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A note on the admissibility of smooth simple RG-modules

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Abstract – Let G be a p-adic reductive group and R be a noetherian Jacobson algebra over the ring $\mathbb{Z}_l$ of l-adic integers with $l \neq p$. In this note, we show that every smooth irreducible R-linear representation of G is admissible using the finiteness result of Dat, Helm, Kurinczuk and Moss for Hecke algebras over R.

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Unless mentioned otherwise, all rings are commutative with unity. A p-adic reductive group is the group of rational points of a reductive group defined over a non-archimedean local field of residue characteristic $p > 0$. Let R be a ring and G be a p-adic reductive group. Let RG denote the group algebra. An RG-module π is called *smooth* if every $v \in \pi$ is fixed by some compact open subgroup in G, and *admissible* if for every compact open subgroup $K \subseteq G$, π^K is a finitely generated R-module.

For a compact open subgroup $K \subseteq G$, let $H_R(G, K)$ denote the Hecke algebra of compactly supported R-valued K-biinvariant functions on G equipped with the convolution product and let $Z_R(G, K)$ denote its center. The Hecke algera $H_R(G, K)$ is an associative R-algebra with unity. The following theorem is the joint work [1] of Dat, Helm, Kurinczuk and Moss:

Theorem 1. *For any noetherian $\mathbb{Z}_l$-algebra R with $l \neq p$ and any compact open subgroup $K \subseteq G$, the Hecke algebra $H_R(G, K)$ is a finitely generated module over $Z_R(G, K)$, and $Z_R(G, K)$ is a finitely generated R-algebra.*

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As an application of Theorem 1, we prove the following result:

Theorem 2. *If R is a noetherian Jacobson $\mathbb{Z}_l$-algebra with $l \neq p$, then any smooth simple RG-module π is admissible.*

Recall that a ring is called *Jacobson* if every prime ideal is the intersection of the maximal ideals which contain it. The examples of noetherian Jacobson $\mathbb{Z}_l$-algebras include all fields that are $\mathbb{Z}_l$-algebras, as well as finitely generated algebras over such fields, such as $\mathbb{F}_l[T_1, \ldots, T_n]$ or finite rings such as $\mathbb{Z}/l^m\mathbb{Z}$ for which Theorem 2 is a new result. When $R = \mathbb{C}$, Theorem 2 is a classical result in the representation theory of p-adic groups. The case when R is any field of characteristic not equal to p is given in [4, Proposition 4.10].

We remark that the authors of [1] expect Theorem 1 to be true for any noetherian $\mathbb{Z}[\frac{1}{p}]$-algebra R, in which case Theorem 2 would also hold for any noetherian Jacobson $\mathbb{Z}[\frac{1}{p}]$-algebra R (in particular for $R = \mathbb{Z}[\frac{1}{p}]$).

Theorem 2 follows from the following corollary to Theorem 1:

Corollary to Theorem 1. *Let R be a noetherian Jacobson $\mathbb{Z}_l$-algebra with $l \neq p$, and let M be a simple (left) module over $H_R(G, K)$. Then M is a finitely generated R-module.*

Proof. To ease the notation, let us write $H = H_R(G, K)$ and $Z = Z_R(G, K)$. Choose a surjective map $H \twoheadrightarrow M$ of R-modules. Let $\mathfrak{m}$ be the kernel of the surjection $H \twoheadrightarrow M$ and $\mathfrak{m}_Z := \mathfrak{m} \cap Z$. Note that $\mathfrak{m}_Z$ is a two-sided ideal of Z because Z is the center of H. We claim that $\frac{Z}{\mathfrak{m}_Z}$ is a field. Let $\bar{z} := z + \mathfrak{m}_Z \in \frac{Z}{\mathfrak{m}_Z}$ be a non-zero element. Then $\bar{z}$ is also non-zero in $\frac{H}{\mathfrak{m}}$. So the left H-submodule $H\bar{z}$ of $\frac{H}{\mathfrak{m}}$ generated by $\bar{z}$ is equal to $\frac{H}{\mathfrak{m}}$ because $\frac{H}{\mathfrak{m}}$ is simple. Therefore, there exists $h \in H$ such that $\bar{h}\bar{z} = \bar{1}$ in $\frac{H}{\mathfrak{m}}$.

Consider the $\frac{Z}{\mathfrak{m}_Z}$-algebra $\frac{Z}{\mathfrak{m}_Z}[\bar{h}]$ generated by $\bar{h}$. It is commutative because $\frac{Z}{\mathfrak{m}_Z}$ is commutative. Moreover, as $\frac{H}{\mathfrak{m}}$ is a finitely generated $\frac{Z}{\mathfrak{m}_Z}$-module and $\frac{Z}{\mathfrak{m}_Z}$ is noetherian, $\frac{Z}{\mathfrak{m}_Z}[\bar{h}]$ is a finitely generated $\frac{Z}{\mathfrak{m}_Z}$-module. Hence, we have that $\bar{h}$ is integral over $\frac{Z}{\mathfrak{m}_Z}$, i.e.

$$\bar{h}^n + \bar{a}_{n-1}\bar{h}^{n-1} + \cdots + \bar{a}_0 = 0,$$

for some $n \in \mathbb{N}$ and $\bar{a}_{n-1}, \bar{a}_{n-2}, \ldots, \bar{a}_0 \in \frac{Z}{\mathfrak{m}_Z}$. Multiplying both sides of the above by $\bar{z}^{n-1}$ and then using that $\frac{Z}{\mathfrak{m}_Z}$ commutes with $\bar{h}$, we obtain that

$$\bar{h} + \bar{a}_{n-1} + \bar{a}_{n-2}\bar{z} + \cdots + \bar{a}_0\bar{z}^{n-1} = 0.$$

Hence, $\bar{h} = -(\bar{a}_{n-1} + \bar{a}_{n-2}\bar{z} + \cdots + \bar{a}_0\bar{z}^{n-1}) \in \frac{Z}{\mathfrak{m}_Z}$.

Now, the field $\frac{Z}{\mathfrak{m}_Z}$ is a finitely generated R-algebra. One of the characterizations of Jacobson rings implies that $\frac{Z}{\mathfrak{m}_Z}$ is a finitely generated R-module [2, Theorem 10]. Since $\frac{H}{\mathfrak{m}}$ is finite over $\frac{Z}{\mathfrak{m}_Z}$, we get that $\frac{H}{\mathfrak{m}} \cong M$ is also a finitely generated R-module. ∎

Proof of Theorem 2. Since G has a fundamental system of neighborhoods of identity consisting of open pro-p subgroups, it is enough to show that π^K is a finitely generated R-module for $K \subseteq G$ an open pro-p subgroup. Let $K \subseteq G$ be an open pro-p subgroup such that $\pi^K \neq 0$. Since π is simple and $p \in R^\times$, π^K is a simple $H_R(G, K)$-module by [6, I.6.3]. Hence, π^K is a finitely generated R-module by the corollary to Theorem 1. ∎

Remark 3. The requirement for R to be Jacobson in the corollary to Theorem 1 is necessary. Indeed, if R is a commutative ring and if all simple modules over $H_R(G, K)$ are finitely generated R-modules for all p-adic reductive groups G and compact open subgroups K, then R is Jacobson. The following proof of this converse statement was communicated to us by M.-F. Vignéras: By Satake [5, §8], if G is a classical simple group with trivial center and $K \subseteq G$ a natural maximal compact subgroup, then $H_R(G, K)$ is a polynomial ring over R in m variables, where m is the rank of a maximal split torus in G. Thus, a finitely generated R-algebra A is a quotient of some $H_R(G, K)$. If A is a field, then A is a simple module over $H_R(G, K)$, and hence a finitely generated R-module by assumption. This means that R is Jacobson.

Remark 4. Let $R = \mathbb{Z}_l$ with $l \neq p$ and $G = \mathrm{GL}_2(\mathbb{Q}_p)$. As $\mathbb{Z}_l$ is not Jacobson, Remark 3 suggests that G admits a smooth irreducible R-representation that is not admissible. Indeed, let $K = \mathrm{GL}_2(\mathbb{Z}_p)$. By [3, Proposition 2.1], $H = H_R(G, K) \cong R[T_0, T_0^{-1}, T_1]$. One can make $M := \mathbb{Q}_l$ into a simple H-module by defining the action via the surjective map $H \twoheadrightarrow M$ which takes T_0 to 1 and T_1 to l^{-1}. However, note that M is not a finitely generated R-module. By choosing a prime l so that the pro-order of K is invertible in R, there exists a smooth simple RG-module π such that $\pi^K \cong M$ as H-modules [6, I.4.4 and I.6.3]. Since π^K is not a finite R-module, π is non-admissible.

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