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Refinement of Gautschi's harmonic mean inequality for the gamma function

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Abstract – In 1974, W. Gautschi proved that

$$1 < \frac{2}{1/\Gamma(x) + 1/\Gamma(1/x)} \quad \text{for} \quad 0 < x \neq 1.$$

Here, we present the following refinement:

$$1 < \Gamma\Big(\frac{2}{x+1/x}\Big) < \frac{2}{1/\Gamma(x) + 1/\Gamma(1/x)}, \quad 0 < x \neq 1.$$

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1. Introduction and statement of the main result

In 1974, W. Gautschi [13] presented a remarkable inequality for the famous gamma function of Euler. He proved that for all positive real numbers x the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is greater than or equal to 1, that is,

$$1 \leq \frac{2}{1/\Gamma(x) + 1/\Gamma(1/x)}, \quad x > 0. \tag{1.1}$$

The validity of (1.1) was conjectured by V. R. Rao Uppuluri, who asked Gautschi in a private communication (April, 1972) for a proof.

Inequality (1.1) found much attention and several extensions, improvements and numerous related results were discovered; we refer to the works Alzer [1–7], Alzer and

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Jameson [8], Alzer and Salem [9], Das and Swaminathan [12], Gautschi [14], Giordano and Laforgia [15], Jameson and Jameson [16], Kershaw and Laforgia [17], Laforgia and Sismondi [18], and the references cited therein. The aim of this paper is to prove the following new refinement of (1.1).

THEOREM. *For all positive real numbers* $x \neq 1$, *we have*

$$1 < \Gamma\Big(\frac{2}{x+1/x}\Big) < \frac{2}{1/\Gamma(x)+1/\Gamma(1/x)}. \tag{1.2}$$

These inequalities can be written in a slightly more elegant form as

$$1 < \Gamma\big(H(x,1/x)\big) < H\big(\Gamma(x),\Gamma(1/x)\big), \quad 0 < x \neq 1, \tag{1.3}$$

where

$$H(x,y) = \frac{2}{1/x+1/y}$$

denotes the unweighted harmonic mean of x and y. This mean value plays an important role in various fields, for example, physics, economics and computer science. The harmonic mean and the two other classical mean values, the arithmetic mean $A(x,y) = (x+y)/2$ and the geometric mean $G(x,y) = \sqrt{xy}$, are known as the three Pythagorean means. They are connected by the formula $H(x,y) = G^2(x,y)/A(x,y)$. For more information on these and other means we refer to the monograph Bullen, Mitrinović, Vasić [11].

We note that the right-hand side of (1.3) is closely related to the concept of MN-convexity. A function $f : I \to (0,\infty)$, where $I \subset (0,\infty)$ is an interval, is called MN-convex, if

$$f\big(M(x,y)\big) \le N\big(f(x), f(y)\big)$$

for all $x, y \in I$. Here, M and N are means. Details on this subject can be found in Anderson et al. [10].

In the next section, we collect some helpful lemmas. Our proof of the Theorem is given in Section 3. The numerical computations have been carried out using the computer software Maple 13 (Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario).

2. Lemmas

The following eight lemmas are important to prove the Theorem. A proof for the first lemma can be found in Alzer [3]. We denote by $\psi = \Gamma'/\Gamma$ the digamma function.

LEMMA 1. *Let $g(x) = x\psi(x)$. The function*

$$g'(x) = \psi(x) + x\psi'(x)$$

is strictly increasing on $(0, \infty)$ *with* $g'(r_0) = 0$ *for* $r_0 = 0.21609\ldots$.

In what follows, $x_0 = 1.46163\ldots$ is the only positive zero of ψ.

LEMMA 2. *If* $1/x_0 \le x \le 1$, *then*

$$g'(1/x) - x_0\psi^2(1/x) > 0.$$

PROOF. From Lemma 1 we conclude that $x \mapsto g'(1/x)$ is decreasing on $[1/x_0, 1]$. Moreover, $x \mapsto -\psi^2(1/x)$ is decreasing on $[1/x_0, 1]$. It follows that

$$g'(1/x) - x_0\psi^2(1/x) \ge g'(1) - x_0\psi^2(1) = 0.58\ldots. \quad \blacksquare$$

LEMMA 3. *If* $0.9 \le x \le 1$, *then*

$$\frac{\psi'(x) - \psi^2(x)}{\Gamma(x)} > 1.25.$$

PROOF. Let $\sigma(x) = \psi'(x) - \psi^2(x) - 1.25\Gamma(x)$, and let $0.9 \le r \le x \le s \le 1$. Since ψ', ψ^2 and Γ are decreasing on $[0.9, 1]$, we obtain

$$\sigma(x) \ge \psi'(s) - \psi^2(r) - 1.25\Gamma(r) = Y(r, s), \quad \text{say.}$$

We set

$$r_k = 0.9 + \frac{k}{200} \quad \text{and} \quad s_k = 0.9 + \frac{k+1}{200}.$$

Then

$$[0.9, 1] = \bigcup_{k=0}^{19} [r_k, s_k].$$

It follows that if $x \in [0.9, 1]$, then there exists an integer $m \in \{0, 1, \ldots, 19\}$ such that $0.9 \le r_m \le x \le s_m \le 1$. Since

$$Y(r_k, s_k) > 0, \quad k = 0, 1, \ldots, 19,$$

we obtain $\sigma(x) > 0$ for $x \in [0.9, 1]$. $\blacksquare$

LEMMA 4. *If* $0.9 \le x \le 1$, *then*

$$\frac{-1}{x^4\Gamma(1/x)}\big(\psi^2(1/x) - 2x\psi(1/x) - \psi'(1/x)\big) > 0.111. \tag{2.1}$$

PROOF. Let $1 \le t \le 1.12$, $\alpha = 0.111$, and

$$\Delta(t) = \psi^2(t) + \frac{2}{t}\big(-\psi(t)\big) - \psi'(t) + \frac{\alpha}{t^4}\Gamma(t).$$

The functions ψ^2, $-\psi$, ψ' and Γ are decreasing and positive on $[1, 1.12]$. Let $1 \le r \le t \le s \le 1.12$. Then

$$\Delta(t) \le \psi^2(r) + \frac{2}{r}\big(-\psi(r)\big) - \psi'(s) + \frac{\alpha}{r^4}\Gamma(r) = W(r, s), \quad \text{say.}$$

Since

$$W\Big(1 + \frac{k}{100}, 1 + \frac{k+1}{100}\Big) < 0, \quad k = 0, 1, \ldots, 11,$$

we get $\Delta(t) < 0$ for $t \in [1, 1.12]$. This implies that (2.1) is valid for $x \in [0.9, 1]$. ■

LEMMA 5. *The function*

$$A(x) = \Gamma\Big(\frac{2}{x + 1/x}\Big)$$

is strictly decreasing on $(0, 1]$.

PROOF. Let $0 < x < 1$. Since $0 < 2/(x + 1/x) < 1$, we obtain

$$A'(x) = \frac{2(1 - x^2)}{(1 + x^2)^2}\Gamma'\Big(\frac{2}{x + 1/x}\Big) < 0.$$ ■

LEMMA 6. *The function*

$$B(x) = \frac{1}{\Gamma(x)} + \frac{1}{\Gamma(1/x)}$$

is strictly increasing on $(0, 1]$.

PROOF. If $h(x) = g(x)/\Gamma(x) = x\psi(x)/\Gamma(x)$, then

$$xB'(x) = h(1/x) - h(x).$$

We consider two cases.

Case 1. $0 < x \le 1/x_0 = 0.68\ldots$. Since

$$\psi(x) < 0 \le \psi(1/x),$$

we obtain

$$h(x) < 0 \le h(1/x).$$

This leads to $B'(x) > 0$.

Case 2. $1/x_0 < x < 1$. Let $C(x) = xB'(x)$. Then it holds that

$$\begin{aligned}-\frac{\Gamma(x)}{x}C'(x) &= \Gamma(x)\Big(\frac{1}{x}h'(x) + \frac{1}{x^3}h'(1/x)\Big)\\ &= \frac{1}{x}g'(x) - \psi^2(x) + w(x)\Big(g'(1/x) - \frac{1}{x}\psi^2(1/x)\Big),\end{aligned}$$

where

$$w(x) = \frac{1}{x^3}\cdot\Gamma(x)\cdot\frac{1}{\Gamma(1/x)}.$$

Let $1/x_0 \le r \le x \le s \le 1$. Applying Lemma 1 gives

$$\frac{1}{x}g'(x) \ge \frac{1}{s}g'(r) \quad\text{and}\quad g'(1/x) \ge g'(1/s). \tag{2.2}$$

The function ψ^2 is decreasing on $(0, x_0)$, so we obtain

$$-\psi^2(x) \ge -\psi^2(r) \quad\text{and}\quad \frac{1}{x}\psi^2(1/x) \le \frac{1}{x}\psi^2(1/s) \le x_0\psi^2(1/s). \tag{2.3}$$

Moreover, since w is decreasing on $[1/x_0, 1]$, we get

$$w(x) \ge w(s). \tag{2.4}$$

Using (2.2), (2.3), (2.4) and Lemma 2 gives

$$-\frac{\Gamma(x)}{x}C'(x) \ge \frac{1}{s}g'(r) - \psi^2(r) + w(s)\Big(g'(1/s) - x_0\psi^2(1/s)\Big) = Z(r,s), \quad \text{say.}$$

We have

$$Z(1/x_0, 0.9) = 0.64\ldots \quad\text{and}\quad Z(0.9, 1) = 0.98\ldots.$$

This implies that

$$C'(x) < 0 \quad\text{and}\quad C(x) > C(1) = 0.$$

Thus, $B'(x) > 0$. ■

Lemma 7. *If* $0.9 \le x < 1$, *then*

$$\frac{B'(x)}{1-x} > 1.361.$$

Proof. Let $0.9 \le x < 1$ and $\beta = 1.361$. We define

$$J(x) = B'(x) - \beta(1-x).$$

Then

$$J'(x) = B''(x) + \beta = p(x) + q(x) + \beta$$

with

$$p(x) = \frac{\psi^2(x) - \psi'(x)}{\Gamma(x)} \text{ and } q(x) = \frac{1}{x^4\Gamma(1/x)}\big(\psi^2(1/x) - 2x\psi(1/x) - \psi'(1/x)\big).$$

Applying Lemma 3 and Lemma 4 gives

$$-p(x) > 1.25 \quad \text{and} \quad -q(x) > 0.111.$$

Hence,

$$J'(x) < -1.25 - 0.111 + 1.361 = 0.$$

It follows that $J(x) > J(1) = B'(1) = 0.$ ∎

LEMMA 8. *The functions*

$$U(x) = \frac{2(1+x)}{(1+x^2)^2} \quad \textit{and} \quad V(x) = -\psi\Big(\frac{2x}{1+x^2}\Big)$$

are positive and decreasing on $[0.9, 1]$.

PROOF. Let $x \in [0.9, 1]$. Then

$$U'(x) = \frac{-2}{(1+x^2)^3}(3x^2 + 4x - 1) < 0 \text{ and } V'(x) = -\frac{2(1-x^2)}{(1+x^2)^2}\psi'\Big(\frac{2x}{1+x^2}\Big) \le 0.$$

∎

3. Proof of the Theorem

Since

$$0 < \frac{2}{x + 1/x} < 1, \quad 0 < x \neq 1,$$

we conclude that

$$1 = \Gamma(1) < \Gamma\Big(\frac{2}{x+1/x}\Big).$$

To prove the second inequality in (1.2), we define

$$F(x) = A(x)B(x)$$

with

$$A(x) = \Gamma\Big(\frac{2}{x+1/x}\Big) \quad \text{and} \quad B(x) = \frac{1}{\Gamma(x)} + \frac{1}{\Gamma(1/x)}.$$

Since $F(x) = F(1/x)$, it suffices to prove that

$$F(x) < 2, \quad 0 < x < 1.$$

We consider three cases.

Case 1. $0 < x < 0.1$. We have

$$F(x) = a(x)\big(b(x) + c(x)\big)$$

with

$$a(x) = \frac{1}{2}\Gamma\Big(1 + \frac{2x}{1+x^2}\Big), \quad b(x) = \frac{1+x^2}{\Gamma(1+x)}, \quad \text{and} \quad c(x) = \frac{1+x^2}{x^2\Gamma(1+1/x)}.$$

Since

$$a'(x) = \frac{1-x^2}{(1+x^2)^2}\Gamma'\Big(1 + \frac{2x}{1+x^2}\Big) < 0,$$

we obtain $a(x) < a(0) = 1/2$. The function

$$b(x) = (1+x^2) \cdot \frac{1}{\Gamma(1+x)}$$

is increasing, which implies that $b(x) \le b(0.1) = 1.06\ldots$. Using

$$t + \frac{1}{t} < \Gamma(t), \quad t > 10,$$

gives $c(x) < 1$. Thus,

$$F(x) < \frac{1}{2}(1.1 + 1) < 2.$$

Case 2. $0.1 \le x \le 0.9$. Let $0.1 \le r \le x \le s \le 0.9$. Applying Lemma 5 and Lemma 6 gives

$$F(x) \le A(r)B(s).$$

Since

$$A\Big(0.1 + \frac{k}{100}\Big)B\Big(0.1 + \frac{k+1}{100}\Big) < 2 \quad \text{for} \quad k = 0, 1, \ldots, 79,$$

we conclude that $F(x) < 2$.

Case 3. $0.9 < x < 1$. We have

$$\frac{F'(x)}{(1-x)\Gamma(2x/(1+x^2))} = \frac{B'(x)}{1-x} - U(x)V(x)B(x)$$

with

$$U(x) = \frac{2(1+x)}{(1+x^2)^2} \quad \text{and} \quad V(x) = -\psi\Big(\frac{2x}{1+x^2}\Big).$$

Applying Lemma 6, Lemma 7 and Lemma 8 yields

$$\frac{B'(x)}{1-x} - U(x)V(x)B(x) > 1.361 - U(0.9)V(0.9)B(1) = 0.0007\ldots.$$

This leads to

$$F'(x) > 0 \quad \text{and} \quad F(x) < F(1) = 2.$$

The proof of the Theorem is complete. ∎

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