I_{E'})$$

in $(X/D)_{m\text{-}q\text{-crys}}$, $\mathcal{F}(E')$ is defined as in the case of $\mathcal{F}(E_i)$, and there exists a natural isomorphism

$$\mathcal{F}(E_i) \underset{E_i}{\widehat{\otimes}} E' \xrightarrow{\simeq} \mathcal{F}(E')$$

by definition.

Next we define $\mathcal{F}(E)$ for general $(E, I_E) \in (X/D)_{m\text{-}q\text{-crys}}$. Take $(E_i)_i$ for E as in the first paragraph. Then there exists a cover $(E, I_E) \to \prod_i (E_i, \overline{I_E E_i})$ in $(X/D)_{m\text{-}q\text{-crys}}$. Then we define $\mathcal{F}(E)$ as the kernel of $\prod_i \mathcal{F}(E_i) \rightrightarrows \prod_{i,j} \mathcal{F}(E_i \widehat{\otimes}_E E_j)$: as

$$\mathcal{F}(E_i) \underset{E_i}{\widehat{\otimes}} \left(E_i \underset{E}{\widehat{\otimes}} E_j \right) \cong \mathcal{F}\left(E_i \underset{E}{\widehat{\otimes}} E_j \right) \cong \mathcal{F}(E_j) \underset{E_j}{\widehat{\otimes}} \left(E_i \underset{E}{\widehat{\otimes}} E_j \right)$$

and each $\mathcal{F}(E_i)$ belongs to $\mathcal{M}_\Delta(E_i)$, we see that $\mathcal{F}(E) \in \mathcal{M}_\Delta(E)$ and that

$$\mathcal{F}(E) \underset{E}{\widehat{\otimes}} E_i \cong \mathcal{F}(E_i).$$

For a morphism $(E, I_E) \to (E', I_{E'})$ in $(X/D)_{m\text{-}q\text{-crys}}$, we have the isomorphisms

$$\mathcal{F}(E_i) \underset{E_i}{\widehat{\otimes}} \left(E_i \underset{E}{\widehat{\otimes}} E' \right) \xrightarrow{\simeq} \mathcal{F}\left(E_i \underset{E}{\widehat{\otimes}} E' \right),$$

$$\mathcal{F}\left(E_i \underset{E}{\widehat{\otimes}} E_j \right) \widehat{\otimes}_{E_i \widehat{\underset{E}{\otimes}} E_j} \left(E_i \underset{E}{\widehat{\otimes}} E_j \underset{E}{\widehat{\otimes}} E' \right) \xrightarrow{\simeq} \mathcal{F}\left(E_i \underset{E}{\widehat{\otimes}} E_j \underset{E}{\widehat{\otimes}} E' \right),$$

and these induce the isomorphism

$$\mathcal{F}(E) \underset{E}{\widehat{\otimes}} E' \xrightarrow{\simeq} \mathcal{F}(E').$$

Thus, $\mathcal{F} \in \mathcal{C}_\Delta((X/D)_{m\text{-}q\text{-crys}})$. So the functor $\mathrm{Str}_\Delta(\mathcal{S}_{m,D}) \to \mathcal{C}_\Delta((X/D)_{m\text{-}q\text{-crys}})$ can be defined by

$$((M_\sigma)_{\sigma \in \mathcal{S}_{m,D}}, (\varphi_{\sigma \sigma'})_{\sigma \to \sigma'}) \mapsto \mathcal{F}.$$

The two functors we constructed are quasi-inverse to each other. ∎

Assume that (D, I) is a q-PD pair with $q = 1$ in D. Then we can prove the result for $\mathcal{C}((X/D)_{m\text{-}q\text{-crys}})$ in the same way as Proposition 3.22.

COROLLARY 3.23. *Assume that (D, I) is a q-PD pair with $q = 1$ in D. Then there is an equivalence of categories*

$$\mathcal{C}((X/D)_{m\text{-}q\text{-crys}}) \xrightarrow{\simeq} \mathrm{Str}(\mathcal{S}_{m,D}).$$

4. The m-prismatic site and the $(m-1)$-q-crystalline site

In [5], Bhatt and Scholze showed that the q-crystalline cohomology of a smooth formal scheme X is isomorphic to the prismatic cohomology of the pullback of X by the Frobenius lift ϕ on the base ring. As the latter is isomorphic to the level 1-prismatic cohomology of X by our result, it is then natural to wonder whether the m-prismatic site can be compared with the $(m-1)$-q-crystalline site. In this section we prove that there exists an equivalence between the category of crystals on the m-prismatic site and that on the $(m-1)$-q-crystalline site.

CONSTRUCTION 4.1. Let (D, I) be a q-PD pair, let $J_q = (\phi^{m-1})^{-1}(I)$ and let $J_{\mathbb{\Delta}} = (\phi^m)^{-1}([p]_q D)$. Note that $I \subseteq \phi^{-1}([p]_q D)$ by [5, Corollary 16.8]. In particular, we have $J_q \subseteq J_{\mathbb{\Delta}}$. Let X be a p-adic formal scheme smooth and separated over D/J_q, and let $\widetilde{X}$ be $X \,\hat{\times}_{\mathrm{Spf}(D/J_q)} \mathrm{Spf}(D/J_{\mathbb{\Delta}})$, where the completion is the classical p-completion. Then we have a diagram:

$$
\begin{array}{ccccc}
\mathrm{Spf}(D) & \longleftarrow & \mathrm{Spf}(D/J_q) & \longleftarrow & X \\
\simeq \uparrow & & \uparrow & \square & \uparrow \\
\mathrm{Spf}(D) & \longleftarrow & \mathrm{Spf}(D/J_{\mathbb{\Delta}}) & \longleftarrow & \widetilde{X}.
\end{array}
$$

We can define a functor α from the $(m-1)$-q-crystalline site of X over (D, I) to the m-prismatic site of $\widetilde{X}$ over $(D, [p]_q D)$ in the following way: for an object

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_{E,q}) \to X)$$

of $(X/(D, I))_{(m-1)\text{-}q\text{-crys}}$, we define the object $\alpha(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_{E,q}) \to X)$ of $(\widetilde{X}/(D, [p]_q D))_{m\text{-}\mathbb{\Delta}}$ by

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_{E,\mathbb{\Delta}}) \xrightarrow{f} \widetilde{X}),$$

where the right map f is defined as

$$\mathrm{Spf}(E/J_{E,\mathbb{\Delta}}) \xrightarrow{g} \mathrm{Spf}(E/J_{E,q}) \underset{\mathrm{Spf}(D/J_q)}{\hat{\times}} \mathrm{Spf}(D/J_{\mathbb{\Delta}}) \to X \underset{\mathrm{Spf}(D/J_q)}{\hat{\times}} \mathrm{Spf}(D/J_{\mathbb{\Delta}}) = \widetilde{X}.$$

Here, the first map is induced by the natural surjection of the corresponding rings. This defines the functor

$$\alpha\colon (X/(D,I))_{(m-1)\text{-}q\text{-crys}} \to (\widetilde{X}/(D,[p]_q D))_{m\text{-}\mathbb{\Delta}}.$$

One can check that α is continuous. So α gives a morphism of topoi

$$\hat{\alpha}\colon (\widetilde{X}/(D,[p]_q D))^{\sim}_{m\text{-}\mathbb{\Delta}} \to (X/(D,I))^{\sim}_{(m-1)\text{-}q\text{-crys}}.$$

THEOREM 4.2. *The map $\hat{\alpha}$ induces an equivalence of categories of crystals*

$$\hat{\alpha}_*\colon \mathcal{C}_{\mathbb{\Delta}}((\widetilde{X}/(D,[p]_q D))_{m\text{-}\mathbb{\Delta}}) \xrightarrow{\simeq} \mathcal{C}_{\mathbb{\Delta}}((X/(D,I))_{(m-1)\text{-}q\text{-crys}}).$$

The same holds true for $\mathcal{C}^{\mathrm{fp}}$ and $\mathcal{C}^{\mathrm{tors}}$.

PROOF. By Proposition 3.22, we see that

$$\mathcal{C}_{\mathbb{\Delta}}((X/(D,I))_{(m-1)\text{-}q\text{-crys}}) \xrightarrow{\simeq} \mathrm{Str}_{\mathbb{\Delta}}(\mathcal{S}_{m-1,D,X}),$$

where $\mathcal{S}_{m-1,D,X}$ is the category $\mathcal{S}_{m-1,D}$ for X constructed in Definition 3.21. It remains to show that there is a natural equivalence of categories

$$\mathcal{C}_{\mathbb{\Delta}}((\widetilde{X}/(D,[p]_q D))_{m\text{-}\mathbb{\Delta}}) \xrightarrow{\simeq} \mathrm{Str}_{\mathbb{\Delta}}(\mathcal{S}_{m-1,D,X}).$$

First, we define the functor $\mathcal{C}_{\mathbb{\Delta}}((\widetilde{X}/(D,[p]_q D))_{m\text{-}\mathbb{\Delta}}) \to \mathrm{Str}_{\mathbb{\Delta}}(\mathcal{S}_{m-1,D,X})$. For any object $(\sigma\colon T \to R) \in \mathcal{S}_{m-1,D,X}$, we define $\widetilde{R}' = R \widehat{\otimes}_{D/J_q} D/J_{\mathbb{\Delta}}$. If we describe the kernel of the surjection $f_\sigma\colon D[x_t]^{\wedge}_{t\in T} \to R$ as the ideal $(J_q,(y_w)_{w\in W})$ as in Definition 3.21, then the kernel of the surjection $D[x_t]^{\wedge}_{t\in T} \to R \to \widetilde{R}'$ is the ideal $(J_{\mathbb{\Delta}},(y_w)_{w\in W})$. We can construct the q-PD envelope S_σ as in Definition 3.21. Then S_σ can be regarded as an object of $(X/(D,I))_{(m-1)\text{-}q\text{-crys}}$ as in the proof of Proposition 3.10. As $\phi^{m-1}(y_w)$ belongs to $I_{S_\sigma,q}$, we see that $y_w \in J_{S_\sigma,q}$. By the containment $J_{S_\sigma,q} \subseteq J_{S_\sigma,\mathbb{\Delta}}$, it follows that $y_w \in J_{S_\sigma,\mathbb{\Delta}}$. This gives a map $\widetilde{R}' \to S_\sigma/J_{S_\sigma,\mathbb{\Delta}}$. So we may also regard S_σ as an object of $(\widetilde{X}/(D,[p]_q D))_{m\text{-}\mathbb{\Delta}}$. Thus, we can define the functor

$$\mathcal{C}_{\mathbb{\Delta}}((\widetilde{X}/(D,[p]_q D))_{m\text{-}\mathbb{\Delta}}) \to \mathrm{Str}_{\mathbb{\Delta}}(\mathcal{S}_{m-1,D,X})$$

by

$$\mathcal{F} \mapsto \left((\mathcal{F}(S_\sigma))_{\sigma\in\mathcal{S}_{m-1,D,X}}, \left(\mathcal{F}(S_\sigma) \underset{S_\sigma}{\widehat{\otimes}} S_{\sigma'} \xrightarrow{\simeq} \mathcal{F}(S_{\sigma'}) \right)_{\sigma\to\sigma'} \right).$$

Next we define the functor

$$\mathrm{Str}_{\mathbb{\Delta}}(\mathcal{S}_{m-1,D,X}) \to \mathcal{C}_{\mathbb{\Delta}}((\widetilde{X}/(D,[p]_q D))_{m\text{-}\mathbb{\Delta}})$$

as follows. Given $((M_\sigma)_{\sigma \in \mathcal{S}_{m-1,D,X}}, (\varphi_{\sigma\sigma'})_{\sigma \to \sigma'}) \in \mathrm{Str}_{\underline{\mathbb{\Delta}}}(\mathcal{S}_{m-1,D,X})$ and

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to \widetilde{X}) \in (\widetilde{X}/(D, [p]_q D))_{m\text{-}\underline{\mathbb{\Delta}}},$$

we choose an affine open formal subscheme $\mathrm{Spf}(\widetilde{R}) \subseteq \widetilde{X}$ such that $\mathrm{Spf}(E/J_E) \to \widetilde{X}$ factors through $\mathrm{Spf}(\widetilde{R})$. Since the map $\widetilde{X} \to X$ is a closed immersion defined by the ideal $J_{\underline{\mathbb{\Delta}}}$ which is a nil ideal in D/J_q, there is an open formal subscheme $U \subseteq X$ which lifts $\mathrm{Spf}(\widetilde{R})$. Then the map $U \to \mathrm{Spf}(D/J_q)$ is a smooth morphism which lifts the morphism $\mathrm{Spf}(\widetilde{R}) \to \mathrm{Spf}(D/J_{\underline{\mathbb{\Delta}}})$. By Lemma 2.9, we see that U is affine. We denote the formal scheme U by $\mathrm{Spf}(R)$. Then we have a diagram:

$$
\begin{array}{ccccc}
\mathrm{Spf}(R) & \hookrightarrow & X & \longrightarrow & \mathrm{Spf}(D/J_q) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Spf}(E/J_E) & \longrightarrow & \mathrm{Spf}(\widetilde{R}) \hookrightarrow \widetilde{X} & \longrightarrow & \mathrm{Spf}(D/J_{\underline{\mathbb{\Delta}}}).
\end{array}
$$

There exists an open cover $\mathrm{Spf}(R) = \bigcup_j \mathrm{Spf}(\widehat{R_{g_i}})$ such that each $\widehat{R_{g_i}}/p$ is standard smooth over $D/(p, J_q)$ as in the proof of Proposition 3.22. Then the fiber product $\mathrm{Spf}(\widehat{R_{g_i}} \, \widehat{\otimes}_R \, \widetilde{R})$ is an affine open of $\mathrm{Spf}(\widetilde{R})$. We denote the formal scheme $\mathrm{Spf}(\widehat{R_{g_i}} \, \widehat{\otimes}_R \, \widetilde{R})$ by $\mathrm{Spf}(\widetilde{R}_i)$. We also have the fiber product $\mathrm{Spf}(\widetilde{R}_i \, \widehat{\otimes}_{\widetilde{R}} \, E/J_E)$ which is an affine open of $\mathrm{Spf}(E/J_E)$. We denote the formal scheme $\mathrm{Spf}(\widetilde{R}_i \, \widehat{\otimes}_{\widetilde{R}} \, E/J_E)$ by $\mathrm{Spf}(\overline{E_i})$. By Lemma 2.10, there is a unique bounded prism $(E_i, [p]_q E_i) \in (\widetilde{X}/(D, [p]_q D))_{m\text{-}\underline{\mathbb{\Delta}}}$ for which the corresponding formal scheme $\mathrm{Spf}(E_i)$ is an affine open formal subscheme of $\mathrm{Spf}(E)$, and lifts $\mathrm{Spf}(\overline{E_i})$.

First, we define $\mathcal{F}(E_i)$ for such E_i. By construction, there exists an object in $\mathcal{S}_{m-1,D,X}$ of the form $\sigma \colon T \to \widehat{R_{g_i}}$. Let S_σ be the q-PD envelope constructed from σ in Definition 3.21 (2). Explicitly, the ring S_σ is given by $D\{x_t, \frac{\phi^m(y_w)}{[p]_q}\}^{\wedge}_{t \in T, w \in W}$, where $(y_w)_{w \in W}$ is a sequence of elements in $D[x_t]^{\wedge}_{t \in T}$ in Definition 3.21 (1). Then, noting that $(\phi^m(y_w))_{w \in W}$ is $(p, [p]_q)$-completely regular relative to D by an argument similar to that in Corollary 2.13, we see that S_σ is the prismatic envelope of

$$(D\{x_t\}^{\wedge}_{t \in T}, ([p]_q, (\phi^m(y_w))_{w \in W})).$$

In particular, we can regard S_σ as an object of $(\widetilde{X}/(D, [p]_q D))_{m\text{-}\underline{\mathbb{\Delta}}}$, and then we can construct a morphism $S_\sigma \to E_i$ in $(\widetilde{X}/(D, [p]_q D))_{m\text{-}\underline{\mathbb{\Delta}}}$ as in the proof of Proposition 2.11. We define

$$\mathcal{F}(E_i) := M_\sigma \, \underset{S_\sigma}{\widehat{\otimes}} \, E_i \in \mathcal{M}_{\underline{\mathbb{\Delta}}}(E_i).$$

We can also define $\mathcal{F}(E)$ for general $(E, [p]_q E) \in (\widetilde{X}/(D, [p]_q D))_{m\text{-}\underline{\mathbb{\Delta}}}$ as in the proof of Proposition 3.22.

We can prove that the presheaf $\mathcal{F}$ is well defined and that it defines an object of $\mathcal{C}_{\underline{\mathbb{\Delta}}}((\widetilde{X}/(D, [p]_q D))_{m\text{-}\underline{\mathbb{\Delta}}})$ as in the proof of Proposition 3.22. So the functor

$$\text{Str}_{\underline{\mathbb{\Delta}}}(\mathcal{S}_{m-1,D,X}) \to \mathcal{C}_{\underline{\mathbb{\Delta}}}((\widetilde{X}/(D, [p]_q D))_{m\text{-}\underline{\mathbb{\Delta}}})$$

is defined by

$$((M_\sigma)_{\sigma \in \mathcal{S}_{m-1,D,X}}, (\varphi_{\sigma\sigma'})_{\sigma \to \sigma'}) \mapsto \mathcal{F}.$$

The two functors we constructed are quasi-inverse to each other. Hence, the category $\mathcal{C}_{\underline{\mathbb{\Delta}}}((\widetilde{X}/(D, [p]_q D))_{m\text{-}\underline{\mathbb{\Delta}}})$ is equivalent to the category $\text{Str}_{\underline{\mathbb{\Delta}}}(\mathcal{S}_{m-1,D,X})$, as desired. ∎

5. Relation to the Frobenius descent

The equivalences of categories of crystals proved in the previous sections are modeled on the Frobenius descent, which is due to Berthelot: there exists an equivalence between the category of crystals on the m-crystalline site and that on the usual crystalline site. However, the Frobenius descent was proved by identifying the crystals with the stratifications. It did not follow from a certain equivalence of topoi because the m-crystalline site was not suitable enough to apply the site-theoretic argument.

In this section, we first give an alternative, site-theoretic proof of the Frobenius descent in a certain setting. Our strategy is to suitably modify the definition of the m-crystalline site without changing the category of crystals. Then we apply the site-theoretic argument in the previous sections to this modified site. Next, assuming that (D, I) is a q-PD pair with $q = 1$ in D and $p \in I$, we use the modified version of the m-crystalline site to prove that the equivalence between the category of crystals on the m-q-crystalline site and that on the usual q-crystalline site in Section 3 is compatible with the Frobenius descent.

First, we recall the definition of the m-PD ring.

DEFINITION 5.1. For a $\mathbf{Z}_{(p)}$-ring D and an ideal J of D, an m-PD structure on J is a PD ideal (I, γ) of D satisfying the following conditions:

(1) $J^{(p^m)} + pJ \subseteq I \subseteq J$, where $J^{(p^m)}$ is the ideal generated by x^{p^m} for all elements x of J.

(2) The PD structure γ is compatible with the unique one on $p\mathbf{Z}_{(p)}$.

We call the triple (J, I, γ) an m-PD ideal of D and the quadruple (D, J, I, γ) an m-PD ring.

REMARK 5.2. Let us make some remarks on Definition 5.1:

(1) We warn the reader that the use of the letters I, J in Definition 5.1 is reversed, compared to the one in [2]. We prefer to use the notation in Definition 5.1 because it is compatible with the one in Section 3.

(2) When $m = 0$, the notion of an m-PD structure on an ideal J is nothing but a PD structure on J compatible with the unique one on $p\mathbf{Z}_{(p)}$.

Next we recall the definition of the m-crystalline site. Note that in this paper, we consider the site which is "affine", "big", with respect to the "flat topology", and possibly over the "p-adic base". So it is not exactly the same as the original definition.

DEFINITION 5.3. Let (D, J, I, γ) be a p-torsion-free p-complete m-PD ring or an m-PD ring in which p is nilpotent, and suppose that the ideal I contains p. Let X be a scheme smooth and separated over D/J. We define the *m-crystalline site* $(X/D)_{m\text{-crys}}$ of X over D as follows. Objects are maps $(D, J, I, \gamma) \to (E, J_E, I_E, \gamma_E)$ of m-PD rings together with a map $\text{Spec}(E/J_E) \to X$ over D/J satisfying the following conditions:

(1) There exists some $n \geq 0$ such that $p^n E = 0$.

(2) $\text{Spec}(E/J_E) \to X$ factors through some affine open $\text{Spec}(R) \subseteq X$.

We will often denote such an object by

$$(\text{Spec}(E) \leftarrow \text{Spec}(E/J_E) \to X) \in (X/D)_{m\text{-crys}}$$

or (E, J_E, I_E, γ_E) if no confusion arises. A morphism

$$(\text{Spec}(E') \leftarrow \text{Spec}(E'/J_{E'}) \to X) \to (\text{Spec}(E) \leftarrow \text{Spec}(E/J_E) \to X)$$

is a map of m-PD rings $(E, J_E, I_E, \gamma_E) \to (E', J_{E'}, I_{E'}, \gamma_{E'})$ over (D, J, I, γ) such that the induced morphism $\text{Spec}(E'/J_{E'}) \to \text{Spec}(E/J_E)$ is compatible with the maps $\text{Spec}(E'/J_{E'}) \to X$, $\text{Spec}(E/J_E) \to X$. When we denote such an object by (E, J_E, I_E, γ_E), we will write

$$(E, J_E, I_E, \gamma_E) \to (E', J_{E'}, I_{E'}, \gamma_{E'})$$

(not $(E', J_{E'}, I_{E'}, \gamma_{E'}) \to (E, J_E, I_E, \gamma_E)$) for a morphism from $(E', J_{E'}, I_{E'}, \gamma_{E'})$ to (E, J_E, I_E, γ_E).

A map $(E, J_E, I_E, \gamma_E) \to (E', J_{E'}, I_{E'}, \gamma_{E'})$ in $(X/D)_{m\text{-crys}}$ is a cover if it is a faithfully flat map and satisfies $J_{E'} = J_E E'$, $I_{E'} = I_E E'$.

We define the categories of crystals on $(X/D)_{m\text{-crys}}$ as follows.

DEFINITION 5.4. We keep the notation above:

(1) Let $\mathcal{C}^{\mathrm{qcoh}}((X/D)_{m\text{-crys}})$ be the category of abelian sheaves $\mathcal{F}$ on the m-crystalline site $(X/D)_{m\text{-crys}}$ such that, for any object (E, J_E, I_E, γ_E) in $(X/D)_{m\text{-crys}}$, $\mathcal{F}(E, J_E, I_E, \gamma_E)$ is an E-module, and for any morphism

$$(E, J_E, I_E, \gamma_E) \to (E_1, J_{E_1}, I_{E_1}, \gamma_{E_1})$$

in $(X/D)_{m\text{-crys}}$, $\mathcal{F}(E, J_E, I_E, \gamma_E) \to \mathcal{F}(E_1, J_{E_1}, I_{E_1}, \gamma_{E_1})$ is compatible with the module structures in the usual sense and the canonical map

$$\mathcal{F}(E, J_E, I_E, \gamma_E) \underset{E}{\otimes} E_1 \to \mathcal{F}(E_1, J_{E_1}, I_{E_1}, \gamma_{E_1})$$

is an isomorphism of E_1-modules.

(2) Let $\mathcal{C}^{\mathrm{tors}}((X/D)_{m\text{-crys}})$ be the category of abelian sheaves defined by

$$\mathcal{C}^{\mathrm{tors}}((X/D)_{m\text{-crys}}) = \bigcup_n \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}}),$$

where $\mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}})$ is the full subcategory of $\mathcal{C}^{\mathrm{qcoh}}((X/D)_{m\text{-crys}})$ consisting of p^n-torsion objects. We see that $\mathcal{C}^{\mathrm{tors}}((X/D)_{m\text{-crys}})$ is also a full subcategory of $\mathcal{C}^{\mathrm{qcoh}}((X/D)_{m\text{-crys}})$.

(3) Let $\mathcal{C}^{\mathrm{fp}}((X/D)_{m\text{-crys}})$ be the full subcategory of $\mathcal{C}^{\mathrm{qcoh}}((X/D)_{m\text{-crys}})$ consisting of objects $\mathcal{F}$ such that $\mathcal{F}(E)$ is a finite projective E-module for any (E, J_E, I_E, γ_E) in $(X/D)_{m\text{-crys}}$.

REMARK 5.5. Our definition of an m-crystalline site differs from the one in [2] or in [14] the following sense:

(1) The objects in our definition are "affine" in the sense that the left two schemes appearing in the diagram

$$(\mathrm{Spec}(E) \leftarrow \mathrm{Spec}(E/J_E) \to X)$$

are always affine, while in the usual definition, one considers objects of the form

$$(T \leftarrow U \to X)$$

(with the conditions in our definition), where the schemes T, U are not necessarily affine. We also assume that the morphism $\mathrm{Spec}(E/J_E) \to X$ in the diagram above factors through some affine open $\mathrm{Spec}(R) \subseteq X$, but this condition is not imposed in the usual definition.

(2) Our site is "big" in the sense that, in the definition of an object

$$(\mathrm{Spec}(E) \leftarrow \mathrm{Spec}(E/J_E) \xrightarrow{f} X),$$

the map f may be any map of schemes, while in the usual definition, the morphism $U \to X$ in the diagram in (1) is assumed to be an open immersion.

(3) The topology in our definition is defined by flat covers, while the topology in the usual definition is defined by Zariski covers.

(4) The base ring D can be a p-torsion-free p-complete m-PD ring in our definition, while the base ring D is only allowed to be an m-PD ring in which p is nilpotent in [14].

By a standard argument, we see that the difference in (1) does not change the associated topos and the category of crystals $\mathcal{C}^{\mathrm{qcoh}}((X/D)_{m\text{-crys}})$, and that the differences in (2) and (3) do not change the category of crystals $\mathcal{C}^{\mathrm{qcoh}}((X/D)_{m\text{-crys}})$ (hence the categories $\mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}})$, $\mathcal{C}^{\mathrm{tors}}((X/D)_{m\text{-crys}})$ and $\mathcal{C}^{\mathrm{fp}}((X/D)_{m\text{-crys}})$ are also unchanged). For the difference in (4), we note that if D is a p-torsion-free p-complete m-PD ring, then there are equivalences of categories

$$\mathcal{C}^{\mathrm{qcoh}}((X/(D/p^n D))_{m\text{-crys}}) \simeq \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}}),$$
$$\mathcal{C}^{\mathrm{qcoh}}((X/D)_{m\text{-crys}}) \simeq \varprojlim_n \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}})$$
$$\simeq \varprojlim_n \mathcal{C}^{\mathrm{qcoh}}((X/(D/p^n D))_{m\text{-crys}}).$$

So one can recover the categories of crystals on $(X/(D/p^n D))_{m\text{-crys}}$'s from that on $(X/D)_{m\text{-crys}}$.

Next we review the Frobenius descent functor from the m-crystalline site to the usual crystalline site.

CONSTRUCTION 5.6. Let (D, I, γ) be a p-torsion-free p-complete PD ring or a PD ring in which p is nilpotent. Suppose that the ideal I contains p. Assume that there exists a Cartesian diagram

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\ f\ } & \mathrm{Spec}(D/I) \\
\uparrow & \square \quad (\phi^m)^* \uparrow & \\
X' & \longrightarrow & \mathrm{Spec}(D/I),
\end{array}
$$

where f is a smooth and separated map, and ϕ is the Frobenius on D/I. Note that in this situation, (D, I, I, γ) is an m-PD ring. So we can define the m-crystalline site $(\widetilde{X}/(D, I, I, \gamma))_{m\text{-crys}}$.

We define a functor $\tilde{\sigma}$ from $(\widetilde{X}/(D, I, I, \gamma))_{m\text{-crys}}$ to $(X'/(D, I, \gamma))_{\text{crys}}$ in the following way: for an object $(\mathrm{Spec}(E) \leftarrow \mathrm{Spec}(E/J_E) \to \widetilde{X})$ of $(\widetilde{X}/(D, I, I, \gamma))_{m\text{-crys}}$, we define the object $\tilde{\sigma}(\mathrm{Spec}(E) \leftarrow \mathrm{Spec}(E/J_E) \to \widetilde{X})$ of $(X'/(D, I, \gamma))_{\text{crys}}$ by

$$(\mathrm{Spec}(E) \leftarrow \mathrm{Spec}(E/I_E) \xrightarrow{f} X'),$$

where the right map f is defined as

$$\mathrm{Spec}(E/I_E) \xrightarrow{g} \mathrm{Spec}(E/J_E) \underset{\mathrm{Spec}(D/I),(\phi^m)^*}{\times} \mathrm{Spec}(D/I)$$
$$\to \widetilde{X} \underset{\mathrm{Spec}(D/I),(\phi^m)^*}{\times} \mathrm{Spec}(D/I).$$

Here, the first map g is induced by the map of rings

$$E/J_E \underset{D/I,\phi^m}{\otimes} D/I \to E/I_E, \quad e \otimes d \mapsto \phi^m(e)d.$$

This defines the functor $\tilde{\sigma}\colon (\widetilde{X}/(D, I, I, \gamma))_{m\text{-crys}} \to (X'/(D, I, \gamma))_{\text{crys}}$.

PROPOSITION 5.7. *The functor $\tilde{\sigma}$ is cocontinuous.*

PROOF. We can prove the result in the same way as Proposition 2.8. ∎

By Proposition 5.7, we obtain a morphism of topoi

$$(\widetilde{X}/(D, I, I, \gamma))^{\sim}_{m\text{-crys}} \to (X'/(D, I, \gamma))^{\sim}_{\text{crys}}.$$

By abuse of notation, we will denote it by $\tilde{\sigma}$. The morphism of topoi $\tilde{\sigma}$ induces a pullback functor of categories of crystals

$$\tilde{\sigma}^*\colon \mathcal{C}^{\mathrm{qcoh}}((X'/(D, I, \gamma))_{\text{crys}}) \to \mathcal{C}^{\mathrm{qcoh}}((\widetilde{X}/(D, I, \gamma))_{m\text{-crys}}),$$

which we call the Frobenius descent functor. The Frobenius descent of Berthelot implies that $\tilde{\sigma}^*$ is an equivalence: in fact, in the case where p is nilpotent in D, it follows by gluing its local version in [3, Corollaire 2.3.7]. In the case where D is a p-torsion-free p-complete ring, it follows from the last equivalence of Remark 5.5 (see also [3, Théorème 4.1.3]).

In this section we first give an alternative, site-theoretic proof of the equivalence $\tilde{\sigma}^*$ in a certain setting, which uses techniques similar to those in the previous sections. We will be particularly interested in the case where D is a p-torsion-free p-complete ring.

We need to suitably modify the m-crystalline site for our purposes. First, we recall the infinitesimal invariance of the category of crystals on the m-crystalline site.

CONSTRUCTION 5.8. Let (D, J, I, γ) be a p-torsion-free p-complete m-PD ring or an m-PD ring in which p is nilpotent. Suppose that the ideal I contains p. Let (J_1, I_1, γ_1) be an m-PD subideal of (J, I, γ). Assume that there exists a Cartesian diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ f\ } & \mathrm{Spec}(D/J_1) \\
\uparrow & \square & \uparrow \\
X & \longrightarrow & \mathrm{Spec}(D/J),
\end{array}
$$

where f is a smooth and separated map. Then we have a cocontinuous functor of sites $\pi \colon (X_1/(D, J_1, I_1, \gamma_1))_{m\text{-crys}} \to (X/(D, J, I, \gamma))_{m\text{-crys}}$ defined by

$$
(\mathrm{Spec}(E) \leftarrow \mathrm{Spec}(E/J_E) \to X_1) \mapsto \Big(\mathrm{Spec}(E) \leftarrow X \underset{X_1}{\times} \mathrm{Spec}(E/J_E) \to X \Big).
$$

This is well defined by the proof of [14, Proposition 2.11]. So we obtain a morphism of topoi $(X_1/(D, J_1, I_1, \gamma_1))^{\sim}_{m\text{-crys}} \to (X/(D, J, I, \gamma))^{\sim}_{m\text{-crys}}$. By abuse of notation, we will denote it by π. Note that the inverse image functor π^* is equal to the functor $i^{(m)}_{\mathrm{cris}\,*}$ in [14, Proposition 2.11], except that we work with the big site with respect to the flat topology while Miyatani worked with the small site with respect to the Zariski topology.

Then the following holds true, which is sometimes called the infinitesimal invariance of the category of crystals on the m-crystalline site:

PROPOSITION 5.9. *The map π^* induces an equivalence of categories of crystals*

$$
\mathcal{C}^{\mathrm{qcoh}}((X/(D, J, I, \gamma))_{m\text{-crys}}) \xrightarrow{\ \simeq\ } \mathcal{C}^{\mathrm{qcoh}}((X_1/(D, J_1, I_1, \gamma_1))_{m\text{-crys}}).
$$

PROOF. In the case where p is nilpotent in D, the proposition follows from [14, Corollary 2.14] (the difference between our site and Miyatani's does not cause any problem because Remark 5.5 ensures that the categories of crystals are the same). In the case where D is a p-torsion-free p-complete ring, the proposition follows from the previous case and the last equivalence of Remark 5.5. ∎

Next we relate the functor $\tilde{\sigma}$ to a functor which is more similar to the functors studied in the previous sections, by using the infinitesimal invariance of the category of crystals on the m-crystalline site.

CONSTRUCTION 5.10. Let (D, I, γ) be a p-torsion-free p-complete PD ring with $p \in I$. Set

$$
J = \mathrm{Ker}(D \twoheadrightarrow D/I \xrightarrow{\ \phi^m\ } D/I).
$$

Then (D, J, I, γ) is an m-PD ring. Assume that there exists a diagram with Cartesian squares

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\;f\;} & \mathrm{Spec}(D/I) \\
\uparrow & \square & \uparrow \\
X & \longrightarrow & \mathrm{Spec}(D/J) \\
\uparrow & \square \quad {}_{(\phi^m)^*}\uparrow & \\
X' & \longrightarrow & \mathrm{Spec}(D/I),
\end{array}
$$

where f is a smooth and separated map. Then we have functors of sites as above:

$$
\pi \colon (\tilde{X}/(D, I, I, \gamma))_{m\text{-crys}} \to (X/(D, J, I, \gamma))_{m\text{-crys}},
$$
$$
\tilde{\sigma} \colon (\tilde{X}/(D, I, I, \gamma))_{m\text{-crys}} \to (X'/(D, I, \gamma))_{\text{crys}}.
$$

We can also define the functor σ from $(X/(D, J, I, \gamma))_{m\text{-crys}}$ to $(X'/(D, I, \gamma))_{\text{crys}}$ in the following way: for an object $(\mathrm{Spec}(E) \leftarrow \mathrm{Spec}(E/J_E) \to X)$ of $(X/D)_{m\text{-crys}}$, we define the object $\sigma(\mathrm{Spec}(E) \leftarrow \mathrm{Spec}(E/J_E) \to X)$ of $(X'/D)_{\text{crys}}$ by

$$
(\mathrm{Spec}(E) \leftarrow \mathrm{Spec}(E/I_E) \xrightarrow{\;f\;} X'),
$$

where the right map f is defined as

$$
\mathrm{Spec}(E/I_E) \xrightarrow{\;g\;} \mathrm{Spec}(E/J_E) \underset{\mathrm{Spec}(D/J),(\phi^m)^*}{\times} \mathrm{Spec}(D/I)
$$
$$
\to X \underset{\mathrm{Spec}(D/J),(\phi^m)^*}{\times} \mathrm{Spec}(D/I).
$$

Here, the first map g is induced by the map of rings

$$
E/J_E \underset{D/J,\phi^m}{\otimes} D/I \to E/I_E, \quad e \otimes d \mapsto \phi^m(e)d.
$$

This defines the functor $\sigma \colon (X/(D, J, I, \gamma))_{m\text{-crys}} \to (X'/(D, I, \gamma))_{\text{crys}}$. We can check that $\sigma \circ \pi = \tilde{\sigma}$. Hence we have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{C}^{\mathrm{qcoh}}((X'/(D, I, \gamma))_{\text{crys}}) & \xrightarrow{\;\sigma^*\;} & \mathcal{C}^{\mathrm{qcoh}}((X/(D, J, I, \gamma))_{m\text{-crys}}) \\
& \searrow{}_{\tilde{\sigma}^*} & \simeq\Big\downarrow{}^{\pi^*} \\
& & \mathcal{C}^{\mathrm{qcoh}}((\tilde{X}/(D, I, I, \gamma))_{m\text{-crys}}).
\end{array}
$$

So it is enough to consider the functor σ to study the Frobenius descent. The functor σ is much closer to the functors between the sites of level m and level 0 in the previous sections.

However, the m-crystalline site is not suitable to apply the site-theoretic argument in the previous sections; for any object (E, J_E, I_E, γ_E) of $(X/D)_{m\text{-crys}}$, p is nilpotent in E, and the data (J_E, γ_E) is not uniquely determined by the pair (E, I_E). But we can overcome this difficulty by defining a variant of the m-crystalline site. Fix a non-negative integer m.

DEFINITION 5.11. Let (D, I, γ) be a PD ring with $p \in I$. We define

$$J_{\max} := \mathrm{Ker}(D \twoheadrightarrow D/I \xrightarrow{\phi^m} D/I).$$

Then $(J_{\max}, I, \gamma)$ is an m-PD ideal and for any m-PD ideal of the form (J, I, γ), we have $J \subseteq J_{\max}$. A *maximal m-PD ring* is an m-PD ring (D, J, I, γ) satisfying $J = J_{\max}$.

Note that, for a p-torsion-free p-complete maximal m-PD ring (D, J, I, γ), the data (J, γ) is uniquely determined by the pair (D, I): indeed, the ideal J is uniquely determined by I and the PD structure is uniquely determined by $\gamma_n(x) = \frac{x^n}{n!}$.

Next we define a variant of the m-crystalline site.

DEFINITION 5.12. Let (D, J, I, γ) be a p-torsion-free p-complete maximal m-PD ring with $p \in I$. Let X be a scheme smooth and separated over D/J. We define the *new m-crystalline site* $(X/D)_{m\text{-crys,new}}$ of X over D as follows. Objects are maps $(D, J, I, \gamma) \to (E, J_E, I_E, \gamma_E)$ of p-torsion-free p-complete maximal m-PD rings together with a map $\mathrm{Spec}(E/J_E) \to X$ over D/J satisfying the following condition:

($*$) $\mathrm{Spec}(E/J_E) \to X$ factors through some affine open $\mathrm{Spec}(R) \subseteq X$.

We will often denote such an object by

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spec}(E/J_E) \to X) \in (X/D)_{m\text{-crys,new}}$$

or (E, J_E, I_E, γ_E) if no confusion arises. A morphism

$$(\mathrm{Spf}(E') \leftarrow \mathrm{Spec}(E'/J_{E'}) \to X) \to (\mathrm{Spf}(E) \leftarrow \mathrm{Spec}(E/J_E) \to X)$$

is a map of maximal m-PD rings

$$(E, J_E, I_E, \gamma_E) \to (E', J_{E'}, I_{E'}, \gamma_{E'})$$

over (D, J, I, γ) such that the induced morphism $\mathrm{Spec}(E'/J_{E'}) \to \mathrm{Spec}(E/J_E)$ is compatible with the maps $\mathrm{Spec}(E'/J_{E'}) \to X$, $\mathrm{Spec}(E/J_E) \to X$. When we denote such an object by (E, J_E, I_E, γ_E), we will write $(E, J_E, I_E, \gamma_E) \to (E', J_{E'}, I_{E'}, \gamma_{E'})$ (not $(E', J_{E'}, I_{E'}, \gamma_{E'}) \to (E, J_E, I_E, \gamma_E)$) for a morphism from $(E', J_{E'}, I_{E'}, \gamma_{E'})$ to (E, J_E, I_E, γ_E). A map $(E, J_E, I_E, \gamma_E) \to (E', J_{E'}, I_{E'}, \gamma_{E'})$ in $(X/D)_{m\text{-crys,new}}$ is a cover if it is a p-completely faithfully flat map and satisfies

$I_{E'} = I_E E'$ (but we do not require that $J_{E'} = J_E E'$). (Note that $I_E E'$ is always closed in p-adic topology because $p \in I_E E'$.)

When $m = 0$, we denote the site $(X/D)_{m\text{-crys,new}}$ simply by $(X/D)_{\text{crys,new}}$ and call it the *new crystalline site*.

The new m-crystalline site is much closer to our notion of sites of higher level in the previous sections. We will prove equivalence between the category of crystals on the new m-crystalline site and that on the new crystalline site by the site-theoretic argument as in the previous sections. We first construct a functor between these sites.

CONSTRUCTION 5.13. Let X' be $X \times_{\text{Spec}(D/J),(\phi^m)^*} \text{Spec}(D/I)$. Then we have the functor of sites

$$\sigma_{\text{new}} \colon (X/D)_{m\text{-crys,new}} \to (X'/D)_{\text{crys,new}}$$

that sends an object $(\text{Spf}(E) \leftarrow \text{Spec}(E/J_E) \to X)$ of $(X/D)_{m\text{-crys,new}}$ to an object $(\text{Spf}(E) \leftarrow \text{Spec}(E/I_E) \xrightarrow{f} X')$ of $(X'/D)_{\text{crys,new}}$, where the right map f is defined as

$$\text{Spec}(E/I_E) \xrightarrow{g} \text{Spec}(E/J_E) \underset{\text{Spec}(D/J),(\phi^m)^*}{\times} \text{Spec}(D/I)$$

$$\to X \underset{\text{Spec}(D/J),(\phi^m)^*}{\times} \text{Spec}(D/I).$$

Here, the first map g is induced by the map of rings

$$E/J_E \underset{D/J,\phi^m}{\otimes} D/I \to E/I_E, \quad e \otimes d \mapsto \phi^m(e)d.$$

PROPOSITION 5.14. *The functor σ_{new} is cocontinuous.*

PROOF. We can prove the result in the same way as Proposition 2.8. ∎

By Proposition 5.14, we obtain a morphism of the associated topoi. By abuse of notation, we will denote it by σ_{new}. We have the following result.

PROPOSITION 5.15. *The morphism $\sigma_{\text{new}} \colon \widetilde{(X/D)}_{m\text{-crys,new}} \to \widetilde{(X'/D)}_{\text{crys,new}}$ is an equivalence of topoi.*

PROOF. We can prove that the functor of the sites $\sigma_{\text{new}} \colon (X/D)_{m\text{-crys,new}} \to (X'/D)_{\text{crys,new}}$ is fully faithful, continuous and cocontinuous in the same way as the propositions in Section 2, so the result follows from Proposition 5.18 below and [17, Proposition 4.2.1] (see also [20, Proposition 9.10]). ∎

We will need the following analogs of Lemma 2.10.

LEMMA 5.16. *Let (E, J_E, I_E, γ_E) be an m-PD ring which is an object of $(X/D)_{m\text{-crys}}$. Let $\bar{f} \colon \mathrm{Spec}(\overline{E_i}) \to \mathrm{Spec}(E/J_E)$ be an open immersion. Then there is a unique open immersion $f \colon \mathrm{Spec}(E_i) \to \mathrm{Spec}(E)$ which lifts $\bar{f}$. Moreover, there exists a PD structure γ_{E_i} on $I_E E_i$ such that the corresponding map of rings induces a map $(E, J_E, I_E, \gamma_E) \to (E_i, J_E E_i, I_E E_i, \gamma_{E_i})$ in $(X/D)_{m\text{-crys}}$.*

PROOF. As J_E is a nil ideal, the existence, the uniqueness and the affinity of the lifting follow from Lemma 2.9. We denote the lifting by

$$f \colon \mathrm{Spec}(E_i) \to \mathrm{Spec}(E).$$

As E_i is flat over E, by [4, Corollary 3.22], there exists (uniquely) a PD structure γ_{E_i} on $I_E E_i$ such that the corresponding map of rings induces a map

$$(E, I_E, \gamma_E) \to (E_i, I_E E_i, \gamma_{E_i})$$

of PD rings. This induces a map $(E, J_E, I_E, \gamma_E) \to (E_i, J_E E_i, I_E E_i, \gamma_{E_i})$, as required. ∎

LEMMA 5.17. *Let (E, J_E, I_E, γ_E) be a p-torsion-free p-complete maximal m-PD ring which is an object of $(X/D)_{m\text{-crys,new}}$. Let $\bar{f}$ be an open immersion*

$$\bar{f} \colon \mathrm{Spec}(\overline{E_i}) \to \mathrm{Spec}(E/I_E)$$

or an open immersion

$$\bar{f} \colon \mathrm{Spec}(\overline{E_i}) \to \mathrm{Spec}(E/J_E).$$

Then there is a unique open immersion $f \colon \mathrm{Spf}(E_i) \to \mathrm{Spf}(E)$ which lifts $\bar{f}$. Moreover, there exists a PD structure γ_{E_i} on $I_E E_i$ such that the corresponding map of rings induces a map $(E, J_E, I_E, \gamma_E) \to (E_i, J_{E_i}, I_E E_i, \gamma_{E_i})$ in $(X/D)_{m\text{-crys,new}}$.

PROOF. As the images of J_E and I_E in E/pE are nil ideals, the existence, the uniqueness and the affinity of the lifting follow from Lemma 2.9. We denote the lifting by

$$f \colon \mathrm{Spf}(E_i) \to \mathrm{Spf}(E).$$

Note that $I_E/p^n E$ has naturally the PD structure induced by γ_E. Then, as

$$E/p^n E \to E_i/p^n E_i$$

is flat, by [4, Corollary 3.22], there exists (uniquely) a PD structure on $I_E E_i/p^n E_i$ such that the map $(E/p^n E, I_E/p^n E) \to (E_i/p^n E_i, I_E E_i/p^n E_i)$ is the map of PD

rings. Taking the inverse limit with respect to n, we see that there exists a PD structure γ_{E_i} on $I_E E_i$ such that the map

$$(E, I_E, \gamma_E) \to (E_i, I_E E_i, \gamma_{E_i})$$

is a map of PD rings. Also, the flatness of $E/p^n E \to E_i/p^n E_i$ for all n and the p-torsion-freeness of E imply the p-torsion-freeness of E_i. Hence the map

$$(E, J_E, I_E, \gamma_E) \to (E_i, J_{E_i}, I_E E_i, \gamma_{E_i})$$

(where $J_{E_i} := \mathrm{Ker}(E_i \twoheadrightarrow E_i/I_E E_i \xrightarrow{\phi^m} E_i/I_E E_i)$) is a map in $(X/D)_{m\text{-crys,new}}$. $\blacksquare$

PROPOSITION 5.18. *Let* $(E', J_{E'}, I_{E'}, \gamma_{E'})$ *be an object in* $(X'/D)_{\text{crys,new}}$. *Then there exists an object* (E, J_E, I_E, γ_E) *in* $(X/D)_{m\text{-crys,new}}$ *and a cover of the form*

$$(E', J_{E'}, I_{E'}, \gamma_{E'}) \to \sigma_{\text{new}}(E, J_E, I_E, \gamma_E).$$

PROOF. By Lemma 5.17 and an argument as in the proof of Proposition 2.11, we may assume that the structure morphism $\mathrm{Spec}(E'/I_{E'}) \to X'$ factors through an affine open $\mathrm{Spec}(R') \subseteq X'$ such that R' is of the form $R \otimes_{D/J, \phi^m} D/I$ for some affine open $\mathrm{Spec}(R) \subseteq X$, and that the corresponding map of rings $D/J \to R$ is standard smooth in the sense of [19, Tag 00T6].

By [19, Tag 00T7], there exists a surjection $D[x_1, \ldots, x_n]^\wedge \to R$ whose kernel is the ideal $(J, y_1, \ldots, y_r)$ such that $\overline{y}_1, \ldots, \overline{y}_r \in D/J[x_1, \ldots, x_n]$ form a regular sequence. Since the image of J in D/pD is a nil ideal, the Jacobian criterion for smoothness implies that $\overline{y}_1, \ldots, \overline{y}_r \in D/pD[x_1, \ldots, x_n]$ form a regular sequence and that the quotient $\widetilde{R} := D[x_1, \ldots, x_n]/(p, y_1, \ldots, y_r)$ is smooth over D/pD. By a similar argument to that in the proof of Corollary 2.13, we see that the sequence $y_1^{p^m}, \ldots, y_r^{p^m} \in D[x_1, \ldots, x_n]^\wedge$ is p-completely regular relative to D. Then we can construct a map $D[x_1, \ldots, x_n]^\wedge \to S$ to the p-completed PD envelope with respect to the ideal $(I, y_1^{p^m}, \ldots, y_r^{p^m})$. We denote the PD ideal of S by I_S and let $J_S = \mathrm{Ker}(S \twoheadrightarrow S/I_S \xrightarrow{\phi^m} S/I_S)$. Then we have a map $R \to S/J_S$. Thus, (S, J_S, I_S, γ_S) can be regarded as an object of $(X/D)_{m\text{-crys,new}}$. By definition,

$$S = D[x_1, \ldots, x_n]^\wedge \left[\frac{y_1^{kp^m}}{k!}, \ldots, \frac{y_r^{kp^m}}{k!} \right]^\wedge_{k \in \mathbb{N}}$$

and S is p-torsion-free.

Let $\widetilde{R}'$ be $\widetilde{R} \otimes_{D/pD, \phi^m} D/pD$. Let

$$D/pD[x_1, \ldots, x_n] \to \widetilde{R}', \quad \widetilde{R}' \to R'$$

be the base changes of the surjections $D/pD[x_1, \ldots, x_n] \to \widetilde{R}$, $\widetilde{R} \to R$ along the map $D/pD \xrightarrow{\phi^m} D/pD$ respectively. Then we have a surjection $D[x_1, \ldots, x_n]^\wedge \to R'$ determined as the composition of the map $D[x_1, \ldots, x_n]^\wedge \to D/pD[x_1, \ldots, x_n]$ and the surjection $D/pD[x_1, \ldots, x_n] \to \widetilde{R}' \to R'$ constructed above. On the other hand, the kernel of the map $D[x_1, \ldots, x_n]^\wedge \to \widetilde{R}'$ is $(p, y_1', \ldots, y_r')$, where the $\overline{y_i'}$ are the images of the $\overline{y_i}$ under the map $D/pD[x_1, \ldots, x_n] \to D/pD[x_1, \ldots, x_n]$ sending $\sum_{\underline{j}} \alpha_{\underline{j}} x^{\underline{j}}$ to $\sum_{\underline{j}} \alpha_{\underline{j}}^{p^m} x^{\underline{j}}$ (here we write $\underline{j} := (j_1, \ldots, j_n)$ for the multi-index). So the kernel of the map $D[x_1, \ldots, x_n]^\wedge \to R'$ is $(I, y_1', \ldots, y_r')$. As

$$\overline{y_1}, \ldots, \overline{y_r} \in D/pD[x_1, \ldots, x_n]$$

form a regular sequence, the sequence $\overline{y_1'}, \ldots, \overline{y_r'} \in D/pD[x_1, \ldots, x_n]$ is also a regular sequence. By a similar argument to that in the proof of Corollary 2.13, the sequence $y_1', \ldots, y_r' \in D[x_1, \ldots, x_n]^\wedge$ is p-completely regular relative to D. Then we can construct a map $D[x_1, \ldots, x_n]^\wedge \to S'$ to the p-completed PD envelope with respect to the ideal $(I, y_1', \ldots, y_r')$. By definition,

$$S' = D[x_1, \ldots, x_n]^\wedge \left[\frac{y_1'^k}{k!}, \ldots, \frac{y_r'^k}{k!} \right]^\wedge_{k \in \mathbb{N}}$$

and S' is p-torsion-free.

By the relation between y_i and y_i', we have a map of D-PD rings $(\phi^m)' : S' \to S$ sending x_i to $x_i^{p^m}$. The map $(\phi^m)'$ is p-completely faithfully flat: indeed, since S, S' are p-torsion-free, it suffices to check that $S'/pS' \to S/pS$ is faithfully flat. But this map is the base change of the map $D/pD[x_1, \ldots, x_n] \to D/pD[x_1, \ldots, x_n]$ sending x_i to $x_i^{p^m}$, which is clearly faithfully flat.

Let $(E', J_{E'}, I_{E'}, \gamma_{E'}) \in (X'/D)_{\text{crys,new}}$ be as above. We have a map

$$f_1 : D[x_1, \ldots, x_n]^\wedge \to R' \to E'/I_{E'}.$$

As $D[x_1, \ldots, x_n]^\wedge$ is the completion of a polynomial ring, one can choose a map $f_2 : D[x_1, \ldots, x_n]^\wedge \to E'$ lifting f_1. By the universal property of the PD envelope, f_2 extends uniquely to a D-PD ring map $g : S' \to E'$. If we set $h : E' \to E' \widehat{\otimes}_{S'} S$ to be the base change of $(\phi^m)' : S' \to S$ along g, then by p-complete faithful flatness of $(\phi^m)'$, we see that the same holds true for h.

Then we can prove that the map h defines a morphism

$$(E', J_{E'}, I_{E'}, \gamma_{E'}) \to \sigma_{\text{new}} \left(E' \widehat{\underset{S'}{\otimes}} S, J_{E' \widehat{\underset{S'}{\otimes}} S}, I_{E' \widehat{\underset{S'}{\otimes}} S}, \gamma_{E' \widehat{\underset{S'}{\otimes}} S} \right)$$

in $(X'/D)_{\text{crys,new}}$ as in the proof of Proposition 2.11, so the proposition follows. ∎

The map σ_{new} also induces equivalences between the categories of crystals:

DEFINITION 5.19. Let

$$\mathcal{C}((X/D)_{m\text{-crys,new}})$$

(resp. $\mathcal{C}^{\text{tors}}((X/D)_{m\text{-crys,new}})$, $\mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys,new}})$, $\mathcal{C}^{\text{fp}}((X/D)_{m\text{-crys,new}})$)

be the category of abelian sheaves $\mathcal{F}$ on $(X/D)_{m\text{-crys,new}}$ such that, for any object (E, J_E, I_E, γ_E) in $(X/D)_{m\text{-crys,new}}$,

$$\mathcal{F}(E) \in \mathcal{M}(E) \quad (\text{resp. } \mathcal{M}^{\text{tors}}(E), \mathcal{M}^{p^n\text{-tors}}(E), \mathcal{M}^{\text{fp}}(E)),$$

and for any morphism

$$(E, J_E, I_E, \gamma_E) \to (E_1, J_{E_1}, I_{E_1}, \gamma_{E_1})$$

in $(X/D)_{m\text{-crys,new}}$, $\mathcal{F}(E, J_E, I_E, \gamma_E) \to \mathcal{F}(E_1, J_{E_1}, I_{E_1}, \gamma_{E_1})$ is compatible with the module structures in the usual sense and the canonical map

$$\mathcal{F}(E, J_E, I_E, \gamma_E) \widehat{\underset{E}{\otimes}} E_1 \to \mathcal{F}(E_1, J_{E_1}, I_{E_1}, \gamma_{E_1})$$

is an isomorphism of E_1-modules.

THEOREM 5.20. *The map σ^*_{new} induces an equivalence of categories of crystals*

$$\mathcal{C}((X'/D)_{\text{crys,new}}) \to \mathcal{C}((X/D)_{m\text{-crys,new}}).$$

The same holds true for $\mathcal{C}^{\text{tors}}$, $\mathcal{C}^{p^n\text{-tors}}$ and $\mathcal{C}^{\text{fp}}$.

PROOF. We can prove the result in the same way as Theorem 2.25. ∎

The following corollary recovers the Frobenius descent, namely, we obtain a site-theoretic proof of the Frobenius descent in our setting.

COROLLARY 5.21. *The map σ^* induces an equivalence of categories of crystals*

$$\mathcal{C}^{\text{qcoh}}((X'/D)_{\text{crys}}) \to \mathcal{C}^{\text{qcoh}}((X/D)_{m\text{-crys}}).$$

PROOF. For $n \in \mathbb{N}$, let $\nu_n \colon (X/D)_{m\text{-crys,new}} \to (X/D)_{m\text{-crys}}$ be the functor of sites that sends an object $(\text{Spf}(E) \leftarrow \text{Spec}(E/J_E) \to X)$ to

$$(\text{Spec}(E/p^n E) \leftarrow \text{Spec}((E/p^n E)/(J_E/p^n E)) \to X).$$

The functor $(X'/D)_{\text{crys,new}} \to (X'/D)_{\text{crys}}$ can be defined in the same way. By abuse of notation, we will denote it also by ν_n. Then one can check that they are continuous. So we obtain morphisms of topoi

$$\widetilde{(X/D)}_{m\text{-crys}} \to \widetilde{(X/D)}_{m\text{-crys,new}},$$
$$\widetilde{(X'/D)}_{\text{crys}} \to \widetilde{(X'/D)}_{\text{crys,new}},$$

which we denote by $\hat{\nu}_n$. Then the $\hat{\nu}_{n,*}$ induce functors of categories of crystals

$$\mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}}) \to \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys,new}}),$$
$$\mathcal{C}^{p^n\text{-tors}}((X'/D)_{\text{crys}}) \to \mathcal{C}^{p^n\text{-tors}}((X'/D)_{\text{crys,new}}).$$

Also, we have an equality $\sigma \circ \nu_n = \nu_n \circ \sigma_{\text{new}}$ as morphisms of sites. So we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}^{p^n\text{-tors}}((X'/D)_{\text{crys}}) & \xrightarrow{\ \ \sigma^*\ \ } & \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}}) \\
\downarrow{\scriptstyle \hat{\nu}_{n,*}} & & \downarrow{\scriptstyle \hat{\nu}_{n,*}} \\
\mathcal{C}^{p^n\text{-tors}}((X'/D)_{\text{crys,new}}) & \xrightarrow{\ \sigma_{\text{new}}^*\ } & \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys,new}}),
\end{array}
$$

where the functors σ^*, σ_{new}^* are the pullback functors induced by the morphism of topoi $\sigma, \sigma_{\text{new}}$, which are defined by the cocontinuity of the corresponding functors of sites. As σ_{new}^* is an equivalence by Theorem 5.20 and the $\hat{\nu}_{n,*}$ are equivalences by Lemma 5.22 below, we conclude that the functor

$$\sigma^* \colon \mathcal{C}^{p^n\text{-tors}}((X'/D)_{\text{crys}}) \to \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}})$$

is an equivalence. By taking the projective limit over n and using the last equivalence in Remark 5.5, we obtain the required equivalence

$$\sigma^* \colon \mathcal{C}^{\text{qcoh}}((X'/D)_{\text{crys}}) \to \mathcal{C}^{\text{qcoh}}((X/D)_{m\text{-crys}}). \qquad \blacksquare$$

LEMMA 5.22. *The functors*

$$\hat{\nu}_{n,*} \colon \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}}) \to \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys,new}}) \quad (n \in \mathbf{N}),$$
$$\hat{\nu}_{n,*} \colon \mathcal{C}^{p^n\text{-tors}}((X'/D)_{\text{crys}}) \to \mathcal{C}^{p^n\text{-tors}}((X'/D)_{\text{crys,new}}) \qquad (n \in \mathbf{N})$$

in the proof of Corollary 5.21 are equivalences which are compatible with the natural inclusion functors

$$\iota_n \colon \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}}) \to \mathcal{C}^{p^{n+1}\text{-tors}}((X/D)_{m\text{-crys}}),$$
$$\iota_n \colon \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys,new}}) \to \mathcal{C}^{p^{n+1}\text{-tors}}((X/D)_{m\text{-crys,new}}).$$

For the proof of Lemma 5.22, we will need the notion of stratifications for the new m-crystalline site $(X/D)_{m\text{-crys,new}}$, as in Section 3.

DEFINITION 5.23. Let (D, I), J, X be as above.

(1) We define the category $\mathcal{S}^{\text{new}}_{m,(D,I)}$ in the same way as in Definition 3.21. Namely, objects of $\mathcal{S}^{\text{new}}_{m,(D,I)}$ are maps $\sigma \colon T \to R$, where T is a finite set and $\mathrm{Spec}(R) \subseteq X$ is an affine open subscheme such that the induced map $D/J \to R$ is standard smooth in the sense of [19, Tag 00T6]; moreover, σ is a map satisfying the following conditions:

(a) $\begin{aligned}f_\sigma \colon D_\sigma &:= D[x_t]^{\wedge}_{t \in T} \to R, \\ x_t &\mapsto \sigma(t)\end{aligned}$ is surjective.

(b) There exists a sequence $y_1, \ldots, y_r \in D[x_t]^{\wedge}_{t \in T}$ such that the kernel of f_σ can be described as the ideal $(J, (y_w)_{w \in W})$ (where $W = \{1, \ldots, r\}$), and that $\overline{y}_1, \ldots, \overline{y}_r \in (D/J)[x_t]_{t \in T}$ form a regular sequence.

The morphism and the sum $\sigma \sqcup \sigma'$ for $\sigma, \sigma' \in \mathcal{S}^{\text{new}}_{m,(D,I)}$ are defined in the same way as in Definition 3.21. We will simply write $\mathcal{S}^{\text{new}}_{m,D}$ instead of $\mathcal{S}^{\text{new}}_{m,(D,I)}$ if no confusion arises.

(2) For $\sigma \colon T \to R \in \mathcal{S}^{\text{new}}_{m,D}$, we define S_σ in the same way as in the proof of Proposition 5.18: namely, we define

$$S_\sigma = D[x_t]^{\wedge}\left[\frac{y_w^{kp^m}}{k!}\right]^{\wedge}_{t \in T, w \in W, k \in \mathbb{N}}$$

to be the p-completed PD envelope of $D[x_t]^{\wedge}_{t \in T}$ with respect to the ideal $(I, (y_w^{p^m})_{w \in W})$. Note that it is independent of the choice of the elements y_w ($w \in W$). Indeed, the ideal $(I, (y_w^{p^m})_{w \in W}) = (I, (J, (y_w)_{w \in W})^{(p^m)})$ is independent of the choice. In particular, we see that the construction of S_σ is functorial in $\sigma \in \mathcal{S}^{\text{new}}_{m,D}$.

(3) A *stratification* with respect to $\mathcal{S}^{\text{new}}_{m,D}$ and $\mathcal{M}$ (resp. $\mathcal{M}^{p^n\text{-tors}}$) is a pair

$$((M_\sigma)_{\sigma \in \mathcal{S}^{\text{new}}_{m,D}}, (\varphi_{\sigma\sigma'})_{\sigma \to \sigma'}),$$

where $M_\sigma \in \mathcal{M}(S_\sigma)$ (resp. $\mathcal{M}^{p^n\text{-tors}}(S_\sigma)$) and $\varphi_{\sigma\sigma'} \colon M_\sigma \widehat{\otimes}_{S_\sigma} S_{\sigma'} \xrightarrow{\simeq} M_{\sigma'}$ is an isomorphism of $S_{\sigma'}$-modules satisfying the cocycle condition. We denote the category of stratifications with respect to $\mathcal{S}^{\text{new}}_{m,D}$ and $\mathcal{M}$ (resp. $\mathcal{M}^{p^n\text{-tors}}$) by $\mathrm{Str}(\mathcal{S}^{\text{new}}_{m,D})$ (resp. $\mathrm{Str}^{p^n\text{-tors}}(\mathcal{S}^{\text{new}}_{m,D})$).

We can prove the following proposition in the same way as Proposition 3.22.

PROPOSITION 5.24. *There exist equivalences of categories*

$$\mathcal{C}((X/D)_{m\text{-crys,new}}) \xrightarrow{\simeq} \mathrm{Str}(\mathcal{S}_{m,D}^{\mathrm{new}}),$$

$$\mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys,new}}) \xrightarrow{\simeq} \mathrm{Str}^{p^n\text{-tors}}(\mathcal{S}_{m,D}^{\mathrm{new}}).$$

Using this, we prove Lemma 5.22.

PROOF OF LEMMA 5.22. Let $\mathcal{S}_{m,D}^{\mathrm{new}}$, $\mathrm{Str}^{p^n\text{-tors}}(\mathcal{S}_{m,D}^{\mathrm{new}})$ be as above. Then the category $\mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys,new}})$ is equivalent to the category $\mathrm{Str}^{p^n\text{-tors}}(\mathcal{S}_{m,D}^{\mathrm{new}})$. It remains to show that there is a natural equivalence of categories

$$\mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}}) \xrightarrow{\simeq} \mathrm{Str}^{p^n\text{-tors}}(\mathcal{S}_{m,D}^{\mathrm{new}}).$$

We first define a functor $\mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}}) \to \mathrm{Str}^{p^n\text{-tors}}(\mathcal{S}_{m,D}^{\mathrm{new}})$. For an object $(\sigma\colon T \to R) \in \mathcal{S}_{m,D}^{\mathrm{new}}$, we can construct the p-completed PD envelope S_σ as in Definition 5.23. Then $S_\sigma/p^n S_\sigma$ can be regarded as an object of $(X/D)_{m\text{-crys}}$. Thus, we can define the functor by

$$\mathcal{F} \mapsto \left((\mathcal{F}(S_\sigma/p^n S_\sigma))_{\sigma \in \mathcal{S}_{m,D}^{\mathrm{new}}}, \left(\mathcal{F}(S_\sigma/p^n S_\sigma) \underset{S_\sigma}{\otimes} S_{\sigma'} \xrightarrow{\simeq} \mathcal{F}(S_{\sigma'}/p^n S_{\sigma'}) \right)_{\sigma \to \sigma'} \right).$$

Also, we can define the functor $\mathrm{Str}^{p^n\text{-tors}}(\mathcal{S}_{m,D}^{\mathrm{new}}) \to \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}})$ as follows. Given

$$((M_\sigma)_{\sigma \in \mathcal{S}_{m,D}^{\mathrm{new}}}, (\varphi_{\sigma\sigma'})_{\sigma \to \sigma'}) \in \mathrm{Str}^{p^n\text{-tors}}(\mathcal{S}_{m,D}^{\mathrm{new}})$$

and $(E, J_E, I_E, \gamma_E) \in (X/D)_{m\text{-crys}}$, we choose an affine open $\mathrm{Spec}(R) \subseteq X$ such that $\mathrm{Spec}(E/J_E) \to X$ factors through $\mathrm{Spec}(R)$. As in the proof of Proposition 3.22, there exists an open cover of $\mathrm{Spec}(R)$ by standard opens $\mathrm{Spec}(R_{g_i})$ such that each R_{g_i} is standard smooth over D/J. Then the fiber product $\mathrm{Spec}(R_{g_i} \otimes_R E/J_E)$ is an affine open of $\mathrm{Spec}(E/J_E)$. We denote the scheme $\mathrm{Spec}(R_{g_i} \otimes_R E/J_E)$ by $\mathrm{Spec}(\overline{E_i})$. By Lemma 5.16 there is a unique m-PD ring $(E_i, J_E E_i, I_E E_i, \gamma_{E_i}) \in (X/D)_{m\text{-crys}}$ for which the corresponding scheme $\mathrm{Spec}(E_i)$ is an affine open subscheme of $\mathrm{Spec}(E)$ and lifts $\mathrm{Spec}(\overline{E_i})$, namely, $E_i/J_E E_i = \overline{E_i}$.

First, we define $\mathcal{F}(E_i)$ for such E_i. By construction, there exists an object in $\mathcal{S}_{m,D}^{\mathrm{new}}$ of the form $\sigma\colon T \to R_{g_i}$. Then we can construct a morphism $S_\sigma \to E_i$ as in the proof of Proposition 3.22. We define

$$\mathcal{F}(E_i) := M_\sigma \otimes_{S_\sigma} E_i.$$

We can also define $\mathcal{F}(E)$ for general $(E, J_E, I_E, \gamma_E) \in (X/D)_{m\text{-crys}}$ as in the proof of Proposition 3.22.

We can prove that the presheaf $\mathcal{F}$ is well defined and that it defines an object of $\mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}})$ as in the proof of Proposition 3.22. So the functor

$$\text{Str}^{p^n\text{-tors}}(\mathcal{S}^{\text{new}}_{m,D}) \to \mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}})$$

can be defined by

$$((M_\sigma)_{\sigma \in \mathcal{S}^{\text{new}}_{m,D}}, (\varphi_{\sigma\sigma'})_{\sigma \to \sigma'}) \mapsto \mathcal{F}.$$

The two functors we constructed are quasi-inverse to each other. Hence, the category $\mathcal{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}})$ is equivalent to the category $\text{Str}^{p^n\text{-tors}}(\mathcal{S}^{\text{new}}_{m,D})$, as desired. ∎

Finally, we compare the m-q-crystalline site with our variant of the m-crystalline site. Assume that (D, I) is a q-PD pair with $q = 1$ in D, i.e., a derived p-complete δ-pair over $\mathbf{Z}_p$ satisfying the following conditions:

(1) For any $f \in I$, $f^p \in pI$.

(2) The pair $(D, (p))$ is a bounded prism, i.e., D is p-torsion-free. In particular, D is classically p-complete.

(3) D/I is classically p-complete.

By [5, Remark 16.3], there exists a canonical PD structure γ on I. Now suppose further that $p \in I$. Let $J = (\phi^m)^{-1}(I)$. Then we have the following result.

LEMMA 5.25. *With the notation above,*

$$J^{(p^m)} + pJ \subseteq I \subseteq J.$$

Here, $J^{(p^m)}$ denotes the ideal of D generated by x^{p^m} for all elements x of J. In particular, (J, I, γ) is an m-PD ideal.

PROOF. Set $J_i = (\phi^i)^{-1}(I)$. Then for all i, we have $p \in J_i$. For any $x \in J_i$, we see that $x^p = \phi(x) - p\delta(x) \in J_{i-1}$. So for any $x \in J$, one has $x^{p^m} \in I$. This implies that $J^{(p^m)} + pJ \subseteq I$. On the other hand, for any $x \in I$, we have $\phi^m(x) = p\phi^{m-1}(\frac{\phi(x)}{p}) \in I$. The result follows. ∎

In this situation, there is a functor of sites from the m-q-crystalline site to the new m-crystalline site by forgetting the δ-structure on the q-PD pair.

CONSTRUCTION 5.26. Assume that (D, I) is a q-PD pair with $q = 1$ in D. Suppose that the ideal I contains p. Let X be a scheme smooth and separated over D/J. Then we have the functor of sites

$$\tau \colon (X/D)_{m\text{-}q\text{-crys}} \to (X/D)_{m\text{-crys,new}}$$

that sends an object (E, I_E) of the m-q-crystalline site $(X/D)_{m\text{-}q\text{-crys}}$ to an object (E, J_E, I_E, γ_E) of the new m-crystalline site $(X/D)_{m\text{-crys,new}}$, where $J_E = (\phi^m)^{-1}(I_E)$, and γ_E is the canonical PD structure on I_E defined by [5, Remark 16.3].

PROPOSITION 5.27. *Assume that* (D, I) *is a q-PD pair with $q = 1$ in D. Suppose that the ideal I contains p. Then the functor τ is continuous.*

PROOF. Note that if $q = 1$ in D, then the $(p, [p]_q)$-completion is the same as the p-completion. So we can prove the result in the same way as Proposition 2.7. ■

So we obtain a morphism of topoi

$$\widetilde{(X/D)}_{m\text{-crys,new}} \to \widetilde{(X/D)}_{m\text{-}q\text{-crys}}.$$

We denote it by $\hat{\tau}$. Then the following holds true.

PROPOSITION 5.28. *Assume that* (D, I) *is a q-PD pair with $q = 1$ in D. Suppose that the ideal I contains p. Then $\hat{\tau}_*$ induces an equivalence of categories of crystals*

$$\mathcal{C}((X/D)_{m\text{-crys,new}}) \xrightarrow{\simeq} \mathcal{C}((X/D)_{m\text{-}q\text{-crys}}).$$

PROOF. Let $\mathcal{S}_{m,D}$, $\mathrm{Str}(\mathcal{S}_{m,D})$ be the categories in Definition 3.21. Then the category $\mathcal{C}((X/D)_{m\text{-}q\text{-crys}})$ is equivalent to the category $\mathrm{Str}(\mathcal{S}_{m,D})$. It remains to show that there is a natural equivalence of categories

$$\mathcal{C}((X/D)_{m\text{-crys,new}}) \xrightarrow{\simeq} \mathrm{Str}(\mathcal{S}_{m,D}).$$

We first define a functor

$$\mathcal{C}((X/D)_{m\text{-crys,new}}) \to \mathrm{Str}(\mathcal{S}_{m,D}).$$

For an object $(\sigma\colon T \to R) \in \mathcal{S}_{m,D}$, we can construct the q-PD envelope S_σ as in Definition 3.21. We may also regard S_σ as an object of $(X/D)_{m\text{-crys,new}}$. Thus, we can define the functor by

$$\mathcal{F} \mapsto \left((\mathcal{F}(S_\sigma))_{\sigma \in \mathcal{S}_{m,D}}, \left(\mathcal{F}(S_\sigma) \underset{S_\sigma}{\widehat{\otimes}} S_{\sigma'} \xrightarrow{\simeq} \mathcal{F}(S_{\sigma'}) \right)_{\sigma \to \sigma'} \right).$$

Also, we can define the functor $\mathrm{Str}(\mathcal{S}_{m,D}) \to \mathcal{C}((X/D)_{m\text{-crys,new}})$ as follows. Given $((M_\sigma)_{\sigma \in \mathcal{S}_{m,D}}, (\varphi_{\sigma\sigma'})_{\sigma \to \sigma'}) \in \mathrm{Str}(\mathcal{S}_{m,D})$ and $(E, J_E, I_E, \gamma_E) \in (X/D)_{m\text{-crys,new}}$, we choose an affine open $\mathrm{Spec}(R) \subseteq X$ such that $\mathrm{Spec}(E/J_E) \to X$ factors through $\mathrm{Spec}(R)$. As in the proof of Proposition 3.22, there exists an open cover of $\mathrm{Spec}(R)$ by

standard opens $\mathrm{Spec}(R_{g_i})$ such that each R_{g_i} is standard smooth over D/J. Then the fiber product $\mathrm{Spec}(R_{g_i} \otimes_R E/J_E)$ is an affine open of $\mathrm{Spec}(E/J_E)$. We denote the scheme $\mathrm{Spec}(R_{g_i} \otimes_R E/J_E)$ by $\mathrm{Spec}(\overline{E_i})$. By Lemma 5.17 there is a unique maximal m-PD ring $(E_i, J_{E_i}, I_E E_i, \gamma_{E_i}) \in (X/D)_{m\text{-crys,new}}$ for which the corresponding formal scheme $\mathrm{Spf}(E_i)$ is an affine open formal subscheme of $\mathrm{Spf}(E)$ and lifts $\mathrm{Spec}(\overline{E_i})$, namely, $E_i / J_E E_i = \overline{E_i}$.

First we define $\mathcal{F}(E_i)$ for such E_i. By construction, there exists an object in $\mathcal{S}_{m,D}$ of the form $\sigma\colon T \to R_{g_i}$. As $D\{x_t\}_{t \in T}^{\wedge}$ is the completion of a polynomial ring by [5, Lemma 2.11], one can choose a map $f_1\colon D\{x_t\}_{t \in T}^{\wedge} \to R_{g_i} \to E_i / J_{E_i}$ and a map $f_2\colon D\{x_t\}_{t \in T}^{\wedge} \to E_i$ lifting f_1. On the other hand, we have

$$S_\sigma = D\left\{x_t, \frac{\phi^{m+1}(y_w)}{p}\right\}_{t \in T, w \in W}^{\wedge}$$

by [5, Lemma 16.10]. By assumption, this is the p-completed PD envelope of $D\{x_t\}_{t \in T}^{\wedge}$ with respect to the ideal

$$(I, (\phi^m(y_w))_{w \in W}) = (I, (y_w^{p^m})_{w \in W}).$$

So f_2 extends uniquely to a D-PD ring map $f_3\colon S_\sigma \to E_i$. We define

$$\mathcal{F}(E_i) := M_\sigma \mathbin{\widehat{\underset{S_\sigma}{\otimes}}} E_i \in \mathcal{M}(E_i).$$

We can also define $\mathcal{F}(E)$ for general $(E, J_E, I_E, \gamma_E) \in (X/D)_{m\text{-crys,new}}$ as in the proof of Proposition 3.22.

We can prove that the presheaf $\mathcal{F}$ is well defined and that it defines an object of $\mathcal{C}((X/D)_{m\text{-crys,new}})$ as in the proof of Proposition 3.22. So the functor

$$\mathrm{Str}(\mathcal{S}_{m,D}) \to \mathcal{C}((X/D)_{m\text{-crys,new}})$$

can be defined by

$$((M_\sigma)_{\sigma \in \mathcal{S}_{m,D}}, (\varphi_{\sigma\sigma'})_{\sigma \to \sigma'}) \mapsto \mathcal{F}.$$

The two functors we constructed are quasi-inverse to each other. Hence, the category $\mathcal{C}((X/D)_{m\text{-crys,new}})$ is equivalent to the category $\mathrm{Str}(\mathcal{S}_{m,D})$, as desired. ∎

We have a commutative diagram:

$$
\begin{array}{ccc}
(X/D)_{m\text{-}q\text{-crys}} & \xrightarrow{\ \tau\ } & (X/D)_{m\text{-crys,new}} \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \sigma_{\text{new}}} \\
(X'/D)_{q\text{-crys}} & \xrightarrow{\ \tau\ } & (X'/D)_{\text{crys,new}}.
\end{array}
$$

On the other hand, we have an inclusion $\mathcal{C}((X/D)_{m\text{-}q\text{-crys}}) \subseteq \mathcal{C}_{\underline{\Delta}}((X/D)_{m\text{-}q\text{-crys}})$. So the above square induces the following diagram:

$$
\begin{array}{ccccc}
\mathcal{C}_{\underline{\Delta}}((X/D)_{m\text{-}q\text{-crys}}) & \longleftarrow & \mathcal{C}((X/D)_{m\text{-}q\text{-crys}}) & \xleftarrow[\hat{\tau}_*]{\simeq} & \mathcal{C}((X/D)_{m\text{-crys,new}}) \\
\simeq \uparrow {\scriptstyle \mathrm{C}^*} & & \simeq \uparrow & & \simeq \uparrow {\scriptstyle \sigma^*_{\mathrm{new}}} \\
\mathcal{C}_{\underline{\Delta}}((X'/D)_{q\text{-crys}}) & \longleftarrow & \mathcal{C}((X'/D)_{q\text{-crys}}) & \xleftarrow[\hat{\tau}_*]{\simeq} & \mathcal{C}((X'/D)_{\mathrm{crys,new}}).
\end{array}
$$

In this way, equivalence between the category of crystals on the m-q-crystalline site and that on the usual q-crystalline site in Section 3 is compatible with Frobenius descent.

6. Relation to the results of Xu, Gros–Le Stum–Quirós and Morrow–Tsuji

In this section we discuss relations between our equivalences in Sections 2 and 3 and the results of Xu [20], Gros–Le Stum–Quirós [6–8] and Morrow–Tsuji [15].

First, we establish a relation between our results and the results of Xu. Let k be a perfect field of characteristic p, and let W be the Witt ring of k. We consider the following diagram:

$$
\begin{array}{ccccc}
\mathrm{Spf}(W) & \longleftarrow & \mathrm{Spec}(W/pW) = \mathrm{Spec}(k) & \xleftarrow{\ f\ } & X \\
{\scriptstyle \phi^*} \uparrow & & {\scriptstyle \phi^*} \uparrow & \square & \uparrow \\
\mathrm{Spf}(W) & \longleftarrow & \mathrm{Spec}(k) & \longleftarrow & X',
\end{array}
$$

where the right square is the Cartesian diagram, ϕ^* is the morphism induced by the lift $\phi \colon W \to W$ of Frobenius and f is a smooth and separated map.

Let us now briefly recall some notation and results in [20]. In [20], Xu defined the category $\mathcal{E}'$ (resp. $\underline{\mathcal{E}}$) as follows: Objects are diagrams

$$
(\mathcal{T} \leftarrow T \to U) \quad (\text{resp. } (\mathcal{T} \leftarrow \underline{T} \to U)),
$$

where $\mathcal{T}$ is a flat p-adic formal W-scheme, $T \to \mathcal{T}$ (resp. $\underline{T} \to \mathcal{T}$) is the closed immersion defined by the ideal $p\mathcal{O}_{\mathcal{T}}$ (resp. the ideal $\mathrm{Ker}(\mathcal{O}_{\mathcal{T}} \to \mathcal{O}_{\mathcal{T}}/p\mathcal{O}_{\mathcal{T}} \xrightarrow{\phi} \mathcal{O}_{\mathcal{T}}/p\mathcal{O}_{\mathcal{T}})$ with ϕ the Frobenius) and $T \to U$ is an affine morphism over k to an open subscheme U of X' (resp. X). The notion of a morphism is the obvious one. We endow $\mathcal{E}'$, $\underline{\mathcal{E}}$ with the topology induced by the fppf covers of $\mathcal{T}$ (see [20, 7.13]) and denote these sites by $\mathcal{E}'_{\mathrm{fppf}}$, $\underline{\mathcal{E}}_{\mathrm{fppf}}$ respectively (Xu also considers the topology induced by the Zariski topology on $\mathcal{T}$ in [20, 7.9], which we will not consider here).

We define the categories of crystals by

$$
\mathcal{C}^{\mathrm{tors}}(\mathcal{E}'_{\mathrm{fppf}}) = \bigcup_n \mathcal{C}^{p^n\text{-tors}}(\mathcal{E}'_{\mathrm{fppf}}), \quad \mathcal{C}^{\mathrm{tors}}(\underline{\mathcal{E}}_{\mathrm{fppf}}) = \bigcup_n \mathcal{C}^{p^n\text{-tors}}(\underline{\mathcal{E}}_{\mathrm{fppf}}),
$$

where $\mathcal{C}^{p^n\text{-tors}}(\mathcal{E}'_{\text{fppf}})$, $\mathcal{C}^{p^n\text{-tors}}(\underline{\mathcal{E}}_{\text{fppf}})$ are the categories of p^n-torsion quasi-coherent crystals on $\mathcal{E}'_{\text{fppf}}$, $\underline{\mathcal{E}}_{\text{fppf}}$ respectively. (The categories $\mathcal{C}^{p^n\text{-tors}}(\mathcal{E}'_{\text{fppf}})$, $\mathcal{C}^{p^n\text{-tors}}(\underline{\mathcal{E}}_{\text{fppf}})$ are denoted by $\mathcal{C}^{\text{qcoh}}_{\text{fppf}}(\mathcal{O}_{\mathcal{E}',n})$, $\mathcal{C}^{\text{qcoh}}_{\text{fppf}}(\mathcal{O}_{\underline{\mathcal{E}},n})$ respectively in [20].)

In [20, 9.1], Xu defined a functor $\rho \colon \underline{\mathcal{E}} \to \mathcal{E}'$ that sends $(\mathcal{T} \leftarrow \underline{T} \xrightarrow{f} U)$ to $(\mathcal{T} \leftarrow T \to U')$, where the right map $f \colon T \to U'$ is defined as

$$T \xrightarrow{g} \underline{T} \underset{\text{Spec}(k),\phi^*}{\times} \text{Spec}(k) \to U \underset{\text{Spec}(k),\phi^*}{\times} \text{Spec}(k) =: U'.$$

Here the first map g is induced by the map of sheaves of rings

$$\mathcal{O}_{\underline{T}} \underset{k,\phi}{\otimes} k \to \mathcal{O}_T, \quad t \otimes a \mapsto \phi(t)a.$$

Then, in [20, Theorem 9.2], he proved that the functor ρ induces an equivalence of topoi

$$\rho \colon \tilde{\underline{\mathcal{E}}}_{\text{fppf}} \xrightarrow{\simeq} \tilde{\mathcal{E}}'_{\text{fppf}},$$

and in [20, Theorem 9.12], he proved that ρ induces an equivalence between the categories of crystals

$$\rho^* \colon \mathcal{C}^{p^n\text{-tors}}(\mathcal{E}'_{\text{fppf}}) \xrightarrow{\simeq} \mathcal{C}^{p^n\text{-tors}}(\underline{\mathcal{E}}_{\text{fppf}}).$$

We relate our result in Section 2 to his result. For a pair (E, pE) over (W, pW), we define

$$\underline{E/pE} = E/\text{Ker}(E \to E/pE \xrightarrow{\phi} E/pE)$$

(this is the ring-theoretic version of $\underline{T}$ above). Note that if $(E, I_E) = (E, pE)$ is a δ-pair over (W, pW), then

$$E/\text{Ker}(E \to E/pE \xrightarrow{\phi} E/pE) = E/\text{Ker}(E \xrightarrow{\phi} E \to E/pE) = E/J_E,$$

where J_E is the ideal considered in the level 1-prismatic site, i.e., $J_E = \phi^{-1}(I_E)$. Then we have a commutative diagram:

$$\begin{array}{ccc}
(X/W)_{1\text{-}\underline{\mathbb{\Delta}}} & \xrightarrow{\beta_1} & \underline{\mathcal{E}}_{\text{fppf}} \\
\downarrow{\scriptstyle\rho'} & & \downarrow{\scriptstyle\rho} \\
(X'/W)_{\underline{\mathbb{\Delta}}} & \xrightarrow{\beta_0} & \mathcal{E}'_{\text{fppf}},
\end{array}$$

where ρ is the functor defined by Xu, ρ' is the functor ρ defined in Section 2 and β_0, β_1 are the functors defined as

$\beta_0 \colon (\text{Spf}(E) \hookleftarrow \text{Spec}(E/pE) \to X') \mapsto (\text{Spf}(E) \hookleftarrow \text{Spec}(E/pE) \to X')$,

$\beta_1 \colon (\text{Spf}(E) \hookleftarrow \text{Spec}(E/J_E) \to X) \mapsto (\text{Spf}(E) \hookleftarrow \text{Spec}(E/J_E = \underline{E/pE}) \to X)$.

It is easy to see that the functors β_0, β_1 are continuous. So the functors above induce a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{C}^{p^n\text{-tors}}((X/W)_{1\text{-}\underline{\Delta}}) & \xleftarrow{\hat{\beta}_{1,*}} & \mathcal{C}^{p^n\text{-tors}}(\underline{\mathcal{E}}_{\text{fppf}}) \\
\simeq \uparrow \rho'^* & & \simeq \uparrow \rho^* \\
\mathcal{C}^{p^n\text{-tors}}((X'/W)_{\underline{\Delta}}) & \xleftarrow{\hat{\beta}_{0,*}} & \mathcal{C}^{p^n\text{-tors}}(\mathcal{E}'_{\text{fppf}}).
\end{array}
$$

So our equivalence ρ'^* is compatible with Xu's equivalence ρ^*.

If X/k lifts to $\widetilde{X}/W$ and the relative Frobenius of X lifts to $\widetilde{X}$, then we have the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathrm{MIC}(\widetilde{X}/W)^{\text{qn}} & \xleftarrow[\simeq]{\psi} & \mathcal{C}^{\text{tors}}((X/W)_{\text{crys}}) & \xleftarrow[\simeq]{\lambda} & \mathcal{C}^{\text{tors}}((X/W)_{1\text{-}\underline{\Delta}}) & \xleftarrow{\hat{\beta}_{1,*}} & \mathcal{C}^{\text{tors}}(\underline{\mathcal{E}}_{\text{fppf}}) \\
\simeq \uparrow \varphi & & & & \simeq \uparrow \rho'^* & & \simeq \uparrow \rho^* \\
p\text{-}\mathrm{MIC}(\widetilde{X}'/W)^{\text{qn}} & \xleftarrow[\simeq]{\varpi} & \text{--------} & & \mathcal{C}^{\text{tors}}((X'/W)_{\underline{\Delta}}) & \xleftarrow{\hat{\beta}_{0,*}} & \mathcal{C}^{\text{tors}}(\mathcal{E}'_{\text{fppf}}).
\end{array}
$$

Here, $\mathrm{MIC}(\widetilde{X}/W)$ (resp. $p\text{-}\mathrm{MIC}(\widetilde{X}'/W)$) denotes the category of p-power torsion quasi-coherent $\mathcal{O}_X$-modules with quasi-nilpotent integrable connection (resp. p-connection) relative to W. The functor λ is the composition

$$
\mathcal{C}^{\text{tors}}((X/W)_{1\text{-}\underline{\Delta}}) \to \mathcal{C}^{\text{tors}}((X/W)_{q\text{-crys}}) \to \mathcal{C}^{\text{tors}}((X/W)_{\text{crys,new}}) \to \mathcal{C}^{\text{tors}}((X/W)_{\text{crys}}),
$$

where the first functor is the equivalence

$$
\alpha^* \colon \mathcal{C}^{\text{tors}}((X/W)_{1\text{-}\underline{\Delta}}) \xrightarrow{\simeq} \mathcal{C}^{\text{tors}}((X/W)_{q\text{-crys}})
$$

in Theorem 4.2, the second functor is the inverse of the functor (note that (W, pW) is a q-PD pair with $q = 1$ in W)

$$
\hat{\tau}_* \colon \mathcal{C}^{\text{tors}}((X/W)_{\text{crys,new}}) \xrightarrow{\simeq} \mathcal{C}^{\text{tors}}((X/W)_{q\text{-crys}})
$$

in Proposition 5.28 and the third functor is the inverse of the functor

$$
\hat{\nu}_* \colon \mathcal{C}^{\text{tors}}((X/W)_{\text{crys}}) \xrightarrow{\simeq} \mathcal{C}^{\text{tors}}((X/W)_{\text{crys,new}})
$$

in Lemma 5.22. The functor φ is the equivalence constructed in [18, Proposition 2.5]. The functor ψ is a natural equivalence in the theory of crystalline sites. By composing the equivalences in the above commutative diagram, we obtain an equivalence ϖ from the category of crystals on the prismatic site to the category of modules with integrable

p-connection. We note that the equivalence ϖ has been established in a more general situation by Ogus.

Next we establish a relation between the functors we constructed and the twisted Simpson correspondence by Gros–Le Stum–Quirós. Let (D, I) be a q-PD pair with $I = \phi^{-1}([p]_q D)$. Assume that there exists a pushout diagram

$$
\begin{array}{ccccc}
D & \longrightarrow & D[x]^\wedge & \xrightarrow{\ f\ } & A \\
\downarrow{\scriptstyle \phi} & & & & \downarrow \\
D & \longrightarrow & & & A',
\end{array}
$$

where the map f is (p, I)-completely étale. We give a δ-structure on $D[x]^\wedge$ by $\delta(x) = 0$. By [5, Lemma 2.18], the ring A admits a unique δ-structure compatible with the one on $D[x]^\wedge$. Set

$$
\bar{A} = A \underset{D}{\otimes} D/I,
$$
$$
\bar{A}' = A' \underset{D}{\otimes} D/[p]_q D.
$$

Then by [7, Theorem 4.8 and Proposition 6.9], we have the following commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{C}_{\mathbb{\Delta}}((\bar{A}/D)_{1\text{-}\mathbb{\Delta}}) & \xrightarrow[\hat{\alpha}_*]{\ \simeq\ } & \mathcal{C}_{\mathbb{\Delta}}((\bar{A}/D)_{q\text{-crys}}) & \xrightarrow{\ G\ } & \widehat{\mathrm{Strat}}_q^{(0)}(A/D) \\
{\scriptstyle \simeq}\uparrow{\scriptstyle \rho^*} & & & & {\scriptstyle \simeq}\uparrow{\scriptstyle F^*} \\
\mathcal{C}_{\mathbb{\Delta}}((\bar{A}'/D)_{\mathbb{\Delta}}) & & \xrightarrow{\hspace{3cm} H \hspace{3cm}} & & \widehat{\mathrm{Strat}}_q^{(-1)}(A'/D),
\end{array}
$$

where $\widehat{\mathrm{Strat}}_q^{(0)}(A/D)$ (resp. $\widehat{\mathrm{Strat}}_q^{(-1)}(A'/D)$) is the category of twisted hyper-stratified A-modules of level 0 (resp. A'-modules of level (-1)) defined in [7, Definition 3.9], $\hat{\alpha}_*$ and ρ^* are the functors we constructed, G and H are the functors that appeared in [7, Proposition 6.9] and F^* is an equivalence in [7, Theorem 4.8]. (More precisely, it is not clear to us which category of modules they used in their definition of $\widehat{\mathrm{Strat}}_q^{(0)}(A/D)$ and $\widehat{\mathrm{Strat}}_q^{(-1)}(A'/D)$. We guess that it would be reasonable to use the categories $\mathcal{M}_{\mathbb{\Delta}}(A)$ and $\mathcal{M}_{\mathbb{\Delta}}(A')$ respectively. So the categories of twisted hyper-stratified modules would be related to our equivalences of categories of crystals with notation $\mathcal{C}_{\mathbb{\Delta}}$.)

Thus, we see that our equivalences of categories fit into the diagram in [7, Proposition 6.9]. In particular, our argument gives a direct proof of the equivalence $\hat{\alpha}_* \circ \rho^*$, which they plan to prove indirectly in forthcoming work by showing that G and H are equivalences (see [7, Remark 2 after Proposition 6.9]). Moreover, our proof of the equivalence $\hat{\alpha}_* \circ \rho^*$, which is in the style of [17, 20], answers "the hope" in [7, Remark 3 after Proposition 6.9] to some extent.

Finally, we establish a relation between our results and the results of Morrow–Tsuji in [15]. Let us briefly recall some notation and results in [15]. Let $\mathcal{O}$ be a ring of integers of a characteristic 0 perfectoid field containing all p-power roots of unity, and let

$$A_{\inf} := W(\mathcal{O}^\flat), \ \ \varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}^\flat, \ \ \mu := [\varepsilon] - 1, \ \ \xi := \frac{\mu}{\phi^{-1}(\mu)}, \ \ \tilde{\xi} := \phi(\xi),$$

where ϕ is the Frobenius on $A_{\inf}$ (ϕ is denoted by φ in [15]). Then the pair $(A_{\inf}, (\tilde{\xi}))$ is a bounded prism.

Let R be a smooth $\mathcal{O}$-algebra with a formally étale map $\mathcal{O}[T_1^{\pm 1}, \dots, T_d^{\pm 1}]^\wedge \to R$ called a framing. We can use the framing to define the ring R_∞ with the action

$$\mathrm{Gal}(R_\infty/R) =: \Gamma \cong \mathbb{Z}_p^d,$$

and the rings $A^\square := A_{\inf}^\square(R)$, $A_\infty^\square := A_{\inf}(R_\infty)$. For an $A_{\inf}$-algebra B, we set

$$B^{(1)} := A_{\inf} \otimes_{\phi, A_{\inf}} B.$$

We denote the relative Frobenius $B^{(1)} \to B$ by F. We use the same notation for the base change of an $\mathcal{O}$-algebra along the Frobenius because we can regard any $\mathcal{O}$-algebra as an $A_{\inf}$-algebra.

Let $\mathrm{Rep}_\Gamma^\mu(A^\square)$ (resp. $\mathrm{Rep}_\Gamma^\mu(A^{\square(1)})$) be the category of generalized representations of Γ over $A^\square$ (resp. $A^{\square(1)}$) which is trivial modulo μ, and let $\mathrm{qMIC}(A^\square)$ (resp. $\mathrm{qHIG}(A^{\square(1)})$) be the category of finite projective $A^\square$-modules (resp. $A^{\square(1)}$-modules) with flat q-connection (resp. flat q-Higgs field). We will write $\mathrm{qMIC}(A^\square)^{\mathrm{qnilp}}$ (resp. $\mathrm{qHIG}(A^{\square(1)})^{\mathrm{qnilp}}$) for the full subcategory of $\mathrm{qMIC}(A^\square)$ (resp. $\mathrm{qHIG}(A^{\square(1)})$) consisting of (p, μ)-adically quasi-nilpotent objects (this notation is not used in [15]).

Then, in [15, Section 3], they considered the following commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{C}^{\mathrm{fp}}((\mathrm{Spf}(R^{(1)})/A_{\inf})_{\mathbb{\Delta}}) & \xrightarrow{\mathrm{ev}_{A^{\square(1)}}} & \mathrm{Rep}_\Gamma^\mu(A^{\square(1)}) & \xrightarrow{\simeq} & \mathrm{qHIG}(A^{\square(1)}) \\
& \searrow{\scriptstyle \mathrm{ev}^\phi_{A^{\square(1)}}} & \downarrow{\scriptstyle -\otimes_{A^{\square(1)}, F} A^\square} & & \downarrow \\
& & \mathrm{Rep}_\Gamma^\mu(A^\square) & \xrightarrow{\simeq} & \mathrm{qMIC}(A^\square),
\end{array}
$$

where $\mathrm{ev}_{A^{\square(1)}}$ is the functor defined by evaluation on

$$(\mathrm{Spf}(A^{\square(1)}) \leftarrow \mathrm{Spf}(R^{(1)}) \xrightarrow{=} \mathrm{Spf}(R^{(1)})),$$

and $\mathrm{ev}^\phi_{A^{\square(1)}}$ is the composition of the functor $\mathrm{ev}_{A^{\square(1)}}$ with $- \otimes_{A^{\square(1)}, F} A^\square$. Moreover, the arrows with $\simeq$ are equivalences.

By our result in Section 2, we see that the above diagram extends to the following diagram:

$$
\begin{array}{ccc}
\mathcal{C}^{\mathrm{fp}}((\mathrm{Spf}(R^{(1)})/A_{\mathrm{inf}})_{\mathbb{\Delta}}) \xrightarrow{\ \mathrm{ev}_{A^{\square(1)}}\ } \mathrm{Rep}_{\Gamma}^{\mu}(A^{\square(1)}) \xrightarrow{\ \simeq\ } \mathrm{qHIG}(A^{\square(1)}) \\
\downarrow{\scriptstyle\simeq} \qquad\qquad \downarrow{\scriptstyle -\otimes_{A^{\square(1)},F} A^{\square}} \qquad\qquad \downarrow \\
\mathcal{C}^{\mathrm{fp}}((\mathrm{Spf}(R)/A_{\mathrm{inf}})_{1\text{-}\mathbb{\Delta}}) \xrightarrow{\ \mathrm{ev}_{A^{\square}}\ } \mathrm{Rep}_{\Gamma}^{\mu}(A^{\square}) \xrightarrow{\ \simeq\ } \mathrm{qMIC}(A^{\square}),
\end{array}
$$

where $\mathrm{ev}_{A^{\square}}$ is the functor defined by evaluation on $(\mathrm{Spf}(A^{\square}) \leftarrow \mathrm{Spf}(R) \xrightarrow{=} \mathrm{Spf}(R))$.

In [15, Theorem 3.2], they proved that the functors $\mathrm{ev}_{A^{\square(1)}}, \mathrm{ev}_{A^{\square(1)}}^{\phi}$ are fully faithful and induce equivalences

$$
\mathcal{C}^{\mathrm{fp}}((\mathrm{Spf}(R^{(1)})/A_{\mathrm{inf}})_{\mathbb{\Delta}}) \xrightarrow{\ \simeq\ } \mathrm{qHIG}(A^{\square(1)})^{\mathrm{qnilp}},
$$

$$
\mathcal{C}^{\mathrm{fp}}((\mathrm{Spf}(R^{(1)})/A_{\mathrm{inf}})_{\mathbb{\Delta}}) \xrightarrow{\ \simeq\ } \mathrm{qMIC}(A^{\square})^{\mathrm{qnilp}}.
$$

Thus, we see that the functor $\mathrm{ev}_{A^{\square}}$ is also fully faithful and induces an equivalence

$$
\mathcal{C}^{\mathrm{fp}}((\mathrm{Spf}(R)/A_{\mathrm{inf}})_{1\text{-}\mathbb{\Delta}}) \xrightarrow{\ \simeq\ } \mathrm{qMIC}(A^{\square})^{\mathrm{qnilp}}.
$$

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