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On groups of finite Prüfer rank II

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Abstract – Let G be a group of finite (Prüfer) rank and π any finite set of primes. We prove, in particular, that G contains a characteristic subgroup H of finite index such that for every normal subgroup Y of H of finite index the maximal normal π-subgroup of H/Y lies in the hypercentre of H/Y, so in particular all finite π-images of H are nilpotent.

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A group has finite (Prüfer) rank if there is an integer r such that each of its finitely generated subgroups can be generated by at most r elements, the least such r being its rank. Rank is the interesting concept, but in the main a weaker concept suffices here. Say a group G has finite residual rank r if every finite image of G has finite rank at most r and if r is the least integer with this property. Clearly, groups of finite rank have finite residual rank and there exist divisible abelian groups of every rank, but they all have residual rank 0. Subgroups of finite index in a group of finite residual also have finite residual rank. We prove the following.

Theorem. *Let G be a group of finite residual rank and π any finite set of primes. Then G contains a characteristic subgroup H of finite index such that the following hold:*

(a) *Every π-section of every finite image of H is nilpotent.*

(b) *If $H \geq X \geq Y$ with the index $(H : Y)$ finite, Y normal in X and X/Y a π-group, then X/Y is nilpotent.*

(c) *If X and Y are as in* (b)*, but also with X normal in H, then there exists a positive integer h with $[X, {}_h H] \leq Y$.*

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An important step in the proof of this theorem is the following proposition of independent interest. Notice that no set π is involved; again we do not need the full strength of finite rank. A group G has finite 2-rank r if every elementary abelian 2-section of G has rank at most r and if r is the least integer with this property. Clearly groups of finite rank have finite 2-rank but not conversely. The group G has finite residual 2-rank r if every finite image of G has 2-rank at most r and if r is the least integer with this property.

Proposition. *Let G be a group of finite residual 2-rank. Then G has a normal subgroup N of finite index such that every finite image of N is soluble; in fact, every finite image of every subgroup of N of finite index is soluble.*

The context of the theorem above is the following. Developing ideas of Azarov and Romanovskii [1], we proved in [5, Theorem 1] that for any finite set π of primes, any group G of finite rank has a characteristic subgroup H of finite index such that every finite π-image of H is nilpotent (the main results of [1] are the special cases of this where G is either soluble or finitely generated). Theorem 2 of [5] dealt with this and related properties for groups with finite Hirsch number. Now, with $\langle P',\ L\rangle(\mathrm{AF})$ denoting the smallest class of groups that contains all abelian and all finite groups and is closed under the ascending series operator and the local operator, then $\langle P',\ L\rangle(\mathrm{AF})$ groups of finite rank have finite Hirsch number. An essentially immediate corollary of [5, Theorem 2] is that $\langle P',\ L\rangle(\mathrm{AF})$ groups of finite rank satisfy the conclusions of our theorem above. This current paper answers the obvious question by showing that the $\langle P',\ L\rangle(\mathrm{AF})$ hypothesis is unnecessary. Incidentally, going the other way, using that groups with finite Hirsch number are (locally finite)-by-(of finite rank), it is easy to deduce [5, Theorem 2] from the theorem above.

Lemma 1. *There is a positive integer e such that for any positive integer r, if G is a finite group of 2-rank at most r, then G has at most d non-cyclic composition factors, where $d = t + et + t!$ and $t = [r/2]$. (So certainly $d \leq r + er + r!$.)*

Proof. We may assume $\langle 1\rangle$ is the only soluble normal subgroup of G and $G \neq \langle 1\rangle$. There exists a normal subgroup $E = S_1 \times S_2 \times \cdots \times S_t$ of G, where $C_G(E) = \langle 1\rangle$. Each S_i is simple and hence contains a Klein 4-subgroup. Thus $t \leq r/2$ and if $H = \bigcap_i N_G(S_i)$ and $A_i = \mathrm{Aut}(S_i)$, then G/H embeds into $\mathrm{Sym}(t)$ and H embeds into $A_1 \times A_2 \times \cdots \times A_t$. Set $B_i = \mathrm{Out}(S_i) = A_i/\mathrm{Inn}(S_i)$.

If S_i is alternating, then B_i is abelian. If S_i is of Lie type, then B_i is the product of two abelian groups and as such is metabelian by a theorem of Ito (e.g. [3, Satz VI.4.4]). (For if S_i is of Chevalley type apply [2, Theorem 12.5.1] and note that the images in B_i of the diagonal automorphisms and of the field automorphisms are abelian and the latter centralise the images of the graph automorphisms. If S_i is of twisted type, then [4, Theorem 30] applies directly.)

There are only finitely many possibilities for the S_i not so far covered. There is no need to check the B_i for each of these in detail. Simply set e equal to the product of the orders of the outer automorphism groups of the sporadic simple groups. Then H/E has at most et non-cyclic composition factors and so G has at most $t + et + t!$ non-cyclic composition factors. ■

Proof of the proposition. If N is a normal subgroup of G of finite index, then G/N has at most d non-cyclic composition factors by Lemma 1. Choose N so that G/N has the greatest number of non-cyclic composition factors. Consider subgroups $X \geq Y$ of N with Y normal in X and of finite index in N. Set $Y_G = \bigcap_{x \in G} Y^x \leq N$. Since Y is of finite index in G, so Y_G is normal and of finite index in G. By the choice of N, the group N/Y_G is soluble. Trivially, therefore X/Y is soluble. ■

Lemma 2. *Let G be a group of finite residual rank r, H a characteristic subgroup of G of finite index and N a normal subgroup of H such that H/N is a finite π-group, π some set of primes. If $M = \bigcap_{\phi} N\phi$, as ϕ ranges over* $\operatorname{Aut} G$, *then H/M is also a finite π-group.*

Of course if H/N is soluble, then H/M is also soluble (and this has nothing to do with rank).

Proof. Now $(G : N)$ is finite, so $(G : N_G)$ is finite for $N_G = \bigcap_{g \in G} N^g$. Also H/N_G embeds into the direct product of finitely many copies of H/N and hence is a finite π-group. Further, H/M embeds into a Cartesian product T of copies of H/N_G and T is locally finite of exponent $\exp(H/N)$. In particular, H/M is a locally finite π-group.

Set $K = H/M$, $L = G/M$ and suppose $K \leq T = \prod_I T_i$, where each T_i is isomorphic to H/N_G. Set $T_J = \prod_J T_j$, where J is some cofinite subset of I. Then $(L : K) = (G : H)$ is finite and $K/(K \cap T_J)$ is finite, so $L/(K \cap T_J)_L$ is finite and therefore has finite rank at most r. If $p \in \pi$ and if p^n is the order of a Sylow subgroup of H/N_G, then any p-subgroup of T has a series of length n with elementary abelian factors. Thus a Sylow p-subgroup of $K/(K \cap T_J)_L$ has order at most p^{nr}. This is for every J, so we may pick J to maximise this order and then $(K \cap T_J)_L$ must be a p'-group. This is for any p in π and consequently K is a finite extension of a π'-group. But $K = H/M$ is a π-group. Therefore H/M is finite and the lemma is proved. ■

Lemma 3. *Let N be a normal subgroup of the finite group Q and set $L/N =$* $\operatorname{Fitt}(Q/N)$ *and $M =$* $\operatorname{Fitt}(Q)$. *If $(Q : L) \geq (Q : M)$, then N is nilpotent.*

Proof. Clearly $MN/N \leq L/N$, so $(Q : L) \leq (Q : MN) \leq (Q : M)$. Hence $(Q : L) = (Q : MN) = (Q : M)$, so $L = MN = M$ and N is nilpotent. ■

Lemma 4. *Let π be a finite set of primes and r a positive integer. If G is a finite soluble π-group of rank at most r, then $(G : \mathrm{Fitt}(G))$ divides $k = \prod_p |\mathrm{GL}(r, p)|^2$, where p ranges over π.*

This simple result is [5, Lemma 5]. Also note that if X is a finite soluble group, then every π-section of X is nilpotent if and only if G has a nilpotent Hall π-subgroup.

Proof of the theorem. By the proposition and Lemma 2 we may assume that every finite image of G is soluble. In particular, by Lemma 4 there is an integer k, depending only on r and π, such that if X is any π-section of a finite image of G, then $(X : \mathrm{Fitt}(X)) \leq k$.

Choose a normal subgroup N of G of finite index such that if Q is a Hall π-subgroup of G/N, then $(Q : \mathrm{Fitt}(Q)) \leq k$ is as large as possible. Let $N_0 = N_{\mathrm{Aut}(G)}$, the maximal characteristic subgroup of G in N; note that $(G : N_0)$ is finite by Lemma 2. There is a Hall π-subgroup P of G/N_0 that maps modulo N/N_0 onto Q. By the choice of N we have $(P : \mathrm{Fitt}(P)) \leq (Q : \mathrm{Fitt}(Q))$, so $(P : \mathrm{Fitt}(P)) = (Q : \mathrm{Fitt}(Q))$ and consequently we may replace N by N_0 and assume N is characteristic in G.

Suppose $K \leq N$ is a normal subgroup of G of finite index. If P/K is a Hall π-subgroup of G/K, then PN/N is a Hall π-subgroup of G/N. By the choice of N and by Lemma 3, applied with P/K and $(P \cap N)/K$ for Q and N, the group $(P \cap N)/K$ is nilpotent. Suppose $N \geq X \geq Y$ with $(N : Y)$ finite and Y normal in X. Choose K above to be $Y_G = \bigcap_{g \in G} Y^g \leq Y$ and then choose P so that $(P \cap X)/K$ is a Hall π-subgroup of X/K. By the above, $(P \cap N)/K$ is nilpotent. But then the Hall π-subgroup $(P \cap X)Y/Y$ of X/Y is also nilpotent. Thus every π-section of X/Y is nilpotent. It follows that N and also any characteristic subgroup H of G of finite index in N satisfies (a) and (b) of the theorem.

Let H denote the intersection of the kernels of all homomorphisms of N into the direct product T of $\mathrm{GL}(r, p)$ for $p \in \pi$. Clearly H is characteristic in N and hence in G. Now N/H embeds into a Cartesian product of copies of the finite group T, so N/H is locally finite and has finite exponent dividing the exponent of T. Further, since N has finite residual rank at most r, it follows, as in the proof of Lemma 2, that if the order of a Sylow s-subgroup of T is s^e for some prime s, then each s-subgroup of N/H has order dividing s^{er}. Consequently, H has finite index in N and hence also in G.

Suppose X and Y are as in (c) of the theorem and assume first that X is characteristic in H and hence also in N. Then X/Y_N is a finite π-group by Lemma 2 and hence by (b) is nilpotent. Thus X/Y_N has a characteristic series with elementary abelian factors, each one of which is centralised by H by definition of H.

Consequently, there is some integer h with $[X, {}_h H] \leq Y_N \leq Y$.

Now just assume X is normal in H. Since $(H : Y)$ is finite, X/Y_H is a finite π-group and by Lemma 2 there is a characteristic subgroup L of H of finite index with $L \leq Y_H$ (we are not claiming that H/L is necessarily a π-group). But H/L is at least soluble by the initial reduction, so H/L has a characteristic series each of whose factors is either a π-group or a π'-group. By the X-characteristic case above, each of the π-factors in this series is poly H-central. Since X/Y_H is a π-section of H/L, it has an H-normal series with poly H-central factors. Consequently, X/Y_H is poly H-central; that is, there exists h with $[X, {}_h H] \leq Y_H \leq Y$. The proof of the theorem is complete. ∎

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