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On a generalization of a result of Peskine and Szpiro

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Abstract – Let $(R, \mathfrak{m})$ be a regular local ring containing a field K. Let I be a Cohen–Macaulay ideal of height g. If char $K = p > 0$ then by a result of Peskine and Szpiro the local cohomology modules $H^i_I(R)$ vanish for $i > g$. This result is not true if char $K = 0$. However, we prove that the Bass numbers of the local cohomology module $H^g_I(R)$ completely determine whether $H^i_I(R)$ vanish for $i > g$. The result of this paper has been proved more generally for Gorenstein local rings by Hellus and Schenzel (2008) (Theorem 3.2). However, our result is elementary to prove. In particular, we do not use spectral sequences in our proof.

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1. Introduction

The result of this paper has been proved more generally for Gorenstein local rings by Hellus and Schenzel [2, Theorem 3.2]. *However, our result is elementary to prove. In particular, we do not use spectral sequences in our proof.*

Let $(R, \mathfrak{m})$ be a regular local ring containing a field K. Recall that an ideal I is said to be a Cohen–Macaulay ideal in R if R/I is a Cohen–Macaulay local ring. Motivated by a result of Peskine and Szpiro we make the following definition.

Definition 1.1. An ideal I of R is said to be a *Peskine–Szpiro ideal* of R if the following hold.

(1) I is a Cohen–Macaulay ideal.

(2) $H^i_I(R) = 0$ for all $i \neq \operatorname{height} I$.

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Note that as height I = grade I, we have $H^i_I(R) = 0$ for $i <$ height I. Thus, the only real condition for a Cohen–Macaulay ideal I to be a Peskine–Szpiro ideal is that $H^i_I(R) = 0$ for $i >$ height I. In their fundamental paper [7, Proposition III.4.1] Peskine and Szpiro proved that if char $K = p > 0$ then for all Cohen–Macaulay ideals I the local cohomology modules $H^i_I(R)$ vanish for $i >$ height I. This result is not true if char $K = 0$, for instance, see [4, Example 21.31]. We prove the following surprising result.

THEOREM 1.2. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d containing a field K. Let I be a Cohen–Macaulay ideal of height g. The following conditions are equivalent.*

(i) *I is a Peskine–Szpiro ideal of R.*

(ii) *For any prime ideal P of R containing I,*

$$\mu_i(P, H^g_I(R)) = \begin{cases} 1 & \text{if } i = \text{height } P - g, \\ 0 & \text{otherwise.} \end{cases}$$

Here the j-th Bass number of an R-module M with respect to a prime ideal P is defined as $\mu_j(P, M) = \dim_{k(P)} \operatorname{Ext}^j_{R_P}(k(P), M_P)$, where $k(P)$ is the residue field of R_P. Our result is essentially only an observation.

PROPERTIES 1.3. We need the following remarkable properties of local cohomology modules over regular local rings containing a field (proved by Huneke and Sharp [3] if char $K = p > 0$ and by Lyubeznik [5] if char $K = 0$). Let $(R, \mathfrak{m})$ be a regular ring containing a field K and I an ideal in R. Then the local cohomology modules of R with respect to I have the following properties.

(i) $H^j_{\mathfrak{m}}(H^i_I(R))$ is injective.

(ii) $\operatorname{injdim}_R H^i_I(R) \le \dim \operatorname{Supp} H^i_I(R)$.

(iii) The set of associated primes of $H^i_I(R)$ is finite.

(iv) All the Bass numbers of $H^i_I(R)$ are finite.

Here $\operatorname{injdim}_R H^i_I(R)$ denotes the injective dimension of $H^i_I(R)$. Also, $\operatorname{Supp} M = \{P \mid M_P \neq 0 \text{ and } P \text{ is a prime in } R\}$ is the support of an R-module M.

2. Permanence properties of Peskine–Szpiro ideals

In this section we prove some permanence properties of Peskine–Szpiro ideals. We also show that if $\dim R -$ height $I \le 2$ then a Cohen–Macaulay ideal I is a Peskine–Szpiro ideal.

Proposition 2.1. *Let* $(R, \mathfrak{m})$ *be a regular local ring containing a field* K. *Let* I *be a Peskine–Szpiro ideal of* R. *Let* $g = \operatorname{height} I$.

(1) *Let* P *be a prime ideal in* R *containing* I. *Then* I_P *is a Peskine–Szpiro ideal of* R_P.

(2) *Assume* $g < \dim R$. *Let* $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ *be* R/I*-regular. Then the ideal* $(I + (x))/(x)$ *is a Peskine–Szpiro ideal of* $R/(x)$.

Proof. (1) Note I_P is a Cohen–Macaulay ideal of height g in R_P. Also note that for $i \neq g$ we have

$$H^i_{I_P}(R_P) = H^i_I(R)_P = 0.$$

Thus, I_P is a Peskine–Szpiro ideal of R_P.

(2) Note that $J = (I + (x))/(x)$ is a Cohen–Macaulay ideal of height g in the regular ring $\overline{R} = R/(x)$. The short exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow \overline{R} \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow H^i_I(R) \longrightarrow H^i_J(\overline{R}) \longrightarrow H^{i+1}_I(R) \longrightarrow \cdots .$$

Thus, $H^i_J(\overline{R}) = 0$ for $i > g$. Therefore, J is a Peskine–Szpiro ideal of $\overline{R}$. ■

We now show that Cohen–Macaulay ideals of small dimensions are Peskine–Szpiro. This result is already known. However, we give a proof due to lack of a suitable reference.

Proposition 2.2. *Let* $(R, \mathfrak{m})$ *be a regular local ring containing a field* K. *Let* I *be a Cohen–Macaulay ideal with* $\dim R - \operatorname{height} I \leq 2$. *Then* I *is a Peskine–Szpiro ideal.*

Proof. We first assume $\operatorname{char} K = 0$. Let $\dim R = d$ and $\operatorname{height} I = g$.

If $g = d$ then I is $\mathfrak{m}$-primary. By the Grothendieck vanishing theorem we have $H^i_I(R) = 0$ for $i > d$. So, I is a Peskine–Szpiro ideal.

Now consider the case when $g = d - 1$. Note $\dim R/I = 1$. So, $\dim \widehat{R}/I\widehat{R} = 1$. By the Hartshorne–Lichtenbaum theorem [4, Theorem 14.1], we have that $H^d_{I\widehat{R}}(\widehat{R}) = 0$. By faithful flatness we get $H^d_I(R) = 0$.

Finally, we consider the case when $g = d - 2$. We choose a flat extension $(B, \mathfrak{n})$ of R with $\mathfrak{m}B = \mathfrak{n}$, B complete and $B/\mathfrak{n}$ algebraically closed. We note that B/IB is Cohen–Macaulay and $\dim B/IB = 2$. As B/IB is Cohen–Macaulay, we get that the punctured spectrum $\operatorname{Spec}^\circ(B/IB)$ is connected, see [4, Proposition 15.7]. So, $H^{d-1}_{IB}(B) = 0$ by a result due to Ogus [6, Corollary 2.11]. By faithful flatness we get $H^{d-1}_I(R) = 0$. By an argument similar to the previous case we also get $H^d_I(R) = 0$.

Next, we consider the case when $\operatorname{char} K = p > 0$. The proof in this case follows from [7, Proposition III.4.1]. ■

3. Proof of Theorem 1.2

In this section we prove our main result. The following remarks are relevant.

REMARK 3.1. (1) Notice that for any ideal J of height g we have $\operatorname{Ass} H^g_J(R) = \{P \mid P \supset J \text{ and height } P = g\}$. Also, for any such prime ideal P we have $\mu_0(P, H^g_J(R)) = 1$.

(2) Let I be a Cohen–Macaulay ideal of height g in a regular local ring. Let P be a prime ideal of height $g + r$ and containing I. We note that $\dim H^g_I(R)_P = r$. So, by Properties 1.3 we get $\operatorname{injdim}_{R_P} H^g_I(R)_P \leq r$. Thus, $\mu_i(P, H^g_I(R)) = 0$ for $i > r$.

Let us recall the following result due to Rees, cf. [1, Lemma 3.1.16].

PROPERTIES 3.2. Let S be a commutative ring and let M and N be S-modules. (We note that S needs not be Noetherian. Also, M, N need not be finitely generated as S-modules.) Assume that there exists $x \in S$ such that it is $S \oplus M$-regular and $xN = 0$. Set $T = S/(x)$. Then $\operatorname{Hom}_S(N, M) = 0$ and for $i \geq 1$ we have

$$\operatorname{Ext}^i_S(N, M) \cong \operatorname{Ext}^{i-1}_T(N, M/xM).$$

We now give the proof of our main theorem.

PROOF OF THEOREM 1.2. We first prove (i) $\Rightarrow$ (ii). So, I is a Peskine–Szpiro ideal. We prove our result by induction on $d - g$.

If $d - g = 0$ then I is $\mathfrak{m}$-primary. So, $H^d_I(R) = H^d_{\mathfrak{m}}(R) = E_R(R/\mathfrak{m})$, the injective hull of the residue field. Clearly, $\mu_0(\mathfrak{m}, H^d_I(R)) = 1$ and $\mu_i(\mathfrak{m}, H^d_I(R)) = 0$ for $i \geq 1$.

Now, assume $d - g = 1$. If P is a prime ideal of R containing I with height $P = d - 1$, then by Remark 3.1 we have $\mu_0(P, H^{d-1}_I(R)) = 1$ and $\mu_i(P, H^{d-1}_I(R)) = 0$ for $i \geq 1$. We now consider the case when $P = \mathfrak{m}$. By Remark 3.1 we have $\mu_0(\mathfrak{m}, H^{d-1}_I(R)) = 0$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is R/I-regular. Set $\overline{R} = R/(x)$, $\mathfrak{n} = \mathfrak{m}/(x)$ and $J = I\overline{R} = (I + (x))/(x)$. Then J is $\mathfrak{n}$-primary. The exact sequence $0 \to R \xrightarrow{x} R \to \overline{R} \to 0$ induces the following exact sequence in cohomology,

$$0 \longrightarrow H^{d-1}_I(R) \xrightarrow{x} H^{d-1}_I(R) \longrightarrow H^{d-1}_J(\overline{R}) \longrightarrow 0.$$

Here we have used that I is a Peskine–Szpiro ideal and J is $\mathfrak{n}$-primary. Thus, by Properties 3.2 we have for $i \geq 1$,

$$\mu_i(\mathfrak{m}, H^{d-1}_I(R)) = \mu_{i-1}(\mathfrak{n}, H^{d-1}_J(\overline{R})) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the result follows in this case.

Now consider the case when $d - g \geq 2$. Let P be a prime ideal in R containing I of height $g + r$. We first consider the case when $P \neq \mathfrak{m}$. By Proposition 2.1 we get that I_P is a Peskine–Szpiro ideal of height g in R_P. Also, $\dim R_P - g < d - g$. So, by the induction hypothesis we have

$$\mu_i(P, H^i_I(R)) = \mu_i(PR_P, H^i_{IR_P}(R_P)) = \begin{cases} 1 & \text{if } i = r, \\ 0 & \text{otherwise.} \end{cases}$$

We now consider the case when $P = \mathfrak{m}$. By Remark 3.1 we have $\mu_0(\mathfrak{m}, H^g_I(R)) = 0$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is R/I-regular. Set $\overline{R} = R/(x)$, $\mathfrak{n} = \mathfrak{m}/(x)$ and $J = I\overline{R} = (I + (x))/(x)$. Then J is a height g Peskine–Szpiro ideal in $\overline{R}$, see Proposition 2.1. The exact sequence $0 \to R \xrightarrow{x} R \to \overline{R} \to 0$ induces the following exact sequence in cohomology,

$$0 \longrightarrow H^g_I(R) \xrightarrow{x} H^g_I(R) \longrightarrow H^g_J(\overline{R}) \longrightarrow 0.$$

Here we have used that I is a Peskine–Szpiro ideal in R and J is Peskine–Szpiro ideal in $\overline{R}$. Thus, by Properties 3.2 we have for $i \geq 1$,

$$\mu_i(\mathfrak{m}, H^g_I(R)) = \mu_{i-1}(\mathfrak{n}, H^g_J(\overline{R})) = \begin{cases} 1 & \text{if } i - 1 = d - 1 - g, \\ 0 & \text{otherwise.} \end{cases}$$

For the latter equality we have used the induction hypothesis on the Peskine–Szpiro ideal J (as $\dim \overline{R} - \operatorname{height} J = d - 1 - g$). We note that $i - 1 = d - 1 - g$ is the same as $i = d - g$. Thus, we have

$$\mu_i(\mathfrak{m}, H^g_I(R)) = \begin{cases} 1 & \text{if } i = d - g, \\ 0 & \text{otherwise.} \end{cases}$$

We now prove (ii) $\Rightarrow$ (i). By Peskine and Szpiro's result we may assume $\operatorname{char} K = 0$. We prove the result by induction on $d - g$. If $d - g \leq 2$ then the result holds by Proposition 2.2. So, we may assume $d - g \geq 3$. Let P be a prime ideal in R containing I with $P \neq \mathfrak{m}$. The ideal I_P is a Cohen–Macaulay ideal of height g in R_P satisfying condition (ii) on the Bass numbers of $H^g_{I_P}(R_P)$. As $\dim R_P - g < d - g$, we get by our induction hypothesis that I_P is a Peskine–Szpiro ideal in R_P. Thus, $H^i_{I_P}(R_P) = 0$ for $i > g$. It follows that $\operatorname{Supp} H^i_I(R) \subseteq \{\mathfrak{m}\}$ for $i > g$. Let $k = R/\mathfrak{m}$ and let $E_R(k)$ be the injective hull of k as a R-module. Then, by Properties 1.3 there exist non-negative integers r_i with

$$H^i_I(R) = E_R(k)^{r_i} \quad \text{for } i > g. \tag{3.1}$$

Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is R/I-regular. Set $\overline{R} = R/(x)$, $\mathfrak{n} = \mathfrak{m}/(x)$ and $J = I\overline{R} = (I + (x))/(x)$. Then J is a height g Peskine–Szpiro ideal in $\overline{R}$. The exact sequence $0 \to R \xrightarrow{x} R \to \overline{R} \to 0$ induces the following exact sequence in cohomology,

$$0 \longrightarrow H^g_I(R) \xrightarrow{x} H^g_I(R) \longrightarrow H^g_J(\overline{R}) \longrightarrow H^{g+1}_I(R) \xrightarrow{x} H^{g+1}_I(R) \longrightarrow \cdots. \tag{3.2}$$

We consider two cases.

Case 1: $H^{g+1}_I(R) \neq 0$. We note that $\operatorname{Hom}_R(\overline{R}, E_R(k)) = E_{\overline{R}}(k)$. Thus, the short exact sequence $0 \to R \xrightarrow{x} R \to \overline{R} \to 0$ induces an exact sequence

$$0 \longrightarrow E_{\overline{R}}(k) \longrightarrow E_R(k) \xrightarrow{x} E_R(k) \longrightarrow 0. \tag{3.3}$$

By (3.1) and (3.3) the exact sequence (3.2) breaks up into two exact sequences

$$0 \longrightarrow H^g_I(R) \xrightarrow{x} H^g_I(R) \longrightarrow V \longrightarrow 0, \tag{3.4}$$

$$0 \longrightarrow V \longrightarrow H^g_J(\overline{R}) \longrightarrow E_{\overline{R}}(k)^{r_{g+1}} \longrightarrow 0. \tag{3.5}$$

As J is a Cohen–Macaulay ideal in $\overline{R}$ with $\dim \overline{R}/J = d - 1 - g \geq 2$, we get by Remark 3.1 that $\mathfrak{n} \notin \operatorname{Ass}_{\overline{R}} H^g_J(\overline{R})$. It follows from (3.5) that $\mu_1(\mathfrak{n}, V) \geq r_{g+1} > 0$. By (3.4) and Properties 3.2 we get that

$$\mu_2(\mathfrak{m}, H^g_I(R)) = \mu_1(\mathfrak{n}, V) > 0.$$

So, by our hypothesis we get $d - g = 2$. This is a contradiction as we assumed $d - g \geq 3$.

Case 2: $H^{g+1}_I(R) = 0$. By (3.2) we get a short exact sequence,

$$0 \longrightarrow H^g_I(R) \xrightarrow{x} H^g_I(R) \longrightarrow H^g_J(\overline{R}) \longrightarrow 0.$$

Again by Properties 3.2, we get that the Cohen–Macaulay ideal J of $\overline{R}$ satisfies condition (ii) of our theorem. As $\dim \overline{R} - \operatorname{height} J = d - g - 1$, we get by the induction hypothesis that J is Peskine–Szpiro ideal in $\overline{R}$. Thus, $H^i_J(\overline{R}) = 0$ for $i > g$. Using (3.1) and (3.3) it follows that $H^i_I(R) = 0$ for $i \geq g + 2$. Also, by our assumption, $H^{g+1}_I(R) = 0$. Thus, I is a Peskine–Szpiro ideal of R. ∎

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