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(S_3, S_6)-Amalgams VII.

WOLFGANG LEMPKEN(*) - CHRISTOPHER PARKER(**) - PETER ROWLEY(***)

Introduction.

This paper, which is the last of the series of papers [LPR1], completes the proof of the MAIN THEOREM stated in Section 1 of [LPR1]. More specifically, here we shall be investigating Cases 4 and 5 (as given in Section 12; just as before we continue the section numbering of [LPR1]). In both of these cases we have that for each critical pair (α, α') in Γ, $[Z_\alpha, Z_{\alpha'}] = 1$ and $\alpha \in O(S_3)$. (For notation see Section 1.) Sections 17 and 18 deal with Case 4 and Section 19 is devoted to Case 5. The short Section 20 reviews the main results of [LPR1] and this paper, and shows that together they establish our MAIN THEOREM.

Section 14 concentrates upon the non-central chief factors of G_β within W_β ($\beta \in O(S_6)$), the main conclusions being contained in Theorem 17.3. Note the crucial obstructing role the quadratic fours group $(W_{\alpha'} \cap G_{\alpha+2\alpha+3})Q_\beta/Q_\beta$ ($G_{\beta\alpha+2}$-conjugate to $\langle s_1, t\rangle$ as given in Proposition 2.5(ii)) in the theorem as highlighted in Lemma 17.4(ii). Also, the fact that $Z_\beta \leq W_{\alpha'-2}$ when $b = 7$ (so (17.3.2) doesn't hold) leads us to a lengthy tussle with the situation $b = 7$ before we complete the proof of Theorem 17.3. As ever our old friend the central transvection is never far from the centre of the action; see particularly Section 18 where we build upon the results of the previous section and establish that, for Case 4, $b \in \{3, 5\}$. Case 4 proves to be a slippery customer. For example, we are unable till quite late in the day to establish, in full generality, the following symmetry statement that for $(\alpha, \alpha') \in \mathcal{C}$ we have $V_{\alpha'} \not\leq Q_\beta$ (proved in Lemma 18.7, in fact).

(*) Indirizzo dell'A.: Institute for Experimental Mathematics, University of Essen, Ellernstrasse 29, Essen, Germany.

(**) Indirizzo dell'A.: School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom.

(***) Indirizzo dell'A.: School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9JL, United Kingdom.

That Case 5 is not so elusive as Case 4 is very much due to our having $Y_\beta > Z_\beta$ ($\beta \in O(S_6)$). This gives us extra leverage in the form of the subgroups $F_\alpha = \langle {Y_\beta}^{G_\alpha} \rangle$ and $H_\beta = \langle {F_\alpha}^{G_\beta} \rangle$ (where $\alpha \in \Delta(\beta)$). These subgroups first appeared in Section 12 and were also of use in analyzing Case 2.

Finally, we point out that this paper may be read independently of the other parts with the exception of notation and module data given in Sections 1 and 2, and a small amount of material relating to the case divisions in Section 12.

17. Case 4 - the non-central chief factors in W_β.

Sections 17 and 18 are concerned with Case 4. So, in these sections, we shall be assuming

HYPOTHESIS 17.0. $V_\beta/Z_\beta \cong 4$ *and* $\mathrm{core}_{G_\alpha} V_\beta = V_{\alpha-1} \cap V_\beta = [V_\beta, G_{\alpha\beta}, G_{\alpha\beta}] \cong$ $\cong E(2^3)$.

LEMMA 17.1. $\eta(G_\alpha, U_\alpha) = 3$

PROOF. Since $b > 1$ and $G_{\alpha\beta} = Q_\alpha Q_\beta$ by Lemmas 11.1(iii) and 12.2(ii), $\eta(G_\alpha, U_\alpha/[U_\alpha, Q_\alpha]) = 1$. Because $\mathrm{core}_{G_\alpha} V_\beta = [V_\beta, G_{\alpha\beta}, G_{\alpha\beta}]$ we also obtain $\eta(G_\alpha, [U_\alpha, Q_\alpha]/V_{\alpha-1} \cap V_\beta) = 1$, so giving $\eta(G_\alpha, U_\alpha) = 3$.

LEMMA 17.2. *If* $b > 3$, *then* $\eta(G_\beta, W_\beta) \geq 3$.

PROOF. By combining Lemmas 12.2(ii) and 12.4(i) with [Theorem 1; LPR2] we obtain the lemma.

The main result of this section which gives us a foothold in analysing Case 4 when $b > 5$ is the following theorem.

THEOREM 17.3. *Assume Hypothesis* 17.0 *holds, and* $b > 5$, *and let* $\delta \in O(S_6)$. *Then* $\eta(G_\delta, W_\delta) = 3$ *with the non-central chief factors in* W_δ *being isomorphic natural modules.*

We will present the proof of Theorem 17.3 in a sequence of results. Since we shall suppose the theorem is false and seek a contradiction, we shall assume in Lemma 17.4 and Theorems 17.5 and 17.6 that Hypothesis 17.0 and $b > 5$ holds but not the conclusions of the theorem. An important intermediate step, achieved in Theorem 17.6, is that of finding a critical pair (δ, δ') for which $[Z_{\delta+1}, W_{\delta'}] \neq 1$. Lemma 17.4 contains various observations needed for the proof of Theorem 17.6.

LEMMA 17.4. *Let* $(\alpha, \alpha') \in \mathcal{C}$ *and suppose that* $[Z_\beta, W_{\alpha'}] = 1$. *Put* $X = W_{\alpha'} \cap G_{\alpha+2\alpha+3}$. *Then*

(i) $[W_{\alpha'} : X] = 2$ *and hence there exists* $\alpha' + 3 \in V(\Gamma)$ *such that* $d(\alpha', \alpha' + 3) = 3$ *and* $(\alpha' + 3, \alpha + 3) \in \mathcal{C}$

(ii) $X \leq G_\beta$ *and, in the notation of Proposition* 2.5(ii), XQ_β/Q_β is $G_{\beta\alpha+2}/Q_\beta$-conjugate to $\langle s_1, t\rangle$;

(iii) $Z_{\alpha+2} \leq [X, V_\beta] \cong E(2^3)$;

(iv) $Z_{\alpha+2} \leq W_{\alpha'}$; *and*

(v) $[V_\beta, G_{\beta\alpha+2}] = [X, V_\beta](V_\beta \cap V_{\alpha+3})$.

PROOF. Since $[Z_\beta, W_{\alpha'}] = 1$, $[Z_{\alpha+2}, W_{\alpha'}] = 1$. Therefore $\langle G_{\alpha+2\alpha+3}, W_{\alpha'}\rangle$ normalizes $Z_{\alpha+2}$ and so is a proper parabolic subgroup of $G_{\alpha+3}$. Hence, by Lemma 3.10, $[W_{\alpha'} : X] \leq 2$ and $X \leq Q_{\alpha+2} \leq G_\beta$. Note that, as $W_{\alpha'}$ is abelian, X acts quadratically on V_β, so $|[X, V_\beta]| \leq 2^3$ by Proposition 2.5. If either $X = W_{\alpha'}$ or $|[X, V_\beta]| \leq 2^2$ hold, then Theorem 17.3 follows. Thus $[W_{\alpha'} : X] = 2$ and $|[X, V_\beta]| = 2^3$. We claim that $[X, V_\beta] \neq [V_\beta, G_{\beta\alpha+2}; 2]$. For if not then $[V_\beta, G_{\beta\alpha+2}; 2] = [X, V_\beta] \leq W_{\alpha'}$ and Proposition 2.5(ii) yields that $W_{\alpha'} \leq C_{G_{\alpha+3}}([V_{\alpha+3}, G_{\alpha+2\alpha+3}; 2]) = G_{\alpha+2\alpha+3}$, contradicting $[W_{\alpha'} : X] = 2$. So, since $[X, V_\beta](V_\beta \cap V_{\alpha+3}) \leq [V_\beta, G_{\beta\alpha+2}]$ we obtain (v). If $|XQ_\beta/Q_\beta| \leq 2$, then $[W_{\alpha'} : C_{W_{\alpha'}}(Z_\alpha)] \leq 2^3$ and again we have Theorem 17.3. Bearing in mind that X acts quadratically on V_β, consulting Proposition 2.5(ii) gives that XQ_β/Q_β is $G_{\beta\alpha+2}/Q_\beta$-conjugate to $\langle s_1, t\rangle$ and $Z_{\alpha+2} \leq [X, V_\beta]$. This proves (i), and (ii) and (iii), and (iv) follows from (iii).

THEOREM 17.5. *Assume that for all* $(\delta, \delta') \in \mathcal{C}$, $[Z_{\delta+1}, W_{\delta'}] = 1$. *Then for each* $(\alpha, \alpha') \in \mathcal{C}$ *we have* $V_{\alpha'} \not\leq Q_\beta$.

PROOF. Let $(\alpha, \alpha') \in \mathcal{C}$ be such that $V_{\alpha'} \leq Q_\beta$. Then $V_{\alpha'} \leq G_\alpha$, $V_{\alpha'} \not\leq Q_\alpha$ and $V_{\alpha'}$ interchanges λ and μ where $\Delta(\alpha) = \{\lambda, \mu, \beta\}$. Further, we have $Z_\beta = [V_\beta, V_{\alpha'}]$.

(17.5.1) (i) $U_\alpha \leq Q_{\alpha'-2} \leq G_{\alpha'-1}$.

(ii) If $b > 7$, then $U_\alpha \leq G_{\alpha'}$.

If there exists $(\alpha - 2, \alpha' - 2) \in \mathcal{C}$, then Lemma 17.4(iv) applied to $(\alpha - 2, \alpha' - 2)$ gives $Z_\alpha \leq W_{\alpha'-2} \leq Q_{\alpha'}$, as $b > 5$. Therefore $U_\alpha \leq Q_{\alpha'-2} \leq$

$\leq G_{\alpha'-1}$. Next we verify part (ii). Lemma 17.4(i) yields the existence of $(\alpha'+3, \alpha+3) \in \mathcal{C}$ with $d(\alpha', \alpha'+3) = 3$. Now using Lemma 17.4(iv) on this critical pair gives $Z_{\alpha'+1} \leq W_{\alpha+3}$. Because $b > 7$ by assumption $[W_{\alpha+3}, U_\alpha] = 1$, and so $[Z_{\alpha'+1}, U_\alpha] = 1$. Hence U_α centralizes $Z_{\alpha'-2}Z_{\alpha'} = Z_{\alpha'-1}$ and thus $U_\alpha \leq Q_{\alpha'-1} \leq G_{\alpha'}$.

(17.5.2) If $U_\alpha \leq G_{\alpha'}$, then $[U_\alpha, V_{\alpha'}] \cong E(2^3)$ and $Z_{\alpha'} \leq [U_\alpha, V_{\alpha'}] = C_{V_{\alpha'}}(U_\alpha) \leq U_\alpha$.

Since U_α acts quadratically upon $V_{\alpha'}$, $|[U_\alpha, V_{\alpha'}]| \leq 2^3$ by Proposition 2.5. On the other hand, $V_{\alpha'} \leq G_\alpha$, $V_{\alpha'} \not\leq Q_\alpha$ and Lemma 17.1 imply that $2^3 \leq |[U_\alpha, V_{\alpha'}]|$. Hence $C_{V_{\alpha'}}(U_\alpha) = [U_\alpha, V_{\alpha'}] \cong E(2^3)$ with $Z_{\alpha'} \leq [U_\alpha, V_{\alpha'}] \leq U_\alpha$.

Set $X_\lambda = W_\lambda \cap G_{\alpha'}$ and $X_\mu = W_\mu \cap G_{\alpha'}$.

(17.5.3) If $b > 7$, then $[W_\lambda : X_\lambda] \leq 2$.

First we show that $W_\lambda \leq G_{\alpha'-3}$. If $W_\lambda \not\leq Q_{\alpha'-4}$, then we may find $(\alpha-4, \alpha'-4) \in \mathcal{C}$. Applying Lemma 17.4(iv) to $(\alpha-4, \alpha'-4)$ and using $b > 7$ gives $Z_{\alpha-2} \leq W_{\alpha'-4} \leq Q_{\alpha'}$. But then $Z_\lambda \leq Z_{\alpha-2} \leq Q_{\alpha'}$ forces $Z_\alpha \leq Q_{\alpha'}$. So $W_\lambda \leq Q_{\alpha'-4} \leq G_{\alpha'-3}$ holds.

We now consider two following cases:- $[U_\alpha, V_{\alpha'-2}] = 1$ and $[U_\alpha, V_{\alpha'-2}] \neq 1$. Suppose $[U_\alpha, V_{\alpha'-2}] = 1$. Then orders, (17.5.1)(ii) and (17.5.2) give $V_{\alpha'-2} \cap V_{\alpha'} = C_{V_{\alpha'}}(U_\alpha) \leq U_\alpha$. Hence, as $b > 5$, $W_\lambda \cap Q_{\alpha'-3}$ commutes with $V_{\alpha'-2} \cap V_{\alpha'} = [V_{\alpha'-2}, G_{\alpha'-2\alpha'-1}; 2]$ whence we obtain $W_\lambda \cap Q_{\alpha'-3} \leq G_{\alpha'}$. So $W_\lambda \cap Q_{\alpha'-3} \leq X_\lambda$. Since $\alpha'-3 \in O(S_3)$, we have shown that $[W_\lambda : X_\lambda] \leq 2$ when $[U_\alpha, V_{\alpha'-2}] = 1$. So now we examine the case $[U_\alpha, V_{\alpha'-2}] \neq 1$. Since $V_{\alpha'}$ centralizes $V_{\alpha'-2}$ and interchanges λ and μ, we have $[V_\lambda, V_{\alpha'-2}] \neq 1$. If $V_{\alpha'-2} \leq Q_\lambda$, then $Z_\lambda = [V_\lambda, V_{\alpha'-2}] \leq V_{\alpha'-2} \leq Q_{\alpha'}$, against $(\alpha, \alpha') \in \mathcal{C}$. So $V_{\alpha'-2} \not\leq Q_\lambda$ and therefore $(\xi, \lambda) \in \mathcal{C}$ for some $\xi \in \Delta(\alpha'-2)$. By Lemma 17.4(iv) $Z_{\alpha'-3} \leq W_\lambda$. Because W_λ in abelian this gives $W_\lambda \leq Q_{\alpha'-3}$ and then using $Z_{\alpha'} \leq U_\alpha$ (by (17.5.1)(ii) and (17.5.2)) together with the parabolic argument we conclude that $[W_\lambda : X_\lambda] \leq 2$. This verifies (17.5.3).

Our next objective is to establish a weaker version of (17.5.3) when $b = 7$.

(17.5.4) Suppose $b = 7$ and $Z_\beta \neq Z_{\alpha+5}$. Then $[W_\lambda : X_\lambda] \leq 2$.

By Proposition 2.5(ii)

$$Z_\beta = \left[V_\beta, V_{\alpha'}\right] \leq V_\beta \cap V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'} \; (\ast).$$

We divide the proof of (17.5.4) into the two following cases:- $Z_\beta \leq Z_{\alpha+4}$ and $Z_\beta \not\leq Z_{\alpha+4}$. Beginning with the former, the assumption $Z_\beta \neq Z_{\alpha+5}$ and $(\ast)$ imply that

$$Z_{\alpha+4} = Z_\beta Z_{\alpha+5} \leq V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'}.$$

Also, $Z_\beta \leq Z_{\alpha+4}$ yields $Z_{\alpha+2} = Z_\beta Z_{\alpha+3} = Z_{\alpha+4}$ and hence $\langle G_{\alpha+2\alpha+3}, G_{\alpha+3\alpha+4}\rangle \neq G_{\alpha+3}$. This, when combined with Lemma 17.4(i), yields the existence of $(\alpha'+3, \alpha+3) \in \mathcal{C}$ for which $V_{\alpha'+2}Q_{\alpha+3}/Q_{\alpha+3}$ is not a central transvection of $G_{\alpha+3\alpha+4}/Q_{\alpha+3}$ on $V_{\alpha+3}/Z_{\alpha+3}$. Therefore $[V_{\alpha'+2}, V_{\alpha+3}] \not\leq Z_{\alpha+4}$. Since $[V_{\alpha'+2}, V_{\alpha+3}] \leq V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'}$ (again by Proposition 2.5(ii)), we deduce that

$$E(2^3) \lesssim Z_{\alpha+4}\left[V_{\alpha'+2}, V_{\alpha+3}\right] \leq V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'}.$$

Consequently $V_{\alpha+3} \cap V_{\alpha+5} = V_{\alpha+5} \cap V_{\alpha'}$. Since $[U_\alpha, V_{\alpha+3}] = 1$, $[U_\alpha, Z_{\alpha+6}] = 1$ and so $U_\alpha \leq G_{\alpha'}$ by (17.5.1)(i). Hence, using (17.5.2),

$$V_{\alpha+5} \cap V_{\alpha'} = C_{V_{\alpha'}}(U_\alpha) \leq U_\alpha.$$

Now $[W_\lambda, U_\alpha] = 1$ and Proposition 2.5(ii) (applied twice) force $W_\lambda \leq G_{\alpha'}$, so proving (17.5.4) in this case. Now we assume that $Z_\beta \not\leq Z_{\alpha+4}$. So, by $(\ast)$

$$V_{\alpha+3} \cap V_{\alpha+5} = Z_{\alpha+4}Z_\beta = Z_{\alpha+4}Z_{\alpha+2}.$$

If $Z_{\alpha+5} \leq Z_{\alpha+2}$, then we obtain $Z_{\alpha+2} = Z_{\alpha+4}$ whence $V_{\alpha+3} \cap V_{\alpha+5} = Z_{\alpha+4}Z_{\alpha+2} = Z_{\alpha+2}$, which is impossible. Thus $Z_{\alpha+5} \not\leq Z_{\alpha+2}$ and hence

$$V_{\alpha+3} \cap V_{\alpha+5} = Z_{\alpha+4}Z_{\alpha+2} = Z_{\alpha+5}Z_{\alpha+2}\,(\ast\ast).$$

We claim that $[U_\alpha, V_{\alpha+5}] = 1$. For suppose $[U_\alpha, V_{\alpha+5}] \neq 1$. Then, by Proposition 2.5(ii) and (17.5.1)(i),

$$Z_{\alpha+5} = \left[U_\alpha, V_{\alpha+5}\right] \leq V_\beta \cap V_{\alpha+3} \cap V_{\alpha+5}.$$

Now $Z_{\alpha+2} = Z_\beta Z_{\alpha+3} \leq V_\beta \cap V_{\alpha+3} \cap V_{\alpha+5}$ together with our earlier deduction $Z_{\alpha+5} \not\leq Z_{\alpha+2}$ forces $V_\beta \cap V_{\alpha+3} \cap V_{\alpha+5} \cong E(2^3)$. But then $V_\beta \cap V_{\alpha+3} = V_{\alpha+3} \cap V_{\alpha+5}$ whence Proposition 2.5(ii) gives $W_{\alpha'} \leq C_{G_{\alpha+3}}(V_\beta \cap V_{\alpha+3}) \leq G_{\alpha+2\alpha+3}$, contradicting Lemma 17.4(i). Hence $[U_\alpha, V_{\alpha+5}] = 1$, as claimed. So $U_\alpha \leq G_{\alpha'}$ and therefore

$$Z_{\alpha+6} \leq C_{V_{\alpha'}}(U_\alpha) \leq U_\alpha$$

by (17.5.2). Since $[W_\lambda, U_\alpha] = 1$, we get $[W_\lambda, Z_{\alpha+6}] = 1$ which, by $(\ast\ast)$, implies that W_λ centralizes $V_{\alpha+3} \cap V_{\alpha+5}$. Consequently $W_\lambda \leq G_{\alpha+5}$. Since $[W_\lambda, Z_{\alpha'}] = 1$, employing the parabolic argument gives $[W_\lambda : X_\lambda] \leq 2$ and completes the proof of (17.5.4).

We now show that we do in fact have some critical pairs to which we may apply (17.5.4).

(17.5.5) If $b=7$, then there exists $(\delta,\delta')\in\mathcal{C}$ such that $V_{\delta'}\leq Q_{\delta+1}$ and $Z_{\delta+1}\neq Z_{\delta+5}$.

If $Z_\beta\neq Z_{\alpha+5}$, then we may take $(\delta,\delta')=(\alpha,\alpha')$. So we may assume $Z_\beta=Z_{\alpha+5}$. Thus $Z_{\alpha+2}=Z_{\alpha+4}$ and therefore $\langle G_{\alpha+2\alpha+3},G_{\alpha+3\alpha+4}\rangle\neq G_{\alpha+3}$. Then, by Lemma 17.4(i), we may find $(\alpha'+3,\alpha+3)\in\mathcal{C}$ such that $V_{\alpha'+2}Q_{\alpha+3}/Q_{\alpha+3}$ is not a central transvection of $G_{\alpha+3\alpha+4}/Q_{\alpha+3}$ on $V_{\alpha+3}/Z_{\alpha+3}$. If $V_{\alpha+3}\not\leq Q_{\alpha'+2}$, the there exists $\xi\in\Delta(\alpha+3)$ such that $(\xi,\alpha'+2)\in\mathcal{C}$. Applying Lemma 17.4(v) to $(\xi,\alpha'+2)$ we see that

$$[V_{\alpha+3},G_{\alpha+3\alpha+4}]=[V_{\alpha+3},W_{\alpha'+2}\cap G_{\alpha+4\alpha+5}](V_{\alpha+3}\cap V_{\alpha+5}).$$

But then $V_{\alpha'+2}$ centralizes $[V_{\alpha+3},G_{\alpha+3\alpha+4}]$, a contradiction. So $V_{\alpha+3}\leq Q_{\alpha'+2}$ and $[V_{\alpha'+2},V_{\alpha+3}]=Z_{\alpha'+2}$. Further $Z_{\alpha'+2}\neq Z_{\alpha+5}$, since $V_{\alpha'+2}Q_{\alpha+3}/Q_{\alpha+3}$ is not a central transvection on $V_{\alpha+3}/Z_{\alpha+3}$. Taking $(\delta,\delta')=(\alpha'+3,\alpha+3)$ we see that (17.5.5) holds

(17.5.6) $[X_\lambda,V_{\alpha'}][X_\mu,V_{\alpha'}]\leq U_\alpha\cap G_{\alpha'}$.

Observe that $b>5$ implies $U_\alpha\leq Z(G_\alpha^{[4]})$. So, since $[X_\lambda,V_{\alpha'}]\leq G_\alpha^{[4]}$,

$$[X_\lambda,V_{\alpha'}]\leq C_{V_{\alpha'}}(U_\alpha\cap G_{\alpha'}).$$

Suppose $U_\alpha\leq G_{\alpha'}$ holds. Then, using (17.5.2),

$$[X_\lambda,V_{\alpha'}]\leq C_{V_{\alpha'}}(U_\alpha)\leq U_\alpha\cap G_{\alpha'}.$$

Now we consider the possibility $U_\alpha\not\leq G_{\alpha'}$. From (17.5.1)(i) and Lemma 17.1, we get $[U_\alpha,Z_{\alpha'}]\neq 1$ and $|(U_\alpha\cap G_{\alpha'})Q_{\alpha'}/Q_{\alpha'}|\geq 2^2$. Hence, as $U_\alpha\cap G_{\alpha'}$ acts quadratically on $V_{\alpha'}/Z_{\alpha'}$, Proposition 2.5(ii) gives

$$[U_\alpha\cap G_{\alpha'},V_{\alpha'}]Z_{\alpha'}\geq C_{V_{\alpha'}}(U_\alpha\cap G_{\alpha'})\geq[X_\lambda,V_{\alpha'}].$$

From $[U_\alpha,Z_{\alpha'}]\neq 1$, $Z_{\alpha'}\not\leq G_\alpha^{[4]}$ and so we deduce that

$$[X_\lambda,V_{\alpha'}]=[U_\alpha\cap G_{\alpha'},V_{\alpha'}]\leq U_\alpha\cap G_{\alpha'}.$$

A similar argument proves that $[X_\mu,V_{\alpha'}]\leq U_\alpha\cap G_{\alpha'}$ and so (17.5.6) holds.

(17.5.7) $[W_\lambda:W_\lambda\cap W_\mu]\leq 2$

Since $V_{\alpha'}$ interchanges λ and μ, $V_{\alpha'}$ acts upon $X_\lambda X_\mu$ and $X_\lambda\cap X_\mu$. (Note that X_λ and X_μ normalize each other since $b\geq 5$.) So, by (17.5.6),

$$[X_\lambda X_\mu,V_{\alpha'}]=[X_\lambda,V_{\alpha'}][X_\mu,V_{\alpha'}]\leq U_\alpha\cap G_{\alpha'}\leq X_\lambda\cap X_\mu.$$

Let $x \in V_{\alpha'}$ be such that $\lambda \cdot x = \mu$. If $X_\lambda \cap X_\mu \lneqq X_\lambda$, then $[X_\lambda X_\mu, x] \not\leq X_\lambda \cap X_\mu$ contrary to $[X_\lambda X_\mu, V_{\alpha'}] \leq X_\lambda \cap X_\mu$. Therefore $X_\lambda \cap X_\mu = X_\lambda$, and so $X_\lambda = X_\mu$. If $b > 7$, then (17.5.7) follows from (17.5.3). For $b = 7$, the above argument in conjunction with (17.5.4), (17.5.5) and Lemma 11.1(vii) also yields $[W_\lambda : W_\lambda \cap W_\mu] \leq 2$.

Appealing to Lemma 11.1(vii), (17.5.7) implies that $[W_{\alpha'} : W_{\alpha'} \cap W_{\alpha'-2}] \leq$ $\leq 2 \geq [W_{\alpha'-2} : W_{\alpha'-2} \cap W_{\alpha'-4}]$, whence $[W_{\alpha'} : W_{\alpha'} \cap W_{\alpha'-4}] \leq 2^2$. Since $[Z_\alpha, W_{\alpha'-4}] = 1$, this forces $\eta(G_{\alpha'}, W_{\alpha'}) \leq 2$, against Lemma 17.2. From this contradictory state of affairs we conclude that $V_{\alpha'} \not\leq Q_\beta$.

We are now ready to begin the proof of

THEOREM 17.6. *There exists $(\delta, \delta') \in \mathcal{C}$ such that $[Z_{\delta+1}, W_{\delta'}] \neq 1$.*

PROOF. Arguing for a contradiction we suppose that $[Z_{\delta+1}, W_{\delta'}] = 1$ for all $(\delta, \delta') \in \mathcal{C}$.

(17.6.1) For $(\alpha, \alpha') \in \mathcal{C}$, $[V_\beta, V_{\alpha'}] \leq Z_{\alpha+2} \cap Z_{\alpha'-1}$.

Lemma 17.4(v) gives $[V_\beta, G_{\beta\alpha+2}] = [X, V_\beta](V_\beta \cap V_{\alpha+3})$ (where $X = W_{\alpha'} \cap G_{\alpha+2\alpha+3}$). Since $W_{\alpha'}$ is abelian and $[X, V_\beta] \leq W_{\alpha'}$, together with $[V_{\alpha'}, V_\beta \cap V_{\alpha+3}] = 1$, we get that $V_{\alpha'}$ centralizes $[V_\beta, G_{\beta\alpha+2}]$. Thus $[V_\beta, V_{\alpha'}] \leq Z_{\alpha+2}$. By Theorem 17.5 $V_{\alpha'} \not\leq Q_\beta$ and so there exists $(\alpha'+1, \beta) \in \mathcal{C}$. Then a similar argument gives $[V_\beta, V_{\alpha'}] \leq Z_{\alpha'-1}$.

(17.6.2) For $(\alpha, \alpha') \in \mathcal{C}$ we have $[W_{\alpha'}, V_{\alpha+3}] = Z_{\alpha+4}$.

It follows directly from (17.6.1) that $[W_{\alpha'}, V_{\alpha+3}] \leq Z_{\alpha+4}$. Hence $|W_{\alpha'}Q_{\alpha+3}/Q_{\alpha+3}| \leq 2$ and so, by Lemma 17.4(i), $W_{\alpha'} \cap Q_{\alpha+3} = X$ $(= W_{\alpha'} \cap G_{\alpha+2\alpha+3})$. If $[X, V_{\alpha+3}] = 1$, then X centralizes $[V_\beta, G_{\beta\alpha+2}]$ by Lemma 17.4(v) which implies $|XQ_\beta/Q_\beta| \leq 2$, contradicting Lemma 17.4(ii). So $Z_{\alpha+3} = [X, V_{\alpha+3}] \leq [W_{\alpha'}, V_{\alpha+3}]$. Since $W_{\alpha'} \not\leq Q_{\alpha+3}$, $[W_{\alpha'}, V_{\alpha+3}] \neq Z_{\alpha+3}$ and we have (17.6.2).

(17.6.3) $Z_{\alpha+4} \leq V_{\alpha'}$.

For λ with $\lambda \neq \alpha' - 2$ and $d(\lambda, \alpha') = 2$ we have, as $V_{\alpha+3}$ centralizes $V_{\alpha'} \cap V_\lambda = [V_\lambda, G_{\lambda-1\lambda}; 2]$, that $[V_{\alpha+3}, V_\lambda] \leq V_{\alpha'} \cap V_\lambda$. So, since $W_{\alpha'} =$ $= V_{\alpha'} \prod_{d(\lambda,\alpha')=2} V_\lambda$, we get, using (17.6.2), that

$$Z_{\alpha+4} = \left[W_{\alpha'}, V_{\alpha+3}\right] = \prod_{\substack{d(\lambda,\alpha')=2 \\ \lambda\neq\alpha'-2}} \left[V_\lambda, V_{\alpha+3}\right] \leq V_{\alpha'}.$$

(17.6.4) Let $(\alpha, \alpha') \in \mathcal{C}$ and set $X = W_{\alpha'} \cap G_{\alpha+2\alpha+3}$. Then we have $Z_{\alpha+3} = [V_\beta, V_{\alpha'}] = Z_{\alpha'-2} = V_{\alpha'} \cap [X, V_\beta]$.

By Lemma 17.4(iii) and (17.6.3), $Z_{\alpha+3} \leq V_{\alpha'} \cap [X, V_\beta]$. Also we clearly have $[V_{\alpha'}, V_\beta] \leq V_{\alpha'} \cap [X, V_\beta]$. Now suppose that $|V_{\alpha'} \cap [X, V_\beta]| > 2$, and put $\bar{W}_{\alpha'} = W_{\alpha'}/V_{\alpha'}$. Then, by Lemma 17.4(i), (iii), $|[V_\beta, \bar{X}]| \leq 2 \geq [\bar{W}_{\alpha'} : \bar{X}]$. Hence $\eta(G_{\alpha'}, \bar{W}_{\alpha'}) \leq 2$ and thus $\eta(G_{\alpha'}, W_{\alpha'}) = 3$ by Lemma 17.2, with V_β acting as a transvection upon each of the non-central chief factors within $\bar{W}_{\alpha'}$. By (17.6.1) V_β also acts as a transvection on $V_{\alpha'}/Z_{\alpha'}$ and thus the non-central chief factors in $\overline{W}_{\alpha'}$ are isomorphic natural modules, a contradiction. Therefore we deduce that $|V_{\alpha'} \cap [X, V_\beta]| = 2$ and so $Z_{\alpha+3} = $ $= V_{\alpha'} \cap [X, V_\beta] = [V_\beta, V_{\alpha'}]$. Since $V_{\alpha'} \not\leq Q_\beta$ by Theorem 17.5, a symmetric argument gives $[V_{\alpha'}, V_\beta] = Z_{\alpha'-2}$. This completes the proof of (17.6.4).

(17.6.5) For $(\alpha, \alpha') \in \mathcal{C}$ we have $Z_{\alpha+4} = Z_{\alpha'-1}$.

By (17.6.4) applied to (α, α') we obtain $Z_{\alpha+3} = Z_{\alpha'-2}$. From Lemma 17.4(i) there exists $(\alpha+3, \alpha'+3) \in \mathcal{C}$ and using (17.6.4) on this critical pair gives $Z_{\alpha'} = Z_{\alpha+5}$. Therefore

$$Z_{\alpha+4} = Z_{\alpha+3} Z_{\alpha+5} = Z_{\alpha'-2} Z_{\alpha'} = Z_{\alpha'-1},$$

as required.

(17.6.6) $b > 9$.

Suppose (17.6.6) is false. Then $b = 7$ or 9. Let $(\alpha, \alpha') \in \mathcal{C}$. Just as in (17.6.5) we have $Z_{\alpha+5} = Z_{\alpha'}$ and this rules out $b = 7$. Since $V_{\alpha'} \not\leq Q_\beta$ by Theorem 17.5, we have $(\lambda, \beta) \in \mathcal{C}$ for some $\lambda \in \Delta(\alpha')$ and likewise we deduce that $Z_\beta = Z_{\alpha'-4}$. So, as $b = 9$, $Z_\beta = Z_{\alpha+5} = Z_{\alpha'}$. Therefore

$$Z_{\alpha'} = Z_\beta \leq V_{\alpha'} \cap \left[X, V_\beta\right] = Z_{\alpha+3}$$

by Lemma 17.4(iii) and (17.6.4), whereas $Z_\beta \neq Z_{\alpha+3}$. Thus (17.6.6) holds.

Now we fix $(\alpha, \alpha') \in \mathcal{C}$ and let $\xi \in \Delta(\alpha') \setminus \{\alpha' - 1\}$ be such that

(i) $Z_\xi \neq Z_{\alpha+6}$; and
(ii) $\langle V_\beta, G_{\alpha'\xi} \rangle = G_{\alpha'}$.

By Proposition 2.8(viii) we may choose $\rho \in \Delta(\alpha') \setminus \{\alpha' - 1\}$ such that $\langle V_\beta, G_{\alpha'\rho} \rangle = G_{\alpha'}$. Since $Z_\rho \ntrianglelefteq G_{\alpha'}$ and $[V_\beta, Z_{\alpha+6}] = 1$ clearly $Z_\rho \neq Z_{\alpha+6}$, so we may take $\xi = \rho$.

Put $R_\xi = G_\xi^{[4]}$ and $X = W_{\alpha'} \cap G_{\alpha+2\alpha+3}$.

(17.6.7) $R_\xi \leq Q_{\alpha+5}$.

If $R_\xi \not\leq Q_{\alpha+5}$, then there exists ρ such that $d(\xi,\rho)=4$, $d(\rho,\alpha+5)=b$ and $Z_\rho \not\leq Q_{\alpha+5}$. So $(\rho,\alpha+5)\in\mathcal{C}$. Applying (17.6.5) to $(\rho,\alpha+5)$ yields $Z_\xi = Z_{\alpha+6}$ contrary to the choice of ξ. Thus $R_\xi \leq Q_{\alpha+5}$.

(17.6.8) (i) $R_\xi = (R_\xi \cap G_{\alpha+2\alpha+3})W_{\alpha'}$; and
(ii) $(R_\xi \cap G_{\alpha+2\alpha+3})Q_\beta = XQ_\beta$.

By (17.6.6) ${G_{\alpha'}}^{[5]}$ is abelian. Since $Z_{\alpha+2}\leq W_{\alpha'}$ and $Z_{\alpha+4}=Z_{\alpha'-1}\leq W_{\alpha'}$ by Lemma 17.4(iv) and (17.6.5), R_ξ commutes with both $Z_{\alpha+2}$ and $Z_{\alpha+4}$. Hence, using (17.6.7),

$$R_\xi \leq Q_{\alpha+4} \leq G_{\alpha+3} \text{ and } \langle G_{\alpha+2\alpha+3}, R_\xi\rangle \neq G_{\alpha+3}.$$

Therefore $[R_\xi : R_\xi\cap G_{\alpha+2\alpha+3}]\leq 2$. Since $W_{\alpha'}\leq R_\xi$ and $[W_{\alpha'} : X]=2$ by Lemma 17.4(ii), we have (i). Further, note that $R_\xi\cap G_{\alpha+2\alpha+3}\leq G_\beta$. From $[V_\beta, R_\xi\cap G_{\alpha+2\alpha+3}]\leq {G_{\alpha'}}^{[5]}$ we have that $R_\xi\cap G_{\alpha+2\alpha+3}$ acts quadratically on V_β. Because $X\leq R_\xi\cap G_{\alpha+2\alpha+3}$, Lemma 17.4(ii) implies $(R_\xi\cap G_{\alpha+2\alpha+3})Q_\beta = XQ_\beta$, which completes the proof of (17.6.8).

Since, by Lemma 17.4(iii), $Z_\beta \leq [X, V_\beta]$, (17.6.8)(ii) implies that

$$[V_\beta, R_\xi\cap G_{\alpha+2\alpha+3}] = [V_\beta, X] \leq W_{\alpha'}.$$

Consequently

$$\begin{aligned}[V_\beta, R_\xi] &= [V_\beta, (R_\xi\cap G_{\alpha+2\alpha+3})W_{\alpha'}]\\ &= [V_\beta, R_\xi\cap G_{\alpha+2\alpha+3}]\,[V_\beta, W_{\alpha'}] \leq W_{\alpha'} \leq R_\xi,\end{aligned}$$

by (17.6.8)(i). Hence

$$R_\xi \trianglelefteq \langle V_\beta, G_{\alpha'\xi}\rangle = G_{\alpha'},$$

a contradiction which completes the proof of Theorem 17.6.

We bring the following two groups into the fray:-

$$\begin{aligned} F_\delta &:= [U_\delta, Q_\delta]\\ H_\lambda &:= \langle {F_\delta}^{G_\lambda}\rangle\end{aligned}$$

where $\delta\in O(S_3)$ and $\lambda\in\Delta(\delta)$. These groups will play a somewhat similar role to the F_α and H_β defined in Section 12; note that our present F_δ, H_λ and their counterparts in Section 12 are entirely different groups.

Lemma 17.7. *Let* $(\alpha,\alpha')\in\mathcal{C}$.
(i) $\eta(G_\alpha, F_\alpha) = 2$.

(ii) $F_\alpha V_\beta \ntrianglelefteq G_\beta$.
(iii) $H_\beta \leq [W_\beta, Q_\beta]V_\beta$.

PROOF. Since $\eta(G_\alpha, U_\alpha/[U_\alpha Q_\alpha]) = 1$, (i) follows from Lemma 17.1.

Suppose (ii) is false. From $[Q_\beta, F_\alpha V_\beta] = [Q_\beta, F_\alpha][Q_\beta, V_\beta] = [Q_\beta, F_\alpha]Z_\beta = [Q_\beta, F_\alpha]$, we then get $[Q_\beta, F_\alpha] \trianglelefteq G_\beta$. Since $|[Q_\beta, F_\alpha]| \leq 2^4$ and $Z_\beta \leq [Q_\beta, F_\alpha]$, we deduce that $O^2(G_\beta)$ centralizes $[Q_\beta, F_\alpha]$. If $(V_{\alpha-1} \cap V_\beta) \cap [Q_\beta, F_\alpha] > Z_\beta$, then the uniseriality of $G_{\alpha\beta}$ on V_β/Z_β gives $Z_\alpha \leq [Q_\beta, F_\alpha]$, against Lemma 1.1(ii). Therefore $(V_{\alpha-1} \cap V_\beta) \cap [Q_\beta, F_\alpha] = Z_\beta$. Also, $[F_\alpha, Q_\alpha] \leq V_{\alpha-1} \cap V_\beta$ and thus

$$\begin{aligned}\left[Q_\alpha \cap Q_\beta, F_\alpha V_\beta\right] &= \left[Q_\alpha \cap Q_\beta, F_\alpha\right]\left[Q_\alpha \cap Q_\beta, V_\beta\right] \\ &\leq \left[Q_\beta, F_\alpha\right] \cap \left(V_{\alpha-1} \cap V_\beta\right) = Z_\beta.\end{aligned}$$

Now we may obtain a contradiction as in Lemma 12.5(ii). This proves (ii).

Turning to (iii) we first show that $V_{\alpha-1} \cap [W_\beta, Q_\beta]V_\beta \gneqq V_\beta \cap V_{\alpha-1}$. By [Proof of Theorem 1; LPR2] $(Q_\alpha \cap Q_\beta)Q_{\alpha-1}/Q_{\alpha-1}$ is not contained in the quadratic $E(2^3)$-subgroup of $G_{\alpha-1\alpha}/Q_{\alpha-1}$ (on $V_{\alpha-1}/Z_{\alpha-1}$), and so $[V_{\alpha-1}, Q_\alpha \cap Q_\beta] \not\leq V_{\alpha-1} \cap V_\beta$. Since $[V_{\alpha-1}, Q_\alpha \cap Q_\beta] \leq V_{\alpha-1} \cap [W_\beta, Q_\beta]$, we see that $V_{\alpha-1} \cap [W_\beta, Q_\beta]V_\beta \gneqq V_\beta \cap V_{\alpha-1}$. Hence

$$\left|V_{\alpha-1}\left[W_\beta, Q_\beta\right]V_\beta/\left[W_\beta, Q_\beta\right]V_\beta\right| = 2$$

and therefore $[V_{\alpha-1}, Q_\alpha] \leq [W_\beta, Q_\beta]V_\beta$. Since this holds for all $\alpha - 1 \in \Delta(\alpha)$, $F_\alpha = [U_\alpha, Q_\alpha] \leq [W_\beta, Q_\beta]V_\beta$ and thus $H_\beta \leq [W_\beta, Q_\beta]V_\beta$.

With Theorem 17.6 and Lemma 17.7 to hand we now start the

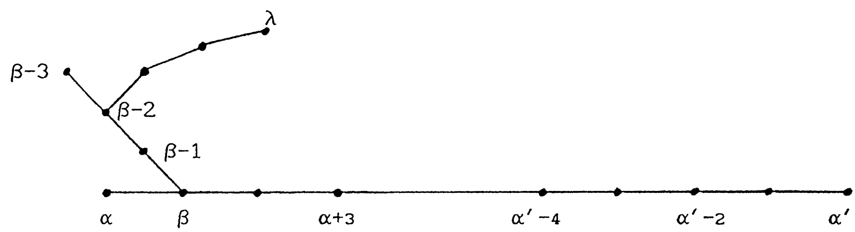
PROOF OF THEOREM 17.3. We suppose the theorem is false and seek a contradiction. By Theorem 17.6 we may choose $(\alpha, \alpha') \in \mathcal{C}$ such that $[Z_\beta, W_{\alpha'}] \neq 1$. In particular, $Z_\beta \not\leq W_{\alpha'}$ and hence $V_{\alpha'} \not\leq Q_\beta$. So there exists $\rho \in \Delta(\alpha')$ such that $(\rho, \beta) \in \mathcal{C}$.

(17.3.1) $W_\beta \not\leq Q_{\alpha'-2}$ and $W_{\alpha'} \not\leq Q_{\alpha+3}$.

Suppose $W_\beta \leq Q_{\alpha'-2}$ holds. By Lemma 17.4(i) applied to (ρ, β) we deduce that $[Z_{\alpha'}, W_\beta] \neq 1$. In particular, $Z_{\alpha'} \not\leq W_\beta$. So, since $W_\beta \cap G_{\alpha'}$ acts quadratically upon $V_{\alpha'}$, $|[W_\beta \cap G_{\alpha'}, V_{\alpha'}]| \leq 2^2$. But $[W_\beta : W_\beta \cap G_{\alpha'}] \leq 2$ whence $\eta(G_\beta, W_\beta) = 3$ with all non-central chief factors within W_β being isomorphic natural modules. Thus we must have $W_\beta \not\leq Q_{\alpha'-2}$. A similar argument shows that $W_{\alpha'} \not\leq Q_{\alpha+3}$.

From now until (17.3.9) we assume, in addition, that $b > 7$.

By (17.3.1) there exists $(\beta-3, \alpha'-2) \in \mathcal{C}$ with $d(\beta, \beta-3) = 3$. We display the part of Γ that will be of interest to us (λ has yet to be introduced).



(17.3.2) $Z_\beta \not\leq W_{\alpha'-2}$.

Because $b > 7$, $Z_\beta \leq W_{\alpha'-2}$ would imply $[Z_\beta, W_{\alpha'}] = 1$ contrary to the choice of (α, α').

(17.3.3) (i) $W_{\alpha'-2} \not\leq G_{\beta\beta-1}$.
(ii) $[W_{\alpha'-2} : W_{\alpha'-2} \cap Q_\beta] \geq 2^2$.

(i) Suppose $W_{\alpha'-2} \leq G_{\beta\beta-1}$ holds. Since $[W_{\alpha'-2}, Z_{\beta-1}] \leq Z_\beta$ and, by (17.3.2), $Z_\beta \not\leq W_{\alpha'-2}$, $[W_{\alpha'-2}, Z_{\beta-1}] = 1$ and so $W_{\alpha'-2} \leq Q_{\beta-1} \leq G_{\beta-2}$. Thus $|[W_{\alpha'-2}, V_{\beta-2}]| \leq 2^3$ which implies $\eta(G_{\alpha'-2}, W_{\alpha'-2}) = 3$ with all non-central chief factors within $W_{\alpha'-2}$ being isomorphic natural modules, a contradiction. Therefore $W_{\alpha'-2} \not\leq G_{\beta\beta-1}$.

(ii) Assume that $[W_{\alpha'-2} : W_{\alpha'} \cap Q_\beta] \leq 2$ holds. Since $Z_\beta \not\leq W_{\alpha'-2}$, $[W_{\alpha'-2} \cap Q_\beta, V_\beta] = 1$. So $[W_{\alpha'} \cap Q_\beta, Z_{\beta-1}] = 1$ and thus $W_{\alpha'-2} \cap Q_\beta \leq$ $\leq Q_{\beta-1} \leq G_{\beta-2}$. Since the theorem is supposed false we must have $|[W_{\alpha'-2} \cap Q_\beta, V_{\beta-2}]| = 2^3$. Also from $[W_{\alpha'-2} \cap Q_\beta, V_\beta] = 1$ we have

$$V_\beta \cap V_{\beta-2} \leq C_{V_{\beta-2}}\big(W_{\alpha'-2} \cap Q_\beta\big).$$

Consequently, as $W_{\alpha'-2} \cap Q_\beta$ acts quadratically on $V_{\beta-2}$ and not as a transvection on $V_{\beta-2}/Z_{\beta-2}$,

$$V_\beta \cap V_{\beta-2} = C_{V_{\beta-2}}\big(W_{\alpha'-2} \cap Q_\beta\big) = \big[W_{\alpha'-2} \cap Q_\beta, V_{\beta-2}\big] \leq W_{\alpha'-2}.$$

Therefore $[V_\beta \cap V_{\beta-2}, W_{\alpha'-2}] = 1$ whence $W_{\alpha'-2} \leq G_{\beta\beta-1}$ by Proposition 2.5(ii), contrary to part (i). So we have proved (ii).

(17.3.4) (i) $V_\beta \cap V_{\alpha+3} = Z_\beta[W_{\alpha'-2}, V_\beta] = C_{V_\beta}(W_{\alpha'-2})$; and
(ii) $|[W_{\alpha'-2}, V_\beta]| = 2^2$.

By (17.3.3)(ii) $W_{\alpha'-2}$ acts as at least a quadratic fours group on V_β/Z_β.

Therefore $V_\beta \cap V_{\alpha+3} = C_{V_\beta}(W_{\alpha'-2})$ and $V_\beta \cap V_{\alpha+3} = Z_\beta[W_{\alpha'-2}, V_\beta]$. Because $[W_{\alpha'-2}, V_\beta] \le W_{\alpha'-2}$ (17.3.2) implies that $|[W_{\alpha'-2}, V_\beta]| = 2^2$, and we have (17.3.4).

(17.3.5) $$\left[Z_{\beta-2}, W_{\alpha'-2}\right] \neq 1.$$

If we have $[Z_{\beta-2}, W_{\alpha'-2}] = 1$, then Lemma 17.4(iv) applied to the critical pair $(\beta-3, \alpha'-2)$ gives $Z_{\beta-1} \le W_{\alpha'-2}$. Hence $Z_\beta \le Z_{\beta-1} \le W_{\alpha'-2}$, contradicting (17.3.2). Therefore $[Z_{\beta-2}, W_{\alpha'-2}] \neq 1$.

By (17.3.5) we have $(\beta-3, \alpha'-2) \in \mathcal{C}$ with $[Z_{\beta-2}, W_{\alpha'-2}] \neq 1$. So we may repeat the procedure that produced $(\beta-3, \alpha'-2)$ from (α, α'), this time starting with $(\beta-3, \alpha'-2)$ to obtain $(\lambda, \alpha'-4) \in \mathcal{C}$ with $d(\lambda, \beta-2) = 3$ and $[Z_{\beta-2}, W_{\alpha'-2}] \neq 1$. As a consequence all the results obtained for $(\beta-3, \alpha'-2)$ also hold for $(\lambda, \alpha'-4)$. In particular,

(17.3.6) (i) $Z_{\beta-2} \not\le W_{\alpha'-4}$ (analogue of (17.3.2)); and
(ii) $|[W_{\alpha'-4}, V_{\beta-2}]| = 2^2$ (analogue of (17.3.4)(ii)).

(17.3.7) $[W_{\alpha'-4}, V_{\beta-2}] \le V_{\alpha'-4}$.

This follows from the fact that $W_{\alpha'-4} = V_{\alpha'-4} \prod_{d(\mu,\alpha'-4)=2} V_\mu$ and, for each μ, $[V_\mu, V_{\beta-2}] \le V_{\alpha'-4} \cap V_\mu$.

(17.3.8) $$\left[W_{\alpha'-4}, V_{\beta-2}\right] = \left[W_{\alpha'-2}, V_\beta\right] \le V_\beta \cap V_{\beta-2}.$$

From $[V_\beta, W_{\alpha'-4}] = 1$, $[V_\beta \cap V_{\beta-2}, W_{\alpha'-4}] = 1$ and so $W_{\alpha'-4} \le G_{\beta-2\beta-1}$ and $[W_{\alpha'-4}, V_{\beta-2}] \le V_\beta \cap V_{\beta-2} \le V_\beta$. Now $b > 7$ gives $[W_{\alpha'-2}, W_{\alpha'-4}] = 1$ and hence

$$\left[W_{\alpha'-4}, V_{\beta-2}\right] \le C_{V_\beta}\left(W_{\alpha'-2}\right) = V_\beta \cap V_{\alpha+3},$$

using (17.3.4)(i). If $[W_{\alpha'-4}, V_{\beta-2}] \not\le [W_{\alpha'-2}, V_\beta]$, then, as $[V_\beta \cap V_{\alpha+3} : [W_{\alpha'-2}, V_\beta]] = 2$ by (17.3.4)(ii),

$$V_\beta \cap V_{\alpha+3} = \left[W_{\alpha'-4}, V_{\beta-2}\right]\left[W_{\alpha'-2}, V_\beta\right].$$

Then, using (17.3.7), we deduce that

$$\begin{aligned} Z_\beta \le Z_{\alpha+2} \le V_\beta \cap V_{\alpha+3} &= \left[W_{\alpha'-4}, V_{\beta-2}\right]\left[W_{\alpha'-2}, V_\beta\right] \\ &\le V_{\alpha'-4}\left[W_{\alpha'-2}, V_\beta\right] \le W_{\alpha'-2}, \end{aligned}$$

against (17.3.2). Therefore $[W_{\alpha'-4}, V_{\beta-2}] \le [W_{\alpha'-2}, V_\beta]$ and hence $[W_{\alpha'-4}, V_{\beta-2}] = [W_{\alpha'-2}, V_\beta]$ by (17.3.4)(ii) and (17.3.6)(ii). This establishes (17.3.8).

We now unveil the desired contradiction. Combining (17.3.4)(i) and (17.3.8) gives

$$V_\beta \cap V_{\alpha+3} = Z_\beta\bigl[W_{\alpha'-2}, V_\beta\bigr] \le Z_\beta\bigl(V_{\beta-2} \cap V_\beta\bigr) \le V_{\beta-2} \cap V_\beta.$$

Thus $V_\beta \cap V_{\alpha+3} = V_{\beta-2} \cap V_\beta$ by orders. But then $W_{\alpha'-2}$ centralizes $V_{\beta-2} \cap V_\beta$ whence $W_{\alpha'-2} \le G_{\beta\beta-1}$ by Proposition 2.5(ii), contradicting (17.3.3)(i). From this we conclude that

(17.3.9) $b = 7$.

Before making essential use of (17.3.9) we observe the following generation result for W_β. For $\tau \in \Delta(\beta)$, put $\Lambda(\beta, \tau) = \{\gamma \in \Delta(\beta) | Z_\gamma \not\le [V_\beta, G_{\beta\tau}]\}$.

(17.3.10) Let $\tau \in \Delta(\beta)$ and $x \in G_{\beta\tau} \setminus Q_\beta$. If $[W_\beta/V_\beta : C_{W_\beta/V_\beta}(x)] \le 2^3$, then

$$W_\beta = \bigl\langle U_\gamma | \gamma \in \Lambda(\beta, \tau)\bigr\rangle U_\tau.$$

From Lemma 17.7 $H_\beta \le [W_\beta, Q_\beta]V_\beta$. Also we note that $[U_\alpha, Q_\beta, Q_\beta] \le$ $\le [U_\alpha, Q_\alpha] \le H_\beta$ and $[U_\alpha, Q_\alpha, Q_\beta, Q_\beta] \le V_\beta$. Thus $W_\beta/[W_\beta, Q_\beta]V_\beta$, $[W_\beta, Q_\beta]V_\beta/H_\beta$, $H_\beta/[H_\beta, Q_\beta]$ and $[H_\beta, Q_\beta]V_\beta/V_\beta$ are all $GF(2)(G_\beta/Q_\beta)$-modules. Now $H_\beta/[H_\beta, Q_\beta]V_\beta$ is generated by an involution centralized by $G_{\alpha\beta}$ and $[H_\beta, Q_\beta]V_\beta/V_\beta$ is either trivial or generated by an involution centralized by $G_{\alpha\beta}$. Since $\eta(G_\beta, W_\beta/V_\beta) \ge 2$, our assumption implies

$$\bigl[H_\beta/V_\beta : C_{H_\beta/V_\beta}(x)\bigr] \le 2^2$$

and so Proposition 2.15 applies to both these sections. Thus $H_\beta/[H_\beta, Q_\beta]V_\beta$ and $[H_\beta, Q_\beta]V_\beta/V_\beta$ are both quotients of $\binom{4}{1} \oplus 1$. Proceeding as in Lemma 5.17 gives

$$H_\beta = \bigl\langle \bigl[U_\gamma, Q_\gamma\bigr] | \gamma \in \Lambda(\beta, \tau)\bigr\rangle \bigl[U_\tau, Q_\tau\bigr]$$

The same arguments apply to W_β/H_β, and so (17.3.10) holds.

We shall make repeated use, often without reference, of the following facts.

(17.3.11) Let $(\delta, \delta') \in \mathcal{C}$.

(i) $[V_{\delta+1}, V_{\delta'}] \le V_{\delta+1} \cap V_{\delta+3} \cap V_{\delta+5} \cap V_{\delta'}$.

(ii) $[V_{\delta+3}, W_{\delta'}] \le V_{\delta+3} \cap V_{\delta+5} \cap V_{\delta'}$.

(iii) $V_{\delta+1} \cap V_{\delta+3} \ne V_{\delta+3} \cap V_{\delta+5}$. Further, if $V_{\delta'} \not\le Q_{\delta+1}$, then $V_{\delta+3} \cap$ $\cap V_{\delta+5} \ne V_{\delta+5} \cap V_{\delta'}$.

Part (i) is a consequence of $[V_{\delta+1}, V_{\delta+5}] = [V_{\delta'}, V_{\delta+3}] = 1$, and (ii) follows from (i). If $V_{\delta+1} \cap V_{\delta+3} = V_{\delta+3} \cap V_{\delta+5}$, then, using Proposition 2.5(ii),

$W_{\delta'} \leq G_{\delta+1}$. Hence $|[W_{\delta'}, V_{\delta+1}]| \leq 2^3$, a contradiction. Therefore $V_{\delta+1} \cap \cap V_{\delta+3} \neq V_{\delta+3} \cap V_{\delta+5}$ and (iii) holds.

Our next goal is (17.3.25), which asserts that $|W_{\alpha'}Q_{\alpha+3}/Q_{\alpha+3}| = 2$ for any $(\alpha, \alpha') \in \mathcal{C}$ for which $[Z_\beta, W_{\alpha'}] \neq 1$.

The results (17.3.12), (17.3.13) and (17.3.14) prepare the ground for the proof of (17 3.25). For the duration of these results (δ, δ') is assumed to be a critical pair for which $[Z_{\delta+1}, W_{\delta'}] \neq 1$ and $|W_{\delta'}Q_{\delta+3}/Q_{\delta+3}| \neq 2$. (Recall from (17.3.1) that $W_{\delta'} \not\leq Q_{\delta+3}$ and so $|W_{\delta'}Q_{\delta+3}/Q_{\delta+3}| > 2$.) Also recall that $V_{\delta'} \not\leq Q_{\delta+1}$.

(17.3.12) (i) $[W_{\delta'}, V_{\delta+3}] = V_{\delta+3} \cap V_{\delta+5} \cap V_{\delta'} \cong E(2^2)$.

(ii) $Z_{\delta+3} \not\leq V_{\delta'}$.

(iii) $[W_{\delta'} \cap Q_{\delta+3}, V_{\delta+3}] = 1$.

By assumption $|W_{\delta'}Q_{\delta+3}/Q_{\delta+3}| > 2$ and therefore $|[W_{\delta'}, V_{\delta+3}/Z_{\delta+3}]| \geq 2^2$. Thus parts (i) and (ii) follow from (17.3.11)(ii), (iii). Because

$$\left[W_{\delta'} \cap Q_{\delta+3}, V_{\delta+3}\right] \leq Z_{\delta+3} \cap V_{\delta'},$$

(ii) implies (iii).

(17.3.13) $V_{\delta'}Q_{\delta+1}/Q_{\delta+1}$ is a non-central transvection of $G_{\delta+1\delta+2}/Q_{\delta+1}$ (acting on $V_{\delta+1}/Z_{\delta+1}$) and $|[V_{\delta+1}, V_{\delta'}]| = 2$.

First we demonstrate that $|[V_{\delta+1}, V_{\delta'}]| = 2$. Put $V = V_{\delta+1} \cap V_{\delta+3} \cap \cap V_{\delta+5} \cap V_{\delta'}$. If $|[V_{\delta+1}, V_{\delta'}]| > 2$, then, by (17.3.11) (i), (iii), $[V_{\delta+1}, V_{\delta'}] = V$. Thus, by (17.13.12)(ii), $Z_{\delta+3}V = V_{\delta+1} \cap V_{\delta+3}$ and consequently $W_{\delta'}$ centralizes $[V_{\delta+3}/Z_{\delta+3}, G_{\delta+2\delta+3}; 2]$. This forces $W_{\delta'} \leq \leq G_{\delta+2\delta+3}$, and so $[W_{\delta'} : W_{\delta'} \cap G_{\delta+1}] \leq 2$. Since $[W_{\delta'}, Z_{\delta+1}] \neq 1$ and $W_{\delta'}$ acts quadratically on $V_{\delta+1}$, we must have $[W_{\delta'} \cap G_{\delta+1}, V_{\delta+1}] = = [V_{\delta+1}, V_{\delta'}] \leq V_{\delta'}$. But this gives the impossible $\eta(G_{\delta'}, W_{\delta'}/V_{\delta'}) \leq 1$. So $|[V_{\delta+1}, V_{\delta'}]| = 2$. Observe that $V_{\delta'}Q_{\delta+1}/Q_{\delta+1}$ being a central transvection of $G_{\delta+1\delta+2}/Q_{\delta+1}$ on $V_{\delta+1}/Z_{\delta+1}$ gives $[V_{\delta+1}, V_{\delta'}] \leq Z_{\delta+2}$. By (17.3.12)(ii) $[V_{\delta+1}, V_{\delta'}] \neq Z_{\delta+3}$ and hence $Z_{\delta+2} = [V_{\delta+1}, V_{\delta'}]Z_{\delta+3}$, which yields $[Z_{\delta+1}, W_{\delta'}] = 1$, whereas $[Z_{\delta+1}, W_{\delta'}] \neq 1$. This proves (17.3.13).

From $[V_{\delta+1}, V_{\delta'}] \neq Z_{\delta+3}$, $[V_{\delta+1}, V_{\delta'}] \leq V_{\delta+1} \cap V_{\delta+3} \cap V_{\delta+5}$ and $V_{\delta+1} \cap \cap V_{\delta+3} \neq V_{\delta+3} \cap V_{\delta+5}$, we have that $(V_{\delta+1} \cap V_{\delta+3})(V_{\delta+3} \cap V_{\delta+5}) \cong E(2^4)$. Let $E_{\delta+3}$ be such that $G_{\delta+3\delta+4} \geq E_{\delta+3} > Q_{\delta+3}$ and $E_{\delta+3}$ centralizes $(V_{\delta+1} \cap V_{\delta+3})(V_{\delta+3} \cap V_{\delta+5})/Z_{\delta+3}$; note that $E_{\delta+3}$ induces a transvection on $V_{\delta+3}/Z_{\delta+3}$.

(17.3.14) (i) $W_{\delta'} \cap E_{\delta+3} \leq G_{\delta+1}$ and $[W_{\delta'} \cap E_{\delta+3}, V_{\delta+1}] \leq V_{\delta+1} \cap V_{\delta+3}$.

(ii) $[W_{\delta'} : W_{\delta'} \cap E_{\delta+3}] \leq 2^2$.

From $[W_{\delta'} \cap E_{\delta+3}, V_{\delta+1} \cap V_{\delta+3}] \leq [W_{\delta'}, V_{\delta+3}] \cap Z_{\delta+3}$ and (17.3.12)(i), (ii) we deduce that $[W_{\delta'} \cap E_{\delta+3}, V_{\delta+1} \cap V_{\delta+3}] = 1$, which yields part (i). Because $W_{\delta'}Q_{\delta+3}/Q_{\delta+3}$ is contained in the quadratic $E(2^3)$-subgroup of $G_{\delta+3\delta+4}/Q_{\delta+3}$ acting on $V_{\delta+3}/Z_{\delta+3}$ clearly we have $[W_{\delta'} : W_{\delta'} \cap E_{\delta+3}] \leq 2^2$.

Now, until the end of (17.3.24), let (α, α') denote a critical pair for which $[Z_\beta, W_{\alpha'}] \neq 1$ and $|W_{\alpha'}Q_{\alpha+3}/Q_{\alpha+3}| > 2$ (so (17.3.12)-(17.3.14) apply to (α, α')). By (17.3.14), as $Z_\beta \not\leq W_{\alpha'}$ and $[V_\beta, V_{\alpha'}] \leq V_\beta \cap V_{\alpha+3}$, $[W_{\alpha'}/V_{\alpha'} : C_{W_{\alpha'}/V_{\alpha'}}(x)] \leq 2^3$ for any $x \in V_\beta$. Employing (17.3.10) we then obtain

(17.3.15) $\quad W_{\alpha'} = \big\langle V_\tau | \tau \in \Delta(\gamma) \text{ for } \gamma \in \Lambda(\alpha', \alpha+6)\big\rangle U_{\alpha+6}.$

Our next assertion concerns critical pairs $(\tau+1, \alpha+3)$ where $\tau + 1 \in \Delta^{[2]}(\gamma)$ and $\gamma \in \Lambda(\alpha', \alpha+6)$.

(17.3.16) Let $\gamma \in \Lambda(\alpha', \alpha+6)$ and suppose there exists $\tau \in \Delta(\gamma)$ such that $V_\tau \not\leq Q_{\alpha+3}$ (so $(\tau+1, \alpha+3) \in \mathcal{C}$ for some $\tau+1 \in \Delta(\tau)$). Then

(i) $V_{\alpha+3} \not\leq Q_\tau$; and
(ii) $[W_\tau, Z_{\alpha+3}] \neq 1$.

If $V_{\alpha+3} \leq Q_\tau$, then

$$Z_\tau = \big[V_{\alpha+3}, V_\tau\big] \leq V_{\alpha+5} \cap V_{\alpha'},$$

which gives

$$Z_\gamma = Z_\tau Z_{\alpha'} \leq V_{\alpha+5} \cap V_{\alpha'} \leq \big[V_{\alpha'}, G_{\alpha+6\alpha'}\big].$$

This is against $\gamma \in \Lambda(\alpha', \alpha+6)$, and so $V_{\alpha+3} \not\leq Q_\tau$.

Turning to part (ii), assume that $[W_\tau, Z_{\alpha+3}] = 1$. By part (i) there exists $\xi \in \Delta(\alpha+3)$ for which $(\xi, \tau) \in \mathcal{C}$. Then applying Lemma 17.4(v) to (ξ, τ) yields

$$\big[V_{\alpha+3}, G_{\alpha+3\alpha+4}\big] = \big[W_\tau \cap G_{\alpha+3}, V_{\alpha+3}\big]\big(V_{\alpha+3} \cap V_{\alpha+5}\big).$$

From $[W_{\alpha'}, V_{\alpha+3}] \leq V_{\alpha'}$ we clearly get

$$\big[W_{\alpha'}, V_{\alpha+3}, W_\tau \cap G_{\alpha+3}\big] = 1.$$

Furthermore, $[W_{\alpha'}, W_\tau \cap G_{\alpha+3}] \leq V_{\alpha'}$ and so

$$\big[W_{\alpha'}, W_\tau \cap G_{\alpha+3}, V_{\alpha+3}\big] = 1.$$

Appealing to the Three Subgroup Lemma gives

$$\big[V_{\alpha+3}, W_\tau \cap G_{\alpha+3}, W_{\alpha'}\big] = 1$$

and therefore $W_{\alpha'}$ centralizes $[V_{\alpha+3}, G_{\alpha+3\alpha+4}]$, contrary to $|W_{\alpha'}Q_{\alpha+3}/Q_{\alpha+3}| > 2$. Hence (17.3.16) holds.

Up until the beginning of (17.3.22) we assume that $\gamma \in \Lambda(\alpha', \alpha+6)$ and that there exists $\tau \in \Delta(\gamma)$ such that $V_\tau \not\leq Q_{\alpha+3}$ and $V_\tau Q_{\alpha+3}/Q_{\alpha+3}$ is NOT a non-central transvection on $V_{\alpha+3}/Z_{\alpha+3}$. Let $\tau+1 \in \Delta(\tau)$ be such that $(\tau+1, \alpha+3) \in \mathcal{C}$. By (17.3.16)(i) we may find $\xi \in \Delta(\alpha+3)$ such that $(\xi, \tau) \in \mathcal{C}$. The choice of τ together with (17.3.16)(ii) and (17.3.13) (applied to (ξ, τ)) imply that

(17.3.17) $|W_\tau Q_{\alpha+5}/Q_{\alpha+5}| = 2$.

(17.3.18) (i) $V_\tau Q_{\alpha+3}/Q_{\alpha+3}$ is the central transvection of $G_{\alpha+3\alpha+4}/Q_{\alpha+3}$ (on $V_{\alpha+3}/Z_{\alpha+3}$).

(ii) $V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'} \cap V_\tau = [V_{\alpha+3}, V_\tau] = Z_{\alpha+5}$ and $Z_{\alpha+6} = V_{\alpha+5} \cap \cap V_{\alpha'} \cap V_\tau$.

Set $V = V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'} \cap V_\tau$, and assume that $|V| > 2$. From (17.3.17) and (17.3.16) we infer that $[W_\tau : W_\tau \cap G_{\alpha+3}] \leq 2^2$ and $[W_\tau \cap G_{\alpha+3}, V_{\alpha+3}] = V \leq V_\tau$. Recalling that $V_{\alpha+3} \not\leq Q_\tau$, we conclude that the two G_τ-non-central chief factors of W_τ/V_τ are isomorphic natural modules. Transferring our attention to (α, α') where, by (17.3.13), we have that V_β acts as a transvection on $V_{\alpha'}/Z_{\alpha'}$, and consequently $[W_{\alpha'} : C_{W_{\alpha'}}(x)] \geq 2^5$ for every $x \in V_\beta \setminus Q_{\alpha'}$. However (17.3.14) gives $[W_{\alpha'} : C_{W_{\alpha'}}(x)] \leq 2^4$ for each $x \in V_\beta$. Hence $|V| = 2$, and so $V = [V_{\alpha+3}, V_\tau]$. The choice of τ now yields part (i).

By (i) $[V_{\alpha+3}, V_\tau] \leq Z_{\alpha+4}$. If $[V_{\alpha+3}, V_\tau] \neq Z_{\alpha+5}$, then we have

$$Z_{\alpha+3} \leq Z_{\alpha+4} = \big[V_{\alpha+3}, V_\tau\big] Z_{\alpha+5} \leq V_{\alpha'},$$

which is untenable by (17.3.12). Hence $[V_{\alpha+3}, V_\tau] = Z_{\alpha+5}$. So $Z_{\alpha+6} = Z_{\alpha+5}Z_{\alpha'} \leq V_{\alpha+5} \cap V_{\alpha'} \cap V_\tau$, and we have proved (ii).

(17.3.19) $\big[W_{\alpha+3}, Z_\tau\big] \neq 1$.

Suppose $[W_{\alpha+3}, Z_\tau] = 1$ holds. Then, using Lemma 17.4(v), we see that $V_{\alpha+3}Q_\tau/Q_\tau$ acts as a central transvection of $G_{\gamma\tau}/Q_\tau$ on V_τ/Z_τ. Hence, by (17.3.18)(ii), $Z_{\alpha+5} = [V_{\alpha+3}, V_\tau] \leq Z_\gamma$. But then $Z_\gamma = Z_{\alpha+6} \leq [V_{\alpha'}, G_{\alpha+6\alpha'}]$, against $\gamma \in \Lambda(\alpha', \alpha+6)$. This verifies (17.3.19).

(17.3.20) $\big|W_{\alpha+3}Q_{\alpha'}/Q_{\alpha'}\big| > 2$.

If (17.3.20) is false, then $|W_{\alpha+3}Q_{\alpha'}/Q_{\alpha'}| = 2$. Because $[W_{\alpha+3}, Z_\tau] \neq 1$

by (17.3.19), $|[W_{\alpha+3} \cap G_\tau, V_\tau]| \leq 2^2$ and hence we deduce that

$$Z_{\alpha'} = \left[W_{\alpha+3} \cap Q_{\alpha'}, V_{\alpha'}\right] \leq V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'}.$$

But then consulting (17.3.18)(ii) this gives

$$Z_{\alpha'} \leq V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'} \cap V_\tau = Z_{\alpha+5},$$

which is impossible. Therefore (17.3.20) holds.

(17.3.21) $\left[W_{\alpha'} \cap Q_{\alpha+3}, W_{\alpha+3}\right] \leq V_{\alpha+3} \cap V_{\alpha+5}$.

Let $\xi \in \Delta^{[2]}(\alpha+3)$. Since $[W_{\alpha'} \cap Q_{\alpha+3}, V_{\alpha+3}] = 1$ by (17.3.12)(iii), $[W_{\alpha'} \cap Q_{\alpha+3}, V_\xi] \leq V_\xi \cap V_{\alpha+3}$. Because $|W_{\alpha'}Q_{\alpha+3}/Q_{\alpha+3}| > 2$ it follows that $V_{\alpha+3} \cap V_{\alpha+5} = C_{V_{\alpha+3}}(W_{\alpha'})$ and so $[W_{\alpha'} \cap Q_{\alpha+3}, V_\xi] \leq V_{\alpha+3} \cap V_{\alpha+5}$. This proves (17.3.21).

Combining (17.3.20) with (17.3.21) and the fact that $|V_{\alpha+3} \cap V_{\alpha+5}| = 2^3$ we deduce that $\eta(G_{\alpha'}, W_{\alpha'}) = 3$ and $|W_{\alpha'}Q_{\alpha+3}/Q_{\alpha+3}| = 2^3$. Moreover, the $G_{\alpha'}$-non central chief factors in $W_{\alpha'}$ are all natural modules. But $W_{\alpha'}$ acting quadratically on $W_{\alpha+3}$ and $|W_{\alpha'}Q_{\alpha+3}/Q_{\alpha+3}| = 2^3$ implies that the $G_{\alpha+3}$-non central chief factors in $W_{\alpha+3}$ are all isomorphic, a contradiction. This contradiction, together with (17.3.15), yields

(17.3.22) $W_{\alpha'}Q_{\alpha+3}/Q_{\alpha+3}$ is generated by the two non-central transvections of $G_{\alpha+3\alpha+4}/Q_{\alpha+3}$ (on $V_{\alpha+3}/Z_{\alpha+3}$).

Because of our assumption that the theorem is false, (17.3.14) and (17.3.22) imply that $E_{\alpha+3}/Q_{\alpha+3}$ must be the central transvection of $G_{\alpha+3\alpha+4}/Q_{\alpha+3}$ (on $V_{\alpha+3}/Z_{\alpha+3}$) and so $(V_\beta \cap V_{\alpha+3})(V_{\alpha+3} \cap V_{\alpha+5}) = [V_{\alpha+3}, G_{\alpha+3\alpha+4}]$. Hence, by Proposition 2.8(vi), $V_\beta \cap V_{\alpha+3} \cap V_{\alpha+5} = Z_{\alpha+4}$ and so $[V_\beta, V_{\alpha'}] \leq Z_{\alpha+4}$. If $[V_\beta, V_{\alpha'}] \neq Z_{\alpha+5}$, then $Z_{\alpha+4} = [V_\beta, V_{\alpha'}]Z_{\alpha+5} \leq V_{\alpha'}$, against (17.3.12)(ii). Thus

(17.3.22) $\left[V_\beta, V_{\alpha'}\right] = Z_{\alpha+5}$.

We now seek to make use of (17.3.15) again. So, for the next statement, let $\gamma \in \Lambda(\alpha', \alpha+6)$ be such that $V_\tau \not\leq Q_{\alpha+3}$ for some $\tau \in \Delta(\gamma)$. Recall from (17.3.16) that $[W_\tau, Z_{\alpha+3}] \neq 1$ and that there exists $\xi \in \Delta(\alpha+3)$ such that $(\xi, \tau) \in \mathcal{C}$.

(17.3.24) (i) If $|W_\tau Q_{\alpha+5}/Q_{\alpha+5}| > 2$, then $[V_{\alpha+3}, V_\tau] = Z_{\alpha'}$.
(ii) If $|W_\tau Q_{\alpha+5}/Q_{\alpha+5}| = 2$, then $[V_{\alpha+3}, V_\tau] \leq Z_{\alpha+6} = V_{\alpha+5} \cap V_{\alpha'} \cap V_\tau$.

If $|W_\tau Q_{\alpha+5}/Q_{\alpha+5}| > 2$, then (ξ, τ) satisfies the same conditions as (α, α') and hence the analogue of (17.3.22) holds. Thus $[V_{\alpha+3}, V_\tau] = Z_{\alpha'}$, and we have (i). While $|W_\tau Q_{\alpha+5}/Q_{\alpha+5}| = 2$ implies, as $[W_\tau, Z_{\alpha+3}] \neq 1$, that $|[W_\tau \cap G_{\alpha+3}, V_{\alpha+3}]| \leq 2^2$. Consequently $[W_\tau \cap Q_{\alpha+5}, V_{\alpha+5}] \neq 1$ and so

$$Z_{\alpha+5} = \left[W_\tau \cap Q_{\alpha+5}, V_{\alpha+5}\right] \leq \left[W_\tau, V_{\alpha+5}\right] \leq V_{\alpha+5} \cap V_{\alpha'} \cap V_\tau .$$

Hence, using (17.3.11)(iii), we obtain $Z_{\alpha+6} = V_{\alpha+5} \cap V_{\alpha'} \cap V_\tau$, so giving (ii).

(17.3.25) Let $(\alpha, \alpha') \in \mathcal{C}$ be such that $[Z_\beta, W_{\alpha'}] \neq 1$. Then $|W_{\alpha'} Q_{\alpha+3}/Q_{\alpha+3}| = 2$.

We assume (17.3.25) is false and argue for a contradiction; so (17.3.15)-(17.3.24) are available to us. In particular, (17.3.15) and (17.3.24) yield that $[V_{\alpha+3}, W_{\alpha'}] \leq Z_{\alpha+6}$, and thus

$$Z_{\alpha+6} = \left[V_{\alpha+3}, W_{\alpha'}\right] = V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'} .$$

Suppose there exists $\gamma \in \Lambda(\alpha', \alpha+6)$ such that $V_\tau \not\leq Q_{\alpha+3}$ for some $\tau \in \Delta(\gamma)$ and $|W_\tau Q_{\alpha+5}/Q_{\alpha+5}| = 2$. Then, by (17.3.24)(ii), $Z_{\alpha+6} = V_{\alpha+5} \cap$ $\cap V_{\alpha'} \cap V_\tau$ and hence

$$V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'} \cap V_\tau = Z_{\alpha+6} .$$

Since $|W_\tau Q_{\alpha+5}/Q_{\alpha+5}| = 2$ and $[W_\tau, Z_{\alpha+3}] \neq 1$, we may argue as in (17.3.18) to obtain a contradiction. Therefore for all $\gamma \in \Lambda(\alpha', \alpha+6)$ with $V_\tau \not\leq Q_{\alpha+3}$ (some $\tau \in \Delta(\gamma)$), $|W_\tau Q_{\alpha+5}/Q_{\alpha+5}| > 2$. But then (17.3.15) and (17.3.24)(i) give $[V_{\alpha+3}, W_{\alpha'}] = Z_{\alpha'}$, contrary to $|W_{\alpha'} Q_{\alpha+3}/Q_{\alpha+3}| > 2$. This proves (17.3.25).

We continue to let $(\alpha, \alpha') \in \mathcal{C}$ be such that $[Z_\beta, W_{\alpha'}] \neq 1$.

(17.3.26) (i) $[W_{\alpha'} : W_{\alpha'} \cap G_\beta] = 2^2$ and $[W_{\alpha'} \cap G_\beta, V_\beta] = C_{V_\beta \cap V_{\alpha+3}}(W_{\alpha'}) \cong E(2^2)$.
(ii) $Z_{\alpha+4} = V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'}$.

Since $W_{\alpha'} \cap G_\beta$ acts quadratically on V_β and $[Z_\beta, W_{\alpha'}] \neq 1$, $|[W_{\alpha'} \cap G_\beta, V_\beta]| \leq 2^2$. Then (17.3.25) forces $|[W_{\alpha'} \cap G_\beta, V_\beta]| = 2^2$ and $[W_{\alpha'} : W_{\alpha'} \cap G_\beta] = 2^2$. So $W_{\alpha'} \cap Q_{\alpha+3} \not\leq Q_{\alpha+2}$ and then $[W_{\alpha'} \cap G_\beta, V_\beta] \leq$ $\leq V_\beta \cap V_{\alpha+3}$ by the core argument, giving part (i). From $W_{\alpha'} \cap Q_{\alpha+3} \not\leq Q_{\alpha+2}$,

$$Z_{\alpha+3} = \left[V_{\alpha+3}, W_{\alpha'} \cap Q_{\alpha+3}\right] \leq \left[V_{\alpha+3}, W_{\alpha'}\right] \leq V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'}$$

and hence part (ii) follows using (17.3.11)(iii).

(17.3.27) $\left[V_\beta, V_{\alpha'}\right] = Z_{\alpha+3} \leq Z_{\alpha+4} = Z_{\alpha+6}$.

We begin by proving that $Z_{\alpha'} \leq Z_{\alpha+4}$. If $Z_{\alpha'} \not\leq Z_{\alpha+4}$, then, by (17.3.26)(ii),

$$V_{\alpha+5} \cap V_{\alpha'} = Z_{\alpha+4} Z_{\alpha'} .$$

We have that $[W_\beta, V_{\alpha+5} \cap V_{\alpha'}] \neq 1$, for otherwise we get $W_\beta \leq G_{\alpha'}$ with $|[W_\beta, V_{\alpha'}]| \leq 2^3$ which, as $V_{\alpha'} \not\leq Q_\beta$, then yields the theorem. So $[Z_{\alpha'}, W_\beta] \neq 1$. Consequently (17.3.26) also applies to (ρ, β) yielding

$$V_\beta \cap V_{\alpha+3} \cap V_{\alpha+5} = Z_{\alpha+4} = V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'}.$$

So, by (17.3.26)(i), $[W_{\alpha'} \cap G_\beta, V_\beta] = Z_{\alpha+4} \leq V_{\alpha'}$. Putting $\bar{W}_{\alpha'} = W_{\alpha'}/V_{\alpha'}$, (17.3.26)(i) implies that $[\bar{W}_{\alpha'} : C_{\bar{W}_{\alpha'}}(V_\beta)] \leq 2^2$ and so V_β acts as a transvection on each of the two non-central $G_{\alpha'}$-chief factors in $\bar{W}_{\alpha'}$. Hence $[V_\beta : V_\beta \cap Q_{\alpha'}] = 2$. Because $[Z_{\alpha'}, W_\beta] \neq 1$, $[V_{\alpha'}, V_\beta \cap Q_{\alpha'}] = 1$ whence $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] = 2$. Now $[Z_\beta, W_{\alpha'}] \neq 1$ gives $[V_\beta, V_{\alpha'} \cap Q_\beta] = 1$. But then $|[V_\beta, V_{\alpha'}]| = 2$ and the theorem holds. Thus we have proven that $Z_{\alpha'} \leq Z_{\alpha+4}$ and hence $Z_{\alpha+4} = Z_{\alpha+6}$.

To conclude the proof of (17.3.27) we must show that $[V_\beta, V_{\alpha'}] = Z_{\alpha+3}$. Suppose $[V_\beta, V_{\alpha'}] \neq Z_{\alpha+3}$. Then (17.3.26)(i) forces

$$\big[W_{\alpha'} \cap G_\beta, V_\beta\big] = \big[V_\beta, V_{\alpha'}\big] Z_{\alpha+3} \leq \big[V_\beta, V_{\alpha'}\big] Z_{\alpha+4} = \big[V_\beta, V_{\alpha'}\big] Z_{\alpha+6} \leq V_{\alpha'}.$$

Observe that V_β acts as a transvection upon $V_{\alpha'}/Z_{\alpha'}$, since $[V_\beta, V_{\alpha'}] \leq$ $\leq V_{\alpha+3} \cap V_{\alpha+5} \cap V_{\alpha'} = Z_{\alpha+4} = Z_{\alpha+6}$. We again deduce that the theorem holds. Thus $[V_\beta, V_{\alpha'}] = Z_{\alpha+3}$.

(17.3.28) $\quad W_\beta = \big\langle U_\gamma | \gamma \in \Delta(\beta) \text{ and } (\gamma, \alpha') \in \mathcal{C} \big\rangle U_{\alpha+2}$.

From (17.3.27) $[V_{\alpha'}, [V_\beta, G_{\beta\alpha+2}]] = 1$ and so $[V_\beta, G_{\beta\alpha+2}] = V_\beta \cap Q_{\alpha'}$. Now (17.3.26)(i) and (17.3.10) imply (17.3.28).

(17.3.29) $\quad W_\beta \leq G_{\alpha+5\alpha+6}$.

Suppose (17.3.29) is false. Then by (17.3.28) there exists $\gamma \in \Delta(\beta)$ such that $(\gamma, \alpha') \in \mathcal{C}$ and $U_\gamma \not\leq G_{\alpha+5\alpha+6}$. So we may find $\gamma - 1 \in \Delta(\gamma)$ such that $V_{\gamma-1} \not\leq G_{\alpha+5\alpha+6}$; thus $(\gamma - 2, \alpha + 5) \in \mathcal{C}$ for some $\gamma - 2 \in \Delta(\gamma - 1)$. Note that $V_{\alpha+5} \not\leq Q_{\gamma-1}$, since $Z_{\gamma-1} \not\leq V_{\alpha+5}$ (as $(\gamma, \alpha') \in \mathcal{C}$). Hence there exists $\xi \in \Delta(\alpha + 5)$ for which $(\xi, \gamma - 1) \in \mathcal{C}$. Assume that $[Z_{\alpha+5}, W_{\gamma-1}] = 1$. Then Lemma 17.4(v) applies to give

$$\big[V_{\alpha+5}, G_{\alpha+4\alpha+5}\big] = \big[V_{\alpha+5}, W_{\gamma-1} \cap G_{\alpha+3\alpha+4}\big]\big(V_{\alpha+3} \cap V_{\alpha+5}\big),$$

whence $V_{\gamma-1}Q_{\alpha+5}/Q_{\alpha+5}$ is a central transvection of $G_{\alpha+4\alpha+5}/Q_{\alpha+5}$ on $V_{\alpha+5}/Z_{\alpha+5}$. On the other hand, if $[Z_{\alpha+5}, W_{\gamma-1}] \neq 1$ (17.2.27) (applied to $(\xi, \gamma - 1)$) also yields that $V_{\gamma-1}Q_{\alpha+5}/Q_{\alpha+5}$ acts as a central transvection of $G_{\alpha+4\alpha+5}/Q_{\alpha+5}$ on $V_{\alpha+5}/Z_{\alpha+5}$. However, (17.3.27) implies that $\langle G_{\alpha+4\alpha+5}, G_{\alpha+5\alpha+6} \rangle \neq G_{\alpha+5}$ and consequently $V_{\gamma-1} \leq G_{\alpha+5\alpha+6}$, after all. From this contradiction we deduce that (17.3.29) holds.

The final contradiction is now before us for, by (17.3.27), we have $[Z_{\alpha'}, W_\beta] = 1$ and so Lemma 17.1(i) (applied to (ρ, β)) predicts that $W_\beta \not\leq G_{\alpha+5\alpha+6}$. This is contrary to (17.3.29), and so, at long last, the proof of Theorem 17.3 is complete.

18. Case 4 - bounding b

Given the title of this section it is no surprise that our main result here is

THEOREM 18.1. *Assume Hypothesis* 17.0 *holds. Then* $b \in \{3, 5\}$.

The proof of this result will be given in a series of lemmas. Since our proof is again by contradiction we shall suppose for this section that Hypothesis 17.0 holds with $b > 5$.

LEMMA 18.2. *Let* $(\alpha, \alpha') \in \mathcal{C}$. *Then*

(i) $[[U_\alpha, Q_\alpha], Q_\alpha \cap Q_\beta] \leq Z_\alpha$;

(ii) $(Q_\alpha \cap Q_\beta)Q_{\alpha-1}/Q_{\alpha-1}$ *is contained in the non-quadratic* $E(2^3)$*-subgroup of* $G_{\alpha-1}/Q_{\alpha-1}$*, but is not contained in the quadratic* $E(2^3)$*-subgroup of* $G_{\alpha-1\alpha}/Q_{\alpha-1}$ *(on* $V_{\alpha-1}/Z_{\alpha-1}$*); and*

(iii) $V_\beta Q_{\alpha'}/Q_{\alpha'}$ *acts as the central transvection (of* $G_{\alpha'-1\alpha'}/Q_{\alpha'}$*) on each of the non-central chief factors within* $W_{\alpha'}$.

(iv) $\eta(G_\beta, H_\beta) = \eta(G_\beta, [W_\beta, Q_\beta]V_\beta) = 2$.

PROOF. Since $b > 5$, $\eta(G_\beta, W_\beta) = 3$ with the non-central chief factors in W_β being isomorphic natural modules by Theorem 17.3. Let ${G_{\alpha\beta}}^{(1)} \leq G_\beta$ with $G_{\alpha\beta} \leq {G_{\alpha\beta}}^{(1)}$ be the parabolic subgroup for which $(V_\beta/Z_\beta)|_{{G_{\alpha\beta}}^{(1)}} \cong \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and set $C_\beta = C_{Q_\beta}(V_\beta)$. Observe that $Q_\beta/C_\beta \cong (V_\beta/Z_\beta)^* \cong V_\beta/Z_\beta \cong 4$, $[Q_\beta : Q_\alpha \cap Q_\beta] = 2$, $Q_\alpha \cap Q_\beta \trianglelefteq G_{\alpha\beta}$ and $Q_\alpha \cap Q_\beta \geq C_\beta$. Hence $Q_\alpha \cap Q_\beta \trianglelefteq G^{(1)}_{\alpha\beta}$. Recall that $[Q_\alpha, [U_\alpha, Q_\alpha]] = V_\beta \cap V_{\alpha-1}$. Hence there exists $E \trianglelefteq G_\beta$ with $V_\beta \leq E \leq H_\beta V_\beta$ such that $H_\beta V_\beta/E$ is a non-central chief factor of G_β. Observe that $[U_\alpha, Q_\alpha]E/E \leq C_{H_\beta V_\beta/E}(G_{\alpha\beta})$. Consequently $[[U_\alpha, Q_\alpha], O^2({G_{\alpha\beta}}^{(1)})] \leq V_\beta$ and thence

$$\left[\left[U_\alpha, Q_\alpha\right], O^2\left({G_{\alpha\beta}}^{(1)}\right)\right] \leq \left[V_\beta, Q_\alpha\right] \leq \left[U_\alpha, Q_\alpha\right].$$

Therefore

$$[U_\alpha, Q_\alpha] \trianglelefteq \langle G_{\alpha\beta}, O^2(G_{\alpha\beta}{}^{(1)})\rangle = G_{\alpha\beta}{}^{(1)}.$$

Then we obtain

$$[[U_\alpha, Q_\alpha], Q_\alpha \cap Q_\beta] \trianglelefteq G_{\alpha\beta}{}^{(1)}.$$

Since $[[U_\alpha, Q_\alpha], Q_\alpha \cap Q_\beta] \le V_{\alpha-1} \cap V_\beta \ntrianglelefteq G_{\alpha\beta}{}^{(1)}$, we conclude that

$$[[U_\alpha, Q_\alpha], Q_\alpha \cap Q_\beta] \le Z_\alpha,$$

and we have proved (i).

From (i) we have

$$[[V_{\alpha-1}, Q_\alpha], Q_\alpha \cap Q_\beta] \le Z_\alpha$$

which together with $Z_\alpha/Z_{\alpha-1} = C_{V_{\alpha-1}/Z_{\alpha-1}}(G_{\alpha\alpha-1})$ yields the first part of (ii). The latter part of (ii) follows by using the same argument as in [Proof of Theorem 1; LPR2].

Suppose (iii) is false. Then, since $V_\beta \le Q_{\alpha'-2} \cap Q_{\alpha'-1}$, part (ii) and Proposition 2.5(ii) yield $[\bar{F} : C_{\bar{F}}(V_\beta)] \ge 2^2$ for each non-central chief factor $\bar{F}$ of $G_{\alpha'}$ within $W_{\alpha'}$. Thus $[W_{\alpha'} : C_{W_{\alpha'}}(x)] \ge 2^6$ for any $x \in V_\beta \setminus Q_{\alpha'}$. Now $W_{\alpha'} \le Q_{\alpha+4} \cap Q_{\alpha+5}$, part (ii) and $W_{\alpha'}$ acting quadratically on $V_{\alpha+3}/Z_{\alpha+3}$ imply that $[W_{\alpha'} : W_{\alpha'} \cap Q_{\alpha+3}] \le 2^2$. Likewise we deduce that $[W_{\alpha'} \cap Q_{\alpha+3} \cap \cap Q_{\alpha+2} : W_{\alpha'} \cap Q_{\alpha+3} \cap Q_{\alpha+2} \cap Q_\beta] \le 2^2$ and so $[W_{\alpha'} : W_{\alpha'} \cap Q_\beta] \le 2^5$. Clearly $[W_{\alpha'} \cap Q_\beta, V_\beta] \ne 1$ and hence $Z_\beta \le W_{\alpha'}$. Using the parabolic argument (Lemma 3.10) gives $[W_{\alpha'} : W_{\alpha'} \cap Q_\beta] \le 2^5$ and so $[W_{\alpha'} : C_{W_{\alpha'}}(Z_\alpha)] \le 2^5$, a contradiction. Thus we have established (iii).

Finally, because $\eta(G_\beta, W_\beta) = 3$ by Theorem 17.3 and $\eta(G_\beta, W_\beta/[W_\beta, Q_\beta]V_\beta) \ge 1$, Lemma 17.7(ii), (iii) yields part (iv).

For $(\alpha, \alpha') \in \mathcal{C}$ it is often very helpful to know things like $V_{\alpha'} \not\le Q_\beta$ and $U_\alpha \le Q_{\alpha'-2}$. The former statement is established in Lemma 18.7. Then Lemma 18.7 enables us in Lemma 18.8 to prove that $U_\alpha \le Q_{\alpha'-2}$. However, in order to prove Lemma 18.7 we need the weaker form of Lemma 18.8 given in Lemma 18.4.

Lemma 18.3. *Suppose that* $(\alpha, \alpha') \in \mathcal{C}$ *and* $U_\alpha \not\le Q_{\alpha'-2}$. *Then* $[U_\alpha, V_{\alpha'-2}] = Z_\beta$ *and* $Z_\beta Z_{\alpha'-2} = Z_{\alpha'-3}$.

Proof. Using Lemma 18.2(iii) we see that $[U_\alpha, V_{\alpha'-2}] \le Z_\alpha \cap Z_{\alpha'-3}$. Since $U_\alpha \not\le Q_{\alpha'-2} [U_\alpha, V_{\alpha'-2}] \not\le Z_{\alpha'-2}$. In particular $[U_\alpha, V_{\alpha'-2}] \ne 1$ and so, as $[U_\alpha, V_{\alpha'-2}] \le Q_{\alpha'}$, $[U_\alpha, V_{\alpha'-2}] = Z_\beta$ and then we also have $Z_\beta Z_{\alpha'-2} = Z_{\alpha'-3}$.

LEMMA 18.4. *Suppose that* $(\alpha, \alpha') \in \mathcal{C}$ *and* $V_{\alpha'} \leq Q_\beta$. *Then* $U_\alpha \leq G_{\alpha'-2\alpha'-1}$.

PROOF. If $U_\alpha \leq Q_{\alpha'-2}$, then clearly the lemma holds so we may suppose that $U_\alpha \not\leq Q_{\alpha'-2}$ and argue for a contradiction. From $V_{\alpha'} \leq Q_\beta$ and Lemma 18.2(iii) $Z_\beta = [V_\beta, V_{\alpha'}] \leq Z_{\alpha'-1}$. By Lemma 18.3 $Z_\beta \neq Z_{\alpha'-2}$ and so $Z_{\alpha'-1} = Z_\beta Z_{\alpha'-2}$. Thus

(18.4.1) (i) $Z_{\alpha'-1} = Z_{\alpha'-3}$; and

(ii) $\langle G_{\alpha'-3\alpha'-2}, G_{\alpha'-2\alpha'-1}\rangle$ is a proper parabolic of $G_{\alpha'-2}$ which restricts to $\begin{pmatrix}1\\2\\1\end{pmatrix}$ on $V_{\alpha'-2}/Z_{\alpha'-2}$.

Now, by Lemma 18.2(iii), $U_\alpha Q_{\alpha'-2}/Q_{\alpha'-2}$ is a central transvection of $G_{\alpha'-3\alpha'-2}/Q_{\alpha'-2}$ on $V_{\alpha'-2}/Z_{\alpha'-2}$ and so $U_\alpha \leq G_{\alpha'-2\alpha'-1}$ by (18.4.1)(ii), a contradiction.

Before we can begin the proof of Lemma 18.9 we need two further results concerning F_α which will also be important later on.

LEMMA 18.5. *For* $(\alpha, \alpha') \in \mathcal{C}$ *we have* $F_\alpha \leq G_{\alpha'}$.

PROOF. Suppose $[F_\alpha, V_{\alpha'-2}] = 1$. Then there exists $\alpha - 1 \in \Delta(\alpha) \setminus \{\beta\}$ and $\gamma \in \Delta(\alpha' - 2)$ such that $[[V_{\alpha-1}, Q_\alpha], Z_\gamma] \neq 1$. In particular, $[V_{\alpha-1}, Z_\gamma] \neq 1$. So, since $Z_{\alpha-1} \not\leq V_{\alpha'-2}$, $(\gamma, \alpha - 1) \in \mathcal{C}$. Hence $[[V_{\alpha-1}, Q_\alpha], Z_\gamma] \leq Z_{\alpha-1}$ by Lemma 18.2 (iii). But then $Z_{\alpha-1} = [[V_{\alpha-1}, Q_\alpha], Z_\gamma] \leq V_{\alpha'-2}$, a contradiction. Thus $[F_\alpha, V_{\alpha'-2}] = 1$, which implies that $F_\alpha \leq G_{\alpha'}$.

LEMMA 18.6. *Let* $(\alpha, \alpha') \in \mathcal{C}$. *Then* $F_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$.

PROOF. Observe that, by Lemma 18.2(iii), $V_\beta Q_{\alpha'} = Z_\alpha Q_{\alpha'}$. So $Z_\alpha \leq F_\alpha$ implies that $V_\beta Q_{\alpha'} \leq F_\alpha Q_{\alpha'}$. Thus if the lemma is false we have $|F_\alpha Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$. From Lemma 18.2(iii) $[W_{\alpha'} : W_{\alpha'} \cap Q_{\alpha+3}] \leq 2$. Assume for the moment that $[W_{\alpha'}, Z_\beta] = 1$. Then $W_{\alpha'} \cap Q_{\alpha+3} \leq Q_{\alpha+2}$. Hence $(W_{\alpha'} \cap Q_{\alpha+3})Q_\beta/Q_\beta$ is contained in the non-quadratic $E(2^3)$ group on V_β/Z_β. Combining this with $W_{\alpha'} \cap Q_{\alpha+3}$ acting quadratically on V_β gives $|(W_{\alpha'} \cap Q_{\alpha+3})Q_\beta/Q_\beta| \leq 2^2$. Hence we obtain $[W_{\alpha'} : W_{\alpha'} \cap Q_\alpha \cap Q_\beta] \leq 2^4$. Now consider the case $[W_{\alpha'}, Z_\beta] \neq 1$. Then $[W_{\alpha'} : W_{\alpha'} \cap Q_{\alpha+3} \cap Q_{\alpha+4}] \leq 2^2$ and, as above, $|(W_{\alpha'} \cap Q_{\alpha+3} \cap Q_{\alpha+2})Q_\beta/Q_\beta| \leq 2^2$. So $[W_{\alpha'} : W_{\alpha'} \cap Q_\beta] \leq 2^4$. Since $Z_\beta \not\leq W_{\alpha'}$ we must have $[W_{\alpha'} \cap Q_\beta, V_\beta] = 1$ and hence $W_{\alpha'} \cap Q_\beta \leq Q_\alpha$. Thus, in both cases, $[W_{\alpha'} : W_{\alpha'} \cap Q_\alpha \cap Q_\beta] \leq 2^4$.

Using Lemma 18.2(i) gives

$$\left[F_\alpha, W_{\alpha'} \cap Q_\alpha \cap Q_\beta\right] \leq \left[F_\alpha, Q_\alpha \cap Q_\beta\right] \leq Z_\alpha.$$

Since $F_\alpha \leq G_{\alpha'}$ (by Lemma 18.5) and $(\alpha, \alpha') \in \mathcal{C}$ we then infer that $|[F_\alpha, W_{\alpha'} \cap Q_\alpha \cap Q_\beta]| \leq 2$. Therefore $[W_{\alpha'} : C_{W_{\alpha'}}(x)] \leq 2^5$ for all $x \in F_\alpha$. But $|F_\alpha Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$ then dictates that $\eta(G_{\alpha'}, W_{\alpha'}) \leq 2$, a contradiction. Hence $F_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$ holds.

At last we are able to achieve the following "symmetry" result.

Lemma 18.7. *Let $(\alpha, \alpha') \in \mathcal{C}$. Then $V_{\alpha'} \not\leq Q_\beta$.*

Proof. Let $(\alpha, \alpha') \in \mathcal{C}$ and assume that $V_{\alpha'} \leq Q_\beta$. By Lemma 18.4 $U_\alpha \leq G_{\alpha'-2\alpha'-1}$. Combining Lemmas 18.2(iii) and 18.6 gives $[F_\alpha, V_{\alpha'}] \leq Z_{\alpha'-1}$. Because $V_{\alpha'} \leq G_\alpha, V_{\alpha'} \not\leq Q_\alpha$ and $\eta(G_\alpha, F_\alpha) = 2$, we obtain $Z_{\alpha'-1} = [F_\alpha, V_{\alpha'}] \leq F_\alpha$. Hence $U_\alpha \leq Q_{\alpha'-1} \leq G_{\alpha'}$. Then $V_{\alpha'} \not\leq Q_\alpha$ and $\eta(G_\alpha, U_\alpha) = 3$ imply that $|U_\alpha Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$ and $E(2^3) \cong [U_\alpha, V_{\alpha'}] \leq U_\alpha$. Let $\alpha - 1 \in \Delta(\alpha) \setminus \{\beta\}$. So, since $b > 5$, $W_{\alpha-1}$ centralizes both $Z_{\alpha'-2}$ and $Z_{\alpha'}$. Using Lemma 3.10 at $\alpha' - 4$ and $\alpha' - 2$ gives $[W_{\alpha-1} : W_{\alpha-1} \cap G_{\alpha'}] \leq 2^2$. From $W_{\alpha-1} \cap G_{\alpha'} \geq U_\alpha$ and $b > 5$ we get

$$\left[W_{\alpha-1} \cap G_{\alpha'}, V_{\alpha'}\right] = \left[U_\alpha, V_{\alpha'}\right] \leq U_\alpha.$$

Now arguing as in (17.5.4) and using Lemma 11.1(vii) we deduce that $[W_{\alpha'} : W_\alpha \cap W_{\alpha'-4}] \leq 2^4$.

Observe that, as $V_\beta Q_{\alpha'} = Z_\alpha Q_{\alpha'} \leq V_{\alpha-1} Q_{\alpha'}$ and $V_{\alpha'} \not\leq Q_\alpha$, $U_\alpha Q_{\alpha'} = = V_{\alpha-1} Q_{\alpha'}$. So $|V_{\alpha-1} Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$. We now investigate $[V_{\alpha-1}, W_{\alpha'-4}]$. Let $\rho \in V(\Gamma)$ be such that $d(\rho, \alpha' - 4) = 2$. If $\rho \in \Delta(\alpha' - 5)$, then $[V_{\alpha-1}, V_\rho] = 1$. Now assume that $\rho \notin \Delta(\alpha' - 5)$ and that $V_\rho \not\leq Q_{\alpha-1}$. Hence we have a $(\rho + 1, \alpha - 1) \in \mathcal{C}$ for some $\rho + 1 \in \Delta(\rho)$ and so $[V_\rho, V_{\alpha-1}] \leq Z_\alpha$ by Lemma 18.2(iii). Thus $[V_{\alpha-1}, W_{\alpha'-4}] \leq Z_\alpha$. Since $Z_\alpha \not\leq W_{\alpha'}$ we must have $|[V_{\alpha-1}, W_{\alpha'} \cap W_{\alpha'-4}]| \leq 2$. Hence $[W_{\alpha'} : C_{W_{\alpha'}}(x)] \leq 2^5$ for all $x \in V_{\alpha-1}$ which then forces $\eta(G_{\alpha'}, W_{\alpha'}) \leq 2$, a contradiction. Thus, for any $(\alpha, \alpha') \in \mathcal{C}$ we have $V_{\alpha'} \not\leq Q_\beta$.

Lemma 18.8. *Let $(\alpha, \alpha') \in \mathcal{C}$. Then $U_\alpha \leq Q_{\alpha'-2} \cap Q_{\alpha'-1}$ ($\leq G_{\alpha'}$).*

Proof. Let $\Delta(\alpha) = \{\alpha - 1, \lambda, \beta\}$ and suppose $V_{\alpha'-2} \not\leq Q_{\alpha-1}$. Hence $(\rho, \alpha - 1) \in \mathcal{C}$ for $\rho \in \Delta(\alpha' - 2)$. So, by Lemma 18.7, $V_{\alpha-1} \not\leq Q_{\alpha'-2}$. Now let $\rho^* \in \Delta(\alpha' - 2)$ be such that $\langle V_{\alpha-1}, G_{\alpha'-2\rho^*}\rangle = G_{\alpha'-2}$. Because $(\alpha, \alpha') \in \mathcal{C}$, $[V_{\alpha'-2} \cap Q_{\alpha-1}, V_{\alpha-1}] = 1$; also note that this gives $[V_{\alpha'-2}, V_{\alpha-1}] = [Z_{\rho^*}, V_{\alpha-1}]$, contrary to $Z_{\rho^*} \not\trianglelefteq G_{\alpha'-2}$. Therefore $(\rho^*, \alpha - 1) \in \mathcal{C}$.

By Lemma 18.6 $F_{\rho^*}Q_{\alpha-1} = V_{\alpha'-2}Q_{\alpha-1} = Z_{\rho^*}Q_{\alpha-1}$ and so by Lemma 18.2(iii)

$$Z_\alpha \geq \big[F_{\rho^*}, V_{\alpha-1}\big] \geq \big[Z_{\rho^*}, V_{\alpha-1}\big] = \big[V_{\alpha'-2}, V_{\alpha-1}\big].$$

If $[F_{\rho^*}, V_{\alpha-1}] = [V_{\alpha'-2}, V_{\alpha-1}]$, then we obtain $F_{\rho^*}V_{\alpha'-2} \trianglelefteq \langle V_{\alpha-1}, G_{\alpha'-2\rho^*}\rangle = G_{\alpha'-2}$, against Lemma 17.7(ii). Thus $Z_\alpha = [F_{\rho^*}, V_{\alpha-1}] \leq W_{\alpha'-2} \leq Q_{\alpha'}$ (as $b > 5$), a contradiction. So we conclude that $V_{\alpha'-2} \leq Q_{\alpha-1}$ and hence $[V_{\alpha'-2}, V_{\alpha-1}] = 1$. Similarly $[V_{\alpha'-2}, V_\lambda] = 1$ and so, since $[V_{\alpha'-2}, V_\beta] = 1$, $[V_{\alpha'-2}, U_\alpha] = 1$, which gives the lemma.

Lemma 18.9. *Let* $(\alpha, \alpha') \in \mathcal{C}$. *Then* $|[V_\beta, V_{\alpha'}]| = 2$. *Moreover,* $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1 = [V_{\alpha'} \cap Q_\beta, V_\beta]$, $V_\beta \cap Q_{\alpha'} = [V_\beta, G_{\beta\alpha+2}]$ *and* $V_{\alpha'} \cap Q_\beta = [V_{\alpha'}, G_{\alpha'\alpha'-1}]$.

Proof. First we prove that $|[V_\beta, V_{\alpha'}]| = 2$. By Lemmas 18.2(iii) and 18.7 $[V_\beta, V_{\alpha'}] \leq Z_{\alpha'-1} \cap Z_{\alpha+2}$. Suppose that $|[V_\beta, V_{\alpha'}]| > 2$. Then $|[V_\beta, V_{\alpha'}]| = 2^2$ and $[V_\beta, V_{\alpha'}] = Z_{\alpha'-1} = Z_{\alpha+2}$. If $W_{\alpha'} \leq G_\beta$, then, as $W_{\alpha'}$ acts quadratically on V_β, $|[W_{\alpha'}, V_\beta]| \leq 2^3$. But then, as $[V_\beta, V_{\alpha'}] \leq V_{\alpha'}$, $|[W_{\alpha'}/V_{\alpha'}, V_\beta]| \leq 2$ contradicting $\eta(G_{\alpha'}, W_{\alpha'}) = 3$. Thus $W_{\alpha'} \not\leq G_\beta$. Consequently, from $Z_{\alpha+2} = [V_\beta, V_{\alpha'}]$, $W_{\alpha'} \not\leq Q_{\alpha+3}$; also note that $W_{\alpha'} \cap Q_{\alpha+3} \leq Q_{\alpha+3} \cap Q_{\alpha+2}$. So, using Lemma 18.2(iii), $|[W_{\alpha'} : W_{\alpha'} \cap Q_{\alpha+3}]| = 2$. Clearly $[V_\beta, V_{\alpha'}] \leq [V_\beta, W_{\alpha'} \cap Q_{\alpha+3}]$ and so $\eta(G_{\alpha'}, W_{\alpha'}) = 3$ forces $|[V_\beta, W_{\alpha'} \cap Q_{\alpha+3}]| = 2^3$. If $[V_\beta, W_{\alpha'} \cap Q_{\alpha+3}] = V_\beta \cap V_{\alpha+3}$, then $W_{\alpha'} \leq C_{G_{\alpha'}}([V_{\alpha+3}; G_{\alpha+2\alpha+3}; 2]) \leq G_{\alpha+2\alpha+3}$ whence $W_{\alpha'} \leq G_\beta$ which has been ruled out. Hence

$$\big[V_\beta, G_{\beta\alpha+2}\big] = \big(V_\beta \cap V_{\alpha+3}\big)\big[V_\beta, W_{\alpha'} \cap Q_{\alpha+3}\big]$$

which is centralized by $V_{\alpha'}$. By Lemma 18.7 we may argue similarly to deduce that V_β centralizes $[V_{\alpha'}, G_{\alpha'-1\alpha'}]$. Therefore $V_{\alpha'}$ acts as an involution on V_β with $[V_\beta : C_{V_\beta}(V_{\alpha'})] = 2$ which yields $|[V_\beta, V_{\alpha'}]| = 2$, a contradiction. Hence we must have $|[V_\beta, V_{\alpha'}]| = 2$.

Combining $|[V_\beta, V_{\alpha'}]| = 2$ with Lemma 18.7 gives $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = [V_{\alpha'} \cap Q_\beta, V_\beta] = 1$. So $C_{V_\beta/Z_\beta}(V_{\alpha'}) = (V_\beta \cap Q_{\alpha'})/Z_\beta$ and hence $V_\beta \cap Q_{\alpha'} = [V_\beta, G_{\beta\alpha+2}]$ by Lemma 8.2(iii); likewise we obtain $V_{\alpha'} \cap Q_\beta = [V_{\alpha'}, G_{\alpha'\alpha'-1}]$.

Lemma 18.10. *Let* $(\alpha, \alpha') \in \mathcal{C}$. *Then*

(i) *there exists* $\lambda \in \Delta(\beta)$ *such that* $(\lambda, \alpha') \in \mathcal{C}$ *and* $\langle G_{\lambda\beta}, V_{\alpha'}\rangle = G_\beta$;
(ii) $Z_\beta \not\leq V_{\alpha'}$ *and* $Z_{\alpha'} \not\leq V_\beta$; *and*
(iii) $[F_\alpha, V_{\alpha'} \cap Q_\beta] = 1$.

PROOF. (i) By Lemma 18.7 we may choose $\lambda \in \Delta(\beta)$ such that $\langle G_{\lambda\beta}, V_{\alpha'} \rangle = G_\beta$. Since $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$ by Lemma 18.9 and $Z_\lambda \ntrianglelefteq G_\beta$, we must have $(\lambda, \alpha') \in \mathcal{C}$.

(ii) Assume that $Z_{\alpha'} \leq V_\beta$ and let λ be as in part (i). Using Lemma 18.6 gives $[F_\lambda, V_{\alpha'}] \leq [V_\beta, V_{\alpha'}]Z_{\alpha'} \leq V_\beta$ and then $F_\lambda V_\beta \trianglelefteq G_\beta$, against Lemma 17.7(ii). So $Z_{\alpha'} \nleq V_\beta$ and similarly (because of Lemma 18.7) $Z_\beta \nleq V_{\alpha'}$.

(iii) Lemma 18.10 implies that $V_{\alpha'} \cap Q_\beta \leq Q_\alpha$ and hence $[F_\alpha, V_{\alpha'} \cap Q_\beta] \leq Z_\alpha$ by Lemma 18.2(i). Since $(\alpha, \alpha') \in \mathcal{C}$, $[F_\alpha, V_{\alpha'} \cap Q_\beta] \leq Z_\beta$ and then $Z_\beta \nleq V_{\alpha'}$ yields $[F_\alpha, V_{\alpha'} \cap Q_\beta] = 1$, as required.

LEMMA 18.11. *Let $(\alpha, \alpha') \in \mathcal{C}$ and let $\lambda \in \Delta(\alpha')$ be such that $(\lambda, \beta) \in \mathcal{C}$. Suppose that $X \leq G_{\alpha'\lambda}$ with $X \nleq Q_\lambda$ and that $[X, V_\beta] = 1$. Then $U_\lambda Q_\beta = V_{\alpha'} Q_\beta$.*

PROOF. Since $(\lambda, \beta) \in \mathcal{C}$, Lemma 18.8 gives $U_\lambda \leq G_\beta$. So $[U_\lambda, V_\beta \cap Q_{\alpha'}] \leq V_\beta$ and hence X centralizes $[U_\lambda, V_\beta \cap Q_{\alpha'}]$. Let $\lambda + 1 \in \Delta(\lambda) \setminus \{\alpha'\}$. Since $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$ and $X \nleq Q_\lambda$ we note that

$$\big[U_\lambda, V_\beta \cap Q_{\alpha'}\big] = \big[V_{\lambda+1}, V_\beta \cap Q_{\alpha'}\big].$$

Also the core argument gives $[V_{\lambda+1}, V_\beta \cap Q_{\alpha'}] \leq V_{\alpha'} \cap V_{\lambda+1}$. Combining $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$ and Lemma 18.2(ii) gives that $(V_\beta \cap Q_{\alpha'})Q_{\lambda+1}/Q_{\lambda+1}$ is contained in the non-quadratic $E(2^3)$-subgroup of $G_{\lambda\lambda+1}/Q_{\lambda+1}$ (on $V_{\lambda+1}/Z_{\lambda+1}$).

Assume $V_\beta \cap Q_{\alpha'} \nleq Q_{\lambda+1}$. If $(V_\beta \cap Q_{\alpha'})Q_{\lambda+1}/Q_{\lambda+1}$ acts as a transvection on $V_{\lambda+1}/Z_{\lambda+1}$, then, in fact, it must be a central transvection. So $[V_\beta \cap Q_{\alpha'}, V_{\lambda+1}] \leq Z_\lambda$. From $(\lambda, \beta) \in \mathcal{C}$ and $Z_{\alpha'} \nleq V_\beta$ (by Lemma 18.10(ii)) this gives $[V_\beta \cap Q_{\alpha'}, V_{\lambda+1}] = 1$. On the other hand, if $(V_\beta \cap Q_{\alpha'})Q_{\lambda+1}/Q_{\lambda+1}$ does not act as a transvection on $V_{\lambda+1}/Z_{\lambda+1}$, then, bearing in mind that $(\lambda, \beta) \in \mathcal{C}$, $|[V_\beta \cap Q_{\alpha'}, V_{\lambda+1}]| = 2^2$. Calling on Lemmas 18.2(i), 18.9 and 18.10 gives $[U_\lambda, V_\beta \cap Q_{\alpha'}] \leq Z_{\alpha+2}$. Then $Z_{\alpha+2} = [U_\lambda, V_\beta \cap Q_{\alpha'}]$ by orders. Therefore

$$Z_\beta \leq Z_{\alpha+2} = \big[U_\lambda, V_\beta \cap Q_{\alpha'}\big] = \big[V_{\lambda+1}, V_\beta \cap Q_{\alpha'}\big] \leq V_{\alpha'} \cap V_{\lambda+1}.$$

But $Z_\beta \nleq V_{\alpha'}$ by Lemma 8.10(ii). Thus we deduce that $V_\beta \cap Q_{\alpha'} \leq Q_{\lambda+1}$. Now $(\lambda, \beta) \in \mathcal{C}$ dictates that $1 = [V_{\lambda+1}, V_\beta \cap Q_{\alpha'}] = [U_\lambda, V_\beta \cap Q_{\alpha'}]$ which establishes the lemma.

LEMMA 18.12. *Let $(\alpha, \alpha') \in \mathcal{C}$ and let $\lambda \in \Delta(\alpha')$ be such that $(\lambda, \beta) \in \mathcal{C}$. If $[F_\alpha \cap Q_{\alpha'}, Z_\lambda] \neq 1$, then $U_\lambda \cap Q_\beta \nleq Q_\alpha$.*

PROOF. Applying Lemma 18.11 with $X = F_\alpha \cap Q_{\alpha'}$ ($\not\leq Q_\lambda$ as $[F_\alpha \cap Q_{\alpha'}, Z_\lambda] \neq 1$) and using Lemma 18.2(iii) gives $[U_\lambda : U_\lambda \cap Q_\beta] = 2$. If $U_\lambda \cap Q_\beta \leq Q_\alpha$, then Lemma 18.2(i) implies that $[F_\alpha, U_\lambda \cap Q_\beta] \leq Z_\alpha$ whence $[F_\alpha, U_\lambda \cap Q_\beta] \leq Z_\beta$ as $(\alpha, \alpha') \in \mathcal{C}$. Since $F_\alpha \cap Q_{\alpha'} \not\leq Q_\lambda$ this gives $\eta(G_\lambda, U_\lambda) \leq 2$ contrary to Lemma 17.1. Therefore $U_\lambda \cap Q_\beta \not\leq Q_\alpha$.

The long sought final contradiction is now in sight. Let $(\alpha, \alpha') \in \mathcal{C}$. By Lemmas 18.7 and 18.10(i) we may select $\xi \in \Delta(\alpha')$ such that $(\xi, \beta) \in \mathcal{C}$ and $\langle V_\beta, G_{\alpha'\xi} \rangle = G_{\alpha'}$. Applying Lemmas 17.7(ii), 18.2(iii) and 18.6 we have

$$[F_\xi, V_\beta] = [V_{\alpha'}, V_\beta][F_\xi \cap Q_\beta, V_\beta] = [V_{\alpha'}, V_\beta]Z_\beta$$

with $[F_\xi \cap Q_\beta, V_\beta] = Z_\beta$. Let $\lambda^* \in \Delta(\beta)$ be such that $[F_\xi \cap Q_\beta, Z_{\lambda^*}] \neq 1$. Then $(\lambda^*, \alpha') \in \mathcal{C}$, else $Z_{\lambda^*} \leq V_\beta \cap Q_{\alpha'}$ which, applying Lemma 18.10(iii) to (ξ, β), is centralized by F_ξ. So we use Lemma 18.12 with $(\alpha, \alpha') = (\xi, \beta)$ and $\lambda = \lambda^*$ to conclude that $U_{\lambda^*} \cap Q_{\alpha'} \not\leq Q_\xi$. Now using Lemma 18.11 with $X = U_{\lambda^*} \cap Q_{\alpha'}$ on (α, α') with $\lambda = \xi$ we get $U_\xi Q_\beta = V_{\alpha'} Q_\beta$. Hence

$$[U_\xi, V_\beta] \leq [V_{\alpha'}, V_\beta]Z_\beta = [F_\xi, V_\beta] \leq H_{\alpha'}.$$

Consequently

$$H_{\alpha'} U_\xi \trianglelefteq \langle V_\beta, G_{\alpha'\xi} \rangle = G_{\alpha'}.$$

So, since $W_{\alpha'} = \langle U_\xi{}^{G_{\alpha'}} \rangle$, $W_{\alpha'} = H_{\alpha'} U_\xi$. Because $[Q_\xi, U_\xi] = F_\xi \leq H_{\alpha'}$, $\eta(G_{\alpha'}, W_{\alpha'} H_{\alpha'}) = 0$ and hence, by Lemma 17.7(iii),

$$\eta(G_{\alpha'}, W_{\alpha'}) = \eta(G_{\alpha'}, H_{\alpha'}) = 2,$$

a contradiction.

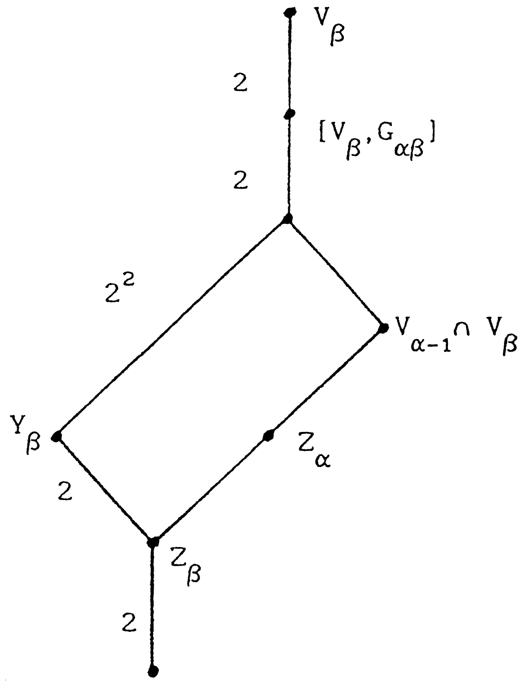
The proof of Theorem 18.1 is complete.

19. Case 5 - the last lap

Finally we consider the last of the cases which were listed in Section 12. Thus in this section we shall be assuming.

HYPOTHESIS 19.0. $V_\beta / Z_\beta \cong \binom{4}{1}$ *and* $\mathrm{core}_{G_\alpha} V_\beta = V_\beta \cap V_{\alpha-1} \cong E(2^3)$.

Recall, from Lemma 12.6, that $Z_\alpha = [V_\beta, G_{\alpha\beta}; 3]$ and that the following holds (where $Y_\beta = C_{V_\beta}(O^2(G_\beta))$).



Let $\alpha \in O(S_3)$ and $\beta \in \Delta(\alpha)$. In the ensueing arguments we shall use the following two subgroups that were introduced in Section 12.

$$F_\alpha := \langle {Y_\beta}^{G_\alpha} \rangle.$$

$$H_\beta := \langle {F_\alpha}^{G_\beta} \rangle.$$

LEMMA 19.1. *If* $\alpha \in O(S_3)$ *and* $\beta \in \Delta(\alpha)$, *then*

(i) $[F_\alpha, Q_\alpha] \le Z_\alpha$, $\eta(G_\alpha, F_\alpha) = 2$, $H_\beta \ne F_\alpha V_\beta$ *and* $\eta(G_\beta, H_\beta) \ge 2$; *and*
(ii) $\eta(G_\alpha, U_\alpha) \ge 4$.

PROOF. (i) See Lemma 12.5.

(ii) By Lemma 1.2(v) $\eta(G_\alpha, U_\alpha/[U_\alpha, Q_\alpha]) \ne 0$. Since $[U_\alpha, Q_\alpha] \ne [V_\beta, Q_\alpha]$ (as $[V_\beta, Q_\alpha] > \mathrm{core}_{G_\alpha} V_\beta$), we also have $\eta(G_\alpha, [U_\alpha, Q_\alpha]/[U_\alpha, Q_\alpha; 2]) \ne 0$. Because $[U_\alpha, Q_\alpha; 2] \ne 1$, $Z_\alpha \le [U_\alpha, Q_\alpha; 2]$ and therefore $\eta(G_\alpha, U_\alpha) \ge 3$. Now we assume that $\eta(G_\alpha, U_\alpha) = 3$ and seek to uncover a contradiction.

Set $R = V_\beta \cap V_{\alpha-1}$.

(19.1.1) $\big[U_\alpha, Q_\alpha; 2\big] = \big[V_\beta, Q_\alpha; 2\big] \le R$.

Since $[U_\alpha, Q_\alpha; 3] \ge Z_\alpha$, our assumption that $\eta(G_\alpha, U_\alpha) = 3$ forces $\eta(G_\alpha, [U_\alpha, Q_\alpha; 2]/[U_\alpha, Q_\alpha; 3]) = 0$ whence we obtain $[V_\beta, Q_\alpha; 2] = = [U_\alpha, Q_\alpha; 2] \trianglelefteq G_\alpha$. This establishes (19.1.1).

(19.1.2) $\big|\big[V_\beta, Q_\alpha\big]/R\big| = 2^2$ and $\big[V_\beta, Q_\alpha\big]/R$ is centralized by $G_{\alpha\beta}$.

From Proposition 2.5(i) we have $|[V_\beta, Q_\alpha]/R| = 2^2$. Since $G_{\alpha\beta} = Q_\alpha Q_\beta$ and $[Q_\beta, V_\beta] = Z_\beta$, (19.1.2) follows from (19.1.1).

(19.1.3) $[U_\alpha, Q_\alpha]/R \lesssim 2 \oplus 1 \oplus 1$ (as an S_3-module) with $\eta(G_\alpha, [U_\alpha, Q_\alpha]/R) = 1$.

Observe that $[U_\alpha, Q_\alpha]/R$ is a $GF(2)(G_\alpha/Q_\alpha)$-module by (19.1.1). Now (19.1.3) is a consequence of (19.1.2) and $\eta(G_\alpha, U_\alpha) = 3$.

Because $E(2^2) \cong [V_\beta, Q_\alpha]/R \le C_{[U_\alpha, Q_\alpha]/R}(G_{\alpha\beta})$ (19.1.3) yields

$$\big(\big[V_\beta, Q_\alpha\big]/R\big) \cap C_{[U_\alpha, Q_\alpha]/R}(G_\alpha) \ne 1,$$

which contradicts the fact that $R = \mathrm{core}_{G_\alpha} V_\beta$ and so we conclude that $\eta(G_\alpha, U_\alpha) \ge 4$.

The main objective of this section is the verification of

THEOREM 19.2. *Assume Hypothesis* 19.0 *holds. Then* $b = 3$.

We suppose the theorem is false and derive a contradiction in a series of lemmas. So, by Lemma 11.1(iii), we have $b > 3$.

LEMMA 19.3. *For* $(\delta, \delta') \in \mathcal{C}$, $F_\delta \le G_{\delta'}$.

PROOF. Because $V_{\delta'-2} \le G_{\delta-1}$ for $\delta - 1 \in \Delta(\delta) \setminus \{\delta + 1\}$ we have $[V_{\delta'-2}, Y_{\delta-1}] \le V_{\delta'-2} \le Q_\delta$, and so, as $(\delta, \delta') \in \mathcal{C}$, $[V_{\delta'-2}, Y_{\delta-1}] = 1$. Therefore $[F_\delta, V_{\delta'-2}] = 1$ and hence $F_\delta \le G_{\delta'}$.

LEMMA 19.4. *Let* (λ, μ, δ) *be a path of length* 2 *in* Γ *where* $\lambda \in O(S_6)$. *If* $X \le G_\lambda$ *and* $[X, V_\lambda \cap V_\delta] = 1$, *then* $X \le G_\delta$.

PROOF. Put $\bar{V}_\lambda = V_\lambda / Y_\lambda$. Since $\overline{V_\lambda \cap V_\delta} = [\bar{V}_\lambda, G_{\lambda\mu}; 2]$ and $\bar{V}_\lambda \cong 4$, Proposition 2.5(ii) gives $X \le G_{\lambda\mu}$, whence $X \le Q_\mu \le G_\delta$.

From now until the end of Lemma 19.13 we shall assume that $b > 5$.

LEMMA 19.5. *Let* $(\alpha, \alpha') \in \mathcal{C}$, *and suppose* $X \leq G_\alpha$ *with* $X \not\leq Q_\alpha$. *If* $[X, H_{\alpha'-2}] = 1$, *then* $U_\alpha \leq Q_{\alpha'-2}$.

PROOF. We suppose the lemma is false, and seek a contradiction. So there exists $\alpha - 2 \in V(\Gamma)$ with $d(\alpha - 2, \alpha) = 2$ and $(\alpha - 2, \alpha' - 2) \in \mathcal{C}$. Set $\{\alpha - 1\} = \Delta(\alpha - 2) \cap \Delta(\alpha)$. Note that $[V_{\alpha'-2} \cap Q_{\alpha-1}, V_{\alpha-1}] = 1$. For $[V_{\alpha'-2} \cap Q_{\alpha-1}, V_{\alpha-1}] \neq 1$ yields, as $b > 3$,

$$Z_{\alpha-1} = \left[V_{\alpha'-2} \cap Q_{\alpha-1}, V_{\alpha-1}\right] \leq V_{\alpha'-2} \leq Q_{\alpha'},$$

against $(\alpha, \alpha') \in \mathcal{C}$. In particular, $V_{\alpha'-2} \not\leq Q_{\alpha-1}$.

Since $V_{\alpha-1} \not\leq Q_{\alpha'-2}$, we may select $\rho \in \Delta(\alpha' - 2)$ so as

$$\left\langle V_{\alpha-1}, G_{\alpha'-2\rho} \right\rangle = G_{\alpha'-2}.$$

If $Z_\rho \leq Q_{\alpha-1}$, then $Z_\rho \leq V_{\alpha'-2} \cap Q_{\alpha-1}$, which commutes with $V_{\alpha-1}$ and hence $Z_\rho \trianglelefteq \langle V_{\alpha-1}, G_{\alpha'-2\rho}\rangle = G_{\alpha'-2}$, a contradiction. Thus $(\rho, \alpha - 1) \in \mathcal{C}$, and so $F_\rho \leq G_{\alpha-1}$ by Lemma 19.3. Since $F_\rho V_{\alpha'-2} \neq H_{\alpha'-2}$ and $\langle V_{\alpha-1}, G_{\alpha'-2\rho}\rangle = G_{\alpha'-2}$, $[F_\rho, V_{\alpha-1}] \not\leq V_{\alpha'-2}$. Set $Y = [V_{\alpha'-2}, V_{\alpha-1}][F_\rho, V_{\alpha-1}]$. Observe that $Y \leq H_{\alpha'-2} \cap \cap V_{\alpha-1}$ and therefore X centralizes Y. Hence, as $X \not\leq Q_\alpha$ by hypothesis,

$$Y \leq V_{\alpha-1} \cap V_\lambda = \mathrm{core}_{G_\alpha} V_\beta$$

$(\Delta(\alpha) = \{\alpha - 1, \lambda, \beta\})$. From $b > 5$, $H_{\alpha'-2} \leq Q_{\alpha'}$ and thus $Z_{\alpha-1} \not\leq Y$. So, since $Z_{\alpha-1} \leq Z_\alpha \leq V_{\alpha-1} \cap V_\lambda$ and $|V_{\alpha-1} \cap V_\lambda| = 2^3$, we conclude that $|Y| \leq 2^2$.

If $|[V_{\alpha'-2}, V_{\alpha-1}]| = 2^2$, then we obtain

$$\left[F_\rho, V_{\alpha-1}\right] \leq Y = \left[V_{\alpha'-2}, V_{\alpha-1}\right] \leq V_{\alpha'-2},$$

a contradiction. Therefore $|[V_{\alpha'-2}, V_{\alpha-1}]| = 2$. Consequently, as $[V_{\alpha'-2}, V_{\alpha-1}] \not\leq \not\leq Z_{\alpha-1}$, $[V_{\alpha-1} \cap Q_{\alpha'-2}, V_{\alpha'-2}] = 1$ and $[V_{\alpha-1} : V_{\alpha-1} \cap Q_{\alpha'-2}] = 2$. Thus $V_{\alpha-1} \cap \cap Q_{\alpha'-2} \leq Q_\rho$ and so $[V_{\alpha-1} \cap Q_{\alpha'-2}, F_\rho] \leq Z_\rho$. Hence $[V_{\alpha-1} \cap Q_{\alpha'-2}, F_\rho] \leq Z_{\alpha'-2}$ as $(\rho, \alpha - 1) \in \mathcal{C}$. Suppose $[V_{\alpha-1} \cap Q_{\alpha'-2}, F_\rho] = 1$. Then $[V_{\alpha-1} : C_{V_{\alpha-1}}(F_\rho)] \leq 2$. Because $Z_{\alpha-1} \not\leq [F_\rho, V_{\alpha-1}]$ and $F_\rho \leq G_{\alpha-1}$ we obtain $|[F_\rho, V_{\alpha-1}]| = 2$. Since $[Z_\rho, V_{\alpha-1}] \neq 1$ and $[Z_\rho, V_{\alpha-1}] \leq [F_\rho, V_{\alpha-1}]$, we then get $[F_\rho, V_{\alpha-1}] = = [Z_\rho, V_{\alpha-1}] \leq V_{\alpha'-2}$, a contradiction. Therefore $[V_{\alpha-1} \cap Q_{\alpha'-2}, F_\rho] \neq 1$, and so $[V_{\alpha-1} \cap Q_{\alpha'-2}, F_\rho] = Z_{\alpha'-2}$. However this also leads to a contradiction. For we then have $Z_{\alpha'-2} \leq Y$ which, together with $[V_{\alpha-1}, V_{\alpha'-2}] \neq Z_{\alpha'-2}$ and $|Y| \leq 2^2$, forces

$$\left[F_\rho, V_{\alpha-1}\right] \leq Y = \left[V_{\alpha-1}, V_{\alpha'-2}\right] Z_{\alpha'-2} \leq V_{\alpha'-2}.$$

This completes the proof of the lemma.

Lemma 19.6. *For* $(\alpha, \alpha') \in \mathcal{C}$, $V_{\alpha'} \cap G_\alpha \le Q_\alpha$.

Proof. Suppose the lemma is false, and set $\Delta(\alpha) = \{\alpha - 1, \lambda, \beta\}$.

(19.6.1) $U_\alpha \le G_{\alpha'}$.

Since $b > 5$, $[V_{\alpha'} \cap G_\alpha, H_{\alpha'-2}] = 1$ and so using Lemma 19.5 with $X = V_{\alpha'} \cap G_\alpha$ we infer that $U_\alpha \le Q_{\alpha'-2} \le G_{\alpha'-1}$. Now assume that $U_\alpha \not\le Q_{\alpha'-1}$. Then $[U_\alpha \cap Q_{\alpha'}, V_{\alpha'}] = 1$ else $b > 5$ and $Z_{\alpha'} = [U_\alpha \cap Q_{\alpha'}, V_{\alpha'}] \le$ $\le W_\beta$ yield that U_α centralizes $Z_{\alpha'-2}Z_{\alpha'} = Z_{\alpha'-1}$, against $U_\alpha \not\le Q_{\alpha'-1}$. Recalling that $Z_{\alpha'-1}/Z_{\alpha'} = [V_{\alpha'}, G_{\alpha'-1\alpha'}; 3]/Z_{\alpha'}$, $U_\alpha \not\le Q_{\alpha'-1}$ and the core argument imply that $(U_\alpha \cap G_{\alpha'})Q_{\alpha'}/Q_{\alpha'}$ doesn't contain the central transvection of $G_{\alpha'-1\alpha'}/Q_{\alpha'}$ (acting on $V_{\alpha'}/Z_{\alpha'}$). Consequently $[U_\alpha : U_\alpha \cap Q_{\alpha'}] \le 2^3$ and therefore $[U_\alpha : C_{U_\alpha}(V_{\alpha'} \cap G_\alpha)] \le 2^3$. But this, as $V_{\alpha'} \cap G_\alpha \not\le Q_\alpha$, forces $\eta(G_\alpha, U_\alpha) \le 3$, contradicting Lemma 19.1(ii). Thus $U_\alpha \le Q_{\alpha'-1}$, and so (19.6.1) holds.

(19.6.2) $U_\alpha Q_{\alpha'}/Q_{\alpha'}$ is the quadratic $E(2^3)$-subgroup of $G_{\alpha'-1\alpha'}/Q_{\alpha'}$ (acting on $V_{\alpha'}/Z_{\alpha'}$).

Since $b > 5$, U_α acts quadratically upon $V_{\alpha'}$. Now (19.6.2) follows from (19.6.1) and $|[U_\alpha \cap Q_{\alpha'}, V_{\alpha'} \cap G_\alpha]| \le 2$.

(19.6.3) $$\left[W_{\alpha-1} : W_{\alpha-1} \cap W_\lambda\right] \le 2.$$

From $\eta(G_\alpha, U_\alpha) \ge 4$ we get $|[U_\alpha, V_{\alpha'} \cap G_\alpha]| \ge 2^4$ and so, with the aid of (19.6.2),

$$Y_{\alpha'}\left(V_{\alpha'} \cap V_{\alpha'-2}\right) = \left[U_\alpha, V_{\alpha'}\right] = \left[U_\alpha, V_{\alpha'} \cap G_\alpha\right] \le U_\alpha.$$

Hence, as $b > 5$, $[W_{\alpha-1}, V_{\alpha'} \cap V_{\alpha'-2}] = 1$. Therefore $W_{\alpha-1}$ centralizes $Z_{\alpha'-4}Z_{\alpha'-2} = Z_{\alpha'-3}$ and hence, using the parabolic argument, $[W_{\alpha-1} : W_{\alpha-1} \cap G_{\alpha'-2}] \le 2$. Thus $[W_{\alpha-1} : W_{\alpha-1} \cap G_{\alpha'}] \le 2$ by Lemma 19.4. By (19.6.2) and $b > 5$, $(W_{\alpha-1} \cap G_{\alpha'})Q_{\alpha'} = U_\alpha Q_{\alpha'}$ and consequently

$$\left[W_{\alpha-1} \cap G_{\alpha'}, V_{\alpha'}\right] = \left[U_\alpha, V_{\alpha'}\right] \le U_\alpha \le W_{\alpha-1} \cap W_\lambda.$$

Arguing as in (17.5.4) we now obtain (19.6.3).

By Lemma 11.1(vii) and (19.6.3) $[W_{\alpha'} : W_{\alpha'} \cap W_{\alpha'-2}] \le 2$, $[W_{\alpha'-2} : W_{\alpha'-2} \cap$ $\cap W_{\alpha'-4}] \le 2$ and $[W_{\alpha'-4} : W_{\alpha'-4} \cap W_{\alpha'-6}] \le 2$. Thus $[W_{\alpha'} : W_{\alpha'} \cap W_{\alpha'-6}] \le 2^3$ and so $[W_{\alpha'} : C_{W_{\alpha'}}(U_\alpha)] \le 2^3$. But then, by (19.6.2), $\eta(G_{\alpha'}, W_{\alpha'}) \le 1$ which is impossible. This completes the proof of the lemma.

Lemma 19.7. *For* $(\alpha, \alpha') \in \mathcal{C}$ *we have* $V_{\alpha'} \not\le Q_\beta$.

Proof. This follows from Lemma 19.6 since $[Z_\alpha, V_{\alpha'}] \neq 1$.

Lemma 19.8. *Let* $(\alpha, \alpha') \in \mathcal{C}$ *be such that* $[V_{\alpha'} \cap Q_\beta, V_\beta] = Z_\beta$. *Then* $Z_\beta = Z_{\alpha'}$.

Proof. Since $[V_{\alpha'} \cap Q_\beta, V_\beta] = Z_\beta$, there exists $\lambda \in \Delta(\beta)$ such that $[V_{\alpha'} \cap Q_\beta, Z_\lambda] = Z_\beta$. So $V_{\alpha'} \cap Q_\beta \leq G_\lambda$ but $V_{\alpha'} \cap Q_\beta \not\leq Q_\lambda$. Therefore $(\lambda, \alpha') \notin \mathcal{C}$ by Lemma 19.6. Hence $Z_\lambda \leq Q_{\alpha'}$ and so $[Z_\lambda, V_{\alpha'}] \leq Z_{\alpha'}$. Thus

$$Z_\beta = \big[V_{\alpha'} \cap Q_\beta, Z_\lambda\big] \leq \big[Z_\lambda, V_{\alpha'}\big] \leq Z_{\alpha'},$$

so proving the lemma.

Lemma 19.9. *Let* $(\alpha, \alpha') \in \mathcal{C}$. *Then there exists* $\lambda \in \Delta(\beta)$ *such that* $\langle G_{\lambda\beta}, V_{\alpha'} \rangle = G_\beta$ *and* $(\lambda, \alpha') \in \mathcal{C}$.

Proof. By Lemma 19.7 and Proposition 2.8(viii) there exists $\lambda \in \Delta(\beta)$ for which $\langle G_{\lambda\beta}, V_{\alpha'} \rangle = G_\beta$. Suppose $Z_\lambda \leq Q_{\alpha'}$. Then $Z_\lambda \leq V_\beta \cap Q_{\alpha'}$. Since $Z_\lambda \ntrianglelefteq G_\beta$, $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] \neq 1$, and so $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'}$. Choosing $\mu \in \Delta(\alpha')$ such that $(\mu, \beta) \in \mathcal{C}$ and using Lemma 19.8 gives $Z_\beta = Z_{\alpha'}$. Hence

$$\big[Z_\lambda, V_{\alpha'}\big] \leq Z_{\alpha'} = Z_\beta \leq Z_\lambda,$$

which forces the impossible $Z_\lambda \trianglelefteq G_\beta$. Thus $(\lambda, \alpha') \in \mathcal{C}$, as required.

Lemma 19.10. *Let* $(\alpha, \alpha') \in \mathcal{C}$. *If* $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$, *then* $|V_{\alpha'} Q_\beta / Q_\beta| \geq 2^2$.

Proof. Suppose $|V_{\alpha'} Q_\beta / Q_\beta| \leq 2$. Then $[V_{\alpha'} \cap Q_\beta, V_\beta] \neq 1$ and so $[V_{\alpha'} \cap Q_\beta, V_\beta] = Z_\beta$. Hence $Z_\beta = Z_{\alpha'}$ by Lemma 19.8. But then V_β centralizes $(V_{\alpha'} \cap Q_\beta)/Z_{\alpha'}$ which has index at most 2 in $V_{\alpha'}/Z_{\alpha'}$ and this is impossible. Therefore $|V_{\alpha'} Q_\beta / Q_\beta| \geq 2^2$.

Lemma 19.11. *Let* $\alpha \in O(S_3)$ *with* $\beta, \alpha - 1 \in \Delta(\alpha)$, $\beta \neq \alpha - 1$. *If* $\eta(G_\beta, W_\beta) = 2$, *then an elementary abelian subgroup of* $(Q_\alpha \cap Q_\beta) Q_{\alpha-1}/Q_{\alpha-1}$ *has order at most* 2^2.

Proof. Put $R = V_\beta \cap V_{\alpha-1}$. As in [Lemma 4; LPR2] we may choose $E \trianglelefteq G_\beta$ with $W_\beta \geq E \geq V_\beta [W_\beta, Q_\beta]$ so as $\eta(G_\beta, E/V_\beta) = 0$. Also we may argue as in [Proof of Theorem 1; LPR2] to eliminate the possibility $[Q_\alpha \cap Q_\beta, R] \leq Z_{\alpha-1}$. Thus $(Q_\alpha \cap Q_\beta) Q_{\alpha-1}/Q_{\alpha-1}$ does not act as the quadratic $E(2^3)$-subgroup of $G_{\alpha-1}/Q_{\alpha-1}$ on $V_{\alpha-1}/Z_{\alpha-1}$.

Set $X = V_{\alpha-1} \cap E$ and $Q = [Q_\beta, O^2(G_\beta)]$. Arguing as in [Proposition 3; LPR2] we may show, as $V_\beta/Y_\beta \cong 4$, that $[X, Q] \le Y_\beta$. Hence, since $Z_\beta \le X$, $[[X, Q]X : X] \le 2$. Now $Q \le Q_\alpha$ by Lemma 12.4(i) and Q normalizing $[X, Q]X$, together with the core argument yields $[X : R] \le 2$. Therefore $[V_{\alpha-1} : X] \ge 2^2$. Since $[V_{\alpha-1}, Q_\alpha \cap Q_\beta] \le X$, we immediately deduce that $(Q_\alpha \cap Q_\beta)Q_{\alpha-1}/Q_{\alpha-1}$ is not equal to $G_{\alpha-1\alpha}/Q_{\alpha-1}$ nor to the non-quadratic $E(2^3)$-subgroup of $G_{\alpha-1\alpha}/Q_{\alpha-1}$ on $V_{\alpha-1}/Z_{\alpha-1}$. This completes the proof of the lemma.

Lemma 19.12. *For* $(\alpha, \alpha') \in \mathcal{C}$ $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = 2 = |V_{\alpha'} Q_\beta/Q_\beta|$.

Proof. Suppose the lemma is false. Then, in view of Lemmas 19.7 and 19.10, we have $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \ge 2^2 \le |V_{\alpha'} Q_\beta/Q_\beta|$. Without loss of generality, by Lemma 19.9, we may suppose $\langle G_{\alpha\beta}, V_{\alpha'}\rangle = G_\beta$.

(19.12.1) $\eta\big(G_\beta, W_\beta\big) = 2$.

Supposing (19.12.1) is false, we argue for a contradiction. So $\eta(G_\beta, W_\beta) \ge 3$. First we show that $[W_\beta, Z_{\alpha'}] \ne 1$ (and so $Z_{\alpha'} \not\le W_\beta$). If $[W_\beta, Z_{\alpha'}] = 1$, then $[W_\beta : W_\beta \cap G_{\alpha'}] \le 2$, using the parabolic argument. Because $W_\beta \cap G_{\alpha'}$ acts quadratically on $V_{\alpha'}$, $|[W_\beta \cap G_{\alpha'}, V_{\alpha'}]| \le 2^4$. Since $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \ge 2^2$, $|[V_\beta, V_{\alpha'}]| \ge 2^2$ and thus

$$\big[W_\beta/V_\beta : C_{W_\beta/V_\beta}(x)\big] \le 2^3$$

for any $x \in V_{\alpha'}$. This, as $|V_{\alpha'} Q_\beta/Q_\beta| \ge 2^2$, implies $\eta(G_\beta, W_\beta) \le 2$, a contradiction. So $[W_\beta, Z_{\alpha'}] \ne 1$, as asserted.

Set $W = W_\beta \cap G_{\alpha'-1}$. We next prove that $W \le Q_{\alpha'-1}$. By Lemma 19.3 $[F_\alpha, V_{\alpha'}] \le V_{\alpha'}$. If $W \not\le Q_{\alpha'-1}$, then $[F_\alpha, V_{\alpha'}][V_\beta, V_{\alpha'}] \le V_{\alpha'-2} \cap V_{\alpha'}$ by the core argument. Since $|[V_\beta, V_{\alpha'}]| \ge 2^2$ and, by Lemma 19.1(i), $[F_\alpha, V_{\alpha'}] \not\le V_\beta$,

$$V_{\alpha'-2} \cap V_{\alpha'} = \big[F_\alpha, V_{\alpha'}\big]\big[V_\beta, V_{\alpha'}\big] \le W_\beta,$$

contrary to $Z_{\alpha'} \not\le W_\beta$. Thus we must have $W \le Q_{\alpha'-1}$ and therefore $[W_\beta : Y] \le 2^3$ where $Y = W_\beta \cap G_{\alpha'}$. If $[Y, V_{\alpha'}] = [V_\beta, V_{\alpha'}]$, then $[W_\beta/V_\beta : C_{W_\beta/V_\beta}(V_{\alpha'})] \le 2^3$, whence $\eta(G_\beta, W_\beta) \le 2$. Thus $[Y, V_{\alpha'}] \ne [V_\beta, V_{\alpha'}]$ and so, as $Z_{\alpha'} \not\le W_\beta$, we deduce that $|YQ_{\alpha'}/Q_{\alpha'}| = 2^3$, $|[Y, V_{\alpha'}]| = 2^3$ and $|[V_\beta, V_{\alpha'}]| = 2^2$ with $\eta(G_\beta, W_\beta) = 3$. Further, $|[V_\beta, V_{\alpha'}]| = 2^2$ and Propositio-n 2.5(ii) imply that $V_\beta Q_{\alpha'}/Q_{\alpha'} \ne Z(G_{\alpha'-1\alpha'}/Q_{\alpha'})$ and therefore all the non-central chief factors of W_β are isomorphic natural modules.

We now display the required contradiction to the supposition $\eta(G_\beta, W_\beta) \ge 3$. Let $x \in Y$ be such that $xQ_{\alpha'}$ is a transvection on $V_{\alpha'}/Z_{\alpha'}$ and $\langle x\rangle V_\beta Q_{\alpha'} = YQ_{\alpha'}$. Set $X = C_{V_{\alpha'}}(x)$. Since $Z_{\alpha'} \not\le W_\beta$, $[V_{\alpha'} : X] = 2$. Also we have $[X, Y] = [V_\beta, X] \leq V_\beta$ which, as $[W_\beta : Y] \leq 2^3$, implies that X acts as a transvection on at least one of the non-central chief factors of W_β / V_β. Hence X acts as a transvection on V_β / Y_β and so as a transvection on V_β / Z_β. Symmetrically we have $Z_\beta \not\leq W_{\alpha'}$ and consequently $|[X, V_\beta]| = 2$. But, by Proposition 2.5(ix) applied to $V_{\alpha'} / Z_{\alpha'}$ we see that $|[X, V_\beta]| = 2^2$, and thus (19.12.1) holds.

Combining $\eta(G_\beta, H_\beta) \geq 2$ with (19.12.1) clearly gives

(19.12.2) $W_\beta = H_\beta U_{\alpha+2}$.

(19.12.3) $W_\beta \leq Q_{\alpha'-2}$.

We assume that $W_\beta \not\leq Q_{\alpha'-2}$, and argue for a contradiction. So, by (19.12.2) $H_\beta \not\leq Q_{\alpha'-2}$. Hence there exists $\beta - 2 \in V(\Gamma)$ with $d(\beta - 2, \beta) = 2$ and $Y_{\beta-2} \not\leq Q_{\alpha'-2}$. Thus $(\beta - 3, \alpha' - 2) \in \mathcal{C}$ for some $\beta - 3 \in \Delta(\beta - 2)$ and by Lemma 19.9 we may assume $\beta - 3$ is chosen so that $\langle G_{\beta-3\beta-2}, V_{\alpha'-2} \rangle = G_{\beta-2}$. Of course we must have $[Y_{\beta-2}, V_{\alpha'-2}] \neq 1$ and so, since $V_{\alpha'-2} \leq G_{\beta-2}$, we have $[Y_{\beta-2}, V_{\alpha'-2}] = Z_{\beta-2}$. Hence we have $|[V_{\beta-2}, V_{\alpha'-2}]| \geq 2^2$. If $Z_{\beta-2} = Z_{\alpha'-2}$, then $[Y_{\beta-2}, V_{\alpha'-2}] = Z_{\alpha'-2}$ and $Y_{\beta-2} \leq Q_{\alpha'-2}$ a contradiction. Thus, by Lemma 19.8, $[V_{\alpha'-2} \cap Q_{\beta-2}, V_{\beta-2}] = 1 = [V_{\alpha'-2}, V_{\beta-2} \cap Q_{\alpha'-2}]$. Suppose that $|V_{\beta-2} Q_{\alpha'-2} / Q_{\alpha'-2}| = 2$. Then, $V_{\beta-2} Q_{\alpha'-2} = Y_{\beta-2} Q_{\alpha'-2}$ and $[V_{\beta-2}, V_{\alpha'-2}] = [Y_{\beta-2}(V_{\beta-2} \cap Q_{\alpha'-2}), V_{\alpha'-2}] = Z_{\beta-2}$, which contradicts $V_{\alpha'-2} \not\leq Q_{\beta-2}$. Hence we have, from Lemmas 19.10 and 19.11 and (19.12.1), that $|V_{\beta-2} Q_{\alpha'-2} / Q_{\alpha'-2}| = |V_{\alpha'-2} Q_{\beta-2} / Q_{\beta-2}| = 2^2$. Therefore $V_{\beta-2} Q_{\alpha'-2} \geq Y_{\beta-2} Q_{\alpha'-2}$ and so $[V_{\beta-2} : V_{\beta-2} \cap Y_{\beta-2} Q_{\alpha'-2}] = 2$. Hence we have

$$\left[V_{\alpha'-2}, V_{\beta-2} \cap Y_{\beta-2} Q_{\alpha'-2}\right] = \left[V_{\alpha'-2}, Y_{\beta-2}\left(V_{\beta-2} \cap Q_{\alpha'-2}\right)\right] = Z_{\beta-2}.$$

Thus $V_{\alpha'-2}$ induces a transvection on $V_{\beta-2} / Z_{\beta-2}$, which contradicts $|V_{\alpha'-2} Q_{\beta-2} / Q_{\beta-2}| = 2^2$. This completes the proof of (19.12.3).

(19.12.4) $W_\beta \leq Q_{\alpha'-1}$.

By (19.12.3) $W_\beta \leq G_{\alpha'-1}$. Also, from Lemma 19.3, $[F_\alpha, V_{\alpha'}] \leq V_{\alpha'}$. So if $W_\beta \not\leq Q_{\alpha'-1}$, the core argument and $b > 5$ yields $[F_\alpha, V_{\alpha'}][V_\beta, V_{\alpha'}] \leq V_{\alpha'-2} \cap V_{\alpha'}$. Since $|[V_\beta, V_{\alpha'}]| \geq 2^2$ and, by Lemma 19.1(i), $[F_\alpha, V_{\alpha'}] \not\leq V_\beta$, this gives

$$V_{\alpha'-2} \cap V_{\alpha'} = \left[F_\alpha, V_{\alpha'}\right]\left[V_\beta, V_{\alpha'}\right] \leq W_\beta.$$

Hence $[W_\beta, Z_{\alpha'-1}] = 1$, a contradiction. Thus $W_\beta \leq Q_{\alpha'-1}$.

Combining (19.12.3), (19.12.4) and Lemma 19.11 yields $|W_\beta Q_{\alpha'} / Q_{\alpha'}| \leq 2^2$ and hence $W_\beta Q_{\alpha'} = V_\beta Q_{\alpha'}$. But then $|[W_\beta / V_\beta, V_{\alpha'}]| \leq 2$ which, as

$|V_{\alpha'}Q_\beta/Q_\beta| \geq 2^2$, forces the untenable $\eta(G_\beta, W_\beta/V_\beta) = 0$ and completes the proof of Lemma 19.12.

LEMMA 19.13. *Let* $(\alpha, \alpha') \in \mathcal{C}$ *be such that* $\langle G_{\alpha\beta}, V_{\alpha'}\rangle = G_\beta$. *Then*

(i) $F_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$;
(ii) $[F_\alpha \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \not\leq V_\beta$;
(iii) $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1 = [V_{\alpha'} \cap Q_\beta, V_\beta]$;

Let $\rho \in \Delta(\alpha')$ *be such that* $F_\alpha \cap Q_{\alpha'} \not\leq Q_\rho$ *(by part* (ii) *such* ρ *exist).*

(iv) $(\rho, \beta) \in \mathcal{C}$;
(v) $U_\rho \leq G_\beta$; *and*
(vi) $E(2) \cong [U_\rho, V_\beta \cap Q_{\alpha'}] \leq V_{\alpha'} \cap V_{\rho+1}$ $(\rho+1 \in \Delta(\rho)\setminus\{\alpha'\})$, $|U_\rho Q_\beta/Q_\beta| = 2^2$ *and* $[U_\rho \cap Q_\alpha, F_\alpha \cap Q_{\alpha'}] = Z_\beta$.

PROOF. (i) From Lemma 19.6 $V_{\alpha'} \cap Q_\beta \leq Q_\alpha$ and so $[V_{\alpha'} \cap Q_\beta, Y_{\alpha-1}] \leq$ $\leq Z_{\alpha-1}$ $(\alpha - 1 \in \Delta(\alpha) \setminus \{\beta\})$. Thus $[V_{\alpha'} \cap Q_\beta, F_\alpha] \leq Z_\beta$ and if $[V_{\alpha'} \cap Q_\beta, F_\alpha] =$ $= Z_\beta$ then Lemma 19.8 yields that $Z_\beta = Z_{\alpha'}$. Hence F_α centralizes $V_{\alpha'} \cap Q_\beta/Z_{\alpha'}$ and then Lemmas 19.3 and 19.12 imply that $F_\alpha Q_{\alpha'} = Z_\alpha Q_{\alpha'} =$ $= V_\beta Q_{\alpha'}$.

(ii) Using (i) gives

$$\big[F_\alpha, V_{\alpha'}\big] = \big[Z_\alpha, V_{\alpha'}\big]\big[F_\alpha \cap Q_{\alpha'}, V_{\alpha'}\big].$$

By Lemma 19.1(i) $[F_\alpha, V_{\alpha'}] \not\leq V_\beta$ and therefore $[F_\alpha \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \not\leq V_\beta$.

(iii) By (ii) $Z_{\alpha'} \neq Z_\beta$ and so (iii) is a consequence of Lemmas 19.7 and 19.8.

(iv) Note that (iii) gives $[F_\alpha, V_{\alpha'} \cap Q_\beta] = 1$. Thus $Z_\rho \leq Q_\beta$ implies $[F_\alpha, Z_\rho] = 1$, against $F_\alpha \cap Q_{\alpha'} \not\leq Q_\rho$. So (iv) holds.

(v) Observe that $[F_\alpha, H_{\alpha+3}] = 1$ for otherwise we would obtain $Z_{\alpha-1} = Z_\lambda$ for some $\alpha - 1 \in \Delta(\alpha) \setminus \{\beta\}$ and some $\lambda \in \Delta^{[2]}(\alpha+3)$, contrary to $(\alpha, \alpha') \in \mathcal{C}$. Applying Lemma 19.5 to (ρ, β) with $X = F_\alpha \cap Q_{\alpha'}$ yields $U_\rho \leq Q_{\alpha+3}$, since $F_\alpha \cap Q_{\alpha'} \not\leq Q_\rho$. By Lemmas 19.7 and 19.9 there exists $\lambda \in \Delta(\alpha')$ for which $(\lambda, \beta) \in \mathcal{C}$ and $\langle G_{\lambda\alpha'}, V_\beta\rangle = G_{\alpha'}$. Part (ii) applied to this critical pair gives $Z_\beta \leq W_{\alpha'}$. Since $b > 5$, we get $[U_\rho, Z_\beta] = 1$ and consequently $U_\rho \leq Q_{\alpha+2} \leq G_\beta$, as required.

(vi) We first prove that $|U_\rho Q_\beta/Q_\beta| \geq 2^2$. Suppose that $|U_\rho Q_\beta/Q_\beta| \leq 2$. Then $[U_\rho : U_\rho \cap Q_\alpha] \leq 2^2$. Since $[U_\rho \cap Q_\alpha, F_\alpha] \leq Q_{\alpha'}$, $|[U_\rho \cap Q_\alpha, F_\alpha]| \leq 2$ by Lemma 19.1(i). But then, as $F_\alpha \cap Q_{\alpha'} \not\leq Q_\rho$, $\eta(G_\rho, U_\rho) \leq 3$, contradicting Lemma 19.1(iii). Thus we conclude that $|U_\rho Q_\beta/Q_\beta| \geq 2^2$. Next we claim that $[U_\rho, V_\beta \cap Q_{\alpha'}] \neq 1$. For $[U_\rho, V_\beta \cap Q_{\alpha'}] = 1$ together with Lemma 19.12 and part (ii) gives $U_\rho Q_\beta = V_{\alpha'} Q_\beta$, contrary to $|U_\rho Q_\beta/Q_\beta| \geq 2^2$.

Let $\rho + 1 \in \Delta(\rho) \setminus \{\alpha'\}$. By (iii) $V_\beta \cap Q_{\alpha'} \leq Q_\rho$ and so $F_\alpha \cap Q_{\alpha'} \not\leq Q_\rho$ and the core argument yield

$$\left[U_\rho, V_\beta \cap Q_{\alpha'}\right] \leq V_{\alpha'} \cap V_{\rho+1}.$$

If $[U_\rho, V_\beta \cap Q_{\alpha'}] \cap Z_\rho \neq 1$, then, as $(\rho, \beta) \in \mathcal{C}$ and $U_\rho \leq G_\beta$, $[U_\rho, V_\beta \cap Q_{\alpha'}] \cap \cap Z_\rho = Z_{\alpha'}$. But then $Z_{\alpha'} \leq V_\beta$, against part (ii). Hence $[U_\rho, V_\beta \cap Q_{\alpha'}] \cap \cap Z_\rho = 1$ from which we infer that $|[U_\rho, V_\beta \cap Q_{\alpha'}]| = 2$.

Since U_ρ acts quadratically on V_β and $[V_\beta : V_\beta \cap Q_{\alpha'}] = 2$, $|[U_\rho, V_\beta \cap Q_{\alpha'}]| = 2$ and Proposition 2.5(ix) imply that $|U_\rho Q_\beta / Q_\beta| = 2^2$. From $\eta(G_\rho, U_\rho) \geq 4$ we see that $[U_\rho \cap Q_\alpha, F_\alpha \cap Q_{\alpha'}] \neq 1$ and hence that $[U_\rho \cap Q_\alpha, F_\alpha \cap Q_{\alpha'}] = Z_\beta$, so proving (vi).

Lemma 19.14. $b = 5$.

Proof. Assume $b > 5$; thus Lemmas 19.5-19.13 apply. Since $[U_\rho \cap Q_\alpha, Y_{\alpha-1}] = 1$ for $\alpha - 1 \in \Delta(\alpha) \setminus \{\beta\}$, Lemma 19.13(vi) implies that $[U_\rho \cap Q_\alpha, Y_\beta] = Z_\beta$. Lemmas 19.7, 19.12 and 19.13(ii) together yield that $Z_\beta \not\leq V_{\alpha'}$ also. Consequently $[Y_\beta, V_{\alpha'}] = 1$ and so $Y_\beta \leq V_\beta \cap Q_{\alpha'}$. Hence, using Lemma 19.13(vi),

$$Z_\beta = \left[U_\rho \cap Q_\alpha, Y_\beta\right] \leq \left[U_\rho, V_\beta \cap Q_{\alpha'}\right] \leq V_{\alpha'} \cap V_{\rho+1},$$

contradicting $Z_\beta \not\leq V_{\alpha'}$. Thus $b = 5$.

In order to complete the proof of Theorem 19.2 we must now analyse Case 5 when $b = 5$. This is the subject of the next four results; we uncover the final contradiction in Lemma 19.18.

Lemma 19.15. *Let $\lambda \in O(S_6)$.*

(i) *H_λ is abelian.*

(ii) *If $H_\lambda \leq Q_{\lambda+2}$ for some $\lambda + 2 \in V(\Gamma)$ with $d(\lambda, \lambda+2) = 2$, then $H_\lambda \leq Z(W_\lambda)$.*

(iii) *$\eta(G_\lambda, W_\lambda / H_\lambda) \neq 0$; in particular $\eta(G_\lambda, W_\lambda) \geq 3$.*

Proof. The key observation is the following

(19.15.1). Let $\lambda - 2, \lambda + 2 \in V(\Gamma)$ with $d(\lambda - 2, \lambda) = 2 = d(\lambda, \lambda + 2)$ and $d(\lambda - 2, \lambda + 2) = 4$ If $Y_{\lambda-2} \leq Q_{\lambda+2}$, then $[Y_{\lambda-2}, V_{\lambda+2}] = 1$.

Suppose that $[Y_{\lambda-2}, V_{\lambda+2}] \neq 1$. Then there exists $\lambda + 3 \in \in \Delta(\lambda + 2) \setminus \{\lambda + 1\}$ such that $[Y_{\lambda-2}, Z_{\lambda+3}] \neq 1$. So $Y_{\lambda-2} \leq G_{\lambda+3}$ and $Y_{\lambda-2} \not\leq Q_{\lambda+3}$. For $\mu \in \Delta(\lambda + 3)$ we have $Y_\mu \leq V_\mu \leq G_\lambda$ and so if $[Y_\mu, V_\lambda] \neq 1$ we then get $Z_\mu = [Y_\mu, V_\lambda] \leq V_\lambda$. But then $Y_{\lambda-2}$ centralizes

$Z_{\lambda+2}Z_\mu = Z_{\lambda+3}$. Thus $[Y_\mu, V_\lambda] = 1$ and so $[F_{\lambda+3}, V_\lambda] = 1$. Consequently $F_{\lambda+3} \leq G_{\lambda-2}$ and therefore $[Y_{\lambda-2}, F_{\lambda+3}] \leq Z_{\lambda-2}$ which implies that $\eta(G_{\lambda+3}, F_{\lambda+3}) \leq 1$. This contradicts Lemma 12.5(i) and so (19.15.1) holds.

We now establish parts (i) and (ii). If (i) is false, then there exists $\lambda-2, \lambda+2 \in V(\Gamma)$ with $d(\lambda-2,\lambda) = 2 = d(\lambda,\lambda+2)$ and $d(\lambda-2,\lambda+2) = 4$ for which $[Y_{\lambda-2}, Y_{\lambda+2}] \neq 1$. Thus $[Y_{\lambda-2}, Y_{\lambda+2}] = Z_{\lambda-2} = Z_{\lambda+2}$. Since $V_{\lambda+2} \leq$ $\leq G_{\lambda-2}$, $[Y_{\lambda-2}, V_{\lambda+2}] = Z_{\lambda-2}$ whence $[Y_{\lambda-2}, V_{\lambda+2}] = Z_{\lambda+2}$, which forces $Y_{\lambda-2} \leq Q_{\lambda+2}$. Now applying (19.15.1) gives $[Y_{\lambda-2}, V_{\lambda+2}] = 1$, a contradiction. Turning to part (ii), Lemma 11.1(vii) and the assumption that $H_\lambda \leq Q_{\lambda+2}$ for some $\lambda+2 \in V(\Gamma)$ with $d(\lambda,\lambda+2) = 2$ imply that $H_\lambda \leq Q_{\lambda+2}$ for all $\lambda+2 \in V(\Gamma)$ with $d(\lambda,\lambda+2) = 2$. Now (ii) follows immediately from (19.15.1).

Finally, considering part (iii), we assume that $\eta(G_\lambda, W_\lambda/H_\lambda) = 0$. Then $W_\lambda = H_\lambda V_{\lambda+2}$ by Lemma 11.1(vii) and so

$$\begin{aligned}[W_\lambda, W_\lambda] &= [H_\lambda V_{\lambda+2}, H_\lambda V_{\lambda+2}] \\ &= [H_\lambda, V_{\lambda+2}],\end{aligned}$$

using part (i). Since $V_\lambda \leq Z(W_\lambda)$, $[H_\lambda, V_\lambda \cap V_{\lambda+2}] = 1$ and thus

$$[W_\lambda, W_\lambda] = [H_\lambda, V_{\lambda+2}] \leq (V_\lambda \cap V_{\lambda+2})Y_{\lambda+2} \cong E(2^4).$$

For $\lambda-2 \in V(\Gamma)$ with $d(\lambda-2,\lambda) = 2$, $[Y_{\lambda-2}, V_{\lambda+2}] \leq Z_{\lambda-2}$ and therefore $[H_\lambda, V_{\lambda+2}] \leq V_\lambda$. Consequently $[W_\lambda, W_\lambda] \leq Y_\lambda$. But then

$$\begin{aligned}[Y_{\lambda-2}, V_{\lambda+2}] &\leq [W_\lambda, W_\lambda] \cap Z_{\lambda-2} \\ &\leq Y_\lambda \cap Z_{\lambda-2} = 1,\end{aligned}$$

whence $[W_\lambda, W_\lambda] = [H_\lambda, V_{\lambda+2}] = 1$ contradicting $b = 5$! Therefore $\eta(G_\lambda, W_\lambda/H_\lambda) \neq 0$ and Lemma 19.15 is proven.

LEMMA 19.16. *Let* $(\alpha,\alpha') \in \mathcal{C}$. *Then* $(V_\beta \cap V_{\alpha+3})Y_{\alpha+3} \neq (V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha+3}$.

PROOF. Supposing $(V_\beta \cap V_{\alpha+3})Y_{\alpha+3} = (V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha+3}$ we show this leads to a contradiction. From $[W_{\alpha'}, V_{\alpha'}] = 1$ and the parabolic argument we have

(19.16.1) $W_{\alpha'} \leq G_{\alpha+2}$

Since $H_{\alpha'}$ is abelian by Lemma 19.15(i) and $V_\beta \cap V_{\alpha+3} \leq$ $\leq (V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha+3} \leq H_{\alpha'}$, we also obtain

(19.16.2) (i) $H_{\alpha'} \leq Q_{\alpha+2} \leq G_\beta$; and
(ii) $[V_\beta, V_{\alpha'}] \leq [V_\beta, H_{\alpha'}] \leq (V_\beta \cap V_{\alpha+3})Y_\beta$.

(19.16.3) (i) $[V_\beta, H_{\alpha'}] \not\leq V_\beta \cap V_{\alpha+3}$; and
(ii) $Y_\beta \leq H_{\alpha'}$.

Suppose (i) is false. Then

$$[V_\beta, H_{\alpha'}] \leq V_\beta \cap V_{\alpha+3} \leq (V_\beta \cap V_{\alpha+3})Y_{\alpha+3} = (V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha+3}\,.$$

This then yields

$$[H_{\alpha'}/V_{\alpha'}[H_{\alpha'}, Q_{\alpha'}], V_\beta] \leq C_{H_{\alpha'}/V_{\alpha'}[H_{\alpha'},Q_{\alpha'}]}(Q_{\alpha+4}Q_{\alpha'}).$$

Consequently $V_\beta Q_{\alpha'}/Q_{\alpha'}$ acts as the central transvection of $G_{\alpha+4\alpha'}/Q_{\alpha'}$ on the non-central chief factor within $H_{\alpha'}/V_{\alpha'}[H_{\alpha'}, Q_{\alpha'}]$. (Note that $\eta(G_{\alpha'}, H_{\alpha'}/V_{\alpha'}[H_{\alpha'}, Q_{\alpha'}]) \neq 0$ since $H_{\alpha'} \neq V_{\alpha'}F_{\alpha+4}$.) Because $[V_\beta, V_{\alpha'}] \leq$ $\leq [V_\beta, H_{\alpha'}] \leq V_\beta \cap V_{\alpha+3}$ we have $[V_\beta, V_{\alpha'}] \leq V_{\alpha+3} \cap V_{\alpha'}$ and thus $V_\beta Q_{\alpha'}/Q_{\alpha'}$ does not act as a transvection on $V_{\alpha'}/Z_{\alpha'}$. Hence $|[V_\beta, V_{\alpha'}]| \geq 2^2$. Note that this implies $V_{\alpha'} \not\leq Q_\beta$. Since $\eta(G_{\alpha'}, H_{\alpha'}/V_{\alpha'}) \neq 0$, $[V_\beta, H_{\alpha'}] \not\leq V_{\alpha'}$ and thus $[V_\beta, V_{\alpha'}] \neq [V_\beta, H_{\alpha'}]$. Thus we have $|[V_\beta, V_{\alpha'}]| = 2^2$ and $[V_\beta, H_{\alpha'}] =$ $= V_\beta \cap V_{\alpha+3}$. So $Z_{\alpha'} \not\leq [V_\beta, V_{\alpha'}]$ for otherwise we get $V_\beta Q_{\alpha'}/Q_{\alpha'}$ acting as a transvection on $V_{\alpha'}/Z_{\alpha'}$. Therefore $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$ and so, as $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = 2$, $V_{\alpha'}Q_\beta/Q_\beta$ acts as a transvection on V_β/Z_β. Hence $Z_\beta \leq$ $\leq [V_\beta, V_{\alpha'}]$ and thus $Z_{\alpha+2}[V_\beta, V_{\alpha'}] \leq V_{\alpha+3} \cap V_{\alpha'}$. Since $[V_\beta, V_{\alpha'}] \leq V_\beta \cap V_{\alpha+3}$, $V_{\alpha'}Q_\beta/Q_\beta$ cannot act as a central transvection on V_β/Z_β and so $Z_{\alpha+2} \neq [V_\beta, V_{\alpha'}]$, whence $Z_{\alpha+2}[V_\beta, V_{\alpha'}] = V_{\alpha+3} \cap V_{\alpha'}$ Thus we conclude that

$$\begin{aligned}[V_\beta, H_{\alpha'}] = V_\beta \cap V_{\alpha+3} &= Z_{\alpha+2}[V_\beta, V_{\alpha'}] \\ &= V_{\alpha+3} \cap V_{\alpha'} \leq V_{\alpha'},\end{aligned}$$

against $\eta(G_{\alpha'}, H_{\alpha'}/V_{\alpha'}) \neq 0$. This completes the proof of part (i).

Combining $[(V_\beta \cap V_{\alpha+3})Y_\beta : V_\beta \cap V_{\alpha+3}] = 2$ and (19.16.2)(ii) with part (i) gives

$$\begin{aligned}(V_\beta \cap V_{\alpha+3})Y_\beta &= (V_\beta \cap V_{\alpha+3})[V_\beta, H_{\alpha'}] \\ &\leq (V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha+3}[V_\beta, H_{\alpha'}] \leq H_{\alpha'}.\end{aligned}$$

Therefore $Y_\beta \leq H_{\alpha'}$, so proving part (ii).

Employing (19.16.3)(ii) we can now investigate the action of V_β upon $W_{\alpha'}/H_{\alpha'}$.

(19.16.4) Put $\widetilde{W}_{\alpha'} = W_{\alpha'}/H_{\alpha'}$. Then $[\widetilde{W}_{\alpha'} : C_{\widetilde{W}_{\alpha'}}(V_\beta)] = 2$ with $C_{\widetilde{W}_{\alpha'}}(V_\beta)$ normalized by $Q_{\alpha+4}$. In particular, $V_\beta Q_{\alpha'}/Q_{\alpha'}$ acts as the central transvection on the non-central chief factor within $\widetilde{W}_{\alpha'}$ and $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = 2$.

First observe that $C_{W_{\alpha'}}(Y_{\alpha+3}) = W_{\alpha'} \cap C_{G_{\alpha+3}}(Y_{\alpha+3})$ is normalized by $Q_{\alpha+4}$ and that $[W_{\alpha'} : C_{W_{\alpha'}}(Y_{\alpha+3})] \leq 2$. Also we have that $C_{W_{\alpha'}}(Y_{\alpha+3})$ centralizes $(V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha+3} = (V_\beta \cap V_{\alpha+3})Y_{\alpha+3}$. So $C_{W_{\alpha'}}(Y_{\alpha+3})$ centralizes $Z_{\alpha+2}$ whence, by (19.16.1), $C_{W_{\alpha'}}(Y_{\alpha+3}) \leq Q_{\alpha+2} \leq G_\beta$. Further, because $C_{W_{\alpha'}}(Y_{\alpha+3})$ centralizes $V_\beta \cap V_{\alpha+3}$ we have that

$$\left[C_{W_{\alpha'}}(Y_{\alpha+3}), V_\beta\right] \leq \left(V_\beta \cap V_{\alpha+3}\right)Y_\beta.$$

Using (19.16.3)(ii) we see that

$$\left(V_\beta \cap V_{\alpha+3}\right)Y_\beta \leq \left(V_{\alpha+3} \cap V_{\alpha'}\right)Y_{\alpha+3}H_{\alpha'} \leq H_{\alpha'},$$

from which we deduce that $\widetilde{C_{W_{\alpha'}}(Y_{\alpha+3})} \leq C_{\widetilde{W}_{\alpha'}}(V_\beta)$. By Lemma 19.15.(iii) $\widetilde{C_{W_{\alpha'}}(Y_{\alpha+3})} = C_{\widetilde{W}_{\alpha'}}(V_\beta)$ and now (19.16.4) follows.

(19.16.5) $V_\beta Q_{\alpha'}/Q_{\alpha'}$ acts as a transvection on $V_{\alpha'}/Z_{\alpha'}$

Assume that $V_\beta Q_{\alpha'}/Q_{\alpha'}$ does not act as a transvection on $V_{\alpha'}/Z_{\alpha'}$. So $V_\beta Q_{\alpha'}/Q_{\alpha'}$ does not act as a transvection on $V_{\alpha'}/Y_{\alpha'}$ and therefore $(V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha'} = [V_\beta, V_{\alpha'}]Y_{\alpha'}$. From $|[V_\beta, V_{\alpha'}]| \geq 2^2$ we have $V_{\alpha'} \not\leq Q_\beta$; so $(\alpha'+1, \beta) \in \mathcal{C}$ for some $\alpha'+1 \in \Delta(\alpha')$. By (19.16.4) $V_\beta Q_{\alpha'}/Q_{\alpha'}$ acts as a transvection on the non-central chief factor within $W_{\alpha'}/H_{\alpha'}$, and so this non-central chief factor is not isomorphic to $V_{\alpha'}/Y_{\alpha'}$. Applying (19.16.4) to $(\alpha'+1, \beta)$ yields that $V_{\alpha'}Q_\beta/Q_\beta$ acts as a transvection on the non-central chief factor within W_β/H_β and consequently $V_{\alpha'}Q_\beta/Q_\beta$ does not act as a transvection on V_β/Z_β. Thus $(V_\beta \cap V_{\alpha+3})Y_\beta = [V_\beta, V_{\alpha'}]Y_\beta$. If $W_{\alpha'} \leq G_\beta$, then, as $[[V_\beta, V_{\alpha'}], W_{\alpha'}] = 1$, using (19.16.3)(ii) we get

$$\begin{aligned}\left[W_{\alpha'}, V_\beta\right] \leq \left(V_\beta \cap V_{\alpha+3}\right)Y_\beta &= \left[V_\beta, V_{\alpha'}\right]Y_\beta \\ &\leq H_{\alpha'}.\end{aligned}$$

This contradicts the fact that $\eta(G_{\alpha'}, W_{\alpha'}/H_{\alpha'}) \neq 0$. Thus $W_{\alpha'} \not\leq G_\beta$. In particular, (19.16.1) implies that $Z_\beta \not\leq [V_\beta, V_{\alpha'}]$. By (19.16.3)(ii) applied to $(\alpha'+1, \beta)$ $Y_{\alpha'} \leq H_\beta$ and so a symmetrical argument gives $Z_{\alpha'} \not\leq [V_\beta, V_{\alpha'}]$ also. Therefore $[V_\beta, V_{\alpha'} \cap Q_\beta] = 1 = [V_{\alpha'}, V_\beta \cap Q_{\alpha'}]$ which, as $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = = 2 = |V_{\alpha'}Q_\beta/Q_\beta|$, gives the untenable $|[V_\beta, V_{\alpha'}]| = 2$. So we conclude that $V_\beta Q_{\alpha'}/Q_{\alpha'}$ acts as a transvection on $V_{\alpha'}/Z_{\alpha'}$.

Now (19.16.4) and (19.16.5) force $[V_\beta, V_{\alpha'}] \not\leq V_{\alpha+3}$. Consequently, as

$[V_\beta, V_{\alpha'}] \le (V_\beta \cap V_{\alpha+3})Y_\beta$, we deduce that

$$\big(V_\beta \cap V_{\alpha+3}\big)Y_\beta = \big(V_\beta \cap V_{\alpha+3}\big)\big[V_\beta, V_{\alpha'}\big].$$

Since $[V_\beta, V_{\alpha'}] \le V_{\alpha'}$ and $[V_\beta \cap V_{\alpha+3} : V_\beta \cap V_{\alpha+3} \cap V_{\alpha'}] \le 2$, (19.16.2)(ii) implies that $|[V_\beta, H_{\alpha'}/V_{\alpha'}]| \le 2$. So V_β acts as a transvection on the non-central chief factor within $H_{\alpha'}/V_{\alpha'}$ and thus, by (19.16.5), this non-central chief factor is isomorphic to $V_{\alpha'}/Y_{\alpha'}$.

From $[V_\beta, V_{\alpha'}] \le (V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha'}$ we also have $(V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha'} = (V_{\alpha+3} \cap V_{\alpha'})[V_\beta, V_{\alpha'}]$. Therefore

$$V_{\alpha+3}Y_\beta = V_{\alpha+3}\big[V_\beta, V_{\alpha'}\big] = V_{\alpha+3}Y_{\alpha'}.$$

(Recall that $H_{\alpha+3}/V_{\alpha+3} = \langle (Y_\beta V_{\alpha+3}/V_{\alpha+3})^{G_{\alpha+3}} \rangle$.) Applying Proposition 2.15 to $H_{\alpha+3}/V_{\alpha+3}[H_{\alpha+3}, Q_{\alpha+3}]$ gives that $H_{\alpha+3}/X_{\alpha+3} \cong 4$ for some $X_{\alpha+3} \trianglelefteq G_{\alpha+3}$, $H_{\alpha+3} \ge X_{\alpha+3} \ge V_{\alpha+3}[H_{\alpha+3}, Q_{\alpha+3}]$. Put $\bar{H}_{\alpha+3} = H_{\alpha+3}/X_{\alpha+3}$. Then

$$\bar{Y}_\beta = \bar{Y}_{\alpha'} \trianglelefteq \big\langle Q_{\alpha+2}, Q_{\alpha+3}, Q_{\alpha+4} \big\rangle = \big\langle G_{\alpha+2\alpha+3}, G_{\alpha+3\alpha+4} \big\rangle = P,$$

and so $\bar{H}_{\alpha+3}\big|_P \cong \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. But, since $(V_\beta \cap V_{\alpha+3})Y_{\alpha+3} = (V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha+3}$, $(V_{\alpha+3}/Y_{\alpha+3})|_P \cong \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and so $\bar{H}_{\alpha+3}$ and $V_{\alpha+3}/Y_{\alpha+3}$ are not isomorphic $G_{\alpha+3}$-chief factors, contrary to our earlier deduction. This contradiction completes the proof of Lemma 19.16.

LEMMA 19.17. *For $(\alpha, \alpha') \in \mathcal{C}$ we have $[V_\beta, V_{\alpha'}] \le V_{\alpha+3}$.*

PROOF. Assuming $R := [V_\beta, V_{\alpha'}] \not\le V_{\alpha+3}$ we derive a contradiction.
Immediately this yields that

(19.17.1) $$V_{\alpha'} \not\le Q_\beta.$$

Put $P = \langle G_{\alpha+2\alpha+3}, G_{\alpha+3\alpha+4} \rangle$ and $\bar{H}_{\alpha+3} = H_{\alpha+3}/V_{\alpha+3}[H_{\alpha+3}, Q_{\alpha+3}]$; recall that $\eta(G_{\alpha+3}, \bar{H}_{\alpha+3}) \ne 0$.

(19.17.2) (i) $P \ne G_{\alpha+3}$; and
(ii) $Z_{\alpha+2} = Z_{\alpha+4} \le V_\beta \cap V_{\alpha+3} \cap V_{\alpha'}$.

From $R \le (V_\beta \cap V_{\alpha+3})Y_\beta$, $R \le (V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha'}$ and $R \not\le V_{\alpha+3}$ we have

$$V_{\alpha+3}Y_\beta = V_{\alpha+3}R = V_{\alpha+3}Y_{\alpha'}.$$

So $\bar{Y}_\beta \trianglelefteq P$. Since $\bar{H}_{\alpha+3} = \langle \bar{Y}_\beta{}^{G_{\alpha+3}} \rangle$, Propositions 2.6(i) and 2.9(i) give $\bar{H}_{\alpha+3}/\bar{X}_{\alpha+3} \cong 4$ (some $X_{\alpha+3} \trianglelefteq G_{\alpha+3}$). Clearly then $P \neq G_{\alpha+3}$ and this together with Lemma 19.16 yields $Z_{\alpha+3}Y_{\alpha+3} = Z_{\alpha+4}Y_{\alpha+3}$. Because $O^2(P)$ centralizes $Z_{\alpha+2}Y_{\alpha+3}$ and there exists $g \in O^2(P)$ such that $(\alpha+2)\cdot g = \alpha+4$ we have established (19.17.2).

(19.17.3) V_β acts as a transvection upon $V_{\alpha'}/Z_{\alpha'}$ and $V_{\alpha'}$ acts as a transvection upon V_β/Z_β.

Suppose V_β does not act as a transvection upon $V_{\alpha'}/Z_{\alpha'}$. Then V_β does not act as a transvection upon $V_{\alpha'}/Y_{\alpha'}$ and hence R covers $(V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha'}/Y_{\alpha'}$. Consequently $W_\beta \cap G_{\alpha'}$ centralizes $(V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha'}/Y_{\alpha'}$ and therefore

$$\left[W_\beta \cap G_{\alpha'}, V_{\alpha'}\right] \leq \left(V_{\alpha+3} \cap V_{\alpha'}\right)Y_{\alpha'}.$$

From (19.17.2)(ii) we have $|R(V_\beta \cap V_{\alpha+3} \cap V_{\alpha'})| \geq 2^3$, and so $|[(W_\beta \cap G_{\alpha'})/V_\beta, V_{\alpha'}]| \leq 2$. Since $\eta(G_\beta, W_\beta) \geq 3$ by Lemma 19.15(iii) and $[W_\beta : W_\beta \cap G_{\alpha'}] \leq 2$, we deduce that all the non-central G_β-chief factors in W_β are isomorphic. Therefore V_β does not act as a transvection on each of the non-central $G_{\alpha'}$-chief factors in $W_{\alpha'}$. Now

$$\left[W_{\alpha'} \cap G_\beta, V_\beta\right] \leq \left[Q_{\alpha+2}, V_\beta\right] \cong E\left(2^5\right).$$

and

$$R\left(V_\beta \cap V_{\alpha+3} \cap V_{\alpha'}\right) \leq \left[Q_{\alpha+2}, V_\beta\right]$$

imply that $|[(W_{\alpha'} \cap G_\beta)/V_{\alpha'}, V_\beta]| \leq 2^2$ which is incompatible with V_β not acting as a transvection on each of the non-central $G_{\alpha'}$-chief factors in $W_{\alpha'}$. This together with a symmetric argument establishes (19.17.3)

From (19.17.3) we have

(19.17.4) $\left|V_\beta Q_{\alpha'}/Q_{\alpha'}\right| = 2 = \left|V_{\alpha'}Q_\beta/Q_\beta\right|$ and $|R| \leq 2^2$.

(19.17.5) $Z_\beta = Z_{\alpha'}$.

Assume that $Z_\beta \neq Z_{\alpha'}$. Then at least one of $Z_\beta \not\leq R$ and $Z_{\alpha'} \not\leq R$ holds. For otherwise, by (19.17.4), we get $R = Z_\beta Z_{\alpha'} \leq Z_{\alpha+2} = Z_{\alpha+4}$, against $R \not\leq V_{\alpha+3}$. So without loss of generality $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$; as a result we may suppose α is chosen so as $\langle G_{\alpha\beta}, V_{\alpha'} \rangle = G_\beta$. If $F_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$, then, using (19.17.2)(ii)

$$\left[F_\alpha, V_{\alpha'}\right] \leq RZ_{\alpha'} \leq RZ_{\alpha+4} \leq V_\beta,$$

contradicting $\eta(G_\beta, H_\beta/V_\beta) \neq 0$. So $F_\alpha Q_{\alpha'} \neq V_\beta Q_{\alpha'}$. Hence $[V_{\alpha'} \cap Q_\beta, V_\beta] = Z_\beta$ else, as $F_\alpha \leq G_{\alpha'}$, we get $[F_\alpha, V_{\alpha'} \cap Q_\beta] = 1$ which, by (19.17.4), gives $F_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$. By (19.17.1) we must have $|R| = 2^2$ which, as $Z_{\alpha'} \not\leq R$, contradicts (19.17.3). Thus $Z_\beta = Z_{\alpha'}$.

(19.17.6) $|F_\alpha Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$ and (so) $\big[V_{\alpha'} \cap Q_\beta, V_\beta\big] \neq 1$.

If (19.17.6) were false, then $F_\alpha Q_{\alpha'} = Z_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$. Arguing as in (19.17.5) and using the fact that $Z_\beta = Z_{\alpha'}$ we may deduce a contradiction to $\eta(G_\beta, H_\beta/V_\beta) \neq 0$.

Note that $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] \neq 1$ by (19.17.1) and (19.17.6). So, as $[V_{\alpha'}, W_{\alpha'}] = 1$, $V_\beta \cap W_{\alpha'} \lneqq V_\beta \cap Q_{\alpha'}$, whence $|V_\beta \cap W_{\alpha'}| \leq 2^4$. Thus $|[V_\beta, W_{\alpha'} \cap G_\beta]| \leq 2^4$ and consequently $|(W_{\alpha'} \cap G_\beta)Q_\beta/Q_\beta| \leq 2^3$. Therefore $[W_{\alpha'} : W_{\alpha'} \cap Q_\alpha] \leq 2^5$. Using (19.17.5) we deduce that $[W_{\alpha'} \cap Q_\alpha, F_\alpha] \leq Z_{\alpha'}$ and now (19.17.6) forces $\eta(G_{\alpha'}, W_{\alpha'}) \leq 2$. This is the desired contradiction which concludes the proof of Lemma 19.17.

Lemma 19.18. *A contradiction*

Proof. Let $(\alpha, \alpha') \in \mathcal{C}$, and put $R = [V_\beta, V_{\alpha'}]$.

(19.18.1) $H_\beta \cap V_{\alpha+3} = \big(V_\beta \cap V_{\alpha+3}\big)Y_{\alpha+3}$.

If (19.18.1) is false, then, as $Q_{\alpha+2}$ normalizes $H_\beta \cap V_{\alpha+3}$,

$$H_\beta \cap V_{\alpha+3} \geq \big[V_{\alpha+3}, Q_{\alpha+2}\big].$$

Thus, since H_β is abelian by Lemma 19.15.(i), either $H_\beta Q_{\alpha+3}/Q_{\alpha+3}$ acts as the central transvection on $V_{\alpha+3}/Z_{\alpha+3}$ or $H_\beta \leq Q_{\alpha+3}$. If the former possibility holds, then there exists $\beta - 2$ with $d(\beta, \beta-2) = 2$ such that $Y_{\beta-2}Q_{\alpha+3}/Q_{\alpha+3}$ acts as the central transvection on $V_{\alpha+3}/Z_{\alpha+3}$; also $(\beta-3, \alpha+3) \in \mathcal{C}$ for some $\beta - 3 \in \Delta(\beta-2)$. But then, using Lemma 19.17,

$$\big[Y_{\beta-2}, V_{\alpha+3}\big] \leq \big[V_{\beta-2}, V_{\alpha+3}\big] \leq V_\beta \cap V_{\alpha+3},$$

a contradiction. On the other hand $H_\beta \leq Q_{\alpha+3}$ yields $H_\beta \leq Z(W_\beta)$ by Lemma 19.15(ii) and hence $W_\beta Q_{\alpha+3}/Q_{\alpha+3}$ acts as the central transvection on $V_{\alpha+3}/Z_{\alpha+3}$ (recall that $W_\beta \not\leq Q_{\alpha+3}$ by Lemma 11.1(vii), as $b = 5$). So we obtain $(\beta-3, \alpha+3) \in \mathcal{C}$ with $V_{\beta-2}Q_{\alpha+3}/Q_{\alpha+3}$ acting as the central transvection on $V_{\alpha+3}/Z_{\alpha+3}$ and again Lemma 19.17 gives a contradiction. Therefore (19.18.1) holds.

By Lemmas 19.16 and 19.17, since $R \not\leq Z_{\alpha+4}$,

(19.18.2) $Z_{\alpha+4} \neq V_\beta \cap V_{\alpha+3} \cap V_{\alpha'} = RZ_{\alpha+3} \cong E\big(2^2\big)$.

(19.18.3) $Z_{\alpha'} \not\leq H_\beta$

Suppose that $Z_{\alpha'} \leq H_\beta$ holds. Then (19.18.1) forces $Z_{\alpha'} \leq$ $\leq (V_\beta \cap V_{\alpha+3})Y_{\alpha+3}$ Hence, by (19.18.2),

$$V_{\alpha+3} \cap V_{\alpha'} = Z_{\alpha+4}R = Z_{\alpha'}Z_{\alpha+3}R \leq \big(V_\beta \cap V_{\alpha+3}\big)Y_{\alpha+3},$$

against Lemma 19.16. Thus $Z_{\alpha'} \not\leq H_\beta$.

(19.18.4) $V_{\alpha'} \not\leq Q_\beta$.

Assume that $V_{\alpha'} \leq Q_\beta$. Then $R = Z_\beta$ and so $V_\beta Q_{\alpha'}/Q_{\alpha'}$ acts as a transvection on $V_{\alpha'}/Z_{\alpha'}$. Because $[Y_{\alpha-1}, V_{\alpha'} \cap Q_\alpha] = 1$ for $\alpha - 1 \in \Delta(\alpha) \setminus \{\beta\}$ we see that $F_\alpha Q_{\alpha'} = Z_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$. Since $Z_{\alpha'} \not\leq H_\beta$ by (19.18.3), this implies that $[F_\alpha, V_{\alpha'}] = [Z_\alpha, V_{\alpha'}] \leq Z_\alpha$, contradicting $\eta(G_\alpha, F_\alpha/Z_\alpha) = 1$, and so we have (19.18.4).

In view of (19.18.4) we have symmetry and thus

(19.18.5) $Z_\beta \not\leq H_{\alpha'}$.

(19.18.6) $\big|V_{\alpha'}Q_\beta/Q_\beta\big| \geq 2^2 \leq \big|V_\beta Q_{\alpha'}/Q_{\alpha'}\big|$ and $|R| \geq 2^2$.

Since $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$ by (19.18.3), we may choose α so as $\langle G_{\alpha\beta}, V_{\alpha'}\rangle =$ $= G_\beta$. Also, by (19.18.5), $[V_{\alpha'} \cap Q_\beta, V_\beta] = 1$ and thus $[F_\alpha, V_{\alpha'} \cap Q_\beta] = 1$. If $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] = 2$, then we get $F_\alpha Q_{\alpha'} = Z_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$ whence, by (19.18.3), $[F_\alpha, V_{\alpha'}] = R$, which gives $\eta(G_\beta, H_\beta/V_\beta) = 0$. Thus $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] \geq 2^2$ and by symmetry (19.18.6) follows.

(19.18.7) $\big[H_\beta, V_{\alpha+3}\big] = 1$ and $H_\beta \leq G_{\alpha'}$.

We claim that $H_\beta \leq Q_{\alpha+3}$ for $H_\beta \not\leq Q_{\alpha+3}$ implies there exists $(\beta - 3, \alpha + 3) \in \mathcal{C}$ with $Z_{\beta-2} = [Y_{\beta-2}, V_{\alpha+3}] \leq V_{\alpha+3}$ which is ruled out by (19.18.5) applied to $(\beta - 3, \alpha + 3)$. Hence $H_\beta \leq Z(W_\beta)$ by Lemma 19.15(ii) which then yields (19.18.7).

We are now in a position to deduce the desired contradiction. From (19.18.7) $[H_\beta, V_{\alpha'}] \leq (V_{\alpha+3} \cap V_{\alpha'})Y_{\alpha'}$ and then $|R| \geq 2^2$ and $Z_{\alpha'} \not\leq H_\beta$ force $|[H_\beta/V_\beta, V_{\alpha'}]| \leq 2$. Therefore, by (19.18.6), $\eta(G_\beta, H_\beta/V_\beta) = 0$ and we have our contradiction.

Now Theorem 9.2 is a consequence of Lemma 19.18.

20. The main Theorem.

We now survey the route that we have traversed in these seven parts and check that we have indeed reached our destination. In Theorem 4.12(d), 5.1 and 7.7 the non-commuting case was handled giving the conclusion $b \in \{1, 2\}$. Proposition 9.1 considers the commuting case when $\alpha \in O(S_6)$ (where $(\alpha, \alpha') \in \mathcal{C}$) and yields that $b \in \{1, 3\}$. Theorem 12.1 and Lemma 12.6 produce a five case subdivision in the remaining commuting case. Each of these possibilities are analysed in Theorems 13.1, 13.11, 14.1, 18.1 and 19.2, the end result being that $b \in \{3, 5\}$. So the MAIN THEOREM is proven - and we've arrived, finally!

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