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Cyclic forms on DG-Lie algebroids and semiregularity

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Abstract – Given a transitive DG-Lie algebroid $(\mathcal{A}, \rho)$ over a smooth separated scheme X of finite type over a field $\mathbb{K}$ of characteristic zero, we define a notion of connection $\nabla\colon \mathbf{R}\Gamma(X, \operatorname{Ker}\rho) \to \mathbf{R}\Gamma(X, \Omega^1_X[-1] \otimes \operatorname{Ker}\rho)$ and construct an L_∞-morphism between DG-Lie algebras $f\colon \mathbf{R}\Gamma(X, \operatorname{Ker}\rho) \rightsquigarrow \mathbf{R}\Gamma(X, \Omega^{\leq 1}_X[2])$ associated to a connection and to a cyclic form on the DG-Lie algebroid. In this way, we obtain a lifting of the first component of the modified Buchweitz–Flenner semiregularity map in the algebraic context, which has an application to the deformation theory of coherent sheaves on X admitting a finite locally free resolution. Another application is to the deformations of (Zariski) principal bundles on X.

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1. Introduction

Let $\mathcal{F}$ be a coherent sheaf admitting a finite locally free resolution on a smooth variety X over a field $\mathbb{K}$ of characteristic zero. The Buchweitz–Flenner semiregularity map, introduced in [7], and generalising the semiregularity map of Bloch [5], is defined by the formula

$$(1.1)\qquad \sigma\colon \operatorname{Ext}^2_X(\mathcal{F}, \mathcal{F}) \longrightarrow \prod_{q\geq 0} H^{q+2}(X, \Omega^q_X), \quad \sigma(c) = \operatorname{Tr}(\exp(-\operatorname{At}(\mathcal{F})) \circ c),$$

where Tr denotes the trace maps $\operatorname{Tr}\colon \operatorname{Ext}^i_X(\mathcal{F}, \mathcal{F} \otimes \Omega^j_X) \to H^i(X, \Omega^j_X)$ for $i, j \geq 0$, and the exponential of the opposite of the Atiyah class $\operatorname{At}(\mathcal{F}) \in \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F} \otimes \Omega^1_X)$ is

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defined via the Yoneda pairing

$$\mathrm{Ext}^i_X(\mathcal{F},\mathcal{F}\otimes\Omega^i_X)\times\mathrm{Ext}^j_X(\mathcal{F},\mathcal{F}\otimes\Omega^j_X)\longrightarrow\mathrm{Ext}^{i+j}_X(\mathcal{F},\mathcal{F}\otimes\Omega^{i+j}_X)$$
$$(a,b)\longmapsto a\circ b,$$

$$\exp(-\mathrm{At}(\mathcal{F}))\in\prod_{q\geq 0}\mathrm{Ext}^q_X(\mathcal{F},\mathcal{F}\otimes\Omega^q_X).$$

We refer to [20] for a discussion of the role of the Buchweitz–Flenner semiregularity map in deformation theory and of the reason why it is more convenient in this setting to consider the modified Buchweitz–Flenner semiregularity map, obtained as follows. Denote by $\sigma_q\colon\mathrm{Ext}^2_X(\mathcal{F},\mathcal{F})\to H^{q+2}(X,\Omega^q_X)$ the components of the semiregularity map $\sigma=\sum\sigma_q$, for every $q\geq 0$ denote by $\Omega^{\leq q}_X=(\bigoplus_{i=0}^q\Omega^i_X[-i],d_{dR})$ the truncated de Rham complex, and consider the composition

$$\tau_q\colon\mathrm{Ext}^2_X(\mathcal{F},\mathcal{F})\xrightarrow{\sigma_q}H^{q+2}(X,\Omega^q_X)=H^2(X,\Omega^q_X[q])\xrightarrow{i_q}\mathbb{H}^2(X,\Omega^{\leq q}_X[2q]),$$

where the map i_q is induced by the inclusion of complexes $\Omega^q_X[q]\subset\Omega^{\leq q}_X[2q]$. The map τ_q is the q-component of the modified Buchweitz–Flenner semiregularity map.

A lifting to an L_∞-morphism of the first component τ_1 of the modified Buchweitz–Flenner semiregularity map was constructed in [20] in the context of complex manifolds: for every connection of type $(1,0)$ (also called connections compatible with the holomorphic structure, see e.g. [16]) on a finite complex of locally free sheaves $\mathcal{E}$ on a complex manifold X, the existence of an L_∞-morphism between DG-Lie algebras

$$g\colon A^{0,*}_X(\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))\rightsquigarrow\frac{A^{*,*}_X}{A^{\geq 2,*}_X}[2]$$

was proved, whose linear component induces τ_1 in cohomology. As a consequence, recalling that $\mathrm{Ext}^2_X(\mathcal{F},\mathcal{F})$ is the obstruction space for the functor of deformations of a coherent sheaf $\mathcal{F}$, the map τ_1 annihilates all obstructions to deformations of a coherent sheaf admitting a finite locally free resolution. We refer again to [20] for a survey on the existing literature in this regard; we only note here that this fact was proved by Mukai and Artamkin [1] for the 0th component of the Buchweitz–Flenner semiregularity map, in [17] with some mild assumptions for the Bloch semiregularity map, in [7] for curvilinear obstructions for the map σ when the Hodge to de Rham spectral sequence of X degenerates at E_1, and finally in the general case for all obstructions and for the maps τ_q in [28].

The fact that the construction of the L_∞-morphism can also be realised in the algebraic case was outlined in [20, Section 5] and is expanded on here: given a simplicial connection on the finite complex of locally free sheaves $\mathcal{E}$, the map τ_1 can be lifted to an L_∞-morphism

$$g\colon \mathrm{Tot}(\mathcal{U}, \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \rightsquigarrow \mathrm{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2]),$$

where Tot denotes the Thom–Whitney totalisation and $\mathcal{U}$ is an affine open cover of X.

Since $\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$ is the kernel of the anchor map of the transitive DG-Lie algebroid of derivations of pairs $\mathcal{D}^*(X,\mathcal{E})$ of [18], it is natural to generalise this construction to the framework of DG-Lie algebroids. In fact, the main result of this paper, Theorem 3.12, is the construction of an L_∞-morphism between DG-Lie algebras

$$f\colon \mathrm{Tot}(\mathcal{U}, \mathcal{L}) \rightsquigarrow \mathrm{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2]),$$

where $\mathcal{L}$ denotes the kernel of the anchor map of a transitive DG-Lie algebroid, which is equipped with a connection $\nabla\colon \mathrm{Tot}(\mathcal{U},\mathcal{L}) \to \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1]\otimes\mathcal{L})$ and a d_{Tot}-closed cyclic form.

For this reason, in Section 2 we introduce connections on transitive DG-Lie algebroids, which are $\mathbb{K}$-linear operators that play the role of connections of type $(1,0)$ in the construction of the L_∞-morphism. Connections on transitive DG-Lie algebroids are associated to simplicial liftings of the identity; also associated to a simplicial lifting of the identity is the extension cocycle, which generalises the notion of Atiyah cocycle. Section 3 describes cyclic forms on DG-Lie algebroids, cyclic forms induced by DG-Lie algebroid representations and the construction of the L_∞-morphism. Also contained in Section 3 is the following application to the deformation theory of coherent sheaves, analogous to the one obtained for complex manifolds in [20].

Theorem 1.1 (Corollary 3.16). *Let $\mathcal{F}$ be a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme X of finite type over a field $\mathbb{K}$ of characteristic zero. Then every obstruction to the deformations of $\mathcal{F}$ belongs to the kernel of the map*

$$\tau_1\colon \mathrm{Ext}^2_X(\mathcal{F},\mathcal{F}) \longrightarrow \mathbb{H}^2(X, \Omega_X^{\leq 1}[2]).$$

If the Hodge to de Rham spectral sequence of X degenerates at E_1, then every obstruction to the deformations of $\mathcal{F}$ belongs to the kernel of the map

$$\sigma_1\colon \mathrm{Ext}^2_X(\mathcal{F},\mathcal{F}) \longrightarrow H^3(X,\Omega^1_X), \quad \sigma_1(a) = -\mathrm{Tr}(\mathrm{At}(\mathcal{F})\circ a).$$

Lastly, since to every principal bundle one can naturally associate the Atiyah Lie algebroid of [2], Section 4 contains the following application to the deformation theory of (Zariski) principal bundles.

Theorem 1.2 (Corollary 4.12). *Let P be a principal bundle on a smooth separated scheme X of finite type over an algebraically closed field $\mathbb{K}$ of characteristic zero and let*

$$\langle -,-\rangle\colon \mathrm{Tot}(\mathcal{U},\Omega^i_X[-i]\otimes \mathit{ad}(P))\times \mathrm{Tot}(\mathcal{U},\Omega^j_X[-j]\otimes \mathit{ad}(P)) \longrightarrow \mathrm{Tot}(\mathcal{U},\Omega^{i+j}_X[-i-j]),$$

for $i,j\geq 0$, be a d_{Tot}-closed cyclic form. Then every obstruction to the deformations of P belongs to the kernel of the map

$$f_1\colon H^2(X,\mathit{ad}(P))\longrightarrow \mathbb{H}^2(X,\Omega^{\leq 1}_X[2]),\qquad f_1(x)=\langle \mathrm{At}(P),x\rangle,$$

where $\mathrm{At}(P)$ denotes the Atiyah class of the principal bundle P.

Notation

By $\mathbb{K}$ we always denote a characteristic zero field. Given two complexes $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{O}_X$-modules, $\mathcal{F}\otimes\mathcal{G}$ denotes $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$. If $V=\bigoplus V^i$ is either a graded vector space or a graded sheaf, $\overline{v}$ denotes the degree of a homogeneous element $v\in V$. For every integer p, the symbol $[p]$ denotes the shift functor, defined by $V[p]^i=V^{p+i}$. For complexes $\mathcal{E},\mathcal{F}$ of $\mathcal{O}_X$-modules we denote by $\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ the graded sheaf of $\mathcal{O}_X$-linear morphisms

$$\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})=\bigoplus_i \mathcal{H}om^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}),$$

where

$$\mathcal{H}om^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})=\prod_j \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^j,\mathcal{F}^{i+j}).$$

If L is a DG-Lie algebra, $H^*(L)$ always denotes the cohomology of the underlying cochain complex, which inherits a graded bracket from the one on L.

2. DG-Lie algebroids, connections and extension cocycles

The goal of this section is to define $\mathbb{K}$-linear operators

$$\nabla\colon \mathrm{Tot}(\mathcal{U},\mathcal{L})\longrightarrow \mathrm{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{L})$$

called connections on the kernel $\mathcal{L}$ of the anchor map of a transitive DG-Lie algebroid. A short review of the Thom–Whitney totalisation functor Tot is given. In order to construct a connection, we introduce the notion of simplicial lifting of the identity. The section ends with the definition of the extension cocycle associated to a simplicial lifting of the identity, which generalises the notion of Atiyah cocycle. Different notions of Atiyah classes for DG-Lie algebroids have been considered elsewhere in the literature, see e.g. [3, 6, 26].

Let X be a smooth separated scheme of finite type over a field $\mathbb{K}$ of characteristic zero, and let $\Theta_X, \Omega_X^1 = \Omega_{X/\mathbb{K}}^1$ denote its tangent and cotangent sheaves respectively. Often, it will be useful to consider the cotangent sheaf as a trivial complex of sheaves concentrated in degree one, so as to have an inclusion $\Omega_X^1[-1] \to \Omega_X^*$, where $\Omega_X^* = \bigoplus_p \Omega_X^p[-p]$ denotes the de Rham complex.

Definition 2.1. A DG-Lie algebroid over X is a complex of sheaves of $\mathcal{O}_X$-modules $\mathcal{A}$ equipped with a $\mathbb{K}$-bilinear bracket $[-,-]\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which defines a DG-Lie algebra structure on the spaces of sections, and with a morphism of complexes of $\mathcal{O}_X$-modules $\rho\colon \mathcal{A} \to \Theta_X$, called the anchor map, such that the induced map on the spaces of sections is a homomorphism of DG-Lie algebras. Moreover, for any sections a_1, a_2 of $\mathcal{A}$ and f of $\mathcal{O}_X$, the following Leibniz identity holds:

$$[a_1, fa_2] = f[a_1, a_2] + \rho(a_1)(f)a_2.$$

Example 2.2. The sheaf Θ_X is a trivial example of a DG-Lie algebroid concentrated in degree zero, with anchor map given by the identity. A DG-Lie algebroid over $\operatorname{Spec}\mathbb{K}$ is exactly a DG-Lie algebra over the field $\mathbb{K}$. Every sheaf of DG-Lie algebras over $\mathcal{O}_X$ can be considered as a DG-Lie algebroid over X with trivial anchor map.

Definition 2.3. Let $(\mathcal{A}, \rho)$ and $(\mathcal{B}, \sigma)$ be DG-Lie algebroids over X. A morphism of DG-Lie algebroids $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a morphism of complexes of sheaves which preserves brackets and commutes with the anchor maps:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \varphi\ } & \mathcal{B} \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\sigma} & \\ & \Theta_X & \end{array}$$

Let $(\mathcal{A}, \rho)$ be a DG-Lie algebroid over X and assume that $\mathcal{L} = \operatorname{Ker}\rho$ is a finite complex of locally free sheaves. Notice that on $\mathcal{L}$ there is a naturally induced graded Lie bracket: for sections x, y of $\mathcal{L}$,

$$[x, y] := [i(x), i(y)],$$

where $i\colon \mathcal{L} \to \mathcal{A}$ denotes the inclusion. This bracket is $\mathcal{O}_X$-linear, in fact for any sections x, y of $\mathcal{L}$ and f of $\mathcal{O}_X$ one has

$$\begin{aligned}[x, fy] &:= [i(x), i(fy)] = [i(x), fi(y)] = f[i(x), i(y)] + \rho(i(x))(f)y \\ &= f[i(x), i(y)] = f[x, y],\end{aligned}$$

so that $\mathcal{L}$ is a sheaf of DG-Lie algebras over $\mathcal{O}_X$.

DEFINITION 2.4 ([22, Chapter 3]). A DG-Lie algebroid $(\mathcal{A}, \rho)$ over X is transitive if the anchor map $\rho\colon \mathcal{A} \to \Theta_X$ is surjective.

Let now $(\mathcal{A}, \rho)$ be a transitive DG-Lie algebroid over X, consider the short exact sequence of complexes of sheaves

$$0 \longrightarrow \mathcal{L} \xrightarrow{i} \mathcal{A} \xrightarrow{\rho} \Theta_X \longrightarrow 0$$

and tensor it with the shifted cotangent sheaf $\Omega^1_X[-1]$ to obtain the short exact sequence

$$0 \longrightarrow \Omega^1_X[-1] \otimes \mathcal{L} \xrightarrow{\mathrm{Id}\otimes i} \Omega^1_X[-1] \otimes \mathcal{A} \xrightarrow{\mathrm{Id}\otimes \rho} \Omega^1_X[-1] \otimes \Theta_X \longrightarrow 0. \tag{2.1}$$

Because of the isomorphisms

$$\Omega^1_X[-1] \otimes \Theta_X \cong \mathcal{H}om^*_{\mathcal{O}_X}(\Omega^1_X, \Omega^1_X[-1]) \cong \mathcal{H}om^*_{\mathcal{O}_X}(\Omega^1_X, \Omega^1_X)[-1],$$

one can consider $\mathrm{Id}_{\Omega^1} \in \Gamma(X, \Omega^1_X[-1] \otimes \Theta_X)$ as an element of degree one.

DEFINITION 2.5. A *lifting of the identity* is a global section D in $\Gamma(X, \Omega^1_X[-1] \otimes \mathcal{A})$ such that $(\mathrm{Id} \otimes \rho)(D) = \mathrm{Id}_{\Omega^1} \in \Gamma(X, \Omega^1_X[-1] \otimes \Theta_X)$.

Since in general the map $\mathrm{Id} \otimes \rho$ is not surjective on global sections, a lifting of the identity does not always exist. However, a germ of a lifting of the identity, i.e., a preimage of Id_{Ω^1} in $\Omega^1_X[-1] \otimes \mathcal{A}$, always exists.

EXAMPLE 2.6. For particular DG-Lie algebroids, the notion of lifting of the identity can be related to the more familiar notion of algebraic connection. Let $(\mathcal{E}, \delta_{\mathcal{E}})$ be a finite complex of locally free sheaves. Following [18, Section 5], define the complex of derivations of pairs

$$\mathcal{D}^*(X, \mathcal{E}) = \left\{ (h, u) \in \Theta_X \times \mathcal{H}om^*_{\mathbb{K}}(\mathcal{E}, \mathcal{E}) \;\middle|\; \begin{array}{l} u(fe) = fu(e) + h(f)e \\ \text{for all } f \in \mathcal{O}_X \text{ and } e \in \mathcal{E} \end{array} \right\}.$$

The complex $\mathcal{D}^*(X, \mathcal{E})$ is a finite complex of coherent sheaves and the natural map

$$\alpha\colon \mathcal{D}^*(X, \mathcal{E}) \longrightarrow \Theta_X, \quad (h, u) \longmapsto h,$$

which is called the anchor map, is surjective, see [18]. The graded Lie bracket is defined as

$$[(h,u),(h',u')] = ([h,h'],[u,u']),$$

where the (graded) Lie brackets on Θ_X and $\mathcal{H}om^*_{\mathbb{K}}(\mathcal{E},\mathcal{E})$ are the (graded) commutators of the composition products. For $f \in \mathcal{O}_X$ we then have that

$$\begin{aligned}[(h,u), f(h',u')] &= [(h,u),(fh', fu')] = ([h, fh'],[u, fu']) \\ &= \big(h(f)h' + fhh' - fh'h, fuu' + h(f)u' - (-1)^{\bar{u}\bar{u'}} fu'u\big) \\ &= \big(h(f)h' + f[h,h'], h(f)u' + f[u,u']\big) \\ &= f([h,h'],[u,u']) + h(f)(h',u') \\ &= f[(h,u),(h',u')] + \alpha((h,u))(f)(h',u'),\end{aligned}$$

hence $(\mathcal{D}^*(X,\mathcal{E}),\alpha)$ is a transitive DG-Lie algebroid over X. By an algebraic connection on the complex $\mathcal{E}$, we mean the data $\{D^i\}$ of algebraic connections on each $\mathcal{E}^i$, i.e., for every i, a $\mathbb{K}$-linear map $D^i\colon \mathcal{E}^i \to \Omega^1_X \otimes \mathcal{E}^i$ such that for $e \in \mathcal{E}^i$, $f \in \mathcal{O}_X$,

$$D^i(fe) = d_{dR} f \otimes e + fD^i(e).$$

Here, d_{dR} denotes the universal derivation $d_{dR}\colon \mathcal{O}_X \to \Omega^1_X$.

$$D(fe) = d_{dR} f \otimes e + fD(e),$$

where d_{dR} denotes the universal derivation $d_{dR}\colon \mathcal{O}_X \to \Omega^1_X$. A global algebraic connection on $\mathcal{E}$needs not to exist. The kernel of the anchor map α is the sheaf of DG-Lie algebras $\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$, the graded sheaf of $\mathcal{O}_X$-linear endomorphisms of $\mathcal{E}$, with bracket equal to the graded commutator

$$[f,g] = fg - (-1)^{\bar{f}\,\bar{g}} gf,$$

and differential given by

$$g \longmapsto [\delta_{\mathcal{E}}, g] = \delta_{\mathcal{E}} g - (-1)^{\bar{g}} g\delta_{\mathcal{E}}.$$

The short exact sequence in (2.1) in this case is isomorphic to

$$0 \longrightarrow \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\Omega^1_X[-1]\otimes\mathcal{E}) \xrightarrow{g\mapsto(0,g)} \mathcal{J}^*_{\Omega^1} \xrightarrow{(\beta,g)\mapsto\beta} \mathcal{D}er^*_{\mathbb{K}}(\mathcal{O}_X,\Omega^1_X[-1]) \longrightarrow 0,$$

where the complex $\mathcal{J}^*_{\Omega^1}$ is defined as the subcomplex of

$$\mathcal{D}er^*_{\mathbb{K}}(\mathcal{O}_X,\Omega^1_X[-1]) \times \mathcal{H}om^*_{\mathbb{K}}(\mathcal{E},\Omega^1_X[-1]\otimes\mathcal{E})$$

of elements (β, v), with $\beta \in \mathcal{D}er^*_{\mathbb{K}}(\mathcal{O}_X,\Omega^1_X[-1])$ and $v \in \mathcal{H}om^*_{\mathbb{K}}(\mathcal{E},\Omega^1_X[-1]\otimes\mathcal{E})$, such that

$$v(fx) = fv(x) + \beta(f)\otimes x,$$

for all $x \in \mathcal{E}$ and $f \in \mathcal{O}_X$; see also [20, Section 5]. In this case, a lifting of the identity is exactly a global algebraic connection on the complex of sheaves $\mathcal{E}$: via the isomorphism $\Omega^1_X[-1] \otimes \mathcal{D}^*(X, \mathcal{E}) \cong \mathcal{J}^*_{\Omega^1}$ a lifting of the identity D corresponds to $\mathbb{K}$-linear maps $D' \colon \mathcal{E}^i \to \Omega^1_X[-1] \otimes \mathcal{E}^i$ for all i such that $D'(fe) = fD'(e) + d_{dR}(f) \otimes e$ for all $f \in \mathcal{O}_X$ and $e \in \mathcal{E}^i$.

Before defining connections on the kernel of the anchor map of a transitive DG-Lie algebroid, it is useful to give a brief review of the definition and some of the main properties of the Thom–Whitney totalisation functor Tot; for more details see e.g. [11, 12, 17, 24, 27]. The Thom–Whitney totalisation is a functor from the category of semicosimplicial DG-vector spaces to the category of DG-vector spaces. For every $n \geq 0$ consider

$$A_n = \frac{\mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]}{(1 - \sum_i t_i, \sum_i dt_i)},$$

the commutative differential graded algebra of polynomial differential forms on the affine standard n-simplex, and the maps

$$\delta^*_k \colon A_n \longrightarrow A_{n-1}, \quad 0 \leq k \leq n, \qquad \delta^*_k(t_i) = \begin{cases} t_i & i < k, \\ 0 & i = k, \\ t_{i-1} & i > k. \end{cases}$$

DEFINITION 2.7. The Thom–Whitney totalisation of a semicosimplicial DG-vector space V,

$$V : \ V_0 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} V_1 \ \underset{\delta_2}{\overset{\delta_0}{\substack{\longrightarrow \\ \xrightarrow{\delta_1} \\ \longrightarrow}}}\ V_2 \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots ,$$

is the DG-vector space

$$\mathrm{Tot}(V) = \left\{ (x_n) \in \prod_{n \geq 0} A_n \otimes V_n \ \middle| \ (\delta^*_k \otimes \mathrm{Id}) x_n = (\mathrm{Id} \otimes \delta_k) x_{n-1} \text{ for all } 0 \leq k \leq n \right\},$$

with differential induced by the one on $\prod_{n \geq 0} A_n \otimes V_n$. To simplify the notation, we will denote this differential by $d_{\mathrm{Tot}} = d_A + d_V$, where d_A denotes the differential of polynomial differential forms, and d_V the differential on V.

If $f \colon V \to W$ is a morphism of semicosimplicial DG-vector spaces, then the morphism $\mathrm{Tot}(f) \colon \mathrm{Tot}(V) \to \mathrm{Tot}(W)$ is defined as the restriction of the map

$$\prod \mathrm{Id} \otimes f \colon \prod_{n \geq 0} A_n \otimes V_n \longrightarrow \prod_{n \geq 0} A_n \otimes W_n.$$

The Tot functor is exact (see e.g. [10,27]): given semicosimplicial DG-vector spaces V, W, Z and morphisms $f\colon V \to W$, $g\colon W \to Z$ such that for every $n \geq 0$ the sequence

$$0 \longrightarrow V_n \xrightarrow{f} W_n \xrightarrow{g} Z_n \longrightarrow 0$$

is exact, one obtains an exact sequence

$$0 \longrightarrow \operatorname{Tot}(V) \xrightarrow{f} \operatorname{Tot}(W) \xrightarrow{g} \operatorname{Tot}(Z) \longrightarrow 0.$$

Given two semicosimplicial DG-vector spaces V and W, $\operatorname{Tot}(V \times W)$ is naturally isomorphic to $\operatorname{Tot}(V) \times \operatorname{Tot}(W)$. An important consequence is the preservation of multiplicative structures; in particular, we will use the fact that the functor Tot sends semicosimplicial DG-Lie algebras to DG-Lie algebras.

Example 2.8. Let $\mathcal{E}$ be a finite complex of quasi-coherent sheaves on X, and $\mathcal{U} = \{U_i\}$ an open cover of X. Denote by $U_{i_1 \dots i_n} = U_{i_1} \cap \cdots \cap U_{i_n}$, and consider the semicosimplicial DG-vector space of Čech cochains:

$$\mathcal{E}(\mathcal{U}) : \prod_i \mathcal{E}(U_i) \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} \prod_{i,j} \mathcal{E}(U_{ij}) \overset{\delta_0}{\underset{\delta_2}{\substack{\longrightarrow\\ \xrightarrow{\delta_1} \\ \longrightarrow}}} \prod_{i,j,k} \mathcal{E}(U_{ijk}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots .$$

The Whitney integration theorem states that there exists a quasi-isomorphism between the Tot complex $\operatorname{Tot}(\mathcal{U}, \mathcal{E})$ and the complex of Čech cochains $C^*(\mathcal{U}, \mathcal{E})$ of $\mathcal{E}$ (see [32] for the C^∞ version, [9,13,21,27] for the algebraic version used here). Hence, if the open cover $\mathcal{U}$ is affine, the cohomology of $\operatorname{Tot}(\mathcal{U}, \mathcal{E})$ is isomorphic to the hypercohomology of the complex of sheaves $\mathcal{E}$. In this case, the complex $\operatorname{Tot}(\mathcal{U}, \mathcal{E})$ is a model for the module $\mathbf{R}\Gamma(X, \mathcal{E})$ of derived global sections of $\mathcal{E}$, see e.g. [12,25].

Moreover, there is a canonical inclusion of the global sections of $\mathcal{E}$ in the totalisation $\operatorname{Tot}(\mathcal{U}, \mathcal{E})$: in fact, the injection

$$\iota\colon \Gamma(X, \mathcal{E}) \longrightarrow \prod_i \mathcal{E}(U_i)$$

is such that $\delta_0 \iota = \delta_1 \iota$ and therefore for every $a \in \Gamma(X, \mathcal{E})$,

$$(1 \otimes \iota(a), 1 \otimes \delta_0 \iota(a), 1 \otimes \delta_0^2 \iota(a), \dots)$$

belongs to $\operatorname{Tot}(\mathcal{U}, \mathcal{E})$. It is easy to see that if the complex $\mathcal{E}$ has trivial differential, every global section gives a cocycle in $\operatorname{Tot}(\mathcal{U}, \mathcal{E})$.

We are now ready to define connections on $\mathcal{L} = \operatorname{Ker} \rho$, the kernel of the anchor map of a transitive DG-Lie algebroid $(\mathcal{A}, \rho)$ over X. Assume that $\mathcal{L}$ is a finite complex of locally free sheaves and fix an affine open cover $\mathcal{U} = \{U_i\}$ of X. The short exact sequence

$$0 \longrightarrow \Omega^1_X[-1] \otimes \mathcal{L} \xrightarrow{\mathrm{Id}\otimes i} \Omega^1_X[-1] \otimes \mathcal{A} \xrightarrow{\mathrm{Id}\otimes \rho} \Omega^1_X[-1] \otimes \Theta_X \longrightarrow 0$$

gives a short exact sequence of the corresponding semicosimplicial complexes of Čech cochains, and since the functor Tot is exact there is an exact sequence

$$0 \longrightarrow \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L}) \xrightarrow{\mathrm{Id}\otimes i} \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A}) - \cdots$$
$$\cdots \xrightarrow{\mathrm{Id}\otimes \rho} \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X) \longrightarrow 0.$$

Denote by d the differential on $\mathcal{A}$ and $\mathcal{L}$, which can be extended to $\Omega^1_X[-1] \otimes \mathcal{A}$ and to $\Omega^1_X[-1] \otimes \mathcal{L}$ by setting

$$d(\eta \otimes x) = (-1)^{\overline{\eta}} \eta \otimes dx = -\eta \otimes dx.$$

Denote by d_{Tot} the differentials on all the above Tot complexes: for $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ and $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$ the differential d_{Tot} is equal to $d_A + d$, while for the complex $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$ one has that d_{Tot} is just d_A, where d_A is the differential of polynomial differential forms on the affine simplex, see Definition 2.7.

Because of the natural inclusion of global sections in the totalisation remarked in Example 2.8, Id_{Ω^1} belongs to $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$, where it has degree one.

DEFINITION 2.9. A *simplicial lifting of the identity* is an element D of the complex $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$ such that $(\mathrm{Id} \otimes \rho)(D) = \mathrm{Id}_{\Omega^1}$ in $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$.

It is clear that a simplicial lifting of the identity always exists and that D has degree one in $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$.

REMARK 2.10. Notice that via the isomorphism

$$\Omega^1_X[-1] \otimes \Theta_X = \Omega^1_X[-1] \otimes \mathcal{D}er_{\mathbb{K}}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{D}er^*_{\mathbb{K}}(\mathcal{O}_X, \Omega^1_X[-1])$$

we have $(\mathrm{Id} \otimes \rho)(D) = d_{dR} \in \mathrm{Tot}(\mathcal{U}, \mathcal{D}er^*_{\mathbb{K}}(\mathcal{O}_X, \Omega^1_X[-1]))$.

In order to define a connection on $\mathcal{L}$, it is necessary to define a Lie bracket

$$[-,-]\colon \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A}) \times \mathrm{Tot}(\mathcal{U}, \mathcal{L}) \longrightarrow \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L}),$$

induced by the bracket of the following lemma.

LEMMA 2.11. *There exists a well-defined $\mathbb{K}$-bilinear bracket*

$$[-,-]\colon (\Omega^1_X[-1] \otimes \mathcal{A}) \times \mathcal{L} \longrightarrow \Omega^1_X[-1] \otimes \mathcal{L}.$$

PROOF. Denote by $i\colon \mathcal{L}\to\mathcal{A}$ the inclusion, take $\eta\otimes a$ with $\eta\in\Omega^1_X[-1]$ and $a\in\mathcal{A}$, and define for $x\in\mathcal{L}$

$$[\eta\otimes a, x] := \eta\otimes[a, i(x)].$$

Notice that the Leibniz identity in Definition 2.1 implies that

$$[fa_1, a_2] = f[a_1,a_2] - (-1)^{\overline{a_1}\,\overline{a_2}}\rho(a_2)(f)a_1.$$

Hence, the bracket $[\eta\otimes a, x]$ is well defined: for any $f\in\mathcal{O}_X$

$$\begin{aligned}[\eta\otimes fa, x] &= \eta\otimes[fa,x] = \eta\otimes\big(f[a,x]-(-1)^{\bar a\,\bar x}\rho(x)(f)a\big) = \eta\otimes f[a,x]\\ &= f\eta\otimes[a,x] = [f\eta\otimes a, x].\end{aligned}$$

It is clear that $[\eta\otimes a, x]$ belongs to $\Omega^1_X[-1]\otimes\mathcal{L}$:

$$(\mathrm{Id}\otimes\rho)([\eta\otimes a,x]) = (\mathrm{Id}\otimes\rho)(\eta\otimes[a,x]) = \eta\otimes[\rho(a),\rho(x)] = 0. \qquad \blacksquare$$

Since the functor Tot preserves products, the map

$$[-,-]\colon(\Omega^1_X[-1]\otimes\mathcal{A})\times\mathcal{L}\to\Omega^1_X[-1]\otimes\mathcal{L}$$

induces a $\mathbb{K}$-bilinear map

$$[-,-]\colon \mathrm{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{A})\times\mathrm{Tot}(\mathcal{U},\mathcal{L})\longrightarrow\mathrm{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{L}), \tag{2.2}$$

which is defined componentwise as the restriction of

$$\begin{array}{c} A_n\otimes\prod(\Omega^1_X[-1]\otimes\mathcal{A})(U_{i_1\ldots i_n})\times A_n\otimes\prod\mathcal{L}(U_{i_1\ldots i_n})\\ \Big\downarrow{\scriptstyle [-,-]}\\ A_n\otimes\prod(\Omega^1_X[-1]\otimes\mathcal{L})(U_{i_1\ldots i_n}),\end{array}$$

$$[\eta_n\otimes(t_{i_1\ldots i_n}),\phi_n\otimes(u_{i_1\ldots i_n})] = \eta_n\phi_n\otimes([(-1)^{\overline{\phi_n}\,\overline{t_{i_1\ldots i_n}}}t_{i_1\ldots i_n}, u_{i_1\ldots i_n}]),$$

for η_n,ϕ_n in A_n, $t_{i_1\ldots i_n}$ in $(\Omega^1_X[-1]\otimes\mathcal{A})(U_{i_1\ldots i_n})$, and $u_{i_1\ldots i_n}$ in $\mathcal{L}(U_{i_1\ldots i_n})$.

DEFINITION 2.12. A *connection* on $\mathcal{L}$ is the adjoint operator of a simplicial lifting of the identity $D\in\mathrm{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{A})$,

$$\nabla = [D,-]\colon\mathrm{Tot}(\mathcal{U},\mathcal{L})\longrightarrow\mathrm{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{L}),$$

where $[-,-]\colon\mathrm{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{A})\times\mathrm{Tot}(\mathcal{U},\mathcal{L})\to\mathrm{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{L})$ is the bracket in (2.2). It is a $\mathbb{K}$-linear operator.

We will now examine the relationship between connections and particular representatives of extension classes. The short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{A} \xrightarrow{\rho} \Theta_X \longrightarrow 0 \tag{2.3}$$

gives an extension class $[u_\rho] \in \mathrm{Ext}^1_X(\Theta_X, \mathcal{L})$. It is possible to give a representative of $[u_\rho]$ in the totalisation $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ with respect to an affine open cover $\mathcal{U}$ of X.

Definition 2.13. An *extension cocycle* u of the transitive DG-Lie algebroid $\mathcal{A}$ is the differential of a simplicial lifting of the identity D in $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$, $u = d_{\mathrm{Tot}} D$.

Notice that u belongs to $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$:

$$(\mathrm{Id} \otimes \rho)u = (\mathrm{Id} \otimes \rho)d_{\mathrm{Tot}}(D) = d_{\mathrm{Tot}}(\mathrm{Id} \otimes \rho)D = d_{\mathrm{Tot}}\mathrm{Id}_{\Omega^1} = 0,$$

where the last equality is a consequence of the fact that Id_{Ω^1} is a global section and $\Omega^1_X[-1] \otimes \Theta_X$ has trivial differential (see Example 2.8). Note that u has degree two in $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ and that $d_{\mathrm{Tot}} u = d_{\mathrm{Tot}} d_{\mathrm{Tot}} D = 0$.

Using the isomorphisms

$$\Omega^1_X[-1] \otimes \mathcal{L} \cong \mathcal{H}om^*_{\mathcal{O}_X}(\Theta_X[1], \mathcal{L}) \cong \mathcal{H}om^*_{\mathcal{O}_X}(\Theta_X, \mathcal{L})[-1],$$

the cohomology class of u belongs to

$$\begin{aligned} H^2(\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})) &\cong H^2(\mathrm{Tot}(\mathcal{U}, \mathcal{H}om^*_{\mathcal{O}_X}(\Theta_X, \mathcal{L})[-1])) \\ &\cong \mathbb{H}^2(X, \mathcal{H}om^*_{\mathcal{O}_X}(\Theta_X, \mathcal{L})[-1]) \\ &\cong \mathbb{H}^1(X, \mathcal{H}om^*_{\mathcal{O}_X}(\Theta_X, \mathcal{L})) \\ &\cong \mathrm{Ext}^1_X(\Theta_X, \mathcal{L}). \end{aligned}$$

This cohomology class does not depend on the chosen simplicial lifting of the identity: if D and D' are two simplicial liftings of the identity in $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$, we have $(\mathrm{Id} \otimes \rho)(D - D') = \mathrm{Id}_{\Omega^1} - \mathrm{Id}_{\Omega^1} = 0$, so $D - D'$ belongs to $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$, and $d_{\mathrm{Tot}} D$ and $d_{\mathrm{Tot}} D'$ differ by the coboundary $d_{\mathrm{Tot}}(D' - D)$. It is easy to see that the cohomology class of u is trivial if and only if the short exact sequence in (2.3) splits.

Lemma 2.14. *Let* $\nabla\colon \mathrm{Tot}(\mathcal{U}, \mathcal{L}) \to \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ *be a connection on* $\mathcal{L}$*, associated to the simplicial lifting of the identity* $D \in \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$*. Let* $u = d_{\mathrm{Tot}} D$ *be the corresponding extension cocycle. Then, for every* x *in* $\mathrm{Tot}(\mathcal{U}, \mathcal{L})$ *we have*

$$\nabla(d_{\mathrm{Tot}} x) = [u, x] - d_{\mathrm{Tot}} \nabla(x).$$

Proof. Recall that d denotes the differential of $\mathcal{A}$ and $\mathcal{L}$, which can be extended to $\Omega^1_X[-1]\otimes\mathcal{A}$ and to $\Omega^1_X[-1]\otimes\mathcal{L}$ by setting $d(\eta\otimes x)=(-1)^{\overline{\eta}}\eta\otimes dx=-\eta\otimes dx$. It is easy to see that for the $\mathbb{K}$-bilinear map $[-,-]\colon(\Omega^1_X[-1]\otimes\mathcal{A})\times\mathcal{L}\to\Omega^1_X[-1]\otimes\mathcal{L}$ of Lemma 2.11,

$$d[\eta\otimes a,x]=[d(\eta\otimes a)x]+(-1)^{\overline{\eta}+\overline{a}}[\eta\otimes a,dx].$$

A straightforward calculation then shows that for $z\in\mathrm{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{A})$ and $w\in\mathrm{Tot}(\mathcal{U},\mathcal{L})$,

$$d_{\mathrm{Tot}}[z,w]=[d_{\mathrm{Tot}}z,w]+(-1)^{\overline{z}}[z,d_{\mathrm{Tot}}w],$$

and the conclusion follows from the fact $u=d_{\mathrm{Tot}}D$. ■

3. Cyclic forms and L_∞-morphisms

This section describes cyclic forms on DG-Lie algebroids and illustrates how DG-Lie algebroid representations give rise to cyclic forms. We then discuss induced cyclic forms on the Thom–Whitney totalisation and the property of d_{Tot}-closure. The central result is the construction of a L_∞-morphism associated to a connection and to a d_{Tot}-closed cyclic form for a transitive DG-Lie algebroid. This allows us to state the results of [20] for a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme of finite type over a field $\mathbb{K}$ of characteristic zero.

Let $\mathcal{A}$ be a DG-Lie algebroid over a smooth separated scheme X of finite type over a field $\mathbb{K}$ of characteristic zero, with anchor map $\rho\colon\mathcal{A}\to\Theta_X$. Assume that the kernel of the anchor map $\mathcal{L}$ is a finite complex of locally free sheaves. Notice that for any $a\in\mathcal{A}$ and $x\in\mathcal{L}$, the bracket $[a,x]$ belongs to $\mathcal{L}$:

$$\rho([a,x])=[\rho(a),\rho(x)]=0.$$

Definition 3.1. A cyclic bilinear form on a DG-Lie algebroid $(\mathcal{A},\rho)$ is a graded symmetric $\mathcal{O}_X$-bilinear product of degree zero on $\mathcal{L}=\operatorname{Ker}\rho$,

$$\langle-,-\rangle\colon\mathcal{L}\times\mathcal{L}\to\mathcal{O}_X,$$

such that for all sections x,y of $\mathcal{L}$ and a of $\mathcal{A}$

$$\langle[a,x],y\rangle+(-1)^{\overline{a}\,\overline{x}}\langle x,[a,y]\rangle=\rho(a)(\langle x,y\rangle). \tag{3.1}$$

Notice that the definition implies that for all $x,y,z\in\mathcal{L}$

$$\langle x,[y,z]\rangle=\langle[x,y],z\rangle. \tag{3.2}$$

These two properties will be discussed after giving some examples.

In the following two examples, the cyclicity of the forms will follow from Lemma 3.5.

EXAMPLE 3.2. An example of a cyclic form on $(\mathcal{A}, \rho)$ is induced by the Killing form. Consider the adjoint representation as a morphism of sheaves of DG-Lie algebras

$$\mathrm{ad}\colon \mathcal{L} \longrightarrow \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}), \quad a \longmapsto [a, -]$$

and consider the trace map $\mathrm{Tr}\colon \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \to \mathcal{O}_X$, which is a morphism of sheaves of DG-Lie algebras (when considering $\mathcal{O}_X$ as a trivial sheaf of DG-Lie algebras). Then one can define the form

$$\langle -, - \rangle\colon \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{O}_X, \quad (x, y) \longmapsto \mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y).$$

EXAMPLE 3.3. For the DG-Lie algebroid $\mathcal{D}^*(X, \mathcal{E})$ of Example 2.6,

$$0 \longrightarrow \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \longrightarrow \mathcal{D}^*(X, \mathcal{E}) \longrightarrow \Theta_X \longrightarrow 0,$$

a natural bilinear form on the complex $\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ is induced by the trace map $\mathrm{Tr}\colon \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_X$ as

$$\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \times \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \longrightarrow \mathcal{O}_X, \quad \langle f, g \rangle := -\mathrm{Tr}(fg).$$

Example 3.3 explains the definition of cyclic form: (3.2) reflects the cyclicity property of the trace map $\mathrm{Tr}\colon \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_X$, $\mathrm{Tr}(ab) = (-1)^{\bar{a}\bar{b}}\,\mathrm{Tr}(ba)$, while (3.1) is related to the properties of the extension of the trace map to $\mathcal{D}^*(X, \mathcal{E})$, for which we refer to [18].

The Leibniz identity of Definition 2.1,

$$[a, fx] = f[a, x] + \rho(a)(f)x \quad \text{for all } a \in \mathcal{A},\ x \in \mathcal{L}, \text{ and } f \in \mathcal{O}_X,$$

can be restated by noting that for all a in $\mathcal{A}$ the operator $(\rho(a), [a, -])$ belongs to $\mathcal{D}^*(X, \mathcal{L})$ of Example 2.6. Hence, there is a morphism of DG-Lie algebroids

$$\mathrm{ad}\colon \mathcal{A} \to \mathcal{D}^*(X, \mathcal{L}).$$

The morphism $\mathrm{ad}\colon \mathcal{A} \to \mathcal{D}^*(X, \mathcal{L})$ restricts to the morphism $\mathrm{ad}\colon \mathcal{L} \to \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$ of Example 3.2, so that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L} & \xrightarrow{\ i\ } & \mathcal{A} & \xrightarrow{\ \rho\ } & \Theta_X & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{ad}} & & \Big\downarrow{\scriptstyle \mathrm{ad}} & & \Big\| & & \\
0 & \longrightarrow & \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) & \longrightarrow & \mathcal{D}^*(X, \mathcal{L}) & \xrightarrow{\ \alpha\ } & \Theta_X & \longrightarrow & 0
\end{array}$$

This motivates the following definition.

DEFINITION 3.4. A representation of a DG-Lie algebroid $(\mathcal{A}, \rho)$ over X is a morphism of DG-Lie algebroids $\theta\colon \mathcal{A} \to \mathcal{D}^*(X, \mathcal{E})$, where $\mathcal{E}$ is a finite complex of locally free sheaves over X:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad\theta\quad} & \mathcal{D}^*(X,\mathcal{E}) \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & \Theta_X & \end{array}$$

Every representation $\theta\colon \mathcal{A} \to \mathcal{D}^*(X, \mathcal{E})$ induces a form $\langle -, -\rangle_\theta\colon \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X$: for any $x \in \mathcal{L}$ we have $\alpha \circ \theta(x) = \rho(x) = 0$, so that

$$\theta|_{\mathcal{L}}\colon \mathcal{L} \longrightarrow \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}),$$

and using the trace map Tr: $\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_X$ we can define for every pair of sections x, y of $\mathcal{L}$,

$$\langle x, y\rangle_\theta := \operatorname{Tr}(\theta(x)\theta(y)).$$

Forms obtained in this way are cyclic, as shown in the following lemma.

LEMMA 3.5. *For any DG-Lie algebroid representation $\theta\colon \mathcal{A} \to \mathcal{D}^*(X, \mathcal{E})$, the induced form $\langle -, -\rangle_\theta\colon \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X$ is cyclic.*

PROOF. For $a \in \mathcal{A}$ and $x, y \in \mathcal{L}$,

$$\begin{aligned} &\langle [a, x], y\rangle_\theta + (-1)^{\bar{a}\,\bar{x}}\langle x, [a, y]\rangle_\theta \\ &\quad = \operatorname{Tr}(\theta([a, x])\theta(y) + (-1)^{\bar{a}\,\bar{x}}\theta(x)\theta([a, y])) \\ &\quad = \operatorname{Tr}([\theta(a), \theta(x)]\theta(y) + (-1)^{\bar{a}\,\bar{x}}\theta(x)[\theta(a), \theta(y)]) \\ &\quad = \operatorname{Tr}(\theta(a)\theta(x)\theta(y) - (-1)^{\bar{a}(\bar{x}+\bar{y})}\theta(x)\theta(y)\theta(a)) \\ &\quad = \operatorname{Tr}([\theta(a), \theta(x)\theta(y)]). \end{aligned}$$

Notice that if $\bar{a} \neq 0$ then a belongs to $\mathcal{L}$, so that $\theta(a)$ belongs to $\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$, and it is clear that $\operatorname{Tr}([\theta(a), \theta(x)\theta(y)]) = 0$, by the properties of the trace map.

The only remaining non-trivial case is when $\bar{a} = \bar{x} + \bar{y} = 0$. Let $\{e_i^k\}$ with $i = 1, \ldots, n_k$ be a local basis of $\mathcal{L}^k$, and let

$$\theta(a)(e_i^k) = \sum_j A_{ij}^k e_j^k, \quad \theta(x)\theta(y)(e_i^k) = \sum_j B_{ij}^k e_j^k, \quad A_{ij}^k, B_{ij}^k \in \mathcal{O}_X.$$

Then

$$
\begin{aligned}
[\theta(a), \theta(x)\theta(y)](e_i^k) &= \theta(a)\theta(x)\theta(y)(e_i^k) - \theta(x)\theta(y)\theta(a)(e_i^k) \\
&= \theta(a)\Big(\sum_j B_{ij}^k e_j^k\Big) - \theta(x)\theta(y)\Big(\sum_j A_{ij}^k e_j^k\Big) \\
&= \sum_j B_{ij}^k \theta(a)(e_j^k) + \sum_j (\alpha \circ \theta)(a)(B_{ij}^k) e_j^k - \sum_{j,s} A_{ij}^k B_{js}^k e_s^k \\
&= \sum_{j,s} B_{ij}^k A_{js}^k e_s^k + \sum_j \rho(a)(B_{ij}^k) e_j^k - \sum_{j,s} A_{ij}^k B_{js}^k e_s^k .
\end{aligned}
$$

The trace of $[\theta(a), \theta(x)\theta(y)]$ is hence equal to

$$
\begin{aligned}
&\sum_k (-1)^k \Big(\sum_{j,i} B_{ij}^k A_{ji}^k + \sum_i \rho(a)(B_{ii}^k) - \sum_{j,i} A_{ij}^k B_{ji}^k\Big) \\
&\quad = \sum_{k,i} (-1)^k \rho(a)(B_{ii}^k) = \rho(a)\Big(\sum_{k,i} (-1)^k B_{ii}^k\Big) \\
&\quad = \rho(a) \operatorname{Tr}(\theta(x)\theta(y)) = \rho(a)(\langle x, y\rangle_\theta).
\end{aligned}
$$

■

For every $i \geq 0$, let $\Omega_X^i[-i]$ denote the sheaf Ω_X^i seen as a trivial complex concentrated in degree i. Any cyclic form $\langle -, -\rangle \colon \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X$ can be extended to a collection of $\mathcal{O}_X$-bilinear forms

$$
\langle -, -\rangle \colon (\Omega_X^i[-i] \otimes \mathcal{L}) \times (\Omega_X^j[-j] \otimes \mathcal{L}) \longrightarrow \Omega_X^{i+j}[-i-j], \quad i, j \geq 0,
$$

according to the Koszul sign rule, by setting for $x, y \in \mathcal{L}$, $\omega \in \Omega_X^i[-i]$, and $\eta \in \Omega_X^j[-j]$,

$$
\langle \omega \otimes x, \eta \otimes y\rangle = (-1)^{\bar{x} j} \omega \wedge \eta \langle x, y\rangle.
$$

It is immediate to see that this form is cyclic, in the sense that

$$
\langle [b, x], y\rangle + (-1)^{\bar{b}\bar{x}} \langle x, [b, y]\rangle = (\mathrm{Id} \otimes \rho)(b)(\langle x, y\rangle)
$$

for all $b \in \Omega_X^1[-1] \otimes \mathcal{A}$ and $x, y \in \mathcal{L}$, where the bracket is the one of Lemma 2.11, and the anchor map has been extended to $\Omega_X^1[-1] \otimes \mathcal{A}$ by setting $(\mathrm{Id} \otimes \rho)(\omega \otimes a) := \omega \otimes \rho(a)$.

DEFINITION 3.6. A cyclic form $\langle -, -\rangle \colon \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X$ is *d-closed* if for all $z, w \in \mathcal{L}$

$$
\langle dz, w\rangle + (-1)^{\bar{z}} \langle z, dw\rangle = 0.
$$

LEMMA 3.7. *For any DG-Lie algebroid representation $\theta \colon \mathcal{A} \to \mathcal{D}^*(X, \mathcal{E})$ the induced cyclic form $\langle -, -\rangle_\theta \colon \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X$ is d-closed.*

PROOF. Since $\theta|_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$ is a morphism of DG-Lie algebroids, it commutes with differentials: for $x \in \mathcal{L}$,

$$\theta(dx) = d_{\mathcal{H}om^*(\mathcal{E},\mathcal{E})}(\theta(x)) = [d_{\mathcal{E}}, \theta(x)].$$

For x, y sections of $\mathcal{L}$,

$$\begin{aligned}\langle dx, y\rangle_\theta + (-1)^{\overline{x}}\langle x, dy\rangle_\theta &= \operatorname{Tr}(\theta(dx)\theta(y) + (-1)^{\overline{x}}\theta(x)\theta(dy))\\ &= \operatorname{Tr}([d_{\mathcal{E}}, \theta(x)]\theta(y) + (-1)^{\overline{x}}\theta(x)[d_{\mathcal{E}}, \theta(y)])\\ &= \operatorname{Tr}(d_{\mathcal{E}}\theta(x)\theta(y) - (-1)^{\overline{x}+\overline{y}}\theta(x)\theta(y)d_{\mathcal{E}})\\ &= \operatorname{Tr}([d_{\mathcal{E}}, \theta(x)\theta(y)]) = 0.\end{aligned}$$

∎

It follows from the properties of the Thom–Whitney totalisation functor Tot that every collection of cyclic forms

$$\langle -, -\rangle\colon (\Omega^i_X[-i] \otimes \mathcal{L}) \times (\Omega^j_X[-j] \otimes \mathcal{L}) \longrightarrow \Omega^{i+j}_X[-i-j],$$

with $i, j \geq 0$, induces a collection of $\mathbb{K}$-bilinear forms

$$\langle -, -\rangle\colon \operatorname{Tot}(\mathcal{U}, \Omega^i_X[-i] \otimes \mathcal{L}) \times \operatorname{Tot}(\mathcal{U}, \Omega^j_X[-j] \otimes \mathcal{L}) \longrightarrow \operatorname{Tot}(\mathcal{U}, \Omega^{i+j}_X[-i-j]).$$

Recalling Definition 2.7, the required forms are induced componentwise by the restriction of

$$\begin{array}{c} A_n \otimes \prod_{i_1\dots i_n} (\Omega^i_X[-i] \otimes \mathcal{L})(U_{i_1\dots i_n}) \times A_n \otimes \prod_{i_1\dots i_n} (\Omega^j_X[-j] \otimes \mathcal{L})(U_{i_1\dots i_n}) \\ \big\downarrow \\ A_n \otimes \prod_{i_1\dots i_n} \Omega^{i+j}_X[-i-j](U_{i_1\dots i_n}), \end{array}$$

$$\langle \eta_n \otimes (x_{i_1\dots i_n}), \omega_n \otimes (y_{i_1\dots i_n})\rangle := \eta_n\omega_n(\langle(-1)^{\overline{\omega_n}\,\overline{(x_{i_1\dots i_n})}}x_{i_1\dots i_n}, y_{i_1\dots i_n}\rangle),$$

with $x_{i_1\dots i_n}$ in $(\Omega^i_X[-i] \otimes \mathcal{L})(U_{i_1\dots i_n})$, $y_{i_1\dots i_n}$ in $(\Omega^j_X[-j] \otimes \mathcal{L})(U_{i_1\dots i_n})$, and η_n, ω_n in A_n.

Let $(\Omega^*_X = \bigoplus_p \Omega^p_X[-p], d_{dR})$ denote the de Rham complex. In the following, when working with $\operatorname{Tot}(\mathcal{U}, \Omega^*_X)$, the differential is denoted by d_{Tot} if $\Omega^*_X = \bigoplus_p \Omega^p_X[-p]$ is considered as complex with trivial differential, and by $d_{\mathrm{Tot}} + d_{dR}$ if it is considered as a complex with the de Rham differential.

LEMMA 3.8. *The form induced on the totalisation by a cyclic form* $\langle -, -\rangle\colon \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X$ *is cyclic: for all* $b \in \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$ *and* $z, w \in \operatorname{Tot}(\mathcal{U}, \mathcal{L})$ *one has that*

$$\langle [b, z], w\rangle + (-1)^{\overline{b}\,\overline{z}}\langle z, [b, w]\rangle = (\operatorname{Id} \otimes \rho)(b)(\langle z, w\rangle).$$

Moreover, if the form $\langle -,-\rangle\colon \mathcal{L}\times\mathcal{L}\to\mathcal{O}_X$ *is* d*-closed (see Definition 3.6), then for the induced form, for* $z\in \operatorname{Tot}(\mathcal{U},\Omega^i_X[-i]\otimes\mathcal{L})$ *and* $w\in\operatorname{Tot}(\mathcal{U},\Omega^j_X[-j]\otimes\mathcal{L})$*, we have that*

$$\langle d_{\mathrm{Tot}}z,w\rangle+(-1)^{\overline{z}}\langle z,d_{\mathrm{Tot}}w\rangle=d_{\mathrm{Tot}}\langle z,w\rangle.$$

The above condition will be called d_{Tot}-closure.

PROOF. For the first part of the statement, since everything is defined componentwise, it suffices to prove that for every $a\in A_n\otimes\prod(\Omega^1_X[-1]\otimes\mathcal{A})(U_{i_1\dots i_n})$ and every $x,y\in A_n\otimes\prod\mathcal{L}(U_{i_1\dots i_n})$,

$$\langle[a,x],y\rangle+(-1)^{\overline{a}\,\overline{x}}\langle x,[a,y]\rangle=(\mathrm{Id}\otimes\rho)(a)(\langle x,y\rangle)$$

for every $n\geq 0$. By linearity, let $a=\omega_n\otimes z_n$, with $\omega_n\in A_n$ and $z_n\in\prod(\Omega^1_X[-1]\otimes\mathcal{A})(U_{i_1\dots i_n})$; let $x=\eta_n\otimes x_n$ and $y=\phi_n\otimes y_n$, with η_n,ϕ_n in A_n and x_n,y_n in $\prod\mathcal{L}(U_{i_1\dots i_n})$. Then

$$\begin{aligned}
&\langle[a,x],y\rangle+(-1)^{\overline{a}\,\overline{x}}\langle x,[a,y]\rangle\\
&\quad=\langle[\omega_n\otimes z_n,\eta_n\otimes x_n],\phi_n\otimes y_n\rangle\\
&\qquad+(-1)^{\overline{a}\,\overline{x}}\langle\eta_n\otimes x_n,[\omega_n\otimes z_n,\phi_n\otimes y_n]\rangle\\
&\quad=(-1)^{\overline{\eta_n}\,\overline{z_n}}\langle\omega_n\eta_n\otimes[z_n,x_n],\phi_n\otimes y_n\rangle\\
&\qquad+(-1)^{\overline{a}\,\overline{x}}\langle\eta_n\otimes x_n,(-1)^{\overline{\phi_n}\,\overline{z_n}}\omega_n\phi_n\otimes[z_n,y_n]\rangle\\
&\quad=(-1)^{\overline{\phi_n}(\overline{z_n}+\overline{x_n})+\overline{\eta_n}\,\overline{z_n}}\big(\omega_n\eta_n\phi_n\langle[z_n,x_n],y_n\rangle\\
&\qquad+(-1)^{\overline{\omega_n}\,\overline{\eta_n}+\overline{z_n}\,\overline{x_n}}\eta_n\omega_n\phi_n\langle x_n,[z_n,y_n]\rangle\big)\\
&\quad=(-1)^{\overline{\phi_n}(\overline{z_n}+\overline{x_n})+\overline{\eta_n}\,\overline{z_n}}\omega_n\eta_n\phi_n\big(\langle[z_n,x_n],y_n\rangle\\
&\qquad+(-1)^{\overline{x_n}\,\overline{z_n}}\langle x_n,[z_n,y_n]\rangle\big)\\
&\quad=(-1)^{\overline{\phi_n}(\overline{z_n}+\overline{x_n})+\overline{\eta_n}\,\overline{z_n}}\omega_n\eta_n\phi_n(\mathrm{Id}\otimes\rho)(z_n)(\langle x_n,y_n\rangle)\\
&\quad=(\mathrm{Id}\otimes\rho)(a)(\langle x,y\rangle).
\end{aligned}$$

For the second part of the statement, recall that d_{Tot} is the differential on $\operatorname{Tot}(\mathcal{U},\Omega^*_X)$ when considering Ω^*_X as a complex with trivial differential. Again, since everything is defined componentwise, it is sufficient to prove that

$$\begin{aligned}
&\langle d_{\mathrm{Tot}}(\eta_n\otimes x_n),\omega_n\otimes y_n\rangle+(-1)^{\overline{x_n}+\overline{\eta_n}}\langle\eta_n\otimes x_n,d_{\mathrm{Tot}}(\omega_n\otimes y_n)\rangle\\
&\quad=d_{\mathrm{Tot}}\langle\eta_n\otimes x_n,\omega_n\otimes y_n\rangle,
\end{aligned}$$

for $\eta_n, \omega_n \in A_n$, $x_n \in \prod(\Omega^i_X[-i] \otimes \mathcal{L})(U_{i_1\dots i_n})$, and $y_n \in \prod(\Omega^j_X[-j] \otimes \mathcal{L})(U_{i_1\dots i_n})$. Then

$$\begin{aligned}
&\langle d_{\mathrm{Tot}}(\eta_n \otimes x_n), \omega_n \otimes y_n\rangle + (-1)^{\overline{x_n}+\overline{\eta_n}}\langle \eta_n \otimes x_n, d_{\mathrm{Tot}}(\omega_n \otimes y_n)\rangle \\
&\quad = \langle d_{A_n}\eta_n \otimes x_n + (-1)^{\overline{\eta_n}}\eta_n \otimes dx_n, \omega_n \otimes y_n\rangle \\
&\qquad + (-1)^{\overline{x_n}+\overline{\eta_n}}\langle \eta_n \otimes x_n, d_{A_n}\omega_n \otimes y_n + (-1)^{\overline{\omega_n}}\omega_n \otimes dy_n\rangle \\
&\quad = (-1)^{\overline{\omega_n}\,\overline{x_n}} d_{A_n}(\eta_n)\omega_n\langle x_n, y_n\rangle \\
&\qquad + (-1)^{\overline{\eta_n}+\overline{\omega_n}(\overline{x_n}+1)}\eta_n\omega_n\langle dx_n, y_n\rangle \\
&\qquad + (-1)^{\overline{x_n}+\overline{\eta_n}+(\overline{\omega_n}+1)\overline{x_n}}\eta_n d_{A_n}(\omega_n)\langle x_n, y_n\rangle \\
&\qquad + (-1)^{\overline{x_n}+\overline{\eta_n}+\overline{\omega_n}+\overline{\omega_n}\,\overline{x_n}}\eta_n\omega_n\langle x_n, dy_n\rangle \\
&\quad = (-1)^{\overline{\omega_n}\,\overline{x_n}} d_{A_n}(\eta_n\omega_n)\langle x_n, y_n\rangle \\
&\qquad + (-1)^{\overline{\eta_n}+\overline{\omega_n}(\overline{x_n}+1)}\eta_n\omega_n(\langle dx_n, y_n\rangle + (-1)^{\overline{x_n}}\langle x_n, dy_n\rangle) \\
&\quad = (-1)^{\overline{\omega_n}\,\overline{x_n}} d_{A_n}(\eta_n\omega_n)\langle x_n, y_n\rangle \\
&\quad = d_{A_n}\langle \eta_n \otimes x_n, \omega_n \otimes y_n\rangle \\
&\quad = d_{\mathrm{Tot}}\langle \eta_n \otimes x_n, \omega_n \otimes y_n\rangle,
\end{aligned}$$

where d_{A_n} denotes the differential on A_n, the differential graded algebra of polynomial differential forms on the affine n-simplex. ∎

Corollary 3.9. *Let D in $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$ be a simplicial lifting of the identity and let*

$$\nabla = [D, -] \colon \mathrm{Tot}(\mathcal{U}, \mathcal{L}) \longrightarrow \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$$

be its associated connection, as in Definition 2.12. Then for any cyclic form

$$\langle -, -\rangle \colon \mathrm{Tot}(\mathcal{U}, \Omega^i_X[-i] \otimes \mathcal{L}) \times \mathrm{Tot}(\mathcal{U}, \Omega^j_X[-j] \otimes \mathcal{L}) \longrightarrow \mathrm{Tot}(\mathcal{U}, \Omega^{i+j}_X[-i-j]),$$

with $i, j \geq 0$, we have

$$\langle \nabla(x), y\rangle + (-1)^{\overline{x}}\langle x, \nabla(y)\rangle = d_{dR}\langle x, y\rangle$$

for $x, y \in \mathrm{Tot}(\mathcal{U}, \mathcal{L})$.

Proof. It follows from the cyclicity of the form and by Remark 2.10. ∎

The next part is dedicated to defining an L_∞-morphism associated to a connection and to a d_{Tot}-closed cyclic form on a transitive DG-Lie algebroid. We assume that the reader is familiar with the notions and basic properties of DG-Lie algebras and L_∞-morphisms between them; details can be found in [13, 19, 23, 24] and in the references therein. The definition of an L_∞-morphism between a DG-Lie algebra and an abelian DG-Lie algebra, i.e., a DG-Lie algebra with trivial bracket is recalled here, because it will be needed for explicit calculations.

Let V be a graded vector space over a field of characteristic zero. Let σ be a permutation of $\{1,\dots,n\}$, and let $v_1,\dots,v_n$ be homogeneous vectors of V; denote by $\chi(\sigma; v_1,\dots,v_n) = \pm 1$ the antisymmetric Koszul sign, defined by the relation

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \chi(\sigma; v_1,\dots,v_n)\, v_1 \wedge \cdots \wedge v_n$$

in the nth exterior power $V^{\wedge n}$. If the vectors $v_1,\dots,v_n$ are clear from the context we will write $\chi(\sigma)$ instead of $\chi(\sigma; v_1,\dots,v_n)$. Given two non-negative integers p and q, $S(p,q)$ denotes the set of (p,q)-shuffles, the permutations σ of the set $\{1,\dots,p+q\}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p); \quad \sigma(p+1) < \cdots < \sigma(p+q).$$

Recall that the cardinality of $S(p,q)$ is $\binom{p+q}{p}$. Because of the universal property of wedge powers, every linear map $V^{\wedge p} \to W$ will be interpreted as a graded skew-symmetric p-linear map $V \times \cdots \times V \to W$.

Definition 3.10. Let $(V, \delta, [-,-])$ be a DG-Lie algebra and (M, d) an abelian DG-Lie algebra. An L_∞-morphism $g\colon V \to M$ is a sequence of maps $g_n\colon V^{\wedge n} \to M$, $n \geq 1$, with g_n of degree $1-n$ such that, for every n and every homogeneous $v_1,\dots,v_n \in V$, the following conditions (C_i) are satisfied for all $i \in \mathbb{N}$.

$$dg_1(v_1) = g_1(\delta v_1), \tag{C_1}$$

and

$$\begin{aligned} dg_n(v_1,\dots,v_n) &= (-1)^{1-n} \sum_{\sigma\in S(1,n-1)} \chi(\sigma) g_n(\delta v_{\sigma(1)}, v_{\sigma(2)},\dots,v_{\sigma(n)}) \\ &\quad + (-1)^{2-n} \sum_{\sigma\in S(2,n-2)} \chi(\sigma) g_{n-1}([v_{\sigma(1)}, v_{\sigma(2)}], v_{\sigma(3)},\dots,v_{\sigma(n)}). \end{aligned} \tag{C_n}$$

Remark 3.11. Notice that condition (C_1) entails that the linear component g_1 induces a map in cohomology $g_1\colon H^*(V) \to H^*(M)$. It is clear that the cohomology $H^*(M)$ of an abelian DG-Lie algebra M is an abelian graded Lie algebra. Condition (C_2) can be written as

$$g_1([v_1,v_2]) = dg_2(v_1,v_2) + g_2(\delta v_1, v_2) + (-1)^{\overline{v_1}} g_2(v_1, \delta v_2),$$

which implies that the map induced by g_1 in cohomology is a morphism of graded Lie algebras.

Recall that since the functor Tot sends semicosimplicial DG-Lie algebras to DG-Lie algebras, the complex $\operatorname{Tot}(\mathcal{U},\mathcal{L})$ is a DG-Lie algebra. The complex of $\mathcal{O}_X$-modules $\Omega_X^{\leq 1} = \mathcal{O}_X \xrightarrow{d_{dR}} \Omega_X^1$ can be considered as a sheaf of abelian DG-Lie algebras, hence it gives rise to a semicosimplicial abelian DG-Lie algebra; therefore, the complex $\operatorname{Tot}(\mathcal{U},\Omega_X^{\leq 1}[2])$ is an abelian DG-Lie algebra.

THEOREM 3.12. *Let $(\mathcal{A},\rho)$ be a transitive DG-Lie algebroid over a smooth separated scheme X of finite type over a field $\mathbb{K}$ of characteristic zero. Let $\mathcal{L} = \operatorname{Ker}\rho$ be a finite complex of locally free sheaves and let, for $i, j \geq 0$,*

$$\langle -,-\rangle \colon \operatorname{Tot}(\mathcal{U},\Omega_X^i[-i]\otimes\mathcal{L}) \times \operatorname{Tot}(\mathcal{U},\Omega_X^j[-j]\otimes\mathcal{L}) \longrightarrow \operatorname{Tot}(\mathcal{U},\Omega_X^{i+j}[-i-j])$$

be a cyclic form which is d_{Tot}-closed. For every simplicial lifting of the identity $D \in \operatorname{Tot}(\mathcal{U},\Omega_X^1[-1]\otimes\mathcal{A})$ there exists a L_∞-morphism between DG-Lie algebras on the field $\mathbb{K}$,

$$f\colon \operatorname{Tot}(\mathcal{U},\mathcal{L}) \rightsquigarrow \operatorname{Tot}(\mathcal{U},\Omega_X^{\leq 1}[2]),$$

with components

$$\begin{aligned} f_1(x) &= \langle u, x\rangle,\\ f_2(x,y) &= \frac{1}{2}\big(\langle\nabla(x),y\rangle - (-1)^{\bar{x}\bar{y}}\langle\nabla(y),x\rangle\big),\\ f_3(x,y,z) &= -\frac{1}{2}\langle x,[y,z]\rangle,\\ f_n &= 0 \ \forall n\geq 4, \end{aligned}$$

where $\nabla = [D,-]\colon \operatorname{Tot}(\mathcal{U},\mathcal{L}) \to \operatorname{Tot}(\mathcal{U},\Omega_X^1[-1]\otimes\mathcal{L})$ denotes the connection associated to the simplicial lifting of the identity D, and $u = d_{\mathrm{Tot}}D$ its extension cocycle.

PROOF. The strategy of the proof is to check that the conditions (C_n) of Definition 3.10 hold for $n = 1,2,3,4$. In fact, since $f_n = 0$ for $n \geq 4$, the conditions are automatically satisfied for $n \geq 5$.

Denote by d_{Tot} the differential on $\operatorname{Tot}(\mathcal{U},\mathcal{L})$, and by $d_{\mathrm{Tot}} + d_{dR}$ the differential on $\operatorname{Tot}(\mathcal{U},\Omega_X^{\leq 1}[2])$. Condition ($C_1$) requires that

$$f_1(d_{\mathrm{Tot}}x) = (d_{\mathrm{Tot}} + d_{dR})f_1(x);$$

notice, however, that since u belongs to $\operatorname{Tot}(\mathcal{U},\Omega_X^1[-1]\otimes\mathcal{L})$, we have the equality $(d_{\mathrm{Tot}} + d_{dR})f_1 = d_{\mathrm{Tot}}f_1$ in $\operatorname{Tot}(\mathcal{U},\Omega_X^{\leq 1}[2])$. Then, by the d_{Tot}-closure of the cyclic form and by the fact that u is closed,

$$\begin{aligned} f_1(d_{\mathrm{Tot}}x) = \langle u, d_{\mathrm{Tot}}x\rangle &= (-1)^{\bar{u}}d_{\mathrm{Tot}}\langle u,x\rangle - (-1)^{\bar{u}}\langle d_{\mathrm{Tot}}u, x\rangle\\ &= d_{\mathrm{Tot}}\langle u,x\rangle = d_{\mathrm{Tot}}f_1(x). \end{aligned}$$

For $n = 2$ the condition is

$$f_2(d_{\mathrm{Tot}}x, y) + (-1)^{\overline{x}} f_2(x, d_{\mathrm{Tot}}y) = f_1([x, y]) - (d_{\mathrm{Tot}} + d_{dR}) f_2(x, y).$$

By definition of f_2, we have again that $(d_{\mathrm{Tot}} + d_{dR}) f_2 = d_{\mathrm{Tot}} f_2$, and then, using Lemma 2.14,

$$\begin{aligned}
&f_2(d_{\mathrm{Tot}}x, y) + (-1)^{\overline{x}} f_2(x, d_{\mathrm{Tot}}y)\\
&\quad= \frac{1}{2}\big(\langle \nabla(d_{\mathrm{Tot}}x), y\rangle - (-1)^{(\overline{x}+1)\overline{y}}\langle \nabla(y), d_{\mathrm{Tot}}x\rangle\\
&\qquad + (-1)^{\overline{x}}\langle \nabla(x), d_{\mathrm{Tot}}y\rangle - (-1)^{\overline{x}\,\overline{y}}\langle \nabla(d_{\mathrm{Tot}}y), x\rangle\big)\\
&\quad= \frac{1}{2}\big(-\langle d_{\mathrm{Tot}}\nabla(x), y\rangle + \langle [u, x], y\rangle\\
&\qquad - (-1)^{(\overline{x}+1)\overline{y}}\langle \nabla(y), d_{\mathrm{Tot}}x\rangle + (-1)^{\overline{x}}\langle \nabla(x), d_{\mathrm{Tot}}y\rangle\\
&\qquad + (-1)^{\overline{x}\,\overline{y}}\langle d_{\mathrm{Tot}}\nabla(y), x\rangle - (-1)^{\overline{x}\,\overline{y}}\langle [u, y], x\rangle\big)\\
&\quad= \frac{1}{2}\big(\langle u, [x, y]\rangle - (-1)^{\overline{x}\,\overline{y}}\langle u, [y, x]\rangle\\
&\qquad - d_{\mathrm{Tot}}\langle \nabla(x), y\rangle + (-1)^{\overline{x}\,\overline{y}} d_{\mathrm{Tot}}\langle \nabla(y), x\rangle\big)\\
&\quad= \langle u, [x, y]\rangle - \frac{1}{2} d_{\mathrm{Tot}}(\langle \nabla(x), y\rangle - (-1)^{\overline{x}\,\overline{y}}\langle \nabla(y), x\rangle)\\
&\quad= f_1([x, y]) - d_{\mathrm{Tot}} f_2(x, y).
\end{aligned}$$

Condition (C_3) is the following:

$$\begin{aligned}
(d_{\mathrm{Tot}} + d_{dR}) f_3(x, y, z) = {}& f_3(d_{\mathrm{Tot}}x, y, z) - (-1)^{\overline{x}\,\overline{y}} f_3(d_{\mathrm{Tot}}y, x, z)\\
& + (-1)^{\overline{z}(\overline{x}+\overline{y})} f_3(d_{\mathrm{Tot}}z, x, y) - f_2([x, y], z)\\
& + (-1)^{\overline{y}\,\overline{z}} f_2([x, z], y) - (-1)^{\overline{x}(\overline{y}+\overline{z})} f_2([y, z], x),
\end{aligned}$$

and we begin by noting that by the d_{Tot}-closure

$$\begin{aligned}
&f_3(d_{\mathrm{Tot}}x, y, z) - (-1)^{\overline{x}\,\overline{y}} f_3(d_{\mathrm{Tot}}y, x, z) + (-1)^{\overline{z}(\overline{x}+\overline{y})} f_3(d_{\mathrm{Tot}}z, x, y)\\
&\quad= -\frac{1}{2}\big(\langle d_{\mathrm{Tot}}x, [y, z]\rangle - (-1)^{\overline{x}\,\overline{y}}\langle d_{\mathrm{Tot}}y, [x, z]\rangle + (-1)^{\overline{z}(\overline{x}+\overline{y})}\langle d_{\mathrm{Tot}}z, [x, y]\rangle\big)\\
&\quad= -\frac{1}{2}\big(\langle d_{\mathrm{Tot}}x, [y, z]\rangle - (-1)^{\overline{x}\,\overline{y}}\langle [d_{\mathrm{Tot}}y, x], z\rangle + (-1)^{\overline{x}+\overline{y}}\langle [x, y], d_{\mathrm{Tot}}z\rangle\big)\\
&\quad= -\frac{1}{2}\big(\langle d_{\mathrm{Tot}}x, [y, z]\rangle + (-1)^{\overline{x}}\langle [x, d_{\mathrm{Tot}}y], z\rangle + (-1)^{\overline{x}+\overline{y}}\langle x, [y, d_{\mathrm{Tot}}z]\rangle\big)\\
&\quad= -\frac{1}{2}\big(\langle d_{\mathrm{Tot}}x, [y, z]\rangle + (-1)^{\overline{x}}\langle x, d_{\mathrm{Tot}}[y, z]\rangle\big)\\
&\quad= -\frac{1}{2} d_{\mathrm{Tot}}\langle x, [y, z]\rangle = d_{\mathrm{Tot}} f_3(x, y, z).
\end{aligned}$$

On the other hand, by Corollary 3.9,

$$
\begin{aligned}
&- f_2([x,y],z) + (-1)^{\overline{y}\,\overline{z}} f_2([x,z],y) - (-1)^{\overline{x}(\overline{y}+\overline{z})} f_2([y,z],x) \\
&\quad = -\frac{1}{2}\big(\langle \nabla([x,y]), z\rangle - (-1)^{\overline{z}(\overline{x}+\overline{y})}\langle \nabla(z), [x,y]\rangle - (-1)^{\overline{y}\,\overline{z}}\langle \nabla([x,z]), y\rangle \\
&\qquad + (-1)^{\overline{x}\,\overline{y}}\langle \nabla(y), [x,z]\rangle + (-1)^{\overline{x}(\overline{y}+\overline{z})}\langle \nabla([y,z]), x\rangle - \langle \nabla(x), [y,z]\rangle\big) \\
&\quad = -\frac{1}{2}\big(\langle [\nabla(x), y], z\rangle + (-1)^{\overline{x}}\langle [x, \nabla(y)], z\rangle - (-1)^{\overline{z}(\overline{x}+\overline{y})}\langle \nabla(z), [x,y]\rangle \\
&\qquad - (-1)^{\overline{y}\,\overline{z}}\langle [\nabla(x), z], y\rangle - (-1)^{\overline{y}\,\overline{z}+\overline{x}}\langle [x, \nabla(z)], y\rangle \\
&\qquad + (-1)^{\overline{x}\,\overline{y}}\langle \nabla(y), [x,z]\rangle + (-1)^{\overline{x}(\overline{y}+\overline{z})}\langle [\nabla(y), z], x\rangle \\
&\qquad + (-1)^{\overline{x}(\overline{y}+\overline{z})+\overline{y}}\langle [y, \nabla(z)], x\rangle - \langle \nabla(x), [y,z]\rangle\big) \\
&\quad = -\frac{1}{2}\big(\langle \nabla(x), [y,z]\rangle + (-1)^{\overline{x}}\langle x, \nabla([y,z])\rangle\big) \\
&\quad = -\frac{1}{2} d_{dR}\langle x, [y,z]\rangle = d_{dR} f_3(x,y,z).
\end{aligned}
$$

Lastly, condition (C_4) is

$$
\begin{aligned}
&f_3([a_1,a_2],a_3,a_4) + (-1)^{(\overline{a_1}+\overline{a_2})(\overline{a_3}+\overline{a_4})} f_3([a_3,a_4],a_1,a_2) \\
&\quad + (-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})} f_3([a_2,a_3],a_1,a_4) \\
&\quad - (-1)^{\overline{a_3}\,\overline{a_4}+\overline{a_1}\,\overline{a_2}+\overline{a_1}\,\overline{a_4}} f_3([a_2,a_4],a_1,a_3) \\
&\quad - (-1)^{\overline{a_2}\,\overline{a_3}} f_3([a_1,a_3],a_2,a_4) \\
&\quad + (-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})} f_3([a_1,a_4],a_2,a_3) = 0
\end{aligned}
$$

By the graded Jacobi identity we have

$$
\begin{aligned}
&-\frac{1}{2}\big(\langle [a_1,a_2],[a_3,a_4]\rangle - (-1)^{\overline{a_2}\,\overline{a_3}}\langle [a_1,a_3],[a_2,a_4]\rangle \\
&\qquad + (-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})}\langle [a_1,a_4],[a_2,a_3]\rangle \\
&\qquad + (-1)^{(\overline{a_1}+\overline{a_2})(\overline{a_3}+\overline{a_4})}\langle [a_3,a_4],[a_1,a_2]\rangle \\
&\qquad - (-1)^{\overline{a_3}\,\overline{a_4}+\overline{a_1}\,\overline{a_2}+\overline{a_1}\,\overline{a_4}}\langle [a_2,a_4],[a_1,a_3]\rangle \\
&\qquad + (-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})}\langle [a_2,a_3],[a_1,a_4]\rangle\big) \\
&\quad = -(\langle [a_1,a_2],[a_3,a_4]\rangle - (-1)^{\overline{a_2}\,\overline{a_3}}\langle [a_1,a_3],[a_2,a_4]\rangle \\
&\qquad + (-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})}\langle [a_1,a_4],[a_2,a_3]\rangle) \\
&\quad = -(\langle a_1,[a_2,[a_3,a_4]]\rangle - (-1)^{\overline{a_2}\,\overline{a_3}}\langle a_1,[a_3,[a_2,a_4]]\rangle \\
&\qquad + (-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})}\langle a_1,[a_4,[a_2,a_3]]\rangle) \\
&\quad = -\langle a_1, [a_2,[a_3,a_4]] - (-1)^{\overline{a_2}\,\overline{a_3}}[a_3,[a_2,a_4]] - [[a_2,a_3],a_4]\rangle = 0 \qquad \blacksquare
\end{aligned}
$$

We can now state the results of [20] for a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme X of finite type on a field $\mathbb{K}$ of characteristic zero.

REMARK 3.13. It is not very restrictive to require that a coherent sheaf on X has a finite locally free resolution: in fact, by [14, III, Exercises 6.8, 6.9] every coherent sheaf on a smooth, Noetherian, integral, separated scheme admits a finite locally free resolution.

Let $(\mathcal{E}, d_{\mathcal{E}})$ be a finite complex of locally free sheaves. Consider the DG-Lie algebroid of derivations of pairs $\mathcal{D}^*(X, \mathcal{E})$ of Example 2.6 (see also [18]), and the short exact sequence

$$0 \longrightarrow \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \longrightarrow \mathcal{D}^*(X, \mathcal{E}) \xrightarrow{\alpha} \Theta_X \longrightarrow 0;$$

it was noted in Example 2.6 that by tensoring with $\Omega^1_X[-1]$ one obtains

$$0 \longrightarrow \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X[-1] \otimes \mathcal{E}) \xrightarrow{g \mapsto (0,g)} \mathcal{J}^*_{\Omega^1} \xrightarrow{(\beta,g)\mapsto\beta} \mathcal{D}er_{\mathbb{K}}(\mathcal{O}_X, \Omega^1_X[-1]) \longrightarrow 0.$$

Fixing an affine open cover $\mathcal{U}$ of X and applying the Tot functor, we get the short exact sequence

$$0 \longrightarrow \operatorname{Tot}(\mathcal{U}, \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X[-1] \otimes \mathcal{E})) \longrightarrow \operatorname{Tot}(\mathcal{U}, \mathcal{J}^*_{\Omega^1}) - \cdots$$
$$\cdots \to \operatorname{Tot}(\mathcal{U}, \mathcal{D}er_{\mathbb{K}}(\mathcal{O}_X, \Omega^1_X[-1])) \longrightarrow 0.$$

We have already remarked that a lifting of the identity in $\mathcal{J}^*_{\Omega^1}$ is equivalent to a global algebraic connection on every component $\mathcal{E}^i$; hence, a lifting to $\operatorname{Tot}(\mathcal{U}, \mathcal{J}^*_{\Omega})$ of the universal derivation $d_{dR}\colon \mathcal{O}_X \to \Omega^1_X[-1]$ in $\operatorname{Tot}(\mathcal{U}, \mathcal{D}er_{\mathbb{K}}(\mathcal{O}_X, \Omega^1_X[-1]))$ can be termed a *simplicial connection* on the complex of locally free sheaves $\mathcal{E}$. As seen in Example 3.3, a natural cyclic form to consider is the one induced by

$$\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \times \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \longrightarrow \mathcal{O}_X, \quad (a, b) \longmapsto -\operatorname{Tr}(ab),$$

where $\operatorname{Tr}\colon \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_X$ is the usual trace map. Then the L_∞-morphism of Theorem 3.12 yields the following.

COROLLARY 3.14. *Let $\mathcal{E}$ be a finite complex of locally free sheaves on a smooth separated scheme X of finite type over a field $\mathbb{K}$ of characteristic zero. For every simplicial connection $D \in \operatorname{Tot}(\mathcal{U}, \mathcal{J}^*_{\Omega^1})$ there exists an L_∞-morphism between DG-Lie algebras on the field $\mathbb{K}$*

$$g\colon \operatorname{Tot}(\mathcal{U}, \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \rightsquigarrow \operatorname{Tot}(\mathcal{U}, \Omega^{\leq 1}_X[2]),$$

with components

$$\begin{aligned} g_1(x) &= -\operatorname{Tr}(ux),\\ g_2(x,y) &= -\frac{1}{2}\operatorname{Tr}(\nabla(x)y-(-1)^{\bar{x}\,\bar{y}}\nabla(y)x),\\ g_3(x,y,z) &= \frac{1}{2}\operatorname{Tr}(x,[y,z]),\\ g_n &= 0 \quad \forall n\geq 4. \end{aligned}$$

Hence, the applications to deformation theory of [20], stated in the context of complex manifolds, are also valid in the algebraic context, as announced.

Let $\mathcal{F}$ be a coherent sheaf on X admitting a finite locally free resolution, and denote

$$\sigma=\sum_{q\geq 0}\sigma_q\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F})\longrightarrow\prod_{q\geq 0}H^{q+2}(X,\Omega^q_X),\quad \sigma(c)=\operatorname{Tr}(\exp(-\operatorname{At}(\mathcal{F}))\circ c),$$

the Buchweitz–Flenner semiregularity map of [7]. In the previous formula, we have that $\operatorname{At}(\mathcal{F})\in\operatorname{Ext}^1_X(\mathcal{F},\mathcal{F}\otimes\Omega^1_X)$ denotes the Atiyah class of $\mathcal{F}$, the exponential of its opposite

$$\exp(-\operatorname{At}(\mathcal{F}))\in\prod_{q\geq 0}\operatorname{Ext}^q_X(\mathcal{F},\mathcal{F}\otimes\Omega^q_X)$$

is obtained via the Yoneda pairing

$$\begin{aligned} \operatorname{Ext}^i_X(\mathcal{F},\mathcal{F}\otimes\Omega^i_X)\times\operatorname{Ext}^j_X(\mathcal{F},\mathcal{F}\otimes\Omega^j_X)&\longrightarrow\operatorname{Ext}^{i+j}_X(\mathcal{F},\mathcal{F}\otimes\Omega^{i+j}_X)\\ (a,b)&\longmapsto a\circ b, \end{aligned}$$

and Tr denotes the trace maps

$$\operatorname{Tr}\colon\operatorname{Ext}^i_X(\mathcal{F},\mathcal{F}\otimes\Omega^j_X)\longrightarrow H^i(X,\Omega^j_X),\quad i,j\geq 0.$$

For every $q\geq 0$ one can consider the composition

$$\tau_q\colon\operatorname{Ext}^2_X(\mathcal{F},\mathcal{F})\xrightarrow{\sigma_q}H^{q+2}(X,\Omega^q_X)=H^2(X,\Omega^q_X[q])\xrightarrow{i_q}\mathbb{H}^2(X,\Omega^{\leq q}_X[2q]),$$

where $\Omega^{\leq q}_X=(\bigoplus_{i=0}^q\Omega^i_X[-i],d_{dR})$ is the truncated de Rham complex and i_q is induced by the inclusion of complexes $\Omega^q_X[q]\subset\Omega^{\leq q}_X[2q]$. The map τ_q is the *q-component of the modified Buchweitz–Flenner semiregularity map*. It is convenient to also consider the maps

$$\begin{aligned} \sigma_q\colon\operatorname{Ext}^*_X(\mathcal{F},\mathcal{F})&\longrightarrow H^*(X,\Omega^q_X[q])\\ \tau_q\colon\operatorname{Ext}^*_X(\mathcal{F},\mathcal{F})&\xrightarrow{\sigma_q}H^*(X,\Omega^q_X[q])\xrightarrow{i_q}\mathbb{H}^*(X,\Omega^{\leq q}_X[2q]) \end{aligned}$$

defined by the same formulas.

Corollary 3.15. *Let $\mathcal{F}$ be a coherent sheaf admitting a finite locally free resolution $\mathcal{E}$ on a smooth separated scheme X of finite type over a field $\mathbb{K}$ of characteristic zero. Then every simplicial connection on the resolution $\mathcal{E}$ gives a lifting of the map*

$$\tau_1\colon \operatorname{Ext}^*_X(\mathcal{F},\mathcal{F}) \longrightarrow \mathbb{H}^*(X, \Omega_X^{\leq 1}[2])$$

to an L_∞-morphism

$$g\colon \operatorname{Tot}(\mathcal{U}, \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \rightsquigarrow \operatorname{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2]).$$

Recall that to a DG-Lie algebra M over a field $\mathbb{K}$ of characteristic zero we can associate a functor $\operatorname{Def}_M\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$, the functor of Maurer–Cartan solutions modulo gauge action (for more details see e.g. [11,17,18,23,24]). It is well known that the second cohomology $H^2(M)$ of the cochain complex underlying M is an obstruction space for the deformation functor Def_M. Recall also that an L_∞-morphism between DG-Lie algebras $g\colon V \rightsquigarrow M$ gives a morphism of deformation functors $g\colon \operatorname{Def}_V \to \operatorname{Def}_M$ such that the map induced in cohomology commutes with obstruction maps. If the DG-Lie algebra M has trivial bracket, every obstruction in Def_M is trivial, and therefore every obstruction in Def_V belongs to the kernel of the map $g\colon H^2(V) \to H^2(M)$.

Corollary 3.16. *Let $\mathcal{F}$ be a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme X of finite type over a field $\mathbb{K}$ of characteristic zero. Then every obstruction to the deformations of $\mathcal{F}$ belongs to the kernel of the map*

$$\tau_1\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \longrightarrow \mathbb{H}^2(X, \Omega_X^{\leq 1}[2]).$$

If the Hodge to de Rham spectral sequence of X degenerates at E_1, then every obstruction to the deformations of $\mathcal{F}$ belongs to the kernel of the map

$$\sigma_1\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \longrightarrow H^3(X, \Omega^1_X), \quad \sigma_1(a) = -\operatorname{Tr}(\operatorname{At}(\mathcal{F}) \circ a).$$

Proof. If $\mathcal{E}$ is a finite locally free resolution of $\mathcal{F}$, then the DG-Lie algebra $\operatorname{Tot}(\mathcal{U}, \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ controls the deformations of $\mathcal{F}$, see e.g. [11]. According to Corollary 3.15, the map

$$\tau_1\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \longrightarrow \mathbb{H}^2(X, \Omega_X^{\leq 1}[2])$$

lifts to an L_∞-morphism

$$g\colon \operatorname{Tot}(\mathcal{U}, \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \rightsquigarrow \operatorname{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2]),$$

whose linear component g_1 commutes with obstruction maps of the associated deformation functors. By construction, the DG-Lie algebra $\operatorname{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2])$ is abelian and therefore every obstruction of the associated deformation functor is trivial.

If the Hodge to de Rham spectral sequence of X degenerates at E_1 then the inclusion of complexes $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[1]) \to \operatorname{Tot}(\mathcal{U}, \Omega^{\leq 1}_X[2])$ is injective in cohomology, so that $H^3(X, \Omega^1_X) \hookrightarrow \mathbb{H}^2(X, \Omega^{\leq 1}_X[2])$ and the maps σ and τ have the same kernel. ■

Remark 3.17. In the setting of Theorem 3.12, if the cyclic form is induced by a DG-Lie algebroid representation $\theta\colon \mathcal{A} \to \mathcal{D}^*(X, \mathcal{E})$, the L_∞-morphism can be obtained up to a sign from the the L_∞-morphism of Corollary 3.14 as follows. Let $D \in \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$ denote a simplicial lifting of the identity, and denote by

$$\operatorname{Id} \otimes \theta\colon \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A}) \longrightarrow \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{D}^*(X, \mathcal{E})) \cong \operatorname{Tot}(\mathcal{U}, \mathcal{J}^*_\Omega)$$

the induced map on the totalisation. Denoting as usual by α the anchor map of the transitive DG-Lie algebroid $\mathcal{D}^*(X, \mathcal{E})$, it is clear that $(\operatorname{Id} \otimes \theta)(D)$ is a simplicial lifting of the identity in $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{D}^*(X, \mathcal{E}))$:

$$\begin{aligned}(\operatorname{Id} \otimes \alpha)(\operatorname{Id} \otimes \theta)(D) &= \operatorname{Id} \otimes (\alpha \circ \theta)(D) = (\operatorname{Id} \otimes \rho)(D)\\ &= \operatorname{Id}_{\Omega^1} \in \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X).\end{aligned}$$

Let $u = d_{\mathrm{Tot}} D \in \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ denote the extension cocycle associated to D, then

$$(\operatorname{Id} \otimes \theta)(u) = (\operatorname{Id} \otimes \theta)(d_{\mathrm{Tot}} D) = d_{\mathrm{Tot}}(\operatorname{Id} \otimes \theta)(D).$$

Therefore, the L_∞-morphism $f\colon \operatorname{Tot}(\mathcal{U}, \mathcal{L}) \rightsquigarrow \operatorname{Tot}(\mathcal{U}, \Omega^{\leq 1}_X[2])$ associated to D and to $\langle -, - \rangle_\theta$ is the composition of the DG-Lie algebra morphism

$$\theta\colon \operatorname{Tot}(\mathcal{U}, \mathcal{L}) \longrightarrow \operatorname{Tot}(\mathcal{U}, \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$$

and of the L_∞-morphism

$$-g\colon \operatorname{Tot}(\mathcal{U}, \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \rightsquigarrow \operatorname{Tot}(\mathcal{U}, \Omega^{\leq 1}_X[2])$$

associated to the simplicial lifting of the identity $(\operatorname{Id} \otimes \theta)(D)$ and to the cyclic form $(a, b) \mapsto \operatorname{Tr}(ab)$.

4. The L_∞-morphism for the Atiyah Lie algebroid of a principal bundle

Since Lie algebroids arise naturally in connection with principal bundles, we give an application of the L_∞-morphism constructed in Theorem 3.12 to the deformation theory of principal bundles.

Let X be a smooth separated scheme of finite type over an algebraically closed field $\mathbb{K}$ of characteristic zero, let G be an affine algebraic group with Lie algebra $\mathfrak{g}$, and let

$P \to X$ be a principal G-bundle on X. By G-principal bundle we mean a G-fibration which is locally trivial for the Zariski topology, see e.g. [30]. We begin by finding a DG-Lie algebra that controls the deformations of P, using an argument similar to those in [4, 24, 31]. Let $\mathbf{Art}_{\mathbb{K}}$ be the category of Artin local $\mathbb{K}$-algebras with residue field $\mathbb{K}$. For any A in $\mathbf{Art}_{\mathbb{K}}$ denote by $\mathfrak{m}_A$ its maximal ideal and by 0 the closed point in Spec A.

To every semicosimplicial Lie algebra $\mathfrak{h}$ over $\mathbb{K}$,

$$\mathfrak{h}: \ \mathfrak{h}_0 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathfrak{h}_1 \underset{\delta_2}{\overset{\delta_0}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}} \mathfrak{h}_2 \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots,$$

there are associated two functors $Z^1_{\mathfrak{h}}, H^1_{\mathfrak{h}} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$, which here are described in brief; for more details see[12, 24]. The functor of non-abelian cocycles $Z^1_{\mathfrak{h}}$ is defined as

$$Z^1_{\mathfrak{h}}(A) = \{e^x \in \exp(\mathfrak{h}_1 \otimes \mathfrak{m}_A) \mid e^{\delta_1(x)} = e^{\delta_2(x)} e^{\delta_0(x)}\}.$$

For every $A \in \mathbf{Art}_{\mathbb{K}}$ there is a left action of $\exp(\mathfrak{h}_0 \otimes \mathfrak{m}_A)$ on $Z^1_{\mathfrak{h}}(A)$

$$\begin{aligned} \exp(\mathfrak{h}_0 \otimes \mathfrak{m}_A) \times Z^1_{\mathfrak{h}}(A) &\longrightarrow Z^1_{\mathfrak{h}}(A), \\ (e^a, e^x) &\longmapsto e^{\delta_1(a)} e^x e^{-\delta_0(a)}. \end{aligned}$$

The functor $H^1_{\mathfrak{h}} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ is then defined as

$$H^1_{\mathfrak{h}}(A) = \frac{Z^1_{\mathfrak{h}}(A)}{\exp(\mathfrak{h}_0 \otimes \mathfrak{m}_A)}.$$

Consider the Thom–Whitney totalisation functor Tot from semicosimplicial DG-vector spaces to DG-vector spaces (see Definition 2.7), and recall it takes semicosimplicial Lie algebras to DG-Lie algebras. We then have the following result, see [12, 15, 24].

PROPOSITION 4.1. *For every semicosimplicial Lie algebra $\mathfrak{h}$ there exists a natural isomorphism of functors $H^1_{\mathfrak{h}} \cong \operatorname{Def}_{\operatorname{Tot}(\mathfrak{h})}$.*

DEFINITION 4.2 ([4, 8]). An infinitesimal deformation of P over $A \in \mathbf{Art}_{\mathbb{K}}$ is the data of a principal G-bundle $P_A \to X \times \operatorname{Spec} A$ and an isomorphism $\theta : i^*(P_A) \cong P$.

$$\begin{array}{ccc} P & \longrightarrow & P_A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_A} \\ X & \xrightarrow{\ i\ } & X \times \operatorname{Spec} A. \end{array}$$

Two deformations (P_A, θ) and (P'_A, θ') are isomorphic if there exists an isomorphism of principal G-bundles $\lambda : P_A \to P'_A$ such that $\theta = \theta' \circ i^*(\lambda)$.

This defines a functor $\mathrm{Def}_P\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ such that $\mathrm{Def}_P(A)$ is the set of isomorphism classes of deformations of P over $A \in \mathbf{Art}_{\mathbb{K}}$. For every $A \in \mathbf{Art}_{\mathbb{K}}$, the set $\mathrm{Def}_P(A)$ contains the trivial deformation $P \times \operatorname{Spec} A \to X \times \operatorname{Spec} A$.

If M is a DG-Lie algebra such that $\mathrm{Def}_P \cong \mathrm{Def}_M$, where Def_M is the functor of Maurer-Cartan solutions modulo gauge action, one says that M controls the deformations of P.

Fix an open cover $\mathcal{U} = \{U_i\}$ of X such that P is trivial on every U_i, and let $\{g_{ij}\colon U_{ij} \to G\}$ denote the transition functions for P. Let $\mathfrak{g}$ be the Lie algebra of G.

- Let $\operatorname{ad} P = P \times^G \mathfrak{g}$ denote the adjoint bundle of P, with transition functions $\{\mathrm{Ad}_{g_{ij}}\}$, and let (P) denote the sheaf of sections of the vector bundle $\operatorname{ad} P$.
- The group G acts on itself by conjugation; denote by $\mathrm{Ad}\, P = P \times^G G$ the associated bundle corresponding to this action. Recall that $\Gamma(X, \mathrm{Ad}(P)) \cong \mathrm{Gauge}(P)$, where $\mathrm{Gauge}(P)$ is the group of bundle automorphisms of P.

There is a one to one correspondence between first order deformations of P, i.e., deformations over $\mathbb{K}[t]/(t^2) \in \mathbf{Art}_{\mathbb{K}}$, and $H^1(X, \mathcal{ad}(P))$, see e.g. [8,31]. This implies that on every affine open set the deformations of P are trivial.

Lemma 4.3. *Let $P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)$ be the associated bundle induced by the action* $\mathrm{Ad} \otimes \mathrm{Id}\colon G \times \mathfrak{g} \otimes \mathfrak{m}_A \to \mathfrak{g} \otimes \mathfrak{m}_A$. *Then there is an isomorphism*

$$\Gamma(P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)) \cong \Gamma(\mathcal{ad}(P)) \otimes \mathfrak{m}_A.$$

Proof. A section of $P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)$ is the data of

$$\{\omega_i\colon U_i \to \mathfrak{g} \otimes \mathfrak{m}_A \mid \omega_i(p) = (\mathrm{Ad}_{g_{ij}(p)} \otimes \mathrm{Id})\omega_j(p) \text{ for all } p \in U_{ij}\}.$$

Let $t_1, \ldots, t_n$ be a basis of the finite dimensional vector space $\mathfrak{m}_A$, then for every $p \in U_i$ one can write $\omega_i(p) = \sum_k h_{i,k}(p) \otimes t_k$. Since the action of G on $\mathfrak{g} \otimes \mathfrak{m}_A$ is defined as

$$g \cdot (x \otimes t) = \mathrm{Ad}_g(x) \otimes t,$$

the maps $h_{i,k}$ are such that $h_{i,k}(p) = \mathrm{Ad}_{g_{ij}(p)}\, h_{j,k}(p)$ for every $p \in U_{ij}$.

An element of $\Gamma(\mathcal{ad}(P)) \otimes \mathfrak{m}_A$ is a finite sum $\sum_k \eta_k \otimes t_k$, with η_k being sections of $\operatorname{ad} P$, so that each η_k is the data of

$$\{\eta_{k,i}\colon U_i \to \mathfrak{g} \mid \eta_{k,i}(p) = \mathrm{Ad}_{g_{ij}(p)}\, \eta_{k,j}(p) \text{ for all } p \in U_{ij}\}.$$

Then, setting $(\eta_{k,i} \otimes t_k)(p) = \eta_{k,i}(p) \otimes t_k$ for every $p \in U_i$, the data $\{\eta_{k,i} \otimes t_k\colon U_i \to \mathfrak{g} \otimes \mathfrak{m}_A\}$ is exactly a section of $P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)$. ∎

Lemma 4.4. *For every $A \in \mathbf{Art}_{\mathbb{K}}$ there is an isomorphism of groups*

$$\exp(\Gamma(ad(P)) \otimes \mathfrak{m}_A) \cong \begin{Bmatrix} \textit{automorphisms of the trivial} \\ \textit{deformation } P \times \operatorname{Spec} A \end{Bmatrix}.$$

Proof. Denote by $G^0(A)$ the group of morphisms $f\colon \operatorname{Spec} A \to G$ such that $f(0) = \mathrm{Id}_G$, and recall that there is an isomorphism of groups $\exp(\mathfrak{g} \otimes \mathfrak{m}_A) \cong G^0(A)$ (see e.g. [29, Section 10]). The group structure on $G^0(A)$ is induced by the group structure on G, while $\exp(\mathfrak{g} \otimes \mathfrak{m}_A)$ is a group with the Baker–Campbell–Hausdorff product. By Lemma 4.3,

$$\Gamma(ad(P)) \otimes \mathfrak{m}_A \cong \Gamma(P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)),$$

so that we can work with $\exp(\Gamma(P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)))$. Consider the associated bundle $P \times^G G^0(A)$, induced by the adjoint action of G on $G^0(A)$; the isomorphism $\exp(\mathfrak{g} \otimes \mathfrak{m}_A) \cong G^0(A)$ induces an isomorphism $\exp(\Gamma(P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A))) \cong \Gamma(P \times^G G^0(A))$. In fact, a section of $P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)$ is the data of

$$\{\eta_i\colon U_i \to \mathfrak{g} \otimes \mathfrak{m}_A \mid \eta_i(p) = (\mathrm{Ad}_{g_{ij}(p)} \otimes \mathrm{Id})\eta_j(p) \text{ for all } p \in U_{ij}\},$$

and composing with the exponential $\exp\colon \mathfrak{g} \otimes \mathfrak{m}_A \to G^0(A)$ we obtain

$$\{\exp \circ \eta_i\colon U_i \to G^0(A) \mid \exp \circ \eta_i(p) = g_{ij}(p)\exp \circ \eta_j(p) g_{ij}(p)^{-1} \text{ for all } p \in U_{ij}\}.$$

Notice that this data is equivalent to

$$\left\{ \lambda_i\colon U_i \times \operatorname{Spec} A \to G \;\middle|\; \begin{array}{l} \lambda_i(p,0) = \mathrm{Id}_G \text{ for all } p \in U_i, \\ \lambda_i(p) = g_{ij}(p)\lambda_j(p) g_{ij}(p)^{-1} \text{ for all } p \in U_{ij} \end{array} \right\},$$

which is a section of the associated bundle $\mathrm{Ad}(P \times \operatorname{Spec} A) = (P \times \operatorname{Spec} A) \times^G G$, where G acts on itself by conjugation.

For any G-principal bundle Q the global sections of the associated bundle $\mathrm{Ad}(Q) = Q \times^G G$ correspond to bundle automorphisms of Q. Therefore, the $\{\lambda_i\}$ give an element $F \in \mathrm{Gauge}(P \times \operatorname{Spec} A)$, and the condition $\lambda_i(p,0) = \mathrm{Id}_G$ for all $p \in U_i$ is equivalent to the fact that the automorphism F induces the identity when restricted to P, so that F is an automorphism of the trivial deformation. ∎

Proposition 4.5. *Let $\mathcal{U} = \{U_i\}$ be an affine open cover of X and let $ad(P)(\mathcal{U})$ be the semicosimplicial Lie algebra of Čech cochains*

$$\prod_i ad(P)(U_i) \rightrightarrows \prod_{i,j} ad(P)(U_{ij}) \overset{\textstyle\rightarrow}{\rightrightarrows} \prod_{i,j,k} ad(P)(U_{ijk}) \rightrightarrows\!\!\rightrightarrows \cdots .$$

There is a natural isomorphism of functors $H^1_{ad(P)(\mathcal{U})} \to \mathrm{Def}_P$.

Proof. Recall that all deformations of P on an affine open set are trivial, as mentioned above. Fix $A \in \mathbf{Art}_{\mathbb{K}}$; by Lemma 4.4 an element f of $Z^1_{ad(P)(\mathcal{U})}(A)$ is the data for every U_{ij} of isomorphisms $f_{ij}\colon P|_{U_{ij}} \times \operatorname{Spec} A \to P|_{U_{ij}} \times \operatorname{Spec} A$, which restrict to the identity $P|_{U_{ij}} \to P|_{U_{ij}}$ and such that $f_{ik} = f_{ij} f_{jk}$ for all i, j, k.

The last condition means that the $\{f_{ij}\}$ glue to obtain a principal G-bundle $P_A \to X \times \operatorname{Spec} A$ and isomorphisms

$$f_i\colon P_A|_{U_i \times \operatorname{Spec} A} \longrightarrow P|_{U_i} \times \operatorname{Spec} A$$

such that $f_{ij} = f_i f_j^{-1}$. Such isomorphisms coincide when restricted to

$$\overline{f}_i\colon i^*(P_A|_{U_i \times \operatorname{Spec} A}) \longrightarrow P|_{U_i}$$

and hence glue to an isomorphism of principal bundles $i^*(P_A) \to P$. This means that an element of $Z^1_{ad(P)(\mathcal{U})}(A)$ gives a locally trivial deformation of P over $A \in \mathbf{Art}_{\mathbb{K}}$.

An element of $\exp(\prod_i ad(P)(U_i) \otimes \mathfrak{m}_A)$ is again by Lemma 4.4 the data, for every U_i, of automorphisms $\lambda_i\colon P|_{U_i} \times \operatorname{Spec} A \to P|_{U_i} \times \operatorname{Spec} A$ which restrict to the identity $P|_{U_i} \to P|_{U_i}$. Two elements $f = \{f_{ij}\}, h = \{h_{ij}\}$ of $Z^1_{ad(P)(\mathcal{U})}(A)$ are equivalent under the action of $\lambda \in \exp(\prod_i ad(P)(U_i) \otimes \mathfrak{m}_A)$ if and only if $h_{ij} = \lambda_i f_{ij} \lambda_j^{-1}$ for all i, j.

$$\begin{array}{ccc} P_A|_{U_i \times \operatorname{Spec} A} & \xrightarrow{f_i} & P|_{U_i} \times \operatorname{Spec} A \\ \lambda \downarrow & & \downarrow \lambda_i \\ P'_A|_{U_i \times \operatorname{Spec} A} & \xrightarrow{h_i} & P|_{U_i} \times \operatorname{Spec} A \end{array}$$

This can be expressed as $h_i^{-1} \lambda_i f_i = h_j^{-1} \lambda_j f_j$, which means that the $\{\lambda_i\}$ glue to a bundle isomorphism $\lambda\colon P_A \to P'_A$, where P_A is the deformation corresponding to $\{f_{ij}\}$, and P'_A to $\{h_{ij}\}$. Since each λ_i restricts to the identity on $P|_{U_i}$, λ is an isomorphism of deformations. ∎

Corollary 4.6. *If $\mathcal{U} = \{U_i\}$ is an affine open cover of X, there is an isomorphism*

$$\operatorname{Def}_P \cong \operatorname{Def}_{\operatorname{Tot}(\mathcal{U}, ad(P))},$$

i.e., the DG-Lie algebra $\operatorname{Tot}(\mathcal{U}, ad(P))$ *controls the deformations of P.*

Proof. Consequence of Propositions 4.5 and 4.1 . ∎

We now specialise the L_∞-morphism of Section 3 to the Atiyah Lie algebroid of the principal G-bundle P.

A Lie algebroid is a DG-Lie algebroid (see Definition 2.1) concentrated in degree zero. Consider the *Atiyah Lie algebroid* of the principal bundle P introduced in [2], which is a Lie algebroid structure on the sheaf $\mathcal{Q}$ of sections of the vector bundle $Q = T_P/G$, the quotient of the tangent bundle of the total space T_P by the canonical induced G-action. There is a canonical short exact sequence of locally free sheaves over X

$$0 \longrightarrow \mathcal{ad}(P) \longrightarrow \mathcal{Q} \xrightarrow{\rho} \Theta_X \longrightarrow 0, \tag{4.1}$$

where $\mathcal{ad}(P)$ denotes the sheaf of sections of the adjoint bundle $\operatorname{ad} P = P \times^G \mathfrak{g}$ and $\rho\colon \mathcal{Q} \to \Theta_X$ is the anchor map. The vector bundle Q is the bundle of invariant tangent vector fields on P, and the Lie bracket on $\mathcal{Q}$ is induced by the Lie bracket of vector fields.

Definition 4.7 ([2]). A connection on the principal bundle $P \to X$ is a splitting of the exact sequence in (4.1). The *Atiyah class* of P is the extension class $\operatorname{At}_X(P) \in \operatorname{Ext}^1_X(\Theta_X, \mathcal{ad}(P)) \cong H^1(X, \Omega^1_X \otimes \mathcal{ad}(P))$ of the short exact sequence (4.1).

Therefore, the Atiyah class $\operatorname{At}_X(P)$ is trivial if and only if there exists a connection on P.

Let Ω^1_X denote the cotangent sheaf, and $\Omega^1_X[-1]$ the cotangent sheaf considered as a trivial complex of sheaves concentrated in degree one. As in Section 2, one can tensor the short exact sequence (4.1) with $\Omega^1_X[-1]$ to obtain a short exact sequence of complexes of sheaves

$$0 \longrightarrow \Omega^1_X[-1] \otimes \mathcal{ad}(P) \longrightarrow \Omega^1_X[-1] \otimes \mathcal{Q} \xrightarrow{\operatorname{Id}\otimes\rho} \Omega^1_X[-1] \otimes \Theta_X \longrightarrow 0.$$

Fix an affine open cover $\mathcal{U} = \{U_i\}$ of X; as in Section 2 the short exact sequence above induces a short exact sequence of DG-vector spaces

$$0 \longrightarrow \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{ad}(P)) \longrightarrow \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{Q}) - \cdots$$
$$\cdots \xrightarrow{\operatorname{Id}\otimes\rho} \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X) \longrightarrow 0,$$

and we denote by d_{Tot} the differentials of the above complexes.

It is easily seen that a lifting of the identity $\operatorname{Id}_{\Omega^1} \in \Gamma(X, \Omega^1_X[-1] \otimes \Theta_X)$ to $D \in \Gamma(X, \Omega^1_X[-1] \otimes \mathcal{Q})$ is equivalent to a splitting of the exact sequence in (4.1). Hence, in the case of a principal bundle P, a lifting of the identity can be identified with a connection on P. Therefore, we call a preimage of $\operatorname{Id}_{\Omega^1}$ in $\Omega^1_X[-1] \otimes \mathcal{Q}$ a germ of a connection on P, and we use the following terminology.

Definition 4.8. A *simplicial connection* on the principal bundle P is a lifting D in $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{Q})$ of the identity $\operatorname{Id}_{\Omega^1}$ in $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$.

Definition 4.9. The *Atiyah cocycle* of P is

$$u = d_{\mathrm{Tot}} D \in \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathit{ad}(P)).$$

It is natural to use the name Atiyah cocycle instead of extension cocycle of Definition 2.13, because its cohomology class is equal to the Atiyah class of Definition 4.7.

As in Definition 2.12, given a simplicial connection $D \in \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{Q})$ it is possible to define an adjoint operator

$$\nabla = [D, -] \colon \mathrm{Tot}(\mathcal{U}, \mathit{ad}(P)) \longrightarrow \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathit{ad}(P)).$$

A cyclic form on the Atiyah Lie algebroid $\mathcal{Q}$ is a symmetric bilinear form

$$\langle -, - \rangle \colon \mathit{ad}(P) \times \mathit{ad}(P) \to \mathcal{O}_X$$

such that for all $x, y \in \mathit{ad}(P)$ and $q \in \mathcal{Q}$,

$$\langle [q, x], y \rangle + \langle x, [q, y] \rangle = \rho(q)(\langle x, y \rangle),$$

where $\rho \colon \mathcal{Q} \to \Theta_X$ is the anchor map of the Atiyah Lie algebroid $\mathcal{Q}$.

Example 4.10. The cyclic form induced by the adjoint representation of a DG-Lie algebroid of Example 3.2 in this case can be constructed in an equivalent way, starting from the Killing form of the Lie algebra $\mathfrak{g}$ of the group G,

$$K \colon \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g} \longrightarrow \mathbb{K}, \qquad K(g, h) = \mathrm{Tr}(\mathrm{ad}\, g\, \mathrm{ad}\, h).$$

Take x, y in $\mathit{ad}(P)(U)$ and let $U = \bigcup_i U_i$ with U_i open sets trivialising the principal bundle P, then

$$x = \{x_i \colon U_i \to \mathfrak{g} \mid x_i(p) = \mathrm{Ad}_{g_{ij}(p)}\, x_j(p) \text{ for all } p \in U_{ij}\},$$

and analogously for y. Define $\langle x, y \rangle$ as $\{\langle x_i, y_i \rangle \colon U_i \to \mathbb{K}\}$, where for $p \in U_i$

$$\langle x_i, y_i \rangle(p) = K(x_i(p), y_i(p)).$$

This is well defined because the Killing form is invariant under automorphisms of the Lie algebra $\mathfrak{g}$, so that for $p \in U_{ij}$,

$$\begin{aligned} K(x_i(p), y_i(p)) &= K(\mathrm{Ad}_{g_{ij}(p)}\, x_j(p), \mathrm{Ad}_{g_{ij}(p)}\, y_j(p)) \\ &= K(x_j(p), y_j(p)). \end{aligned}$$

Recall that Tot preserves multiplicative structures, hence $\mathrm{Tot}(\mathcal{U}, \mathit{ad}(P))$ is a DG-Lie algebra. In the sequel, $\mathrm{Tot}(\mathcal{U}, \Omega^{\leq 1}_X[2]) = \mathrm{Tot}(\mathcal{U}, \mathcal{O}_X[2] \xrightarrow{d_{dR}} \Omega^1_X[1])$ is considered as a DG-Lie algebra with trivial bracket; its differential is denoted by $d_{\mathrm{Tot}} + d_{dR}$. Theorem 3.12 then yields the following.

Corollary 4.11. *For every simplicial connection D on a principal bundle P on a smooth separated scheme X of finite type over an algebraically closed field $\mathbb{K}$ of characteristic zero, endowed with a d_{Tot}-closed cyclic form*

$$\langle -,-\rangle\colon \mathrm{Tot}(\mathcal{U},\Omega^i_X[-i]\otimes ad(P))\times \mathrm{Tot}(\mathcal{U},\Omega^j_X[-j]\otimes ad(P)) \longrightarrow \mathrm{Tot}(\mathcal{U},\Omega^{i+j}_X[-i-j]),$$

for $i,j\geq 0$, there exists an L_∞-morphism of DG-Lie algebras on the field $\mathbb{K}$

$$f\colon \mathrm{Tot}(\mathcal{U},ad(P)) \rightsquigarrow \mathrm{Tot}(\mathcal{U},\Omega^{\leq 1}_X[2]),$$

with components

$$\begin{aligned} f_1(x) &= \langle u,x\rangle,\\ f_2(x,y) &= \frac{1}{2}\big(\langle\nabla(x),y\rangle-(-1)^{\bar{x}\,\bar{y}}\langle\nabla(y),x\rangle\big),\\ f_3(x,y,z) &= -\frac{1}{2}\langle x,[y,z]\rangle,\\ f_n &= 0\ \forall n\geq 4. \end{aligned}$$

As seen in Remark 3.11, the linear component f_1 of the L_∞-morphism induces a map of graded Lie algebras

$$f_1\colon H^*(\mathrm{Tot}(\mathcal{U},ad(P)))\longrightarrow H^*(\mathrm{Tot}(\mathcal{U},\Omega^{\leq 1}_X[2])),$$

which, since the open cover $\mathcal{U}$ is affine, becomes

$$f_1\colon H^*(X,ad(P))\longrightarrow \mathbb{H}^*(X,\Omega^{\leq 1}_X[2]).$$

Corollary 4.12. *Let P be a principal bundle on a smooth separated scheme X of finite type over an algebraically closed field $\mathbb{K}$ of characteristic zero and let*

$$\langle -,-\rangle\colon \mathrm{Tot}(\mathcal{U},\Omega^i_X[-i]\otimes ad(P))\times \mathrm{Tot}(\mathcal{U},\Omega^j_X[-j]\otimes ad(P)) \longrightarrow \mathrm{Tot}(\mathcal{U},\Omega^{i+j}_X[-i-j]),$$

for $i,j\geq 0$, be a d_{Tot}-closed cyclic form. Then every obstruction to the deformations of P belongs to the kernel of the map

$$f_1\colon H^2(X,ad(P))\longrightarrow \mathbb{H}^2(X,\Omega^{\leq 1}_X[2]),\qquad f_1(x)=\langle \mathrm{At}(P),x\rangle,$$

where $\mathrm{At}(P)$ denotes the Atiyah class of the principal bundle P.

Proof. The proof is analogous to the one of Corollary 3.16: the linear component of the L_∞-morphism of DG-Lie algebras of Corollary 4.11 induces a morphism in cohomology which commutes with the obstruction maps of the associated deformation functors, and the deformation functor associated to an abelian DG-Lie algebra has trivial obstructions. By Corollary 4.6, if $\mathcal{U} = \{U_i\}$ is an affine open cover of X, the DG-Lie algebra $\mathrm{Tot}(\mathcal{U}, ad(P))$ controls the deformations of P and an obstruction space is $H^2(\mathrm{Tot}(\mathcal{U}, ad(P))) \cong H^2(X, ad(P))$. Since the DG-Lie algebra $\mathrm{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2])$ is abelian, we obtain that f_1 annihilates all obstructions. ∎

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