Rend. Sem. Mat. Univ. Padova (Online first)
DOI 10.4171/RSMUP/146

Published by EMS Press

Existence and uniqueness for renormalized solutions to noncoercive nonlinear parabolic equations with unbounded convective term

Thi Tam Dang (*) – Giandomenico Orlandi (**)

Abstract – We study the existence and uniqueness of renormalized solutions for noncoercive nonlinear parabolic equations in the presence of an unbounded convective term.

Mathematics Subject Classification 2020 – 35K61 (primary); 35K55 (secondary).

Keywords – nonlinear parabolic equations, renormalized solutions, Lorentz spaces.

1. Introduction

In this paper, we consider the following nonlinear parabolic equation

$$(1.1) \qquad \begin{cases} u_t - \operatorname{div}[A(x,t,\mathrm{D}u) + B(x,t,u)] = -\operatorname{div} F(x,t) & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a regular bounded domain in $\mathbb{R}^N$ with $N \geq 3$ and $T > 0$, and $Q = \Omega \times (0,T)$. Let us describe our main structural assumptions concerning (1.1). Firstly,

$$(1.2) \qquad F \in L^2(Q) \text{ and } u_0 \in L^2(\Omega).$$

The operator $A = A(x,t,\xi) : Q \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following monotonicity and boundedness conditions:

(*) *Indirizzo dell'A.*: Institut für Mathematik, Leopold Franzens Universität Innsbruck, Technikerstr. 13, 6020 Innsbruck, Austria; thi-tam.dang@uibk.ac.at

(**) *Indirizzo dell'A.*: Dipartimento di Informatica, Università di Verona, Strada Le Grazie 15, 37134 Verona, Italy; giandomenico.orlandi@univr.it

$$(1.3)\qquad |A(x,t,\xi)| \le c|\xi| + d(x,t) \text{ for some } c>0 \text{ and } d \in L^2(Q),$$

$$(1.4)\qquad \langle A(x,t,\xi) - A(x,t,\eta), \xi - \eta\rangle \ge \alpha|\xi-\eta|^2 \text{ for some } \alpha > 0,$$

for a.e. $(x,t) \in Q$, for any $\xi, \eta \in \mathbb{R}^N$, and

$$(1.5)\qquad A(x,t,0) = 0 \text{ for a.e. } (x,t) \in Q.$$

Moreover, we assume that $B = B(x,t,s) : Q \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function fulfilling the following properties:

$$(1.6)\qquad |B(x,t,s) - B(x,t,s')| \le b(x,t)|s-s'|,$$

$$(1.7)\qquad B(x,t,0) = 0,$$

for a.e. $x \in \Omega$, for any $t \in (0,T)$, for any $s, s' \in \mathbb{R}$, and for some suitable nonnegative function b such that

$$(1.8)\qquad b \in L^\infty(0,T; L^{N,\infty}(\Omega)).$$

In the homogeneous case, equation (1.1) is known as a version of Fokker–Planck equation, and it has been studied for instance in [8]. In the literature, many authors extensively paid attention to find a good definition for solutions introducing the notion of renormalized solution and entropy solution. The notion of renormalized solution has been investigated in the elliptic framework (see, e.g., [1] and references therein), then extended to the parabolic case. In [3], the existence of renormalized solutions has been proven in the case $b = 0$. In a series of works by A. Porretta [4, 7, 8], the existence of renormalized solutions is investigated in case the datum b is continuous and belongs to Lebesgue spaces. In this paper, we extend the available results (see e.g., [6]) to the case of noncoercive nonlinear convection-diffusion equations with a possibly unbounded field b lying in a suitable subset of the Lorentz space $L^\infty(0,T; L^{N,\infty}(\Omega))$. Proving the existence of a renormalized solution for equation (1.1) in this framework is challenging due to the lack of coercivity of the operator in divergence form, and the lack of summability for b. Because of the unboundedness of the convective term, an additional assumption on b is necessary. The idea is to decompose the datum b as the sum of a bounded function, and a function belonging to a Lorentz space with a control on its norm. The main point which allows us to go further than previous works is to rephrase (1.1), as in [6], in terms of the following equivalent fixed point equation:

$$(1.9)\qquad \begin{cases} u_t - \operatorname{div}[A(x,t,Du) + (1-\theta(x,t))B(x,t,u)] & \\ \qquad\qquad = -\operatorname{div}(F - \theta(x,t)B(x,t,u)) & \text{in } Q \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\theta(x,t)$ is defined by

$$\theta(x,t) = \frac{T_M(b(x,t))}{b(x,t)},$$

where T_M is the usual truncation operator at level $\pm M$, defined for any $M > 0$ as the real-valued Lipschitz function

$$T_M(s) = \min(M, \max(s, -M)).$$

The main goal of this paper is to prove existence and uniqueness of renormalized solutions to equation (1.9), slightly extending the results of [6]. Our main result is the following.

Theorem 1.1. *Let us assume that conditions* (1.2)-(1.8) *are satisfied. Then there exists a unique renormalized solution of problem* (1.1).

The plan of the paper is as follows: in Section 2, we recall the definition of renormalized solution and the main properties of Lorentz spaces. In Section 3, we extend some known results related to (1.1), in order to carry out a complete proof of Theorem 1.1.

2. Preliminary results

2.1 – *Renormalized solutions*

The purpose of this section is to give a thorough definition of renormalized solution applied to (1.1). The following definition of a renormalized solution is adapted from the existing literature.

Definition 2.1. The real-valued function u defined on $Q = \Omega \times (0,T)$ is a renormalized solution of equation (1.1), if

$$u \in C^0([0,T]; L^2(\Omega)), \tag{2.1}$$

$$T_K(u) \in L^2(0,T; W_0^{1,2}(\Omega)) \text{ for all } K > 0, \tag{2.2}$$

for any positive real number C,

$$T_{K+C}(u) - T_K(u) \to 0 \text{ strongly in } L^2(0,T; W_0^1(\Omega)) \text{ as } K \to +\infty,$$

and

$$\begin{aligned}(S(u))_t &- \operatorname{div}([A(x,t,\mathrm{D}u) + (1-\theta(x,t))B(x,t,u)]S'(u)) \\ &+ [A(x,t,\mathrm{D}u) + (1-\theta(x,t))B(x,t,u)]\mathrm{D}u \cdot S''(u) \\ &= -\operatorname{div}(F - \theta(x,t)B(x,t,u)) \cdot S'(u),\end{aligned}$$

where $S \in C^{\infty}(\mathbb{R})$ is such that $S' \in C_0^{\infty}(\mathbb{R})$, and

$$S(u(t=0)) = S(u_0).$$

2.2 – *Lorentz spaces*

In this section, we give the definition and main properties of Lorentz spaces, following [5,6].

Let Ω be a bounded domain of $\mathbb{R}^N$. For any $p, q \in (1, \infty)$, the Lorentz space $L^{p,q}(\Omega)$ consists of all measurable functions f defined on Ω for which the quantity

$$\|f\|_{p,q} = \left(p \int_0^{\infty} |\Omega_h|^{\frac{q}{p}} h^{q-1}\, \mathrm{d}h\right)^{\frac{1}{q}}$$

is finite, where $\Omega_h = \{x \in \Omega : |g(x)| > h\}$ for any $h > 0$. Here and in what follows $|E|$ stands for the Lebesgue measure of a measurable subset E of $\mathbb{R}^N$.

For $p = q$, the Lorentz space $L^{p,p}(\Omega)$ reduces to the Lebesgue space $L^p(\Omega)$. On the other hand, for $q = \infty$ the class $L^{p,\infty}(\Omega)$ consists of all measurable functions g defined on Ω, for which the quantity

$$\|g\|_{p,\infty} = \sup_{E \subset \Omega} |E|^{\frac{1}{p}-1} \int_E |g|\, \mathrm{d}x$$

is finite. The class $L^{p,\infty}(\Omega)$ coincides with the Marcinkiewicz class of weak-L^p. For the Lorentz spaces, the following inclusions hold:

$$L^r(\Omega) \subset L^{p,q}(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega)$$

whenever $1 \le q < p < r \le \infty$.

The following Hölder-type inequality plays an important role in our proof of Lemma 3.2, which will be mentioned in Section 3.

Lemma 2.1 (Hölder inequality). *For $1 < p < \infty$, $1 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, if $f \in L^{p,q}(\Omega)$ and $g \in L^{p',q'}(\Omega)$, we have the Hölder-type inequality*

$$\int_{\Omega} |f(x)g(x)| dx \le \|f\|_{p,q} \|g\|_{p',q'}.$$

The following theorem turns out to be crucial for the coercive property of the operator, which will be used later on.

THEOREM 2.1 (Sobolev embedding theorem). *Let us assume that* $1 < p < N$, $1 \le q \le p$. *Then any function* $g \in W_0^{1,1}(\Omega)$ *satisfying* $|\nabla g| \in L^{p,q}(\Omega)$ *belongs to* $L^{p^\star,p}(\Omega)$, *where* $p^* = \frac{Np}{N-p}$ *and*

$$\|g\|_{p^*,q} \le S_{N,p}\|\nabla g\|_{p,q},$$

where $S_{N,p} = \omega_N^{-1/N}\frac{p}{N-p}$ *and* ω_N *stands for the measure of the unit ball in* $\mathbb{R}^N$.

3. Existence and uniqueness of renormalized solutions

In this section, we state and prove some useful lemmas for the proof of Theorem 1.1. We adapt to our specific noncoercive situation some tools used for many different types of nonlinear parabolic equations, see e.g., [2, 3, 7, 8]. Let us consider the following auxiliary approximate problem.

$$\text{(3.1)}\qquad \begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \operatorname{div}[A(x,t,\mathrm{D}u^\varepsilon) + (1-\theta(x,t))B(x,t,u^\varepsilon)] & \\ \qquad\qquad\qquad = -\operatorname{div}(F^\varepsilon - \theta(x,t)B(x,t,u^\varepsilon)) & \text{in } Q, \\ u^\varepsilon(t=0) = u_0^\varepsilon & \text{in } \Omega, \end{cases}$$

where $F^\varepsilon = T_{1/\varepsilon}(F)$ and $u_0^\varepsilon = T_{1/\varepsilon}(u_0)$. Since F^ε and u_0^ε belong, respectively, to $L^2(Q)$ and $L^2(\Omega)$, this problem admits a solution

$$u^\varepsilon \in L^2(0,T;W_0^1(\Omega)) \cap C^0(0,T;L^2(\Omega)),$$

as proven in [6]. When ε tends to 0, the sequences F^ε and u_0^ε strongly converge to F and u_0 in $L^2(Q)$ and $L^2(\Omega)$ respectively. For any fixed positive real number K, the asymptotic behavior of the sequence $T_K(u^\varepsilon)$ as ε tends to zero is investigated in [2, 3, 8]. We follow their strategy considering the following lemmas.

LEMMA 3.1. *For any fixed* K, *we have*

$$\text{(3.2)}\qquad \lim_{K\to\infty}\int_{\{K\le|u|\le K+C\}} |\mathrm{D}u|^2\,\mathrm{d}x\mathrm{d}t = 0$$

for any positive real number C.

It has been observed that problem (1.1) is nonconercive, so we need to consider the auxiliary problem (1.9). The idea is that under the assumption that b belongs to a suitable subset of the Lorentz space $L^\infty(0,T;L^{N,\infty}(\Omega))$, we can recover some coercivity for the operator in divergence form. To be explicit, let us define the operator

$$\mathcal{A}_S : L^2(0,T;W_0^{1,2}(\Omega)) \to L^2(0,T;W^{-1,2}(\Omega)),$$

defined for $u, v \in L^2(0,T;W_0^{1,2}(\Omega))$, as follows:

$$\langle \mathcal{A}_S u, v\rangle := \int_0^T \int_\Omega \langle A(x,t,\mathrm{D}S(u)) + (1-\theta(x,t))B(x,t,S(u)), \mathrm{D}S(v)\rangle \,\mathrm{d}x\mathrm{d}t,$$

where S denotes the truncation operator at level K.

Lemma 3.2 (see [6, Lemma 3.1]). *For each $u, v \in L^2(0,T;W_0^{1,2}(\Omega))$, we get*

$$\langle \mathcal{A}_S u - \mathcal{A}_S v, u-v\rangle \geq \frac{\alpha}{2}\int_0^T \|\mathrm{D}S(u) - \mathrm{D}S(v)\|^2_{L^2(\Omega)}\,\mathrm{d}t.$$

In particular,

$$\langle \mathcal{A}_S u, u\rangle \geq \frac{\alpha}{2}\int_0^T \|\mathrm{D}S(u)\|^2_{L^2(\Omega)}\,\mathrm{d}t.$$

Proof. We follow the strategy of [6] and choose $M > 0$ in such a way that

$$\sup_{0<t<T} \|b(x,t) - T_M(b(x,t))\|_{L^{N,\infty}} \leq \frac{\alpha}{2S_{N,2}}. \tag{3.3}$$

Using the coerciveness of operator A in (1.4) and the definition of $\theta(x,t)$ together with (3.3), we get

$$\begin{aligned}
&\langle \mathcal{A}_S u - \mathcal{A}_S v, u-v\rangle \\
&\quad= \int_0^T \langle A(x,t,\mathrm{D}S(u)) - A(x,t,\mathrm{D}S(v)) + (1-\theta(x,t))B(x,t,S(u)) \\
&\qquad\qquad - B(x,t,S(v)), \mathrm{D}S(u) - \mathrm{D}S(v)\rangle\,\mathrm{d}x\mathrm{d}t \\
&\quad\geq \alpha\int_0^T \|\mathrm{D}S(u) - \mathrm{D}S(v)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&\qquad - S_{N,2}\int_0^T \big(\|b(x,t) - T_M(b(x,t))\|_{L^{N,\infty}(\Omega)} \\
&\qquad\qquad\qquad \cdot \|S(u) - S(v)\|_{L^{2^*,2}(\Omega)} \cdot \|\mathrm{D}S(u) - \mathrm{D}S(v)\|_{L^2(\Omega)}\big)\,\mathrm{d}t \\
&\quad\geq \frac{\alpha}{2}\int_0^T \|\mathrm{D}S(u) - \mathrm{D}S(v)\|^2_{L^2(\Omega)}\,\mathrm{d}t,
\end{aligned}$$

which can be obtained by applying Hölder and Sobolev inequalities in the setting of Lorentz spaces. ■

With the coercivity of operator $\mathcal{A}_S$ at hand, we are now in position to prove Lemma 3.1.

Proof of Lemma 3.1. Using $T_{K+C}(u^\varepsilon) - T_K(u^\varepsilon)$ as a test function in (3.1), we obtain

$$
\begin{aligned}
&\int_\Omega [\varphi_{K+C}(u^\varepsilon)(T) - \varphi_K(u^\varepsilon)(T)]\,\mathrm{d}x \\
&\quad + \int_Q [A(\mathrm{D}u^\varepsilon) + (1-\theta(x,t))B(u^\varepsilon)][\mathrm{D}T_{K+C}(u^\varepsilon) - \mathrm{D}T_K(u^\varepsilon)]\,\mathrm{d}x\mathrm{d}t \\
&= \int_Q [F^\varepsilon - \theta(x,t)B(u^\varepsilon)][\mathrm{D}T_{K+C}(u^\varepsilon) - \mathrm{D}T_K(u^\varepsilon)]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_\Omega [\varphi_{K+C}(u_0^\varepsilon) - \varphi_K(u_0^\varepsilon)]\,\mathrm{d}x,
\end{aligned} \tag{3.4}
$$

where

$$\varphi_K(r) = \int_0^r T_K(s)\,\mathrm{d}s.$$

Since the function $\varphi_{K+C}(r) - \varphi_K(r)$ is positive, using the convergence of F^ε to F in $L^1(Q)$, the convergence of u_0^ε to u_0 in $L^1(\Omega)$, the fact that $|T_{K+C}(s) - T_K(s)| \le C$, and the linear growth of φ_K at infinity, together with the coerciveness result of Lemma 3.2, we deduce from (3.4) that

$$
\begin{aligned}
&\frac{\alpha}{2}\lim_{\varepsilon\to 0}\int_Q |\mathrm{D}T_{K+C}(u^\varepsilon) - \mathrm{D}T_K(u^\varepsilon)|^2\,\mathrm{d}x\mathrm{d}t \\
&\le \int_Q (F^\varepsilon - |u^\varepsilon|)[DT_{K+C}(u^\varepsilon) - \mathrm{D}T_K(u^\varepsilon)]\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_\Omega [\varphi_{K+C}(u_0^\varepsilon) - \varphi_K(u_0^\varepsilon)]\,\mathrm{d}x.
\end{aligned}
$$

By using the Young and Poincaré inequalities, we get

$$
\begin{aligned}
&c(\alpha)\lim_{\varepsilon\to 0}|\mathrm{D}T_{K+C}(u^\varepsilon) - \mathrm{D}T_K(u^\varepsilon)|^2\,\mathrm{d}x\mathrm{d}t \\
&\le \int\int_{\{|K|<u^\varepsilon\}} |F^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t + \int_{\{|u_0^\varepsilon|>K\}} |u_0^\varepsilon|\,\mathrm{d}x.
\end{aligned}
$$

When K tends to infinity, the right-hand side tends to 0 since the sequences F^ε and u_0^ε converge strongly, and $\{|u^\varepsilon| > K\}$ has small measure. So u satisfies (3.2). This completes the proof of the lemma. ∎

The following theorem gives estimates on the sequence $\mathrm{D}T_K(u^\varepsilon)$. Here, we extend previous results from [2], where the authors considered the case $b = 0$.

Theorem 3.1. *Assume* (1.2) *to* (1.8). *Let*

$$u^\varepsilon \in L^2(0,T;W_0^{1,2}(\Omega)) \cap C^0([0,T];L^2(\Omega))$$

be a sequence such that

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \operatorname{div}[A(x,t,\mathrm{D}u^\varepsilon) + (1-\theta(x,t))B(x,t,u^\varepsilon)] & \\ \qquad\qquad = -\operatorname{div}\left(F^\varepsilon - \theta(x,t)B(x,t,u^\varepsilon)\right) & \text{in } Q, \\ u^\varepsilon(t=0) = u_0^\varepsilon & \text{in } \Omega, \end{cases}$$

and the following properties hold:

(i) u^ε *converges almost pointwise in* Q *to a measurable function* u,

(ii) F^ε *is a sequence of* $L^2(Q)$ *weakly convergent to* F *in* $L^2(Q)$,

(iii) u_0^ε *is a sequence of* $L^2(\Omega)$ *strongly convergent to* u_0 *in* $L^2(\Omega)$.

Then, for any positive real number K,

$$\int_Q [\mathcal{A}(x,t,u,\mathrm{D}T_K(u^\varepsilon)) - \mathcal{A}(x,t,u,\mathrm{D}T_K(u))]\cdot[\mathrm{D}T_K(u^\varepsilon) - \mathrm{D}T_K(u)] \to 0 \tag{3.5}$$

as $\varepsilon \to 0$, *where* $\mathcal{A}(t,x,u,\mathrm{D}u) := A(x,t,\mathrm{D}u) + (1-\theta(x,t))B(x,t,u)$.

Proof. The proof of Theorem 3.1 can be obtained following closely the strategy in the proof of [2, Theorem 2], with few adaptations due to the unbouded convective term. More precisely, the differences from [2] are stated in the following lemma. ■

Lemma 3.3. *For any positive real number* K, *the following assertions hold:*

(1) $T_K(u^\varepsilon)$ *weakly converges to* $T_K(u)$ *in* $L^2(0,T;W_0^{1,2}(\Omega))$,

(2) $A(t,x,\mathrm{D}T_K(u^\varepsilon))$ *weakly converges in* $L^2(Q)$ *to an element* σ_K *of* $L^2(Q)$,

(3) $\int_Q \left[\mathcal{A}\left(t,x,T_K(u^\varepsilon),\mathrm{D}T_K(u^\varepsilon)\right) - \mathcal{A}\left(t,x,T_K(u^\varepsilon),\mathrm{D}T_K(u)\right)\right] \to 0$ *as* $\varepsilon \to 0$.

Proof of Lemma 3.3. Using $T_K(u^\varepsilon)$ as a test function in problem (3.1), we have

$$\begin{aligned} &\frac{1}{2}\|T_K(u^\varepsilon)(T)\|^2_{L^2(\Omega)} \\ &\qquad + \int_Q [A(x,t,\mathrm{D}T_K u^\varepsilon) + (1-\theta(x,t))B(x,t,T_K u^\varepsilon)]\mathrm{D}T_K(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t \\ &= \frac{1}{2}\|T_K(u_0^\varepsilon)\|^2_{L^2(\Omega)} + \int_Q [F^\varepsilon - \theta(x,t)B(x,t,u^\varepsilon)]\mathrm{D}T_K(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Thanks to the coercivity property stated in Lemma 3.2, this gives

$$
\begin{aligned}
&\frac{1}{2}\|T_K(u^\varepsilon)(T)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_Q |\mathrm{D}T_K(u^\varepsilon)|^2 \,\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{2}\|T_K(u_0^\varepsilon)\|^2_{L^2(\Omega)} + \int_Q F^\varepsilon\, \mathrm{D}T_K(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_Q \theta(x,t)B(x,t,T_K u^\varepsilon)\mathrm{D}T_K(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.6}
$$

The last term in the right-hand side in (3.6) can be estimated as

$$
\begin{aligned}
&\int_\Omega \theta(x,t)B(x,t,T_K u^\varepsilon)\mathrm{D}T_K(u^\varepsilon)\,\mathrm{d}x \\
&= \int_\Omega \frac{T_M(b(x,t))}{b(x,t)}[B(x,t,T_K u^\varepsilon) - B(x,t,0)]\mathrm{D}T_K(u^\varepsilon)\,\mathrm{d}x \\
&\le \int_\Omega T_M(b(x,t))|T_K u^\varepsilon||\mathrm{D}T_K(u^\varepsilon)|\,\mathrm{d}x \\
&\le \int_{\{|u^\varepsilon|\le K\}} T_M(b(x,t))|T_K(u^\varepsilon)||\mathrm{D}T_K(u^\varepsilon)|\,\mathrm{d}x \\
&\quad + \int_{\{|u^\varepsilon|> K\}} T_M(b(x,t))|T_K(u^\varepsilon)||\mathrm{D}T_K(u^\varepsilon)|\,\mathrm{d}x \\
&\le \int_{\{|u^\varepsilon|\le K\}} b(x,t)|T_K(u^\varepsilon)||\mathrm{D}T_K(u^\varepsilon)|\,\mathrm{d}x \\
&\quad + M\int_{\{|u^\varepsilon|> K\}} |T_K(u^\varepsilon)||\mathrm{D}T_K(u^\varepsilon)|\,\mathrm{d}x \\
&\le \|b\|_{L^{N,\infty}(|u^\varepsilon|<K)}\|T_K(u^\varepsilon)\|_{2^*,2}\|\mathrm{D}T_K(u^\varepsilon)\|_2 \\
&\quad + ME_K^N\|T_K(u^\varepsilon)\|_{2^*,2}\|\mathrm{D}T_K(u^\varepsilon)\|_2 \\
&\le \frac{\alpha}{S_{N,2}}\|\mathrm{D}T_K(u^\varepsilon)\|^2_{L^2(\Omega)} + ME_K^N\|\mathrm{D}T_K(u^\varepsilon)\|^2_{L^2(\Omega)}
\end{aligned}
\tag{3.7}
$$

with $E_K^N := \{u^\varepsilon > K\}$, this shows that E_K tends to 0 as $K \to \infty$. Hence, we obtain $M|E_K^N| \le \alpha/8$. Inserting the last expression in (3.7) into the last expression on the right-hand side of (3.6), we get

$$
\begin{aligned}
&\frac{1}{2}\|T_K(u^\varepsilon)(T)\|^2_{L^2(\Omega)} + \frac{\alpha}{4}\int_Q |\mathrm{D}T_K(u^\varepsilon)|^2 \,\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{2}\|T_K(u_0^\varepsilon)\|^2_{L^2(\Omega)} + \frac{2}{\alpha}\|F^\varepsilon\|^2_{L^2(Q)}.
\end{aligned}
\tag{3.8}
$$

We observe that $\mathrm{D}T_K(u^\varepsilon)$ is bounded in $L^2(Q)$ due to estimate (3.8). Proceeding as in [3], it is easy to conclude that $T_K(u^\varepsilon)$ satisfies (1). Conclusion (2) is not new, and has already been proven for example in [2, 3, 8], so we skip its proof.

The proof of estimates (3) can be obtained by proceeding with the same strategy as in [8].

Let us now consider the following renormalized equation:

$$\begin{aligned}&\partial_t S_n(u^\varepsilon) - \operatorname{div}(\mathcal{A}(x,t,u^\varepsilon,\mathrm{D}u^\varepsilon)S_n'(u^\varepsilon)) + E_n^\varepsilon \\ &= -\operatorname{div}(F^\varepsilon - \theta(x,t)B(x,t,u^\varepsilon))S_n'(u^\varepsilon),\end{aligned}$$

where $E_n^\varepsilon = \mathcal{A}(x,t,u^\varepsilon,\mathrm{D}u^\varepsilon)\mathrm{D}u^\varepsilon S_n''(u^\varepsilon)$. Thanks to the coerciveness result from Lemma 3.1 and (3.2), we get

$$\lim_{n\to\infty}\sup_{\varepsilon}\|E_n^\varepsilon\|_{L^1(Q)} = 0. \tag{3.9}$$

Using $T_K(u^\varepsilon) - T_K(u)$ as a test function in the renormalized equation, we get

$$\begin{aligned}&\int_0^T \langle (S_n(u^\varepsilon))_t, T_K(u^\varepsilon) - T_K(u)\rangle \,\mathrm{d}t \\ &\quad + \int_0^T\int_\Omega \mathcal{A}(x,t,u^\varepsilon,\mathrm{D}u^\varepsilon)\,\mathrm{D}(T_K(u^\varepsilon) - T_K(u))S_n'(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T\int_\Omega E_n^\varepsilon(T_K(u^\varepsilon) - T_K(u))\,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^T\int_\Omega F^\varepsilon\,\mathrm{D}(T_K(u^\varepsilon) - T_K(u))S_n'(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t + \omega_n^\varepsilon,\end{aligned} \tag{3.10}$$

where $\omega_n^\varepsilon = \int_0^T\int_\Omega \theta(x,t)B(x,t,T_K(u^\varepsilon))\mathrm{D}(T_K(u^\varepsilon) - T_K(u))S_n'(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t$. We need to prove

$$\limsup_{\varepsilon\to 0}\|\omega_n^\varepsilon\|_{L^1(Q)} = 0. \tag{3.11}$$

We have

$$\begin{aligned}&\int_0^T\int_Q \theta(x,t)B(x,t,T_K(u^\varepsilon))\mathrm{D}(T_K(u^\varepsilon) - T_K(u))S_n'(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T\int_Q \frac{T_M(b(x,t))}{b(x,t)}[B(x,t,T_K(u^\varepsilon)) - B(x,t,0)] \\ &\qquad\qquad \cdot \mathrm{D}(T_K(u^\varepsilon) - T_K(u))S_n'(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t \\ &\le \int_0^T\int_\Omega T_M(b(x,t))|T_K(u^\varepsilon)||\mathrm{D}(T_K(u^\varepsilon) - T_K(u))||S_n'(u^\varepsilon)| \\ &\le 2M\int_{\{|u^\varepsilon|>K\}} C_P|\mathrm{D}u^\varepsilon|^2|S_n'(u^\varepsilon)|.\end{aligned}$$

Thanks to (3.2), and the fact that S_n' has a compact support, (3.11) is proven. The second term of the left-hand side of (3.10) can be treated as follows:

$$\begin{aligned}&\int_0^T\int_\Omega \mathcal{A}(x,t,u^\varepsilon,\mathrm{D}u^\varepsilon)\mathrm{D}(T_K(u^\varepsilon)-T_K(u))S_n'(u^\varepsilon)\\&\geq \int_0^T\int_\Omega \mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u^\varepsilon))\mathrm{D}(T_K(u^\varepsilon)-T_K(u))\,\mathrm{d}x\mathrm{d}t\\&\quad-\int_0^T\int_\Omega \mathcal{A}(x,t,u^\varepsilon,\mathrm{D}u^\varepsilon)S_n'\chi_{\{|u^\varepsilon|>K\}}\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

which yields

$$\begin{aligned}&\int_0^T\int_\Omega \mathcal{A}(x,t,u^\varepsilon,\mathrm{D}u^\varepsilon)\mathrm{D}(T_K(u^\varepsilon)-T_K(u))S_n'(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t\\&\geq \int_0^T\int_\Omega \{\mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u^\varepsilon))-\mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u))\}\\&\qquad\qquad\cdot\mathrm{D}(T_K(u^\varepsilon)-T_K(u))\,\mathrm{d}x\mathrm{d}t\\&\quad-\int_0^T\int_\Omega \mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u^\varepsilon))\mathrm{D}(T_K(u^\varepsilon)-T_K(u))\,\mathrm{d}x\mathrm{d}t\\&\quad+\int_0^T\int_\Omega \mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u))\mathrm{D}(T_K(u^\varepsilon)-T_K(u))\,\mathrm{d}x\mathrm{d}t\\&\quad-\int_0^T\int_\Omega \mathcal{A}(x,t,u^\varepsilon,\mathrm{D}u^\varepsilon)S_n'(u^\varepsilon)\mathrm{D}T_K(u^\varepsilon)\chi_{\{|u^\varepsilon|>K\}}\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.12}$$

Using the facts that S_n' has compact support, $\mathrm{D}T_K(u^\varepsilon)$ is bounded in $L^1(Q)$ for every $K>0$, and that in the limit $\varepsilon\to 0$ the last three terms of the right-hand side of (3.12) go to zero, we deduce

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_0^T\int_\Omega \{\mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u^\varepsilon))-\mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u))\}\\&\qquad\qquad\cdot\mathrm{D}(T_K(u^\varepsilon)-T_K(u))\,\mathrm{d}x\mathrm{d}t\\&\leq \limsup_{\varepsilon\to 0}\int_0^T\int_\Omega \mathcal{A}(x,t,u^\varepsilon,\mathrm{D}u^\varepsilon)\cdot\mathrm{D}(T_K(u^\varepsilon)-T_K(u))S_n'(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.13}$$

Inserting the first part of (3.13) into (3.12), we obtain

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_0^T\int_\Omega \{\mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u^\varepsilon))-\mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u))\}\\&\qquad\qquad\cdot\mathrm{D}(T_K(u^\varepsilon)-T_K(u))\,\mathrm{d}x\mathrm{d}t\\&\leq \limsup_{\varepsilon\to 0}\left\{-\int_0^T\langle (S_n(u^\varepsilon))_t,T_K(u^\varepsilon)-T_K(u)\rangle\,\mathrm{d}t+2K\|E_n^\varepsilon\|_{L^1(Q)}\right\}.\end{aligned}$$

For n large enough, proceeding as in [3], we get

$$\liminf_{\varepsilon\to 0}\int_0^T \langle (S_n(u^\varepsilon))_t, T_K(u^\varepsilon)-T_K(u)\rangle\,\mathrm{d}t \geq 0.$$

From this, together with (3.9) and by letting n tend to infinity, we obtain

$$\limsup_{\varepsilon\to 0}\int_0^T\int_\Omega \{\mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u^\varepsilon))-\mathcal{A}(x,t,T_K(u^\varepsilon),\mathrm{D}T_K(u))\} \cdot \mathrm{D}(T_K(u^\varepsilon)-T_K(u))\,\mathrm{d}x\mathrm{d}t = 0,$$

which concludes the proof. ■

The following lemma is an extension of [3, Lemma 3.2].

LEMMA 3.4. *There exist a subsequence (still indexed by ε) of the sequence u^ε and an element u of $C^0([0,T];L^2(\Omega))$ such that when ε tends to* 0 *and for any fixed K, the following limits hold:*

(1) $u^\varepsilon \to u$ *in* $C^0([0,T];L^2(\Omega))$,

(2) $T_K(u^\varepsilon)\to T_K(u)$ *in* $L^2([0,T];W_0^{1,p}(\Omega))$,

(3) $A(x,t,\mathrm{D}T_K(u^\varepsilon))\to A(x,t,\mathrm{D}T_K(u))$ *weakly in* $L^2(Q)^N$,

(4) $B(x,t,u^\varepsilon)\to B(x,t,u)$ *in* $L^2(Q)$,

(5) $\int_Q \mathcal{A}(x,t,u^\varepsilon,\mathrm{D}T_K(u^\varepsilon))\mathrm{D}T_K(u^\varepsilon)\,\mathrm{d}x\mathrm{d}t \to \int_Q \mathcal{A}(x,t,u,\mathrm{D}T_K(u))\,\mathrm{D}T_K(u)\,\mathrm{d}x\mathrm{d}t.$

PROOF. The strong convergence of u^ε can be obtained by using $1/\sigma T_\sigma(u^\varepsilon-u^\eta)$ as a test function in the difference of the two equations (3.1) concerning u^ε and u^η, and letting σ tend to zero. As a matter of fact, we can prove that u^ε is a Cauchy sequence in $C^0([0,T];L^2(\Omega))$. Thanks to the weak convergence together with the asymptotic behavior of the sequence $T_K(u^\varepsilon)$ as ε tends to zero, which is stated and proved in Lemma 3.3, the strong convergence in (2) holds. The last three conclusions are simply inherited from the strong convergence of u^ε and $T_K(u^\varepsilon)$ along with the results which are mentioned in Lemma 3.3. ■

Having at hand those useful lemmas, we are now in position to prove Theorem 1.1, which is our main result.

PROOF OF THEOREM 1.1. EXISTENCE OF RENORMALIZED SOLUTIONS. The proof of the existence of renormalized solutions follows closely [3, Theorem 3.1], with some slight differences due to the unbounded convective term. Since we know that u^ε belongs to $L^2(0,T;W_0^1(\Omega))$, and $\partial u^\varepsilon/\partial t$ belongs to $L^2(0,T;W_0^{-1,2}(\Omega))$, we have, for any $S \in C^\infty(\mathbb{R})$ with $S' \in C_0^\infty(\mathbb{R})$,

$$\frac{\partial S(u^\varepsilon)}{\partial t} = S'(u^\varepsilon)\frac{\partial u^\varepsilon}{\partial t}.$$

This applied to (3.1) leads to

$$\begin{aligned}&\frac{\partial S(u^\varepsilon)}{\partial t} - \operatorname{div}[S'(u^\varepsilon)A(t,x,\mathrm{D}u^\varepsilon) + (1-\theta(x,t))B(x,t,u^\varepsilon)]\\ &\quad + S''(u^\varepsilon)[A(t,x,\mathrm{D}u^\varepsilon) + (1-\theta(x,t))B(t,x,u^\varepsilon)]\mathrm{D}u^\varepsilon\\ &= -\operatorname{div}(F - \theta(x,t)B(x,t,u^\varepsilon))S'(u^\varepsilon).\end{aligned}$$

In order to pass to the limit in (3.1) as ε tends to zero, define M in such a way that $\operatorname{supp} S' \subset [-M,M]$, so that

$$S'(u^\varepsilon)A(x,t,\mathrm{D}u^\varepsilon) = S'(u^\varepsilon)A(x,t,\mathrm{D}T_M(u^\varepsilon)).$$

Using the facts that u^ε converges to u as ε tends to zero, and that S' is bounded, we may conclude that

$S'(u^\varepsilon)A(x,t,\mathrm{D}u^\varepsilon) \rightharpoonup S'(u)A(x,t,\mathrm{D}T_M(u))$ weakly in $(L^2(Q))^N$, and
$S'(u^\varepsilon)(1-\theta(x,t))B(x,t,u^\varepsilon) \rightharpoonup S'(u)(1-\theta(x,t))B(x,t,u)$ weakly in $(L^2(Q))^N$

as ε tends to zero. As already mentioned, $S'(u)A(x,t,\mathrm{D}T_M(u))$ has been denoted by $S'(u)A(t,x,\mathrm{D}u)$ in equation (2.2).

Let us consider the term

$$\begin{aligned}&S''(u^\varepsilon)[A(t,x,\mathrm{D}u^\varepsilon) + (1-\theta(x,t))B(x,t,u^\varepsilon)]\mathrm{D}u^\varepsilon\\ &= S''(u^\varepsilon)[A(t,x,\mathrm{D}T_M(u^\varepsilon)) + (1-\theta(x,t))B(x,t,u^\varepsilon)]\mathrm{D}T_M(u^\varepsilon).\end{aligned}$$

Thanks to the asymptotic estimates of $T_K(u^\varepsilon)$ in Lemma 3.3,

$$\lim_{\varepsilon\to 0}\int_Q [\mathcal{A}(x,t,u^\varepsilon,\mathrm{D}u^\varepsilon) - \mathcal{A}(x,t,u^\varepsilon,\mathrm{D}T_M(u))][\mathrm{D}T_M(u^\varepsilon) - \mathrm{D}T_M(u)]\,\mathrm{d}x\mathrm{d}t = 0.$$

From the above equality we obtain

$$[\mathcal{A}(x,t,u^\varepsilon,\mathrm{D}u^\varepsilon) - \mathcal{A}(x,t,u,\mathrm{D}T_M(u))]\cdot[\mathrm{D}T_M(u^\varepsilon) - \mathrm{D}T_M(u)] \to 0 \ \text{ in } L^1(Q)$$

as ε tends to zero. Using the facts that $S''(u^\varepsilon)$ is pointwise convergent to $S''(u)$ and S'' is bounded, by applying Egoroff's theorem we get

$$\begin{aligned}&S''(u^\varepsilon)\mathcal{A}(x,t,u^\varepsilon,\mathrm{D}T_M(u^\varepsilon))\mathrm{D}T_M(u^\varepsilon)\\ &\to S''(u)\mathcal{A}(x,t,u,\mathrm{D}T_M(u))\mathrm{D}T_M(u) \ \text{ in } \ L^1(Q)\end{aligned} \tag{3.14}$$

as ε tends to zero. Since u^ε is strongly convergent to u in $C^0([0,T];L^2(\Omega))$ as in Lemma 3.4 (1),

$$\theta(x,t)B(x,t,u^\varepsilon)S'(u^\varepsilon) \to \theta(x,t)B(x,t,u)S'(u) \text{ in } L^1(Q). \tag{3.15}$$

Finally, the pointwise convergence of u^ε together with the strong convergence of F^ε to F yields

$$F^\varepsilon S'(u^\varepsilon) \to FS'(u) \text{ in } L^1(Q). \tag{3.16}$$

From (3.14), (3.15) and (3.16) we deduce that (3.1) is satisfied in the sense of distributions. This completes the proof of the existence result of Theorem 3.1. ∎

The proof of the uniqueness of the renormalized solution turns out to be more involved than in the case treated in [3], due to the unboudedness of the convective term.

Proof of Theorem 1.1. Uniqueness. Uniqueness of the solution will follow as a direct application of the following lemma, which is adapted to our situation from [3]. ∎

Lemma 3.5. *Assume that u_{01} and u_{02} lie in $L^2(\Omega)$, F_1 and F_2 lie in $L^2(Q)$, and that they satisfy*

$$u_{01} \le u_{02} \text{ and } F_{01} \le F_{02}.$$

Then if u_1 and u_2 are two renormalized solutions of problem (1.9) for the respective data (F_1, u_{01}) and (F_2, u_{02}), we have

$$u_1 \le u_2 \text{ a.e. } Q.$$

Proof. The proof follows the strategy of [3], with some slight differences. In [3], uniqueness is proved when the data $b = 0$, and the proof is based on the monotonicity of the operator A in divergence form. As a matter of fact, in our case we cannot rely on the monotonicity of the operator, due to the unbounded convective term. But we can repeat the proof of [3, Lemma 3.4], based on the fact that the operator $\mathcal{A}$ is coercive, as stated in Lemma 3.2.

Define $f^+ = \sup(f,0)$ for any measurable function defined on Q. As in [3], we define in a similar way the function $S_n(r)$ in $C^\infty(\mathbb{R})$ through

$$S_n(r) = \int_0^r h_n(s)\,\mathrm{d}s, \qquad h_n(r) = \begin{cases} 1 & \text{if } |r| \le n-1, \\ h(r-(n-1)\,\mathrm{sgn}(r)) & \text{if } |r| \ge n-1, \end{cases}$$

where $\mathrm{sgn}(r)$ denotes the sign of r.

The function S_n is a smooth approximation of the truncation T_n, and it satisfies $\operatorname{supp} S_n' \subset [-(n+1), n+1]$, $\|S_n'\|_{L^\infty(\mathbb{R})} \leq \|h\|_{L^\infty(\mathbb{R})}$, and $S_n(r) = S_n(T_{n+1})(r)$ for all r in$\mathbb{R}$ and all integers $n \geq 2$.

We use $T_C^+(S_n(u_1) - S_n(u_2))$ as a test function in the difference of equation (2.2) relative to u_1 and u_2. Upon integration over $(0,t)$ and then upon $(0,T)$, we get

$$\begin{aligned} &\int_0^T \int_0^t \int_\Omega \Big\langle \frac{\partial S_n(u_1)}{\partial t} - \frac{\partial S_n(u_2)}{\partial t}, T_C^+(S_n(u_1) - S_n(u_2)) \Big\rangle (s) \,\mathrm{d}x\mathrm{d}s\mathrm{d}t \\ &\quad + I_1^n + I_2^n + I_3^n + I_4^n \\ &= \int_0^T \int_0^t \int_\Omega (F_1 S_n'(u_1) - F_2 S_n'(u_2)) \mathrm{D} T_C^+(S_n(u_1) - S_n(u_2)) \,\mathrm{d}x\mathrm{d}s\mathrm{d}t \\ &\quad + J_1^n + J_2^n, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} I_1^n &= \int_0^T \int_0^t \int_\Omega [S_n'(u_1)[A(\mathrm{D}u_1) + (1-\theta(x,s))B(u_1)] \\ &\qquad\qquad - S_n'(u_2)[A(\mathrm{D}u_2) + (1-\theta(x,s))B(u_2)]] \\ &\qquad\qquad \cdot \mathrm{D}T_C^+(S_n(u_1) - S_n(u_2)) \,\mathrm{d}x\mathrm{d}s\mathrm{d}t, \\ I_2^n &= \int_0^T \int_0^t \int_\Omega S_n''(u_1)[A(\mathrm{D}u_1) + (1-\theta(x,s))B(u_1)] \\ &\qquad\qquad \cdot \mathrm{D}u_1 T_C^+(S_n(u_1) - S_n(u_2)) \,\mathrm{d}x\mathrm{d}s\mathrm{d}t, \\ I_3^n &= -\int_0^T \int_0^t \int_\Omega S_n''(u_2)[A(\mathrm{D}u_2) + (1-\theta(x,s))B(u_1)] \\ &\qquad\qquad \cdot \mathrm{D}u_2 T_C^+(S_n(u_1) - S_n(u_2)) \,\mathrm{d}x\mathrm{d}s\mathrm{d}t, \\ J_1^n &= \int_0^T \int_0^t \int_\Omega \theta(x,s) B(u_1) S'(u_1) \mathrm{D}T_C^+(S_n(u_1) - S_n(u_2)) \,\mathrm{d}x\mathrm{d}s\mathrm{d}t, \\ J_2^n &= -\int_0^T \int_0^t \int_\Omega \theta(x,s) B(u_2) S'(u_2) \mathrm{D}T_C^+(S_n(u_1) - S_n(u_2)) \,\mathrm{d}x\mathrm{d}s\mathrm{d}t. \end{aligned}$$

The first term of (3.17) can be handled in the following way:

$$\begin{aligned} &\int_0^t \Big\langle \frac{\partial S_n(u_1)}{\partial t} - \frac{\partial S_n(u_2)}{\partial t}, T_C^+(S_n(u_1) - S_n(u_2)) \Big\rangle (s) \,\mathrm{d}s \\ &= \int_\Omega \varphi_C([S_n(u_1) - S_n(u_2)]^+)(t) \,\mathrm{d}x - \int_\Omega \varphi_C([S_n(u_{01}) - S_n(u_{02})]^+) \,\mathrm{d}x, \end{aligned}$$

for all t in $[0, T]$. Equation (3.17) then can be rewritten as follows:

$$\int_Q \varphi_C([S_n(u_1) - S_n(u_2)]^+)(t)\,\mathrm{d}x\mathrm{d}t + I_1^n + I_2^n + I_3^n$$
$$\leq \int_0^T \int_0^t \int_\Omega (F_1 S_n'(u_1) - F_2 S_n'(u_2))\mathrm{D}T_C^+(S_n(u_1) - S_n(u_2))\,\mathrm{d}x\mathrm{d}s\mathrm{d}t$$
$$+ T\int_\Omega \varphi_C([S_n(u_{01}) - S_n(u_{02})]^+)\,\mathrm{d}x + J_1^n + J_2^n.$$

In order to complete the proof, the quantities $I_1^n, I_2^n, I_3^n, J_1^n, J_2^n$ will be shown to satisfy

$$\lim_{n\to\infty} I_1^n = 0, \tag{3.18}$$

$$\lim_{n\to\infty} I_2^n = 0, \tag{3.19}$$

$$\lim_{n\to\infty} I_3^n = 0, \tag{3.20}$$

$$\lim_{n\to\infty} J_1^n = 0, \tag{3.21}$$

$$\lim_{n\to\infty} J_2^n = 0. \tag{3.22}$$

We have

- $S_n(u_1) - S_n(u_2) \to u_1 - u_2$ strongly in $L^1(Q)$,
- $(F_1 S_n'(u_1) - F_2 S_n'(u_2))\mathrm{D}T_C^+(S_n(u_1) - S_n(u_2)) \to (F_1 - F_2)\mathrm{D}T_C^+(u_1 - u_2)$ strongly in $L^1(Q)$,
- $S_n(u_{01}) - S_n(u_{02}) \to u_{01} - u_{02}$ strongly in $L^1(\Omega)$.

When n tends to infinity,

$$\int_Q \varphi_C([u_1 - u_2]^+) \leq \int_0^T \int_0^t \int_\Omega (F_1 - F_2)\mathrm{D}T_C^+(u_1 - u_2)\,\mathrm{d}x\mathrm{d}s\mathrm{d}t + T\int_\Omega \varphi_C([u_{01} - u_{02}]^+)\,\mathrm{d}x. \tag{3.23}$$

The right-hand side of (3.23) is nonpositive, thus we have $\varphi_C([u_1 - u_2]^+) \leq 0$. Since C is arbitrary, this implies that $u_1 \leq u_2$ almost everywhere in Q. This will prove Lemma 3.5. In order for the proof to be complete, it remains to prove (3.18)-(3.22). For any measurable function v defined on Q, and any positive real number K, let $\chi_{\{v\leq K\}}$ denote the characteristic function of the measurable set $\{(t,x) : v(t,x) \leq K\}$. Then I_1^n can be rewritten as

$$I_1^n = \int_Q (T-t)\chi_{\{0\leq S_n(u_1)-S_n(u_2)\leq C\}}[S_n'(u_1)\mathcal{A}(u_1) - S_n'(u_2)\mathcal{A}(u_2)] \cdot \mathrm{D}(S_n(u_1) - S_n(u_2))\,\mathrm{d}x\mathrm{d}t,$$

or equivalently as

$$\begin{aligned}
I_1^n = &\int_Q (T-t)\chi_{\{0\le S_n(u_1)-S_n(u_2)\le C\}} \\
&\quad \cdot [\mathcal{A}(x,t,u_1,\mathrm{D}S_n(u_1)) - \mathcal{A}(x,t,u_2,\mathrm{D}S_n(u_2))] \\
&\quad \cdot \mathrm{D}(S_n(u_1) - \mathrm{D}S_n(u_2))\,\mathrm{d}x\mathrm{d}t \\
&+ \int_Q (T-t)\chi_{\{0\le S_n(u_1)-S_n(u_2)\le C\}} \\
&\quad \cdot [S_n'(u_1)\mathcal{A}(x,t,u_1,\mathrm{D}u_1) - \mathcal{A}(x,t,u_1,\mathrm{D}S_n(u_1))] \\
&\quad \cdot \mathrm{D}(S_n(u_1) - \mathrm{D}S_n(u_2))\,\mathrm{d}x\mathrm{d}t \\
&- \int_Q (T-t)\chi_{\{0\le S_n(u_1)-S_n(u_2)\le C\}}[S_n'(u_2)\mathcal{A}(x,t,u_2,\mathrm{D}u_2) - \mathcal{A}(\mathrm{D}S_n(u_2))] \\
&\quad \cdot \mathrm{D}(S_n(u_1) - \mathrm{D}S_n(u_2))\,\mathrm{d}x\mathrm{d}t \\
:= &\ F_1^n + F_2^n + F_3^n.
\end{aligned}$$

Using the asymptotic estimates from Lemma 3.3, it is easy to see that F_1^n is nonnegative, so we only need to show that F_2^n and F_3^n tend to zero when $n \to \infty$. Due to the definition of S_n, we have

$$S_n'(u_1)\mathcal{A}(x,t,u_1,\mathrm{D}u_1) = \mathcal{A}(x,t,u_1,\mathrm{D}S_n(u_1)),$$

almost everywhere except on the subset $\{(x,t) : n \le |u_1(x,t)| \le n+1\}$. Thus F_2^n can be rewritten as

$$\begin{aligned}
F_2^n = &\int_Q (T-t)\chi_{\{0\le S_n(u_1)-S_n(u_2)\le C\}}\chi_{\{n\le|u_1|\le n+1\}} \\
&\quad \cdot [S_n'(u_1)\mathcal{A}(x,t,u_1,\mathrm{D}u_1) - \mathcal{A}(x,t,u_1,\mathrm{D}S_n(u_1))] \\
&\quad \cdot \mathrm{D}(S_n(u_1) - \mathrm{D}S_n(u_2))\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Using the assumptions on S_n, the bound $\|S_n'\|_{L^\infty(\mathbb{R})} \le \|h\|_{L^\infty(\mathbb{R})}$, and the fact that $\operatorname{supp} S_n' \subset [-(n+1), n+1]$, implies that, for any $n > C$,

$$\begin{aligned}
&\chi_{\{0\le S_n(u_1)-S_n(u_2)\le C\}}\chi_{\{n\le|u_1|\le n+1\}}S_n'(u_2) \\
&= \chi_{\{0\le S_n(u_1)-S_n(u_2)\le C\}}\chi_{\{n\le|u_1|\le n+1\}}\chi_{\{n-C\le|u_2|\le n+1\}}S_n'(u_2).
\end{aligned}$$

We obtain

$$\begin{aligned}
F_2^n = &\int_Q (T-t).\chi_{\{0\le S_n(u_1)-S_n(u_2)\le C\}}\cdot\chi_{\{n\le|u_1|\le n+1\}} \\
&\cdot[S_n'(u_1)\mathcal{A}(x,t,u_1,\mathrm{D}u_1) - \mathcal{A}(x,t,u_1,\mathrm{D}S_n(u_1))] \\
&\cdot [S_n'(u_1)\,\mathrm{D}u_1 - S_n'(u_2)\,\mathrm{D}u_2]\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq T \int_Q \chi_{\{n\leq|u_1|\leq n+1\}}\big[S_n'(u_1)[A(\mathrm{D}u_1)+(1-\theta(x,t))B(u_1)] \\
&\qquad\qquad - [A(\mathrm{D}S_n(u_1))+(1-\theta(x,t))B(u_1)]\big] \\
&\qquad \cdot [\|h\|_{L^\infty(\mathbb{R})}|\mathrm{D}u_1| - \chi_{\{n-C\leq|u_2|\leq n+1\}}\|h\|_{L^\infty(\mathbb{R})}||\mathrm{D}u_2|]\,\mathrm{d}x\mathrm{d}t \\
&\leq T \int_Q \chi_{\{n\leq|u_1|\leq n+1\}}[S_n'(u_1)A(\mathrm{D}u_1) - A(\mathrm{D}S_n(u_1))] \\
&\qquad \cdot [\|h\|_{L^\infty(\mathbb{R})}|\mathrm{D}u_1| + \chi_{\{n-C\leq|u_2|\leq n+1\}}\|h\|_{L^\infty(\mathbb{R})}|\mathrm{D}u_2|]\,\mathrm{d}x\mathrm{d}t \\
&\leq T \int_Q \chi_{\{n\leq|u_1|\leq n+1\}}[\|h\|_{L^\infty(\mathbb{R})}(c|\mathrm{D}u_1|+d)+d+c\|h\|_{L^\infty(\mathbb{R})}|\mathrm{D}u_1|] \\
&\qquad \cdot [\|h\|_{L^\infty(\mathbb{R})}|\mathrm{D}u_1| + \chi_{\{n-C\leq|u_2|\leq n+1\}}\|h\|_{L^\infty(\mathbb{R})}|\mathrm{D}u_2|]\,\mathrm{d}x\mathrm{d}t \\
&= T\|h\|_{L^\infty(\mathbb{R})} \int_Q [(1+\|h\|_{L^\infty(\mathbb{R})})d + 2\|h\|_{L^\infty(\mathbb{R})}c\,\chi_{\{n\leq|u_1|\leq n+1\}}|\mathrm{D}u_1|] \\
&\qquad\qquad \cdot \chi_{\{n\leq|u_1|\leq n+1\}}|\mathrm{D}u_1| + \chi_{\{n-C\leq|u_2|\leq n+1\}}\|h\|_{L^\infty(\mathbb{R})}|\mathrm{D}u_2|\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Thanks to Lemma 3.1 we obtain

$$
\lim_{n\to\infty}\int_{\{n\leq|u_1|\leq n+1\}}|\mathrm{D}u_1|^2\,\mathrm{d}x\mathrm{d}t = 0 \text{ and } \lim_{n\to\infty}\int_{\{n\leq|u_2|\leq n+1\}}|\mathrm{D}u_2|^2\,\mathrm{d}x\mathrm{d}t = 0,
$$

thus F_2^n tends to zero.

The proof that F_3^n tends to zero as $n\to+\infty$ is similar to the case of F_2^n by switching the role of u_1 and u_2. Thus $\lim_{n\to\infty} I_1^n = 0$.

The proofs that $\lim_{n\to\infty} I_2^n = 0$ and $\lim_{n\to\infty} I_3^n = 0$ are also similar, so here we just describe the proof of $\lim_{n\to\infty} I_2^n = 0$, and the conclusion $\lim_{n\to\infty} I_3^n = 0$ follows similarly.

By using the coerciveness of operator $\mathcal{A}$, we easily get

$$
\begin{aligned}
|I_2^n| &= \int_Q (T-t)S_n''(u_1)\mathcal{A}(x,t,u_1,\mathrm{D}u_1)\mathrm{D}u_1 T_C^+(S_n(u_1)-S_n(u_2))\,\mathrm{d}x\mathrm{d}t \\
&\leq TC\frac{\alpha}{2}\|h'\|_{L^\infty(\mathbb{R})}\int_Q \chi_{\{n\leq|u_1|\leq n+1\}}|\mathrm{D}u_1|^2\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Thanks to Lemma 3.3, $\lim_{n\to\infty} I_2^n = 0$. Finally, the proofs of $\lim_{n\to\infty} J_1^n = 0$ and $\lim_{n\to\infty} J_2^n = 0$ are similar, so we just do it for J_1^n.

$$
\begin{aligned}
J_1^n &= \int_0^T\int_0^t\int_\Omega \theta(x,s)B(u_1)S'(u_1)\mathrm{D}T_C^+(S_n(u_1)-S_n(u_2))\,\mathrm{d}x\mathrm{d}s\mathrm{d}t \\
&\leq \int_Q (T-t)\chi_{\{0\leq S_n(u_1)-S_n(u_2)\leq C\}}\chi_{\{n\leq|u_1|\leq n+1\}} \\
&\qquad \cdot\frac{T_M(b(x,s))}{b(x,s)}b(x,s)|u_1|\|h\|_{L^\infty(\mathbb{R})}[S_n'(u_1)\mathrm{D}u_1 - S_n'(u_2)\mathrm{D}u_2]\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$\leq TMC_P \|h\|_{L^\infty(\mathbb{R})} \cdot \int_Q \chi_{\{n \leq |u_1| \leq n+1\}} [\chi_{n \leq |u_1| \leq n+1} |Du_1|^2 + \chi_{\{n-C \leq |u_2| \leq n+1\}} |Du_2|] \,dx\,dt,$$

which tends to zero because of (3.2) of Lemma 3.1. This completes the proof of Lemma 3.5. ∎

Funding – This work was partially supported by GNAMPA-INdAM.

References

[1] M. F. Betta – A. Mercado – F. Murat – M. M. Porzio, Existence of renormalized solutions to nonlinear elliptic equations with a lower-order term and right-hand side a measure. *J. Math. Pures Appl. (9)* **82** (2003), no. 1, 90–124. MR 1967494

[2] D. Blanchard, Truncations and monotonicity methods for parabolic equations. *Nonlinear Anal.* **21** (1993), no. 10, 725–743. Zbl 0796.35077 MR 1246504

[3] D. Blanchard – F. Murat, Renormalised solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness. *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), no. 6, 1137–1152. Zbl 0895.35050 MR 1489429

[4] D. Blanchard – A. Porretta, Nonlinear parabolic equations with natural growth terms and measure initial data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30** (2001), no. 3-4, 583–622. Zbl 1072.35089 MR 1896079

[5] L. Boccardo – L. Orsina – A. Porretta, Some noncoercive parabolic equations with lower order terms in divergence form. *J. Evol. Equ.* **3** (2003), no. 3, 407–418.

[6] F. Farroni – G. Moscariello, A nonlinear parabolic equation with drift term. *Nonlinear Anal.* **177** (2018), part. B, 397–412. Zbl 1402.35143 MR 3886581

[7] A. Porretta, Existence results for nonlinear parabolic equations via strong convergence of truncations. *Ann. Mat. Pura Appl. (4)* **177** (1999), 143–172. Zbl 0957.35066 MR 1747629

[8] A. Porretta, Weak solutions to Fokker–Planck equations and mean field games. *Arch. Ration. Mech. Anal.* **216** (2015), no. 1, 1–62. Zbl 1312.35168 MR 3305653

Manoscritto pervenuto in redazione il 4 luglio 2022.