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Groups of order p^3 are mixed Tate

Tudor Pădurariu (*)

Abstract – Let G be a finite group. A natural place to study the Chow ring of the classifying space BG is Voevodsky's triangulated category of motives, inside which Morel–Voevodsky and Totaro have defined motives $M(BG)$ and $M^c(BG)$, respectively. We show that, for any group G of order p^3 over a field of characteristic not equal to p which contains a primitive p^3-th root of unity, the motive $M(BG)$ is a mixed Tate motive. We also show that, for a finite group G over a field of characteristic zero, $M(BG)$ is a mixed Tate motive if and only if $M^c(BG)$ is a mixed Tate motive.

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1. Introduction

1.1 – *Mixed Tate groups*

The group cohomology of a group G can be computed as the cohomology (with twisted coefficients) of the classifying space BG. One would like to understand what part of the group cohomology of G comes from algebraic geometry. Morel–Voevodsky [17] and Totaro [20] defined the motive of a classifying space $M(BG)$ and the motive of a classifying space with compact supports $M^c(BG)$, respectively, as objects in $\mathrm{DM}(k; R)$, Voevodsky's "big" triangulated category of motives over the field k with coefficients in a commutative ring R [22]. One can recover the motivic (co)homology groups of BG as defined by Edidin–Graham [7] by computing the motivic (co)homology groups of these motives.

(*) *Indirizzo dell'A.*: Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027, USA; tpadurariu@gmail.com

Inside DM$(k; R)$, one can define the subcategory of mixed Tate motives DMT$(k; R)$ as the smallest triangulated and closed under arbitrary direct sums subcategory which contains all the objects $R(j)$ with $j \in \mathbb{Z}$. We prove in Theorem 5.1 that for a finite group G the motive $M(BG)$ is *mixed Tate* if and only if $M^c(BG)$ is mixed Tate. We will simply say that a finite group G is mixed Tate if $M^c(BG)$ is in the category DMT$(k; R)$. From now on, we will restrict the discussion in the introduction to finite groups. Our main result the following.

Theorem 1.1. *Let G be a group of order p^3 and let k be a field of characteristic not equal to p which contains a primitive p^3-root of unity. Then $M^c(BG)$ is mixed Tate.*

One is interested in understanding p-groups because one recovers important information about a given finite group by studying its Sylow groups. The precise form of this philosophy which is applicable in our case is [20, Lemma 9.3] which says that BG is mixed Tate with $\mathbb{Z}/p$ or $\mathbb{Z}_{(p)}$ coefficients if BH is, where H is a p-Sylow subgroup of G.

1.2 – *Other properties of finite groups*

A group G is called *stably rational* if it has a faithful representation V such that $V /\!\!/ G$ is stably rational over $\mathbb{C}$. A group G the *weak Chow–Künneth property* if $CH^*(BG) \twoheadrightarrow CH^*(BG_E)$ is surjective for every extension of fields E/k. If G is mixed Tate, then BG is stably rational, satisfies the weak Chow–Küneth property, and has *trivial unramified cohomology*, see [20, Section 9] for definitions and references. We do not know whether any of these properties of a finite group G are equivalent.

1.3 – *Related results*

In all the following examples, we assume that k is a field in which p is invertible and which contains $|G|$-roots of unity, where G is the group studied.

The starting point for studying these properties of a group G are Bogomolov's [2] and Saltman's [19] examples of groups of order p^7 and p^9, respectively, which are not stably rational. Chu–Kang [4] and Chu et. al. [3] showed that for every p-group G of order $\leq p^4$ or 2-group of order $\leq 2^5$ and for every G-representation V, the quotient $V /\!\!/ G$ is rational. This property is stronger than saying that BG is stably rational.

Bogomolov [2] showed (with a further correction in [10]) that every p-group of order $\leq p^4$, for p an odd prime, or $\leq 2^5$ for p equal to 2, has trivial unramified cohomology, and that these are the best possible bounds.

Totaro [20, Section 10] showed that all 2-groups of order $\leq 2^5$ and all p-groups of order $\leq p^4$ have the weak Chow–Künneth property. He also showed [20, Corollary 9.10] that all abelian p-groups are mixed Tate. There are groups of order p^5 for p odd which do not have the weak Chow–Künneth property (see the discussion after [20, Corollary 3.1]) and thus which are not mixed Tate.

In view of these examples, it is worth investigating whether all p-groups of order $\leq p^4$ and all 2-groups of order $\leq 2^5$ are actually mixed Tate. Our methods only apply to p-groups of order $\leq p^3$ and to some groups of order p^4 as explained in Section 4.

1.4 – *Structure of the paper*

In Section 2, we recall the definitions of linear schemes and of the motives $M(X)$ and $M^c(X)$ for a quotient stack X in DM$(k; R)$. In Section 3, we reduce the proof of Theorem 1.1 to Theorem 3.3 and we prove three technical preliminary lemmas. Section 4 contains the proof of Theorem 3.3, which says that for a group G of order p^3 and V an irreducible G-representation of dimension p, the scheme $V /\!\!/ G$ is a linear scheme. The proof is inspired by a result of Chu–Kang [4] that says that $V /\!\!/ G$ is rational for G of order p^3 and V a G-representation. In Section 5, we show that $M(BG)$ is mixed Tate if and only if $M^c(BG)$ is mixed Tate.

2. Definitions and notations

2.1 – Fix p a prime number. Unless otherwise stated, we will denote by k a field of characteristic not equal to p which contains a primitive p^2-root of unity. In Section 5, we assume that the characteristic of k is zero.

All the schemes considered will be separated schemes of finite type over k. One can define the Chow groups $CH_i(X)$ as the group of i-dimensional algebraic cycles modulo rational equivalence [8]. One can further define the higher Chow groups [1], or the motivic (co)homology groups of such a scheme [22], see [20, Section 5] for a brief overview of these topics.

Let A be an affine k-scheme with a linear action of a reductive group G. We denote by $A /\!\!/ G := \mathrm{Spec}(\mathcal{O}_A^G)$ the quotient scheme and by A/G the corresponding quotient stack.

For a finite group G, we denote by $|G|$ the order of G. We denote by $[n]$ the set $\{1, \ldots, n\}$.

2.2 – We will work in the category DM$(k; R)$, the "big" triangulated category of motives over the field k with coefficients in the commutative ring R [20, Section 5], see also the general references [16, 22].

The exponential characteristic of k is 1 if k has characteristic zero and p if k has characteristic $p > 0$. We will assume throughout the paper that the exponential characteristic of k is invertible in R. Voevodsky defined two natural functors from the category of schemes to $\mathrm{DM}(k; R)$, which we will write as M and M^c [22], see also [20, Section 5].

We can associate a motive to any quotient stack $\mathcal{X} = Y/G$, with Y a quasi-projective scheme over k and G an affine group scheme of finite type over k such that there is a G-equivariant ample line bundle on Y, as follows [20, Section 8]. Choose G-representations $V_1 \hookrightarrow V_2 \hookrightarrow \cdots$ of G such that $\mathrm{codim}(S_i$ in $V_i)$ increases to infinity, where S_i is the locus of V_i where G does not act freely. Denote by $M_i(\mathcal{X}) := M(((V_i - S_i) \times Y)/G)$ and define

$$M(\mathcal{X}) = \mathrm{hocolim}(\cdots \to M_2(\mathcal{X}) \to M_1(\mathcal{X})),$$

where the maps are induced by the inclusions $V_i \hookrightarrow V_{i+1}$. To define $M^c(\mathcal{X})$, choose G-representations $\cdots \twoheadrightarrow V^2 \twoheadrightarrow V^1$ with loci S^i having the same property as above. Let $M_i^c(\mathcal{X}) := M_c(((V^i - S^i) \times Y)/G)$. Let n_i be the rank of the bundle V^i. Define

$$M^c(\mathcal{X}) = \mathrm{holim}(\cdots \to M_2^c(\mathcal{X})(-n_2)[-2n_2] \to M_1^c(\mathcal{X})(-n_1)[-2n_1]),$$

where the maps are induced by the projections $V^{i+1} \twoheadrightarrow V^i$. The definitions of $M^c(\mathcal{X})$ and $M(\mathcal{X})$ are independent of the choices of V_i and V^i, see [20, Theorem 8.4] and the discussion in Section 8 therein.

2.3 – A *linear scheme* over k is defined inductively as follows [20, Section 5, pages 2099-2100]: all the affine spaces are linear; if $Z \subset X$ is closed, and X and Z are linear, then $X \setminus Z$ is linear; further, if $X \setminus Z$ and Z are linear, then X is linear [20, page 2099]. There are examples of schemes with mixed Tate motive but which are not linear schemes [9].

Let X be a linear scheme over k and let R be a ring whose exponential characteristic is invertible in R. Then $M^c(X)$ is a mixed Tate motive.

Let I be a finite set, let $X_i \subset X$ be locally closed irreducible subschemes of X, and let $d = \dim(X)$. For $e \leq d$, let Y_e be the union of X_i for $i \in I$ such that $\dim(X_i) = e$. We say that X has a *stratification* $(X_i)_{i \in I}$ if there is a partition of underlying topological spaces

$$X = \bigsqcup_{i \in I} X_i$$

and Y_e is open in $X \setminus \bigsqcup_{f > e} Y_f = \bigsqcup_{g \leq e} Y_g$ for every $e \leq d$.

3. The plan of the proof and preliminaries

3.1 – Theorem 1.1 is known for abelian groups [20, Corollary 9.10]. The two non-abelian groups of order 8 are the dihedral and the quaternion group. Theorem 1.1 holds for them by [20, Corollary 9.7]. It thus suffices to show the following.

Theorem 3.1. *Let p be an odd prime, let k be a field of characteristic not equal to p which contains a primitive p^2-root of unity, and let G be a non-abelian group of order p^3. Then $M^c(BG)$ is mixed Tate.*

There are sufficient conditions on G which imply that G is mixed Tate. For example, by [20, Theorem 9.6] it is enough to show that every proper subgroup $H \subset G$ is mixed Tate and that there exists a faithful representation V of G such that the variety $(V - S) /\!/ G$ is mixed Tate, where S is the closed subset of V where G does not act freely.

For $K \subset G$ a subgroup, let $N_K := \{g \in G \mid gKg^{-1} = K\}$ be the normalizer of K and let $N'_K := N_K/K$.

Lemma 3.2. *Let G be a finite group such that N'_K is abelian for every subgroup $1 < K \subset G$. Let V be a representation of G and let $S \subset V$ be the locus of points with non-trivial stabilizer. Then $(V - S) /\!/ G$ is a linear scheme if and only if $V /\!/ G$ is a linear scheme.*

Proof. It suffices to check that $S /\!/ G$ is a linear scheme. We use induction on $|G|$. The statement is clear if $|G|$ is a prime number, because then G is a cyclic group and S is a subspace of V, and so $S /\!/ G \cong S$ is an affine space.

For $K \subset G$ a subgroup, let $V^K \subset V$ be the subspace of points fixed by K and let

$$V_K := V^K - \bigcup_{K<L\subset G} V^L.$$

If K' is a subgroup of G conjugate to K, the images of $V_K /\!/ N'_K$ and $V_{K'} /\!/ N'_{K'}$ in $V /\!/ G$ are the same. Let I be a set of subgroups of G such that any subgroup K of G is conjugate to a unique group in I. We have that $S = \bigsqcup_{1<K\subset G} V_K$ and there is a stratification

$$S /\!/ G = \bigsqcup_{I} V_K /\!/ N'_K.$$

It suffices to check that $V_K /\!/ N'_K$ is a linear scheme for any $1 < K \subset G$. The group N'_K is abelian, so it satisfies the hypothesis of the lemma. We have that $|N'_K| < |G|$, so by the induction hypothesis we know that $V_K /\!/ N'_K$ is a linear scheme if and only if $V^K /\!/ N'_K$ is a linear scheme. By Lemma 3.4, the quotient $V^K /\!/ N'_K$ is a linear scheme, thus $V_K /\!/ N'_K$ is a linear scheme. ∎

Any non-abelian group of order p^3 has a faithful irreducible representation. Indeed, a p-group has a faithful irreducible representation if and only if its center is cyclic [11, page 29], and $Z(G)$ has order p for any non-abelian group of order p^3. Moreover, all irreducible representations of a group G of order $\leq p^4$ have dimension 1 or p. Any group of order p^3 satisfies the hypothesis of Lemma 3.2 because for every subgroup $1 < K \subset G$, the quotient N_K/K has order 1, p, or p^2, and thus it is abelian. It is thus sufficient to prove the following.

THEOREM 3.3. *Let k be a field of characteristic not equal to p which contains a primitive p^2-root of unity. Let G be a non-abelian group of order p^3 and let V be an irreducible representation of degree p. Then $V /\!\!/ G$ is a linear scheme.*

3.2 – There are two non-abelian groups of order p^3. For a classification of p-groups of order $\leq p^4$ and their representations, see [4].

3.2.1. The first group is $G \cong (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$, which can also be written as

$$G = \langle \sigma, \pi, \tau \mid \sigma^p = \pi^p = \tau^p = 1,\ \sigma\pi = \pi\sigma,\ \sigma\tau = \tau\sigma,\ \tau\pi\tau^{-1} = \sigma\pi \rangle.$$

It has a faithful irreducible representation (ρ, V) which can be written explicitly on a basis $(e_i)_{i=1}^p$ of V as follows:

$$\begin{aligned} \rho(\sigma) &= \operatorname{diag}(\zeta, \ldots, \zeta), \\ \rho(\pi) &= \operatorname{diag}(1, \zeta, \ldots, \zeta^{p-1}), \\ \rho(\tau) &= P, \end{aligned}$$

where P is the matrix which permutes the basis $e_1 \mapsto e_2 \mapsto \cdots \mapsto e_p \mapsto e_1$, and ζ is a primitive p-th root of unity.

3.2.2. The second group is $G \cong \mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$, which can also be written as

$$G = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^p = 1,\ \tau\sigma\tau^{-1} = \sigma^{1+p} \rangle.$$

It has a faithful irreducible representation (ρ, V) given by

$$\begin{aligned} \rho(\sigma) &= \operatorname{diag}\left(\omega, \omega^{1+p}, \ldots, \omega^{1+p(p-1)}\right), \\ \rho(\tau) &= P, \end{aligned}$$

where ω is a primitive p^2-root of unity and P is the permutation matrix defined above.

3.3 – The proof of Theorem 3.3 will be given in Section 4. In the rest of this section, we include two lemmas used in its proof. The first one gives a proof of the already known fact that BG is mixed Tate for G abelian group [20, Corollary 9.10]. Recall that the exponent of a group is defined as the least common multiple of the orders of all elements of the group.

LEMMA 3.4. *Let N be an abelian p-group, and let V be an N-representation over a field k of characteristic not equal to p which contains the p^e-roots of unity, where p^e is the exponent of N. Then* $\operatorname{Spec} k[V]^N$ *is a linear scheme.*

PROOF. As $\operatorname{char} k \neq p$, the representation V decomposes as a sum of one-dimensional representations, and thus we can choose a basis $x_1, \ldots, x_d$ of V on which N acts diagonally. We prove the statement by induction on $|N|$. The base case, when N is the trivial group, is clear. In general, choose $\sigma \in N$ such that $N = \langle\sigma\rangle \oplus M$, where $\langle\sigma\rangle$ denotes the subgroup of N generated by σ. Assume that σ has order p^s. We will use the following stratification,

$$\operatorname{Spec} k[x_1, \ldots, x_d] = \bigsqcup_{J \subset [d]} \operatorname{Spec} k[x_j^{\pm 1} \mid j \in J],$$

where the disjoint union is taken over all sets $J \subset [d]$. This stratification is the partition of the affine space $\mathbb{A}^d_k$ into 2^d schemes P_J with $x_j \neq 0$ for $j \in J$ and $x_j = 0$ for $j \notin J$. We obtain a stratification

$$\operatorname{Spec} k[x_1, \ldots, x_d]^{\langle\sigma\rangle} = \bigsqcup_{J \subset [d]} \operatorname{Spec} k[x_j^{\pm 1} \mid j \in J]^{\langle\sigma\rangle}. \tag{3.1}$$

It is enough to show that

$$\operatorname{Spec} k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{\langle\sigma\rangle} \cong \operatorname{Spec} k[y_j^{\pm 1}], \tag{3.2}$$

where the y_j are monomials in x_i. The analogous statement holds for any stratum on the right hand side of (3.1). Once we show (3.2), we can reduce the problem from N to M for various representations of M.

To find such a decomposition, let $\sigma \cdot x_i = \zeta^{a_i} x_i$, where ζ is a primitive p^s-root of unity chosen such that $a_1 = 1$. Then

$$k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{\sigma} = k\left[x_1^{p^s}, x_2 x_1^{-a_2}, \ldots, x_d x_1^{-a_d}, \frac{1}{x_1^Q x_2 \cdots x_d}\right],$$

where $Q := p^s - a_2 - \cdots - a_d$. The right hand side is included in the left hand side, and $k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ is a free $k[x_1^{p^s}, x_2 x_1^{-a_2}, \ldots, x_d x_1^{-a_d}, \frac{1}{x_1^Q x_2 \cdots x_d}]$-module of rank p^s, so the two sides are indeed equal. ∎

Consider the torus $(\mathbb{G}_m)^p$ with coordinates $w_1, \ldots, w_p$ and let $W \subset (\mathbb{G}_m)^p$ be the subtorus with $w_1 \cdots w_p = 1$. The action of the cyclic group $\mathbb{Z}/p$ of order p which permutes the factors of $(\mathbb{G}_m)^p$ by $w_i \mapsto w_{i+1}$ for $1 \leq i \leq p$, where $w_{p+1} := w_1$, extends to an action of $\mathbb{Z}/p$ on W.

LEMMA 3.5. *The schemes* $S := W /\!\!/ \mathbb{Z}/p$ *and* $T := ((\mathbb{G}_m)^p - W) /\!\!/ \mathbb{Z}/p$ *are linear schemes.*

PROOF. Let τ be a generator of the cyclic group $\mathbb{Z}/p$. Define

$$W_d = 1 + \zeta^d w_1 + \cdots + \zeta^{d(p-1)} w_1 \cdots w_{p-1}$$

for $d = 0, \ldots, p-1$. The stratification we are going to use is

$$S = \bigsqcup_{d=0}^{p-1} S_d,$$

where the schemes S_d are defined as

$$S_d := \operatorname{Spec}\Big(k\Big[w_1^{\pm 1}, \ldots, w_{p-1}^{\pm 1}, \frac{1}{W_d}\Big]\Big/(W_0, \ldots, W_{d-1})\Big)^{\tau}.$$

We will show that every such piece is a linear scheme.

Step 1. We first explain the argument for $S_0 = \operatorname{Spec} k[w_1^{\pm 1}, \ldots, w_{p-1}^{\pm 1}, \frac{1}{W_0}]^{\tau}$. Define

$$s_i := \frac{\prod_{j \leq i} w_j}{W_0},$$

for $i \in \{0, \ldots, p-1\}$, $w_0 := 1$. Observe that $s_0 + \cdots + s_{p-1} = 1$ and that

$$k\Big[w_1^{\pm 1}, \ldots, w_{p-1}^{\pm 1}, \frac{1}{W_0}\Big] \cong k[s_0^{\pm 1}, \ldots, s_{p-1}^{\pm 1}]/(s_0 + \cdots + s_{p-1} - 1).$$

Further, τ acts via $\tau \colon s_0 \mapsto s_1 \mapsto \cdots \mapsto s_{p-1} \mapsto s_0$. To show that

$$\operatorname{Spec}(k[s_0^{\pm 1}, \ldots, s_{p-1}^{\pm 1}]/(s_0 + \cdots + s_{p-1} - 1))^{\tau}$$

is a linear scheme, we linearize the action by introducing the variables

$$v_0 = 1, \quad v_i = s_0 + \zeta^i s_1 + \cdots + \zeta^{i(p-1)} s_{p-1}.$$

Then $\tau v_i = \zeta^{-i} v_i$ and

$$s_i = \frac{v_0 + \zeta^{-i} v_1 + \cdots + \zeta^{-i(p-1)} v_{p-1}}{p}.$$

In this basis, S_0 becomes

$$\operatorname{Spec}\Big(k\Big[v_0,\dots,v_{p-1},\frac{1}{\prod_{i=0}^{p-1}\tau^i(l)}\Big]\Big/(v_0-1)\Big)^\tau \cong \operatorname{Spec} k\Big[v_1,\dots,v_{p-1},\frac{1}{\prod_{i=0}^{p-1}\tau^i(l)}\Big]^\tau,$$

where $l = 1+v_1+\cdots+v_{p-1}$ is the equation of a hyperplane. Now, we can realize S_0 as the complement of a linear scheme inside an affine space. Indeed,

$$\begin{aligned}&\operatorname{Spec} k[v_1,\dots,v_{p-1}]\\ &\quad= \operatorname{Spec} k\Big[v_1,\dots,v_{p-1},\frac{1}{\prod_{i=0}^{p-1}\tau^i(l)}\Big] \sqcup \operatorname{Spec}\Big(k[v_1,\dots,v_{p-1}]\Big/\prod_{i=0}^{p-1}\tau^i(l)\Big)\end{aligned}$$

and τ acts on both terms on the bottom line.

Observe that $\operatorname{Spec}(k[v_1,\dots,v_{p-1}]/\prod_{i=0}^{p-1}\tau^i(l))$ is the union of the hyperplanes $l,\tau(l),\dots,\tau^{p-1}(l)$, which are cyclically permuted by τ. Both $\operatorname{Spec} k[v_1,\dots,v_{p-1}]^\tau$ and $\operatorname{Spec}(k[v_1,\dots,v_{p-1}]/\prod_{i=0}^{p-1}\tau^i(l))^\tau$ are linear schemes, so S_0 is indeed a linear scheme.

Step 2. Fix $0 \le d \le p-1$. The proof that S_d is a linear scheme is similar to the one in Step 1. Define

$$s_i = \frac{\prod_{j\le i} w_j}{W_d},$$

for $i = 0,\dots,p-1$, $w_0 := 1$. Observe that $s_0+\cdots+\zeta^{d(p-1)}s_{p-1} = 1$ and

$$k\Big[w_1^{\pm1},\dots,w_{p-1}^{\pm1},\frac{1}{W_d}\Big] = k[s_0^{\pm1},\dots,s_{p-1}^{\pm1}]/(s_0+\cdots+\zeta^{d(p-1)}s_{p-1}-1).$$

Furthermore, we have that

$$W_e = \frac{s_0+\cdots+\zeta^{e(p-1)}s_{p-1}}{s_0}$$

for $e \le d$, so computations similar to those for S_0 show that

$$S_d \cong \operatorname{Spec}(k[s_0^{\pm1},\dots,s_{p-1}^{\pm1}]/I)^\tau,$$

where I is the ideal generated by $s_0+\zeta^e s_1+\cdots+\zeta^{e(p-1)}s_{p-1}$ for all $0\le e\le d-1$, and by $s_0+\zeta^d s_1+\cdots+\zeta^{d(p-1)}s_{p-1}-1$. Changing the basis to v_j defined as in Step 1, we find out that

$$S_d \cong \operatorname{Spec}\Big(k\Big[v_{d+1},\dots,v_{p-1},\frac{1}{\prod_{i=0}^{p-1}\tau^i(l)}\Big]^\tau\Big).$$

The end of the argument in Step 1 shows that S_d is a linear scheme.

Step 3. The proof that T is a linear scheme is already contained in the above argument. Indeed, introduce the basis

$$v_j = s_0 + \zeta^j s_1 + \cdots + \zeta^{j(p-1)} s_{p-1},$$

for $j = 0, \ldots, p-1$. Then we need to show that

$$\operatorname{Spec}\Big(k\Big[v_0, \ldots, v_{p-1}, \frac{1}{\prod_{i=0}^{p-1} \tau^i(l)}\Big]^{\tau}\Big)$$

is a linear scheme, where τ acts on the v_i by $\tau(v_i) = \zeta^{-i} v_i$ and $l = v_0 + \cdots + v_{p-1}$ is a hyperplane. The same argument as in Step 1 shows this is a linear scheme. ∎

4. Proof of Theorem 3.3

4.1 – In the beginning, we will work in a little more general framework which also covers some groups of order p^4. Thus, assume for the moment that G has order $\leq p^4$ and has an irreducible representation of dimension p. We may assume that V is faithful, and let $\rho\colon G \to \mathrm{GL}(V)$. As ρ is irreducible, it is induced from a one-dimensional representation of a subgroup $N \subset G$, that is, $\rho = \mathrm{Ind}_N^G \psi$ with $\psi\colon N \to \mathrm{GL}(W)$ and with W one-dimensional [14]. As V has dimension p, the subgroup N has index p in G, and so $N \trianglelefteq G$.

Choose representatives $\{1, t, \ldots, t^{p-1}\}$ for the cosets of G/N. The explicit form of ρ is

$$\rho(g) = (\psi(t^{-i} g t^j))_{0 \leq i,j \leq p-1},$$

where $\psi(g) = 0$ if $g \notin N$.

If $Z(G) \not\subset N$, we can choose $t \in Z(G)$. Then $\rho(g) = (\psi(g t^{i-j}))$, so $\rho(g) = \psi(g) I$, for every $g \in N$. As ρ is faithful, this implies that $N \subset Z(G)$, and further that G is abelian, contradicting that G has an irreducible representation of dimension p.

We thus have that $Z(G) \subset N$. In order for ρ to be faithful, $\psi|_{Z(G)}$ needs to be faithful, too, so $Z(G)$ is cyclic.

Using the explicit description of ρ, we have that $\rho(G) \subset T \cdot W$, where T is the group of diagonal matrices and W is the group of permutation matrices. By identifying G with its image $\rho(G)$, G can be written as a semi-direct product $N \rtimes M$, with $M \cong \mathbb{Z}/p$, and N an abelian p-group with $|N| \leq p^3$.

4.2 – The plan is to construct a decomposition of $V /\!\!/ G$ into smaller linear schemes. We isolate one open subset of $V /\!\!/ G$ and decompose its complement in linear schemes. After that, we show that the open subset is itself a linear scheme.

Choose a basis $x_1, \dots, x_p$ of V on which N acts diagonally and which is cyclically permuted by τ, the generator of M. Observe that

$$V /\!\!/ G = \operatorname{Spec} k[x_1, \dots, x_p]^G = \operatorname{Spec}(k[x_1, \dots, x_p]^N)^\tau.$$

As we have already discussed in the proof of Lemma 3.4, there is a stratification

$$\operatorname{Spec} k[x_1, \dots, x_p] = \bigsqcup_{J \subset [p]} \operatorname{Spec} k[x_j^{\pm 1} \mid j \in J],$$

where the disjoint union is taken over all sets $J \subset [p]$. This stratification is the partition of the affine space $\mathbb{A}_k^p$ in the 2^p schemes P_J with $x_j \neq 0$ for $j \in J$ and $x_j = 0$ for $j \notin J$. As N acts linearly on the functions x_i for $1 \leq i \leq p$, we have that

$$\operatorname{Spec} k[x_1, \dots, x_p]^N = \operatorname{Spec} k[x_1^{\pm 1}, \dots, x_p^{\pm 1}]^N \sqcup \bigsqcup_{J < [p]} \operatorname{Spec} k[x_j^{\pm 1} \mid j \in J]^N.$$

By Lemma 3.4, each $\operatorname{Spec} k[x_j^{\pm 1} \mid j \in J]^N$ for $J \subset [p]$ is a linear scheme.

Let $t\colon [p] \to [p]$ be the function $t(x) = x + 1$ for $x \leq p - 1$ and $t(p) = 1$. For $J \subset [p]$, let $t(J) := \{t(x) \mid x \in J\} \subset [p]$. Observe that τ permutes the schemes $S_J = \operatorname{Spec} k[x_j^{\pm 1} \mid j \in J]^N$ by sending S_J to $S_{t(J)}$. Consequently, there is a stratification

$$\operatorname{Spec} k[x_1, \dots, x_p]^G = \operatorname{Spec} k[x_1^{\pm 1}, \dots, x_p^{\pm 1}]^G \sqcup \bigsqcup_A \operatorname{Spec} S_J,$$

where A is a set of representatives of the equivalence classes of the action of t on the set of proper subsets of $[p]$. This means that, in order to show that $V /\!\!/ G$ is a linear scheme, we have to prove that $\operatorname{Spec} k[x_1^{\pm 1}, \dots, x_p^{\pm 1}]^G$ is a linear scheme. We do this in the next subsection.

4.3 – The study of the aforementioned open piece is inspired by [4]. We begin by analyzing the $Z(G)$-invariants. If we can conveniently reduce the dimension of the scheme $\operatorname{Spec} k[x_1^{\pm 1}, \dots, x_p^{\pm 1}]$ on which G acts from p to $p - 1$, for example by finding a G-invariant element among the $Z(G)$-invariants, then the resulting ring will give a natural $\mathbb{Z}[\tau]$-representation on $\mathbb{Z}^{p-1}$. This representation was shown in [4, page 687] to be generated by one element. By a theorem of Reiner [18], this representation is the canonical representation of $\mathbb{Z}[\tau]$ on $\mathbb{Z}[\zeta]$. This reduction can be done for the group $G \cong (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$.

If all elements of N act by the same character of the $Z(G)$-invariants, then we can make a change of variables to reduce to the case of $\operatorname{Spec} k[w_1^{\pm 1}, \dots, w_p^{\pm 1}]^\tau$, where τ cyclically permutes the basis elements w_i. For example, this is the case for $G \cong (\mathbb{Z}/p^2) \rtimes \mathbb{Z}/p$. In both situations, the final ingredient will be Lemma 3.5.

4.3.1. Assume that G has order p^3. Then $Z(G)$ acts on V via multiples of the identity, so

$$k[x_1^{\pm 1}, \dots, x_p^{\pm 1}]^{Z(G)} = k\left[x_1^p, x_1^{-p}, \frac{x_2}{x_1}, \dots, \frac{x_1}{x_p}\right] = k[y_2^{\pm 1}, \dots, y_p^{\pm 1}][y_1^{\pm 1}],$$

for $y_1 = x_1^p, y_i = \frac{x_{i+1}}{x_i}, i = 2, \dots, p$. Assume that we can replace y_1 with a G-invariant monomial z_1 such that

$$k[y_2^{\pm 1}, \dots, y_p^{\pm 1}][y_1^{\pm 1}] = k[y_2^{\pm 1}, \dots, y_p^{\pm 1}][z_1^{\pm 1}].$$

This can be done when $G \cong (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$. Recall the notations from Section 3.2.1. Indeed, in this case $Z(G) = \langle \sigma \rangle$. For the representation (ρ, V) described in Section 3.2.1, π acts on any $y_i, i = 2, \dots, p$, by multiplication with ζ and it fixes y_1, while

$$\tau \colon y_2 \longmapsto \cdots \longmapsto y_p \longmapsto \frac{1}{y_2 \cdots y_p}$$

and $\tau(y_1) = y_1 y_2^p$. If we replace y_1 by $z_1 = y_1 y_2^{p-1} \cdots y_{p-1}^2 y_p$, then z_1 is indeed G-invariant and

$$k[y_2^{\pm 1}, \dots, y_p^{\pm 1}][y_1^{\pm 1}] = k[y_2^{\pm 1}, \dots, y_p^{\pm 1}][z_1^{\pm 1}].$$

Even more, the same argument works for a p-group of cardinality p^4 with $Z(G) \cong \mathbb{Z}/p^2$ and N different from $\mathbb{Z}/p^3$. Indeed, in this case, $N \cong Z(G) \oplus \langle \pi \rangle$, and the $Z(G)$-invariants of $k[x_1^{\pm 1}, \dots, x_p^{\pm 1}]$ are

$$k\left[x_1^{p^2}, x_1^{-p^2}, \frac{x_2}{x_1}, \dots, \frac{x_1}{x_p}\right] = k[y_2^{\pm 1}, \dots, y_p^{\pm 1}][y_1^{\pm 1}],$$

for $y_1 = x_1^{p^2}, y_i = \frac{x_{i+1}}{x_i}$ for $i = 2, \dots, p$. Observe that π acts trivially on y_1 and by a p-root of unity on the others y_i, and that

$$\tau \colon y_2 \longmapsto \cdots \longmapsto y_p \longmapsto \frac{1}{y_2 \cdots y_p}$$

and $\tau(y_1) = y_1 y_2^{p^2}$. In particular, this implies that $y_1 y_2^p \cdots y_p^{p(p-1)}$ is G-invariant, so the above argument works.

4.3.2. Assume $G \cong \mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$. Recall the notations from Section 3.2.2. The center is generated by σ^p. The element σ acts on any $y_i, i = 1, \dots, p$, by multiplication with ζ, while

$$\tau \colon y_2 \longmapsto \cdots \longmapsto y_p \longmapsto \frac{1}{y_2 \dots y_p}$$

and $\tau(y_1) = y_1 y_2^p$. Replace y_1 with $y_1 y_2^{p-1} \dots y_{p-1}^2 y_p$. Then $\sigma(y_1) = \zeta y_1$, and $\tau(y_1) = y_1$. Taking σ-invariants,

$$k[y_1^{\pm 1}, \dots, y_p^{\pm 1}]^\sigma = k\Big[y_1^p, \frac{y_2}{y_1}, \dots, \frac{y_p}{y_1}, \text{their inverses}\Big],$$

which can be further written as $k[w_1^{\pm 1}, \dots, w_p^{\pm 1}]$ for $w_1 = y_1^p$, $w_i = \frac{y_i}{y_1}$, for $i = 2, \dots, p$. Observe that

$$\tau\colon w_2 \longmapsto w_3 \longmapsto \cdots \longmapsto w_p \longmapsto \frac{1}{w_1 \cdots w_p},$$

and thus, by replacing w_1 with $\frac{1}{w_1 \cdots w_p}$, we need to show that $\operatorname{Spec} k[w_1^{\pm 1}, \dots, w_p^{\pm 1}]^\tau$, where τ acts by $\tau\colon w_1 \mapsto \cdots \mapsto w_p \mapsto w_1$, is a linear scheme. This follows from Lemma 3.5. The same argument shows that any group of the form $\mathbb{Z}/p^s \rtimes \mathbb{Z}/p$ is mixed Tate. In particular, this means that any group G of order p^4 and center of order p^2 is mixed Tate.

4.4 – Assume from now on that we are in the situation of Section 4.3.1, in which the dimension of the scheme we want to prove is linear was reduced from p to $p-1$. We will explain how to obtain a $\mathbb{Z}[\tau]$-representation on $\mathbb{Z}^{p-1}$. The argument works for any p-group and V a p-dimensional representation, just in this case we will get a representation of $\mathbb{Z}[\tau]$ on $\mathbb{Z}^p$. In order to compute the τ-invariants of $k[y_2^{\pm 1}, \dots, y_p^{\pm 1}]^N$, write $N = N_1 \oplus N_2$ with N_1 cyclic. As in the proof of the Lemma 3.4, we have that

$$k[y_2^{\pm 1}, \dots, y_p^{\pm 1}]^{N_1} = k[y_2^{a_2}, y_2^{a_3} y_3, \dots, y_2^{a_p} y_p, \text{their inverses}].$$

If we repeat the computation for N_2 instead of N_1, we find that

$$k[y_2^{\pm 1}, \dots, y_p^{\pm 1}]^{N} = k[y_2^{b_2}, y_2^{b_3} y_3, \dots, y_2^{b_p} y_p, \text{their inverses}].$$

Let $z_i := y_2^{b_i} y_i$ for $2 \le i \le p$. Observe that τ acts on z_i in the following way:

$$\begin{aligned}\tau(z_2) &= z_2^{a_{2,2}} z_3^{a_{3,2}},\\ \tau(z_3) &= y_2^{b_{2,3}} z_3^{b_{3,3}} z_4,\end{aligned}$$

for some explicit integer exponents. For any N-invariant z, the element $\tau(z)$ is also N-invariant because

$$n\tau z = \tau n_0 z = \tau z$$

for some $n_0 \in N$. In particular, $\tau(z_2)$ is N-invariant, so $y_2^{b_{3,2}}$ is an integer power of z_2. This implies that $\tau(z_3)$ is a monomial in z_2, z_3, and z_4, and a similar computation shows that this is true for any $2 \leq k \leq p$, namely that there are integer exponents such that

$$\tau(z_k) = z_2^{a_{2,k}} \cdots z_{k+1}^{a_{k+1,k}}.$$

Now, we can construct a $\mathbb{Z}[\mathbb{Z}/p] \cong \mathbb{Z}[\tau]$-representation

$$W := \mathbb{Z}^{p-1} = \mathbb{Z}\log(z_2) \oplus \cdots \oplus \mathbb{Z}\log(z_2)$$

by defining

$$\tau(\log(z_k)) = a_{2,k}\log(z_2) + \cdots + a_{k+1,k}\log(z_{k+1}).$$

By a theorem of Reiner [18], the representation W is isomorphic to an ideal of $\mathbb{Z}[\zeta]$, where ζ is a primitive p-root of unity. Chu–Kang have shown in [4, page 687] that all such representations coming from groups of order $\leq p^3$ are generated by one element, so $W \cong \mathbb{Z}[\zeta]$. Then we can choose monomials w_i in the z_i on which τ acts via

$$\tau\colon w_1 \longmapsto w_2 \longmapsto \cdots \longmapsto w_{p-1} \longmapsto \frac{1}{w_1 \cdots w_{p-1}}$$

and such that

$$k[z_2^{\pm 1}, \ldots, z_p^{\pm 1}] = k[w_1^{\pm 1}, \ldots, w_{p-1}^{\pm 1}].$$

We know that $\operatorname{Spec} k[w_1^{\pm 1}, \ldots, w_{p-1}^{\pm 1}]^{\tau}$ is a linear scheme by Lemma 3.5, so $V /\!\!/ G$ is indeed a linear scheme in our case.

5. More on mixed Tate motives of a classifying space

In this section, we assume that the base field k has characteristic zero.

5.1 – Define the triangulated category of geometrical motives

$$\mathrm{DM}_{\mathrm{gm}}(k; R) \subset \mathrm{DM}(k; R)$$

as the smallest thick subcategory which contains all the motives $M(X)(a)$ for X a separated scheme of finite type over k and a an integer [22], [20, Section 5]. In general, the motive of a quotient stack is not a geometric motive. For example, for a finite non-trivial group G, the Chow groups (with $\mathbb{Z}$-coefficients) $CH^i(BG)$ are non-trivial for infinitely many values of i [23, Theorem 3.1], and thus the motive $M(BG) \in \mathrm{DM}(k, \mathbb{Z})$ is not geometric. For an explicit computation of the motive of a quotient stack, let $k(1)$ be the one-dimensional representation on which $\mathbb{G}_m$ acts with weight one. Observe that $(k(1)^{\oplus(n+1)} - 0)/\mathbb{G}_m \cong \mathbb{P}^n$ "approximate" the motives associated to $\mathbb{G}_m$. We thus have that

$$M(B\mathbb{G}_m) = \bigoplus_{j\geq 0} R(j)[2j], \qquad M^c(B\mathbb{G}_m) = \prod_{j\leq -1} R(j)[2j].$$

None of these motives are geometric.

Even if the motives associated to a quotient stack are not geometric motives, they exhibit some properties which resemble geometric motives. Indeed, recall that for X a proper scheme, $M^c(X) \cong M(X)$, and for X a smooth scheme of pure dimension n over k, $M^c(X) \cong M(X)^*(n)[2n]$ [20, Section 5].

Let $X = Y/G$ be a smooth quotient stack for which we can define motives $M(X)$ and $M^c(X)$, see Section 2.2. There is an isomorphism

$$M(X)^* \cong M^c(X)(-\dim(X))[-2\dim(X)]. \tag{5.1}$$

The isomorphism in (5.1) follows from the fact that the dual of a direct sum in $\mathrm{DM}(k, R)$ is a product, so the dual of a homotopy colimit is a homotopy limit.

Furthermore, the dual of a mixed Tate motive in $\mathrm{DM}(k; R)$ is not necessarily mixed Tate. For example, if k is algebraically closed, $M := \bigoplus_{i\in\mathbb{N}} \mathbb{Z}$ is an element of $\mathrm{DMT}(k; \mathbb{Z})$, but its dual in $\mathrm{DM}(k, \mathbb{Z})$ is $M^* = \prod_{i\in\mathbb{N}} \mathbb{Z}$, which is not an element of $\mathrm{DMT}(k; \mathbb{Z})$ [21, Corollary 4.2].

However, $\mathrm{DMT}_{\mathrm{gm}}(k; R) := \mathrm{DMT}(k; R) \cap \mathrm{DM}_{\mathrm{gm}}(k; R)$ is closed under taking duals [15, Section 5.1]. The main result of this section is the following.

THEOREM 5.1. *Let G be a finite group, let k be a field of characteristic zero, and let R be an arbitrary ring. Then $M^c(BG) \in \mathrm{DMT}(k; R)$ is mixed Tate if and only if $M(BG) \in \mathrm{DMT}(k; R)$ is mixed Tate.*

In light of the above counterexample of a mixed Tate motive whose dual is not mixed Tate, we see that mixed Tate motives of finite groups exhibit finiteness properties. A related result [21, Theorem 3.1] says that any scheme X of finite type over a field k with $M^c(X)$ mixed Tate has finitely generated Chow groups $CH^{\cdot}(X; R)$ as R-modules. This implies that $CH^{\cdot}(BG; R)$ are finitely generated over R, when G is a finite group with BG mixed Tate.

5.2 – We reduce the proof of Theorem 5.1 to the following.

THEOREM 5.2. *Let X be a smooth quotient stack and let E be a $\mathbb{G}_m$-bundle over X. Then $M(X)$ is mixed Tate if and only if $M(E)$ is mixed Tate.*

Totaro has shown in [20, Corollary 8.13] that for a finite group G, $M^c(BG)$ is mixed Tate if and only if $M^c(\mathrm{GL}(n)/G)$ is mixed Tate for a faithful representation

$G \to \mathrm{GL}(n)$. One knows that the category of geometric Tate motives $\mathrm{DMT}_{\mathrm{gm}}(k; R)$ is closed under taking duals, as mentioned above. Recall that for any geometric motive $X \in \mathrm{DM}_{\mathrm{gm}}(k; R)$, the map $X \xrightarrow{\sim} X^{**}$ is an isomorphism [20, Lemma 5.5]. As $\mathrm{GL}(n)/G$ is a smooth scheme, and for any smooth scheme S one has

$$M(S)^* \cong M^c(S)(-\dim(S))[-2\dim(S)],$$

we see that it is enough to prove that $M(BG)$ is mixed Tate if and only if $M(\mathrm{GL}(n)/G)$ is mixed Tate for a faithful representation $G \to \mathrm{GL}(n)$. The strategy is to show the more general result, that for X a quotient stack and E a principal $\mathrm{GL}(n)$-bundle over X, $M(X)$ is mixed Tate if and only if $M(E)$ is mixed Tate. The next lemma inspired by [20, Lemma 7.13], shows that Theorem 5.1 follows from Theorem 5.2.

Lemma 5.3. *Assume that for any smooth quotient stack X and any principal $\mathbb{G}_m$-bundle F over X, $M(X) \in \mathrm{DMT}(k; R)$ if and only if $M(F) \in \mathrm{DMT}(k; R)$. Then, for any smooth quotient stack X and any principal $\mathrm{GL}(n)$-bundle E over X, $M(X) \in \mathrm{DMT}(k; R)$ if and only if $M(E) \in \mathrm{DMT}(k; R)$.*

Proof. Denote by B the subgroup of upper triangular matrices in $\mathrm{GL}(n)$. Then E/B is an iterated projective bundle over X. Recall that $\mathrm{GL}(n)$-bundles are Zariski locally trivial. We obtain the following Leray–Hirsch decomposition for motives,

$$M(E/B) \cong \bigoplus M(X)(a_j)[2a_j],$$

where a_j are the dimensions of the $n!$ Bruhat cells of the flag manifold $\mathrm{GL}(n)/B$, see also the proof of [20, Lemma 7.13].

Now, since $\mathrm{DMT}(k; R)$ is closed under arbitrary direct sums, $M(X) \in \mathrm{DMT}(k; R)$ implies $M(E/B) \in \mathrm{DMT}(k; R)$. Conversely, $\mathrm{DMT}(k; R)$ is thick (see the discussion after [20, Lemma 5.4]), so $M(E/B) \in \mathrm{DMT}(k; R)$ implies $M(X) \in \mathrm{DMT}(k; R)$.

Next, let U be the subgroup of strictly upper triangular matrices in $\mathrm{GL}(n)$. Since $B/U \cong \mathbb{G}_m^n$, E/U is a principal $\mathbb{G}_m^n$-bundle over E/B. Using the assumption on $\mathbb{G}_m$-bundles, we deduce that $M(E/U) \in \mathrm{DMT}(k; R)$ if and only if $M(X) \in \mathrm{DMT}(k; R)$. Finally, U is an extension of copies of the additive group $\mathbb{G}_a$, so $M(E) \cong M(E/U)$, which means that $M(E) \in \mathrm{DMT}(k; R)$ if and only if $M(X) \in \mathrm{DMT}(k; R)$. ∎

5.3 – We will also need the following vanishing result.

Lemma 5.4. *If Y is a smooth quasi-projective scheme, then*

$$\mathrm{Hom}(R(i)[j], M(Y)) = 0,$$

for $j \leq i - 2$.

PROOF. Choose a smooth compactification Z of Y such that the complement $W := Z \setminus Y$ is a divisor with simple normal crossings, which can be done since k has characteristic zero [13, Theorem 3.35]. Then, the Gysin distinguished triangle [22, page 10] gives, for $c = \operatorname{codim} W$,

$$M(W) \longrightarrow M(Z) \longrightarrow M(Y)(c)[2c] \longrightarrow M(W)[1].$$

Taking the dual of this triangle we obtain, for $n = \dim(Y)$,

$$M^c(W)^*(n)[2n-1] \longrightarrow M(Y) \longrightarrow M(Z) \longrightarrow M^c(W)^*(n)[2n].$$

Both $\operatorname{Hom}(R(i)[j], M(Z)[-1])$ and $\operatorname{Hom}(R(i)[j], M(Z))$ are zero because Z is projective. Indeed, in our case $M(Z) \cong M^c(Z)$ and $j \leq i-2$, and it is known that $\operatorname{Hom}(R(i)[j], M^c(Z)) = 0$ for any scheme Z and any integers i and j with $j \leq i-1$ [20, page 16]. Thus, the Hom-long exact sequence obtained from this distinguished triangle gives that

$$\operatorname{Hom}(R(i)[j], M^c(W)^*(n)[2n-1]) \cong \operatorname{Hom}(R(i)[j], M(Y)).$$

Observe that W is proper, so $M(W) \cong M^c(W)$. Further,

$$\operatorname{Hom}(R(i)[j], M^c(W)^*(n)[2n-1]) \cong \operatorname{Hom}(M^c(W), R(n-i)[2n-1-j]).$$

Thus, it is enough to prove

$$\operatorname{Hom}(M^c(W), R(a)[b]) = 0,$$

for $b - a \geq n+1$. Further, $\dim(W) < n$ and W is a divisor with simple normal crossings, so there are at most n divisor through any point of W. To show this, we will use induction on n, the maximal number of divisors which pass through a given point, and then on the number of connected components of W. If $n = 1$ or if W has only one component, then W is smooth; in this case, $M(W) \cong M^c(W)$ and $M(W)^* \cong M(W)(\dim(W))[-2\dim(W)]$. We need to show that

$$\operatorname{Hom}(R(i + \dim(W) - n)[j + 1 + 2(\dim(W) - n)], M^c(W)) = 0,$$

for $j \leq i-2$, where $i = n - a$ and $j = 2n - 1 - b$. This follows from the vanishing property of motivic homology

$$\operatorname{Hom}(R(i)[j], M^c(Z)) = 0$$

for any scheme Z and any integers i and j with $j \leq i-1$ [20, page 16]. In our case, $b - a \geq n+1$ is equivalent to $j \leq i-2$, and we know that $\dim W < n$, thus $i + \dim W - n \geq j + 1 + 2(\dim W - n) + 1$.

For the general case, let U be a smooth connected component of W and let V be the closure of $W \setminus U$ inside W. Then V will be also be a divisor with simple normal crossings such that there are at most n divisors passing through a given point, but it will have less components than W. Further, $T := U \cap V$ will be a divisor with simple normal crossings, with at most $n-1$ divisors passing through any point. By the induction hypothesis, $\mathrm{Hom}(M(T)[1], R(a)[b]) = 0$ for $b-a \geq n$, and $\mathrm{Hom}(M(V)[1], R(a)[b]) = 0$ for $b-a \geq n+1$. Recall that we want to show $\mathrm{Hom}(M(W)[1], R(a)[b]) = 0$ for $b-a \geq n+1$. For this, use the following two distinguished triangles

$$M^c(U) \longrightarrow M^c(W) \longrightarrow M^c(W-U) \longrightarrow M^c(U)[1],$$
$$M^c(T) \longrightarrow M^c(V) \longrightarrow M^c(W-U) \longrightarrow M^c(T)[1].$$

From the second triangle, we get

$$\mathrm{Hom}(M^c(T)[1], R(a)[b]) \longrightarrow \mathrm{Hom}(M^c(W-U), R(a)[b]) - \cdots$$
$$\cdots \to \mathrm{Hom}(M^c(V), R(a)[b]) \longrightarrow \mathrm{Hom}(M^c(T), R(a)[b]).$$

We deduce that $\mathrm{Hom}(M^c(W-U), R(a)[b]) = 0$ for $b-a \geq n+1$. Similarly, we can use the first triangle to deduce that $\mathrm{Hom}(M^c(W), R(a)[b]) = 0$ for $b-a \geq n+1$. ∎

5.4 – In this subsection, we prove Theorem 5.2. We split its proof in a sequence of steps.

5.4.1. Let T be the total space of a line bundle over X such that $T - X \cong E$, where $X \hookrightarrow T$ is embedded as the zero section. We claim that there is a Gysin distinguished triangle

$$M(T-X) \longrightarrow M(T) \longrightarrow M(X)(1)[2] \longrightarrow M(T-X)[1]. \tag{5.2}$$

Indeed, let $X = Y/G$ and $T = W/G$ with Y smooth and W an $\mathbb{A}^1$-bundle over Y. Consider the (smooth) approximations

$$X_i = ((V_i - S_i) \times Y)/G,$$
$$T_i = ((V_i - S_i) \times W)/G.$$

Then we have the Gysin distinguished triangles [22, Theorem 3.5.4]

$$M(T_i - X_i) \longrightarrow M(T_i) \longrightarrow M(X_i)(1)[2] \longrightarrow M(T_i - X_i)[1].$$

The category $\mathrm{DM}(k; R)$ is a model category with arbitrary direct sums and products [20, Subsection 5], so it has an underlying triangulated derivator [5, Theorem 6.11], [12, Appendix 2, page 1075] Thus, the homotopy colimit of distinguished triangles is a distinguished triangle [12, Corollary 11.4], and we thus obtain the Gysin triangle (5.2). Using $M(X) \cong M(T)$, the distinguished triangle (5.2) becomes

$$M(E) \longrightarrow M(X) \longrightarrow M(X)(1)[2] \longrightarrow M(E)[1]. \tag{5.3}$$

5.4.2. The inclusion

$$\mathrm{DMT}(k;R) \hookrightarrow \mathrm{DM}(k;R)$$

has a right adjoint

$$C\colon \mathrm{DM}(k;R) \longrightarrow \mathrm{DMT}(k;R).$$

We will sometimes write $C(Z)$ instead of $C(M(Z))$ for Z a quotient stack. Let U be the cone of $C(E) \to M(E)$ and let W be the cone of $C(X) \to M(X)$. There is a distinguished triangle

$$U \longrightarrow W \longrightarrow W(1)[2] \longrightarrow U[1].$$

Indeed, this triangle is induced from the triangle (5.3), the diagram

$$\begin{array}{ccccccc}
C(E) & \longrightarrow & C(X) & \longrightarrow & C(E)(1)[2] & \longrightarrow & C(E)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M(E) & \longrightarrow & M(X) & \longrightarrow & M(E)(1)[2] & \longrightarrow & M(E)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
U & \longrightarrow & W & \longrightarrow & W(1)[2] & \longrightarrow & U[1]
\end{array}$$

and the 3×3 lemma.

5.4.3. Observe that $C(W) = 0$. Indeed,

$$M(X) \longrightarrow C(X) \longrightarrow W \longrightarrow M(X)[1]$$

and, for any i and j integers,

$$\mathrm{Hom}(R(i)[j], M(X)) \xrightarrow{\cong} \mathrm{Hom}(R(i)[j], C(X)).$$

This implies that W has trivial motivic homology groups.

Then the Tate motive $C(W)$ has trivial homology groups and so $C(W) = 0$. Indeed, because $\mathrm{Hom}(R(a)[b], C(X)) = 0$ and $R(a)[b]$ generate the category $\mathrm{DMT}(k;R)$, we get that $\mathrm{Hom}(M, C(X)) = 0$ for any mixed Tate motive M, and, in particular, that $\mathrm{Hom}(C(X), C(X)) = 0$, so $C(X) = 0$.

5.4.4. We need to show that $U = 0$ if and only if $W = 0$. If $W = 0$, then it is immediate that $U = 0$. Conversely, suppose $U = 0$. In this case,

$$W \cong W(1)[2]. \tag{5.4}$$

In [6, Proposition 7.10], Dugger and Isaksen have shown that one can compute, via a spectral sequence, the motivic homology of $X \otimes M$ from the motivic homology of M and X, for any motive X and any mixed Tate motive M. A related result [20, Theorem 7.2] says that if

$$C(W) \otimes C(M(Z)) \xrightarrow{\cong} C(W \otimes M(Z)),$$

for any Z a smooth projective scheme, then W is mixed Tate. We will use both these results in our argument below.

The plan is the following: it is enough to show that

$$C(W) \otimes C(M(Z)) \xrightarrow{\cong} C(W \otimes M(Z)),$$

for Z a smooth projective scheme. Taking into account that $C(W) \cong 0$, we will need to show that the motivic homology groups of any product $W \otimes M(Z)$ are trivial.

We show that the motive W has a vanishing property similar to the one of M^c of a geometrical motive, namely that $\mathrm{Hom}(R(i)[j], W) = 0$ for $j \leq i - 2$. Even more, we will be able to show that $\mathrm{Hom}(R(i)[j], W \otimes M(Z)) = 0$ for $j \leq i - 2$ and for Z a smooth projective scheme. This will imply that all the motivic homology groups of $W \otimes M(Z)$ are trivial, because $W \cong W(1)[2]$. Consequently, we only need to show

$$\mathrm{Hom}(R(i)[j], W \otimes M(Z)) = 0 \tag{5.5}$$

for $j \leq i - 2$, where Z is a smooth projective scheme.

5.4.5. First, by Lemma 5.4, we have that $\mathrm{Hom}(R(i)[j], M(Y)) = 0$ for $j \leq i - 2$ for a quasi-projective scheme Y. There is a distinguished triangle:

$$M(X \times Z) \longrightarrow C(M(X)) \otimes M(Z) \longrightarrow W \otimes M(Z) \longrightarrow M(X \times Z)[1]. \tag{5.6}$$

It is enough to show

$$\mathrm{Hom}(R(i)[j], M(X \times Z)) = 0,$$
$$\mathrm{Hom}(R(i)[j], C(M(X)) \otimes M(Z)) = 0$$

for $j \leq i - 2$. To show that $\mathrm{Hom}(R(i)[j], M(X \times Z)) = 0$ for $j \leq i - 2$, write $M(X \times Z)$ as the cone of a morphism

$$\bigoplus_{l \in I} M(S_l) \longrightarrow \bigoplus_{l \in I} M(S_l) \longrightarrow M(X \times Z) \longrightarrow \Big(\bigoplus_{l \in I} M(S_l)\Big)[1],$$

where S_l are quasi-projective schemes for l in a set I. Because $R(i)[j]$ is a compact object inside $\mathrm{DM}(k;R)$, we have that

$$\mathrm{Hom}\Big(R(i)[j], \bigoplus_{l\in I} M(S_l)\Big) = \bigoplus_{l\in I} \mathrm{Hom}(R(i)[j], M(S_l)) = 0$$

for $j \le i-2$. Finally,

$$\mathrm{Hom}\Big(R(i)[j], \bigoplus_{l\in I} M(S_l)\Big) \longrightarrow \mathrm{Hom}(R(i)[j], M(X\times Z)) - \cdots$$
$$\cdots \rightarrow \mathrm{Hom}\Big(R(i)[j], \Big(\bigoplus_{l\in I} M(S_l)\Big)[1]\Big),$$

which immediately implies $\mathrm{Hom}(R(i)[j], M(X\times Z)) = 0$ for $j \le i-2$.

5.4.6. To show $\mathrm{Hom}(R(i)[j], C(M(X)) \otimes M(Z)) = 0$ for $i \le j-2$, use the motivic Künneth spectral sequence [20, Theorem 6.1],

$$E_2^{pq} = \mathrm{Tor}_{-p,-q,i}^{H.(k,R(\cdot))}(H.(C(X), R(\cdot)), H.(Z, R(\cdot))) \Longrightarrow H_{-p-q}(C(X)\otimes Z, R(i)),$$

where $\mathrm{Tor}_{-p,-q,i}$ denotes the $(-q,i)$-bigraded piece of Tor_{-p}. The vanishing properties for the motivic homology of $C(M(X))$ and $M(Z)$ imply the desired result. Indeed, assume $i<0$. On the sheet E_2^{pq}, all non-trivial $H.(k,R(\cdot))$-modules are concentrated in the lower left corner $j \le i-2$, $p \le 0$. Every page E_n^{pq} will be concentrated in the same lower left square, which implies the vanishing of motivic homology groups for $C(M(X)) \otimes M(Z)$ for $j \le i-2$. In particular, $\mathrm{Hom}(R(i)[j], C(M(X)) \otimes M(Z)) = 0$ for $j \le i-2$. Using the triangle (5.6) and the discussion in Section 5.4.5, we see that (5.5) holds.

5.4.7. Finally, let i and j be arbitrary integers, and choose $a \le i-j-2$. By (5.4) and (5.5), we have that

$$\mathrm{Hom}(R(i)[j], W\otimes M(Z)) \cong \mathrm{Hom}(R(i+a)[j+2a], W\otimes M(Z)) \cong 0.$$

Thus, the motivic homology of $W \otimes M(Z)$ is trivial for every smooth projective scheme Y. As discussed in Section 5.4.4, this implies that $W \cong 0$, and thus Theorem 5.2 follows.

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