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Finitely Presented Modules over Right Non-Singular Rings.

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Abstract - This paper characterizes the right non-singular rings R for which $M/Z(M)$ is projective whenever M is a cyclically (finitely) presented module. Several related results investigate right semi-hereditary rings.

1. Introduction.

The straightforward attempt to extend the notion of torsion-freeness from integral domains to non-commutative rings encounters immediate difficulties. To overcome these, one can concentrate on either the computational or the homological properties of torsion-free modules. Goodearl and others took the first approach when they introduced the notion of a non-singular module [8]. A right R-module M is *non-singular* if $Z(M) = 0$ where $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of $R\}$ denotes the *singular submodule of* M. On the other hand, M is *singular* if $Z(M) = M$. Moreover, a submodule U of an R-module M is *$\mathcal{S}$-closed* if M/U is non-singular. Finally, R is a *right non-singular ring* if R_R is non-singular.

The right non-singular rings are precisely the rings which have a regular, right self-injective maximal right ring of quotients, which will be denoted by Q^r (see [8] and [11] for details). Following [11, Chapter XI], Q^r is *a perfect left localization of* R if Q^r is flat as a right R-module and the multiplication map $Q^r \otimes_R Q^r \to Q^r$ is an isomorphism. In particular, Q^r is a perfect left localization of R if and only if every finitely generated non-singular right R-module can be embedded into a projective module ([8] and [11]). We call a right non-singular ring with this property *right strongly non-singular*.

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Hattori took the second approach by defining M to be *torsion-free* if $\operatorname{Tor}_1^R(M, R/Rr) = 0$ for all $r \in R$ [9]. The classes of torsion-free and non-singular right R-modules coincide if and only if R is a right Utumi p.p.-ring without an infinite set of orthogonal idempotents [3, Theorem 3.7]. Here, R is a right *p.p.-ring* if all principal right ideals of R are projective. Moreover, a right non-singular ring R is a *right Utumi-ring* if every $\mathcal{S}$-closed right ideal of R is a right annihilator.

Closely related to the notion of torsion-freeness are those of purity and relative divisibility. A sequence of right R-modules is *pure-exact (RD-exact)* if every finitely presented (cyclically presented) module is projective with respect to it. Investigating RD- and pure-projective modules leads to the investigation of the condition that $M/Z(M)$ is projective. The dual question when $Z(M)$ is injective has been addressed in [8, Page 48, Example 24]. Section 2 discusses the question for which rings $M/Z(M)$ is projective for all RD-projective modules M. Theorem 2.1 shows that, provided R has no infinite set of orthogonal idempotents, these are precisely the right Utumi p.p.-rings discussed in [3]. The structure of pure-projective right R-modules was described in [4] in case that R is a right strongly non-singular, right semi-hereditary ring R without an infinite set of orthogonal idempotents. Theorem 2.3 shows that these conditions on R are not only sufficient, but also necessary for the structure-theorem (part b) of Theorem 2.3 to hold.

Prüfer domains can be characterized as the domains with the property that, whenever a torsion-free module M contains a projective submodule U with M/U finitely generated, then M is projective and M/U is finitely presented [7, Chapter VI]. We show that the right non-singular rings having the corresponding property for non-singular modules are precisely the right strongly non-singular, right semi-hereditary rings of finite right Goldie dimension (Theorem 3.1).

Section 4 investigates pure-projective modules over right hereditary rings. As part of our discussion, we obtain a characterization of the right Noetherian right hereditary rings with the restricted right minimum condition which are right strongly non-singular. The last results of this paper demonstrate that right invertible submodules of Q_R^r which were introduced in [11, Chapters II.4 and IX.5] may fail to share many of the important properties of invertible modules over integral domains. For instance, the lattice of finitely generated right ideals over a Prüfer domain is distributive, i.e. $I \cap (J + K) = (I \cap J) + (I \cap K)$ for all finitely generated ideals I, J, and K of R [7, Theorem III.1.1]. Example 4.7 shows that there exists a right strongly non-singular, hereditary, right and left Noetherian ring whose finitely generated right ideals do not have this property.

2. RD-Projective Modules.

Let U be a submodule of a non-singular module M. The *$\mathcal{S}$-closure of U in M* is the submodule V of M which contains U such that $V/U = Z(M/U)$.

THEOREM 2.1. *The following are equivalent for a right non-singular ring R without an infinite set of orthogonal idempotents:*

a) *R is a right Utumi p.p.-ring.*
b) *$M/Z(M)$ is projective for every RD-projective module M.*

PROOF. *a)* $\Rightarrow$ *b)*: Since every RD-projective module is a direct summand of a direct sum of cyclically presented modules, it suffices to verify b) in case that $M \cong R/aR$ for some $a \in R$. Let J be the right ideal of R which contains aR such that $J/aR = Z(R/aR)$. Then, $R/J \cong M/Z(M)$ is a non-singular cyclic module which is projective by [3, Corollary 3.4].

b) $\Rightarrow$ *a)*: Let I be the $\mathcal{S}$-closure of rR for some $r \in R$. Since R/rR is RD-projective, and $R/I \cong (R/rR)/Z(R/rR)$, we obtain that R/I is projective. Thus, I is generated by an idempotent.

By [3, Lemma 3.5], it suffices to show that every $\mathcal{S}$-closed right ideal J of R is generated by an idempotent. For this, select $0 \neq r_0 \in J$. Since J is $\mathcal{S}$-closed in R, it contains the $\mathcal{S}$-closure I_0 of r_0R. By what has been shown so far, $I_0 = e_0R$ for some idempotent e_0 of R. Hence, $J = e_0R \oplus [J \cap (1 - e_0)R]$. If $J \cap (1 - e_0)R \neq 0$, select a non-zero $r_1 = (1 - e_0)r_1 \in J$; and observe that $J \cap (1 - e_0)R$ is $\mathcal{S}$-closed in R. Hence, it contains the $\mathcal{S}$-closure I_1 of r_1R in R. By the previous paragraph, $I_1 = fR$ for some idempotent f of R. Write $f = (1 - e_0)s$ for some $s \in R$, and set $e_1 = f(1 - e_0)$. Since $e_0f = 0$, we have $e_1e_0 = e_0e_1 = 0$ and $e_1^2 = f^2 - f^2e_0 - fe_0f + (fe_0)^2 = f - fe_0 = e_1$. Thus, e_0 and e_1 are non-zero orthogonal idempotents with $e_1R \subseteq fR$. On the other hand, $f = f(1 - e_0)s = e_1s$ yields $fR = e_1R$. Consequently, $R = e_0R \oplus e_1R \oplus [J \cap (1 - e_0 - e_1)R]$. Continuing inductively, we can construct non-zero orthogonal idempotents $e_0, \ldots, e_{n+1} \in J$ as long as $J \cap (1 - e_0 - \ldots - e_n)R \neq 0$. Since R does not contain an infinite family of orthogonal idempotent, this process has to stop, say $J \cap (1 - e_0 - \ldots - e_n)R = 0$. Then, $e_0 + \ldots + e_m$ is an idempotent with $J = (e_0 + \ldots + e_m)R$. □

We now investigate which conditions R has to satisfy to ensure the validity of the structure theorem for pure-projectives in [4]. We want to remind the reader that a right R-module is *essentially finitely generated* if contains an essential, finitely generated submodule.

LEMMA 2.2. *The following are equivalent for a right non-singular ring R:*

a) *R is right semi-hereditary and has finite right Goldie-dimension.*

b) *A finitely generated right R-module M is finitely presented if and only if $p.d.M \leq 1$.*

PROOF. *a)* $\Rightarrow$ *b)*: Since R is right semi-hereditary, every finitely presented module has projective dimension at most 1. Conversely, whenever $M \cong R^n/U$ for some projective module U, then U is essentially finitely generated since R has finite right Goldie dimension. By Sandomierski's Theorem [5, Proposition 8.24], essentially finitely generated projective modules are finitely generated.

b) $\Rightarrow$ *a)*: Clearly, R has to be right semi-hereditary. If R has infinite right Goldie-dimension, then it contains a family $\{I_n\}_{n<\omega}$ of non-zero, finitely generated right ideals whose sum is direct. Since R is right semi-hereditary, each I_n is projective, and the same holds for $\oplus_{n<\omega} I_n$. By b), $R/\oplus_{n<\omega} I_n$ is finitely presented, a contradiction. □

THEOREM 2.3. *The following conditions are equivalent for a right non-singular ring R without an infinite set of orthogonal idempotents:*

a) *R is a (right Utumi), right semi-hereditary ring such that Q^r is a perfect left localization of R.*

b) *A right R-module M is pure-projective if and only if*

i) *$Z(M)$ is a direct summand of a direct sum of finitely generated modules of projective dimension at most* 1.

ii) *$M/Z(M)$ is projective.*

PROOF. A right strongly non-sigular, right semi-hereditary ring without an infinite set of orthogonal idempotents is a right Utumi ring [3, Theorems 3.7 and 4.2].

a) $\Rightarrow$ *b)*: By [3], R has finite right Goldie-dimension. Because of Lemma 2.2, $Z(M)$ is pure-projective if and only if condition i) in b) holds. It remains to show that $M/Z(M)$ is projective whenever M is finitely presented. However, this follows from [11] since $M/Z(M)$ is a finitely generated non-singular module, and Q^r is a perfect left localization of R.

b) $\Rightarrow$ *a*): To see that R is right semi-hereditary, consider a finitely generated right ideal I if R. The $\mathcal{S}$-closure J of I satisfies $J/I = Z(R/I)$. Hence, R/J is projective by b), and J/I has projective dimension at most 1. Since J is projective, this is only possible if I is projective.

To show that R has finite right Goldie-dimension, consider a right ideal of R of the form $a_0R \oplus \ldots \oplus a_nR \oplus \ldots$ where each $a_n \neq 0$. For $m < \omega$, let I_m be the $\mathcal{S}$-closure of $a_0R \oplus \ldots a_mR$. Since R/I_m is projective by b), I_m is generated by an idempotent e_m of R. Write $I_{m+1} = I_m \oplus [I_{m+1} \cap (1 - e_m)R]$. Observe that $[I_{m+1} \cap (1 - e_m)R]$ is generated by an idempotent f of R as a direct summand of R. Setting $d_m = f(1 - e_m)$ yields an idempotent d_m of R such that $e_md_m = d_me_m = 0$, and $I_{m+1} = I_m \oplus d_mR$ as in the proof of Theorem 2.1. Inductively, one obtains an infinite family of orthogonal idempotents $\{d_m | m < \omega\}$ of R, which is not possible. Thus, R has finite right Goldie-dimension; and every $\mathcal{S}$-closed right ideal J of R is the $\mathcal{S}$-closure of a finitely generated right ideal. By b), R/J is projective; and R is a right Utumi-ring since $J = eR$ for some idempotent e of R.

To establish that Q^r is a perfect left localization of R, it suffices to show that every finitely generated non-singular right R-module M is projective. Write $M \cong R^n/U$ and observe that U is essentially finitely generated. Select a finitely generated essential submodule V of U. Then, $U/V = Z(R^n/U)$, and M is projective by b). □

In the following, the injective hull of a module M is denoted by $E(M)$.

COROLLARY 2.4. *Let R be a right semi-hereditary ring of finite right Goldie-dimension such that Q^r is a perfect left localization of R. A right R-module M is pure-projective if and only if $M/Z(M)$ is projective and $Z(M)$ is isomorphic to a direct summand of a direct sum of finitely generated submodules of $(Q^r/R)^n$.*

PROOF. We first show that a finitely generated singular module M has projective dimension at most 1 if and only if it can be embedded into a finite direct sum of copies of Q^r/R. If $p.d.M \leq 1$, then there exist a finitely generated free module F and an essential projective submodule P of F with $M \cong F/P$. Since R has finite right Goldie-dimension, P is essentially finitely generated, and hence itself finitely generated by Sandomierski's Theorem [5, Proposition 8.24]. We can find a finitely generated projective module U such that $P \oplus U$ is a finitely generated free module, say $P \oplus U \cong R^n$. Since M is singular, $P \oplus U$ is an essential submodule of $F \oplus U$. Therefore, $F \oplus U \subseteq E(P \oplus U) = (Q^r)^n$, and $M \subseteq (Q^r/R)^n$.

On the other hand, if M is a finitely generated submodule of $(Q^r/R)^n$, then there is a finitely generated submodule U of $(Q^r)^n$ containing R^n such that $M = U/R^n$. Since R is right semi-hereditary, and since Q^r is a perfect left localization of R, U is projective and $p.d.M \leq 1$. The corollary now follows directly from Theorem 2.3. □

3. Essential Extensions of Projective Modules.

In the commutative setting, Prüfer domains are characterized by conditions b) and c.ii) [7].

THEOREM 3.1. *The following are equivalent for a right non-singular ring R:*

a) *R is a right semi-hereditary ring of finite right Goldie-dimension for which Q^r is a perfect left localization of R.*

b) *Whenever a non-singular module M contains a projective submodule U such that M/U is finitely generated, then M is projective and M/U is finitely presented.*

c) i) *R is a right p.p.-ring.*

ii) *If a finitely generated non-singular right R-module M contains an essential projective submodule U, then M is projective, and M/U is finitely presented.*

PROOF. *a) ⇒ b)*: Let W be the $\mathcal{S}$-closure of U in M. Since M/W is finitely generated as an image of M/U and non-singular, it is projective by a). Hence, $M = W \oplus P$ for some finitely generated projective module P. We may thus assume that M/U is singular.

Since R is right semi-hereditary, $U = \oplus_I U_i$ where each U_i is finitely generated [1]. Because M/U is singular and M is non-singular, U is essential in M. Thus, $E(M) = E(U) = \oplus_I E(U_i)$ in view of the fact that direct sums of non-singular injectives are injective if R has finite right Goldie-dimension [11, Proposition XIII.3.3]. Choose a finitely generated submodule V of M such that $M = U + V$. There is a finite subset J of I such that $V \subseteq \oplus_J E(U_i)$. Then, $W_1 = V + \oplus_J U_i$ is a finitely generated submodule of $\oplus_J E(U_i)$ such that $V \cap (\oplus_{I \setminus J} U_i) = 0$. Consequently, $M = W_1 \oplus \oplus_{I \setminus J} U_i$. But W_1 is projective by a) showing that M is projective and that $M/U = W_1/U$ is finitely presented.

b) ⇒ c): Observe that every finitely generated non-singular module is projective by choosing $U = 0$ in b).

c) $\Rightarrow$ *a)*: Assume that R contains a right ideal U of the form $U = \oplus_{n<\omega} a_n R$ where each $a_n \neq 0$. By part i) of c), U is projective. Choose a right ideal V of R which is maximal with respect to the property $U \cap V = 0$. Since R is right non-singular, V is an $\mathcal{S}$-closed submodule of R and $[R/V]/[U \oplus V/V] \cong R/(U \oplus V)$ is singular. Therefore, the projective module $U \oplus V/V$ is essential in the non-singular module R/V. By c), R/V is projective; and $R/(U \oplus V)$ is finitely presented. Hence, U is finitely generated which is not possible.

To see that R is a right strongly non-singular, right semi-hereditary ring, it suffices to show that a finitely generated non-singular right R-module M is projective. By [11, Proposition XII.7.2], $M \subseteq (Q^r)^n$ for some $n < \omega$. Since R has finite right Goldie dimension and R^n is essential in $(Q^r)^n$, M has finite Goldie-dimension. Therefore, M contains uniform submodules $U_1, \ldots, U_m$ such that $U_1 \oplus \ldots \oplus U_m$ is essential in M. Furthermore, we may assume that each U_i is cyclic, say $U_i = b_i R$. Since M is non-singular, $ann_r(b_i) = \{r \in R \mid b_i r = 0\}$ is not essential in R. Select $c_i \in R$ with $c_i R \cap ann_r(b_i) = 0$. Then, U_i contains a submodule $V_i \cong c_i R$. Since R is a right p.p.-ring, V_i is projective. Hence, M contains the essential projective submodule $V_1 \oplus \ldots \oplus V_m$. By c), M is projective. □

A submodule U of a module M is *tight* if both U and M/U have projective dimension at most 1. A module is *coherent* if all its finitely generated submodules are finitely presented.

COROLLARY 3.2. *Let R be a right semi-hereditary ring of finite right Goldie-dimension such that Q^r is a perfect localization of R.*

a) *A right R-module M of projective dimension at most* 1 *is coherent. Moreover, all its finitely generated submodules are tight.*

b) *If M is singular and a direct sum of countably generated modules, then p.d.$M \leq 1$ if and only if $M \subseteq (Q^r/R)^{(I)}$ for some index-set I.*

PROOF. a) Write $M \cong F/P$ where F and its submodule P are projective. If U is a finitely generated submodule of M, then there is a submodule W of F containing P with $W/P \cong U$. By Theorem 3.1, W is projective and W/P is finitely presented. Clearly, U and M/U have projective dimension at most 1.

b) Without loss of generality, we may assume that M is countably generated. If $p.d.M \leq 1$, then $M = F/P$ where F is projective and $P \cong R^{(I)}$ for some index-set I. Since P is essential in F, we have $F \subseteq E(P) \cong (Q^r)^{(I)}$

by [11, Proposition XIII.3.3] because R has finite right Goldie-dimension. Hence, $M \subseteq (Q^r/R)^{(I)}$.

Conversely, suppose that $M \subseteq (Q^r/R)^{(\omega)}$, and select a submodule U of $(Q^r)^{(\omega)}$ containing $R^{(\omega)}$ such that $M = U/R^{(\omega)}$. Choose $\{u_n \,|n < \omega\} \subseteq U$ such that $U = \Sigma_{n<\omega} u_n R + R^{(\omega)}$ and $u_0 = 0$. Set $V_\ell = R^{(\omega)} + \Sigma_{n=1}^{\ell} u_n R$. By Theorem 3.1, each V_ℓ is projective. Let $W_\ell = V_\ell/R^{(\omega)} \subseteq M$. Then, $W_0 = 0$ and $M = \cup_{n=1}^{\omega} W_n$. Observe that $W_{\ell+1}/W_\ell \cong V_{\ell+1}/V_\ell$ has projective dimension at most 1. By Auslander's Theorem, $p.d.M \leq 1$. □

4. Hereditary Rings.

The first result describes the right strongly non-singular, right Noetherian, right hereditary rings.

PROPOSITION 4.1. *The following conditions are equivalent for a right non-singular ring R of finite right Goldie dimension:*

a) *R is a right strongly non-singular, right hereditary ring without an infinite set of orthogonal idempotents.*

b) *R is a right strongly non-singular, right Noetherian and right hereditary.*

c) i) *R has finite right Goldie dimension.*

ii) *M is pure projective if and only if $M/Z(M)$ is projective, and $Z(M)$ is a direct summand of a direct sum of finitely generated modules.*

PROOF. *a)* $\Rightarrow$ *b)*: By [3, Theorems 3.7 and 4.2], R has finite right Goldie dimension. However, essentially finitely generated projective modules are finitely generated [5, Proposition 8.24]. *b)* $\Rightarrow$ *c)* is obvious in view of Theorem 2.3.

c) $\Rightarrow$ *a)*: Let I be a right ideal of R, and J its $\mathcal{S}$-closure in R. Since R has finite right Goldie dimension, I contains a finitely generated right ideal K as an essential submodule. Thus, J is the $\mathcal{S}$-closure of K, and J/K is the singular submodule of the finitely presented module R/K. By c), R/J is projective, and $R = J \oplus J_1$. Then, $R/I \cong J/I \oplus J_1$. In particular, J/I is a finitely generated singular module, which is pure-projective by c). Hence, J/I is a direct summand of a direct sum of finitely presented modules. Clearly, this sum can be chosen to be finite. Therefore, J/I is finitely presented, and I is finitely generated since J is a direct summand of R. Once we have shown that every finitely generated non-singular right R-

module M is projective, we will have established that R is a right hereditary ring with the property that Q^r is a perfect left localization of R.

There exists a finitely generated free module F and a submodule U of F such that $M \cong F/U$. Since R has finite right Goldie-dimension, U contains a finitely generated essential submodule V. Because, F/U is non-singular, U/V is the singular submodule of the finitely presented module F/V. By c), $F/U \cong (F/V)/(U/V)$ is projective. □

COROLLARY 4.2. *The following are equivalent for a right non-singular ring R with finite right Goldie-dimension:*

a) *R is a right Noetherian, right hereditary ring which satisfies the restricted right minimum condition such that Q^r is a perfect left localization of R.*

b) *M is pure projective if and only if $M/Z(M)$ is projective and $Z(M)$ is a direct summand of a direct sum of finitely generated Artinian modules.*

PROOF. *a)* ⇒ *b)*: Since R has the restricted minimum condition, every finitely generated singular right module is Artinian.

b) ⇒ *a)*: Let I be an essential right ideal of R. Since R has finite right Goldie-dimension, I contains a finitely generated essential right ideal J. By c), the finitely presented module R/J is a direct summand of a (finite) direct sum of finitely generated Artinian modules. But this is only possible if R/J is Artinian. But then, R/I is Artinian too. Arguing as in the proof of Proposition 4.1, we obtain that R is a right strongly non-singular, right hereditary. □

By [8, Proposition 5.27], a right hereditary, right Noetherian ring which is the product of prime rings and rings Morita equivalent to lower triangular matrix rings over a division algebra is right strongly non-singular and has the restricted right minimum condition, i.e. R/I is Artinian for every essential right ideal I of R.

Let U be a subset of Q^r, and set $(R:U)_r = \{q \in Q^r \mid Uq \subseteq R\}$ and $(R:U)_\ell = \{q \in Q^r \mid qU \subseteq R\}$.

THEOREM 4.3. *The following are equivalent for a ring R:*

a) *R is a right Noetherian right hereditary ring satisfying the restricted right minimum condition such that Q^r is a perfect left localization of R.*

b) *R is a left Noetherian left hereditary ring satisfying the restricted left minimum condition such that* Q^ℓ *is a perfect right localization of R.*

c) *R is a right and left Noetherian, hereditary, right and left Utumi-ring.*

d) $R = R_1 \times \ldots \times R_n$ *where each* R_i *is either a prime right and left Noetherian hereditary ring or Morita equivalent to a lower triangular matrix ring over a division algebra.*

Proof. *a)* $\Rightarrow$ *c)*: By [3, Theorem 4.2], $Q^r = Q^\ell$ is a semi-simple Artinian ring, and R is right and left Utumi. Furthermore, R is left semi-hereditary by [3, Theorem 5.2]. It remains to show that R is left Noetherian.

Suppose that R contains a left ideal I which is not finitely generated. Without loss of generality, we may assume that I is essential in R. Since $Q^r = Q^\ell$ is semi-simple Artinian, R has finite left Goldie-dimension [11, Theorem XII.2.5], and I contains a finitely generated essential left ideal J_0. Since I is not finitely generated, we can find an ascending chain $J_0 \subseteq \ldots \subseteq J_n \subseteq \ldots$ of finitely generated essential left ideals inside I with $J_n \neq J_{n+1}$ for all n.

Since Q^r is an injective left R-module being the maximal left ring of quotients of R, every map $\phi : J_i \to Q^r$ is right multiplication by some $q \in Q^r$, which is uniquely determined by ϕ since J_i is essential. Therefore, we can identify $\mathrm{Hom}_R(J_i, R)$ and $J_i^* = (R : J_i)_r$. Moreover, J_i^* is a finitely generated projective right R-module because J_i is projective since R is left semi-hereditary. Furthermore, $J_i^{**} = (R : J_i^*)_\ell$ satisfies $J_i^{**} = J_i$. To see this, observe that $J_i \subseteq J_i^{**}$ by definition. Conversely, J_i has a finite projective basis since it is finitely generated and projective. There are $a_1, \ldots, a_k \in J_i$ and $q_1, \ldots, q_k \in J_i^*$ such that $y = yq_1a_1 + \ldots + yq_ka_k$ for all $y \in J_i$. Since Q^r also is the maximal left ring of quotients of R, it is non-singular as a left R-module. Because J_i is an essential left ideal, $1 = q_1a_1 + \ldots + q_ka_k$. If $z \in J_i^{**}$, then $zq_i \in R$, and $z = (zq_1)a_1 + \ldots (zq_k)a_k \in J_i$.

We obtain a descending chain $J_0^* \supseteq \ldots \supseteq J_n^* \supseteq \ldots \supseteq R^* = R$ of finitely generated submodules of Q^r_R. Since Q^r/R is singular, J_0^*/R is a finitely generated singular right R-module. By the restricted right minimum condition for R, J_0^*/R is Artinian. Consequently, there is m with $J_m^* = J_{m+1}^*$, from which one obtains $J_m = J_{m+1}$.

c) $\Rightarrow$ *d)*: By [3, Theorem 5.2], the classes of torsion-free, non-singular and flat right R-modules coincide. Because of [3, Theorem 5.5], R has the desired form.

d) $\Rightarrow$ *a*): By [3, Theorem 5.5], R is a right and left Noetherian hereditary ring for which the classes of torsion-free, flat and non-singular modules coincide. Thus, Q^r is a perfect left localization of R by [3, Theorem 5.2]. Finally, R satisfies the restricted right minimum condition by [5].

Since condition c) is right-left symmetric, the equivalence of a) and b) follows immediately. $\square$

A submodule U of the right R-module Q^r is *right invertible* if there exist $u_1, \ldots, u_n \in U$ and $q_1, \ldots, q_n \in (R:U)_\ell$ with $u_1q_1 + \ldots u_nq_n = 1$ [11, Chapters ii.4 and IX.5]. Over an integral domain R, a right invertible module U has the additional property that $U(R:U)_\ell = R$ which may fail in the non-commutative setting:

EXAMPLE 4.4. There exists a right and left Artinian, hereditary ring R such that Q^r is a perfect right and left localization of R, which contains an essential right ideal I with $(R:I)_\ell = Q^r$ and $(R:I)_\ell I = I$. Moreover, there exist $r_1, r_2, s_1, s_2 \in I$ and $q_1, q_2, p_1, p_2 \in (R:I)_\ell$ satisfying $1 = r_1p_1 + r_2p_2 = = s_1q_1 + s_2q_2$ such that $r_1p_1, r_2p_2 \in R$, but $s_1q_1, s_2q_2 \notin R$. Thus, $I(R:I)_\ell \not\subseteq R$.

PROOF. Let F be a field of characteristic different from 2, and R be the lower triangular matrix ring over F. Clearly, R is a right and left Artinian ring. According to [8, Theorem 4.7], R is a right hereditary ring, which is also left hereditary by [5, Corollary 8.18]. Finally, by [8, Proposition 2.28 and Theorem 2.30], the maximal right and left ring of quotients of R is $Q = Mat_2(F)$. Inside R, we consider the idempotents $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Let $I = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$, a two-sided ideal of R which is essential as a right ideal because $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ for all $a, b, c \in F$. Since $I = Q^r e_1$, we have $(R:I)_\ell = Q^r$ and $(R:I)_\ell I = I$.

Finally, consider the elements $s_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ and $s_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ of I and $q_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ c & 0 \end{pmatrix}$ and $q_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ c & 0 \end{pmatrix}$ of Q. It is easy to see that $s_1q_1, s_2q_2 \notin R$ although $s_1q_1 + s_2q_2 = 1$. On the other hand, setting $r_1 = p_1 = e_1 \in I$, $r_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in I$, and $p_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ yields $r_1p_1, r_2p_2 \in R$ and $r_1p_1 + r_2p_2 = 1$. $\square$

In view of the previous example, we define a submodule U of Q^r_R to be *strongly invertible* if there is a submodule M of ${}_RQ^r$ such that $MU = UM = R$.

LEMMA 4.5. *Let R be a right and left non-singular, right and left Utumi-ring with maximal right and left ring of quotients Q. A submodule U of Q^r is strongly invertible if and only if it satisfies the following conditions:*

i) *U is also a submodule of ${}_RQ$.*
ii) *U_R is a finitely generated projective generator of $\mathcal{M}_R$.*
iii) *${}_RU$ is a finitely generated projective generator of ${}_R\mathcal{M}$.*

PROOF. Suppose that U is strongly invertible, and choose a submodule X of ${}_RQ^r$ with $XU = UX = R$. Then, $X \subseteq (R : U)_\ell \cap (R : U)_r$ and $(R : U)_\ell U = R = U(R : U)_r$. Moreover, $RU = (UX)U = U(XU) = = UR = U$ yields that U is a submodule of ${}_RQ^r$ too. By symmetry, X is also a submodule of Q^r_R. Because of this, $(R : U)_\ell$ and $(R : U)_r$ are submodules of both Q^r_R and ${}_RQ^r$ too. Therefore, $U(R : U)_\ell = = U(R : U)_\ell R = U(R : U)_\ell U(R : U)_r = U(R : U)_r = R$. By symmetry, $(R : U)_r U = R$.

By what has been shown in the last paragraph, we can write $1 = u_1q_1 + \ldots + u_nq_n$ with $u_1, \ldots, u_n \in U$ and $q_1, \ldots, q_n \in (R : U)_\ell$. Let $\phi_i : U \to R$ be left multiplication by q_i. As in [11, Proposition IX.5.2], the set $\{(u_1, \phi_1), \ldots, (u_n, \phi_n)\}$ is a projective basis for U. Select $v_1, \ldots, v_m \in U$ and $p_1, \ldots, p_m \in (R : U)_\ell$ with $1 = p_1v_1 + \ldots + p_mv_m$. Define $\psi : U^m \to R$ by $\psi(x_1, \ldots, x_m) = \Sigma_{i=1}^m p_ix_i$. Then, ψ is onto, and $U^m = R \oplus W$, i.e. U is a generator of $\mathcal{M}_R$. By symmetry, ${}_RU$ is a finitely generated projective generator of ${}_R\mathcal{M}$.

Conversely, assume that U satisfies the three conditions. Observe that $(R : U)_\ell$ and $(R : U)_r$ are submodules of both Q^r_R and ${}_RQ^r$. Since it is a projective generator of $\mathcal{M}_R$, there is $\ell < \omega$ such that $U^\ell = R \oplus W$. Let $\pi : U^\ell \to R$ be a projection with kernel W, and $\delta_j : U \to U^\ell$ be the embedding into the j^{th}-coordinate. The map $\pi\delta_j : U \to R$ is left multiplication by some $q_j \in Q^r$ since Q^r is a right self-injective ring. Clearly, since π is onto, there are $u_1, \ldots, u_\ell \in U$ with $q_1u_1 + \ldots + q_\ell u_\ell = 1$. Since $q_1, \ldots, q_\ell \in (R : U)_\ell$, we have $(R : U)_\ell U = R$. By symmetry, $U(U : R)_r = R$. Now, $U(R : U)_\ell = U(R : U)_\ell U(R : U)_r = R$ yields $(R : U)_\ell \subseteq (R : U)_r$. In the same way, $(R : U)_\ell = (R : U)_r$, and U is strongly invertible. □

PROPOSITION 4.6. *Let R be a right and left non-singular, right and left Utumi-ring. If I is a two-sided ideal of R such that ${}_RI$ and I_R are finitely generated projective generators of ${}_R\mathcal{M}$ and $\mathcal{M}_R$ respectively, then R/I is projective with respect to all RD-exact sequences.*

PROOF. The proof of [7, Lemmas I.7.2 and I.7.4] can be adapted to show that R/I is projective with respect to all RD-exact sequences of R-modules provided there are $r_1, \ldots, r_n \in R$ and $q_1, \ldots, q_n \in (R : I)_\ell$ such that $r_1q_1 + \ldots + r_nq_n = 1$ and $r_1q_1, \ldots, r_nq_n \in R$. However, this is guaranteed by Lemma 4.5. $\square$

Finally, the lattice of finitely generated right ideals over right strongly nonsingular, hereditary right and left Noetherian rings may not be distributive:

EXAMPLE 4.7. There exists a right strongly non-singular, hereditary, right and left Noetherian ring R for which the lattice of right ideals is not distributive.

PROOF. Let R be the ring considered in Example 4.4, whose notation will be used in the following. Consider the right ideals $J = e_1R = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$ and $K = e_2R = \begin{pmatrix} 0 & 0 \\ Q & Q \end{pmatrix}$. If $I = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} Q & 0 \\ Q & Q \end{pmatrix} = \{\begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} \mid a \in F\}$, then $I \cap J = I \cap K = 0$, while $I \cap (J + K) = I$. $\square$

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