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Generalized Derivations with Power Central Values on Multilinear Polynomials on Right Ideals.

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ABSTRACT - Let K be a commutative ring with unity, R a prime K-algebra, with extended centroid C and right Utumi quotient ring U, g a non-zero generalized derivation of R. Suppose that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over K, I is a non-zero right ideal of R and $m \geq 1$, a fixed integer. We prove the following results:

If $g(f(r_1, \ldots, r_n))^m = 0$, for all $r_1, \ldots, r_n \in I$, then one of the following holds:

(1) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I;
(2) $g(x) = ax$ for all $x \in R$, where $a \in U$ such that $aI = 0$;
(3) $g(x) = ax + [q, x]$ for all $x \in R$, where $a, q \in U$ such that $aI = 0$ and $[q, I]I = 0$.

If there exist $a_1, \ldots, a_n \in I$ such that $g(f(a_1, \ldots, a_n))^m \neq 0$ and $g(f(r_1, \ldots, r_n))^m \in Z(R)$, for all $r_1, \ldots, r_n \in I$, then one of the following holds:

(1) $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I;
(2) $f(x_1, \ldots, x_n)$ is central valued on R;
(3) $g(x) = ax$ for $a \in C$ and $f(x_1, \ldots, x_n)$ is power central valued on R;
(4) R satisfies s_4, the standard identity in four variables.

1. Introduction.

Throughout this paper, R is always a prime ring with center $Z(R)$, extended centroid C, and right Utumi quotient ring U. An additive mapping $d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. The study of derivations of prime rings was initiated by Posner [20]. By a

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generalized derivation on R one usually means an additive map $g : R \to R$ such that, for any $x, y \in R$, $g(xy) = g(x)y + xd(y)$, for some derivation d in R. Obviously any derivation is a generalized derivation. Moreover, other basic examples of generalized derivations are the following: (i) $g(x) = ax + xb$, for $a, b \in R$; (ii) $g(x) = ax$, for some $a \in R$. Many authors have studied generalized derivations in the context of prime and semiprime rings (see [13], [10], [17]).

In [7], Giambruno and Herstein proved that if d is a derivation of R such that $d(x)^n = 0$ for all $x \in R$, then $d = 0$. Following this line of investigation in [8] Herstein showed that if $d(x)^n \in Z(R)$, for all $x \in R$, then either $d = 0$ or R satisfies the standard identity s_4. Later in [23], Wong considered a similar condition in case the derivation acts on a multilinear polynomial $f(x_1, \ldots, x_n)$. The conclusion was that either $f(x_1, \ldots, x_n)$ is central in R or R satisfies s_4. Recently in [2], Chang extended Wong's result to the case when the $ad(f(x_1, \ldots, x_n))^m = 0$, for all $x_1, \ldots, x_n$ in a non-zero right ideal I of R, $a \in R$ and m is a fixed integer. He concluded that either $aI = 0$ or $d(I)I = 0$ or $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I. At this point the natural question is what happens in case the derivation is replaced by a generalized derivation. More recently in [[22], Theorem 5], Wei proved that if R is a prime ring, I a non-zero ideal of R, m a fixed positive integer and g is a generalized derivation such that $g(x)^m = 0$, for all $x \in I$, then $g = 0$. Finally in a very recent paper [21] Wang generalized above theorem to the case when the generalized derivation acts on all evaluations of a multilinear polynomial in a prime ring. More precisely he proved that if g is a generalized derivation of a prime K-algebra R and $f(x_1, \ldots, x_n)$ is a multilinear polynomial over K, such that $g(f(r_1, \ldots, r_n))^m = 0$ for all $r_1, \ldots, r_n \in R$, then either there exists an element $\lambda \in C$ such that $g(x) = \lambda x$ for all $x \in R$, or $f(x_1, \ldots, x_n)$ is central valued on R, except when R satisfies s_4 the standard identity in four variables.

In this paper we will continue this line of investigation. Our aim is to extend the Wang's result to the case when the multilinear polynomial is evalued on one-sided ideals of the prime K-algebra R. We will prove that:

THEOREM 1. *Let K be a commutative ring with unity, R a prime K-algebra, with extended centroid C, g a non-zero generalized derivation of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over K, I a non-zero right ideal of R and $m \geq 1$, a fixed integer. If $g(f(r_1, \ldots, r_n))^m = 0$, for all $r_1, \ldots, r_n \in I$, then one of the following holds:*

(i) *$[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I;*
(ii) *$g(x) = ax$ for all $x \in R$, where $a \in U$ such that $aI = 0$;*

(iii) $g(x) = ax + [q, x]$ *for all* $x \in R$, *where* $a, q \in U$ *such that* $aI = 0$ *and* $[q, I]I = 0$.

THEOREM 2. *Let K be a commutative ring with unity, R a prime K-algebra, with extended centroid C, g a non-zero generalized derivation of R. Suppose that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over K, I is a non-zero right ideal of R and $m \geq 1$, a fixed integer such that $g(f(r_1, \ldots, r_n))^m \in Z(R)$, for all $r_1, \ldots, r_n \in I$. If there exist $a_1, \ldots, a_n \in I$ such that $g(f(a_1, \ldots, a_n))^m \neq 0$, then one of the following holds:*

(i) $f(x_1, \ldots, x_n)x_{n+1}$ *is an identity for* I;
(ii) $f(x_1, \ldots, x_n)$ *is central valued on* R;
(iii) $g(x) = ax$ *for* $a \in C$ *and* $f(x_1, \ldots, x_n)$ *is power central valued on* R;
(iv) R *satisfies* s_4, *the standard identity in four variables.*

2. Preliminaries.

In all that follows, unless stated otherwise, K will be a commutative ring with unity and R will be a prime K-algebra. The related object we need to mention is the right Utumi quotient U of a ring R (sometimes, as in [1], U is called the maximal right ring of quotients).

The definitions, the axiomatic formulations and the properties of this quotient ring U can be found in [1].

In any case, when R is a prime ring, all that we need about U is that

1) $R \subseteq U$;
2) U is a prime ring;
3) The center of U, denoted by C, is a field which is called the extended centroid of R.

We make also a frequent use of the theory of generalized polynomial identities and differential identities (see [1], [12], [16], [19]). In particular we need to recall that when R is prime and I a non-zero right ideal of R, then I, IR and IU satisfy the same generalized polynomial identities [4].

In [13] T.K. Lee extended the definition of a generalized derivation as follows; by a generalized derivation we mean an additive mapping $g : I \to U$ such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in I$, where I is a dense right ideal of R and d is a derivation from I into U. Moreover Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U and obtained the following results:

THEOREM. (Theorem 4, [13]). *Every generalized derivation g on a dense right ideal of a semiprime ring R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$, for some $a \in U$ and a derivation d on U.*

More detail about generalized derivations can be found in [13], [10], [17].

REMARK 3. It is well known that every derivation of R can be uniquely extended to a derivation of U [16]. Moreover since R is prime ring , we may assume $K \subseteq C$ and so for any $\alpha \in K$ one has $d(\alpha \cdot 1) \in C$.

We will use the following notation:

$$f(x_1, \ldots, x_n) = \alpha x_1 \cdot x_2 \ldots \cdot x_n + \sum_{\sigma \neq 1} \alpha_\sigma x_{\sigma(1)} \cdot x_{\sigma(2)} \ldots \cdot x_{\sigma(n)}$$

for some $\alpha, \alpha_\sigma \in C$ and moreover we denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$. Thus we write $d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$, for all $r_1, r_2, \ldots, r_n \in R$.

REMARK 4. We will also write a multilinear polynomial $f(x_1, \ldots, x_n)$ as follows:

$$f(x_1, \ldots, x_n) = \sum_i t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) x_i$$

where t_i are multilinear polynomials in $n-1$ variables, and x_i never appears in any monomials of t_i.

Before the beginning of our proofs, we would like to recall Wang's results, more precisely we refer to Theorem 2 and Theorem 4 in [21]. All that we need here is to remind the conclusions contained in [21] in the case g is an inner generalized derivation of R. We summarize these reduced results in the following Facts 1,2 and 3:

FACT 1. *Let $a, b \in R$ such that $(af(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)b)^m = 0$, for all $x_1, \ldots, x_n \in R$. Then either $a = -b \in Z(R)$ or $f(x_1, \ldots, x_n)$ is central valued on R.*

FACT 2. *As a special simple case of Fact 1 we have that if $f(x_1, \ldots, x_n)$ is a non-central multilinear polynomial over K such that $(af(x_1, \ldots, x_n))^m = 0$, for all $x_1, \ldots, x_n \in R$, then $a = 0$.*

FACT 3. *Let R be a prime ring and $a, b \in R$, and $m \geq 1$ be a fixed integer. Suppose that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over K which is not vanishing on R, such that $(af(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)b)^m \in Z(R)$ for all $x_1, \ldots, x_n \in R$. Then one of the following holds:*

(i) *$f(x_1, \ldots, x_n)$ is central-valued on R;*
(ii) *$a = -b \in Z(R)$;*
(ii) *R satisfies s_4;*
(iv) *$a, b \in Z(R)$ and $f(x_1, \ldots, x_n)$ is power central valued on R.*

3. The nilpotent case.

We begin with the following:

LEMMA 1. *Let R be a prime K-algebra, I a non-zero right ideal of R, $a \in R$, $m \geq 1$, a fixed integer, $f(x_1, \ldots, x_n)$ a multilinear polynomial over K such that $(af(r_1, \ldots, r_n))^m = 0$, for all $r_1, \ldots, r_n \in I$. Then one of the following holds:*

(i) *$[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I;*
(ii) *$aI = 0$.*

PROOF. Suppose that conclusions (i) and (ii) do not occur. We prove that, in this case, we get a contradiction. Let $b_1, \ldots, b_{n+2}, w \in I$ such that

$$[f(b_1, \ldots, b_n), b_{n+1}]b_{n+2} \neq 0 \quad \text{and} \quad aw \neq 0 \tag{1}$$

Since $aI \neq 0$, there exists a non-central element u of I such that $au \neq 0$. Hence $(af(ux_1, \ldots, ux_n))^m$ is a non-trivial generalized polynomial identity for R. Thus R is a GPI-ring, so that RC has a non-zero socle H with non-zero right ideal $J = IH$. Moreover H is a simple ring with minimal right ideals (see [19]). Note that $J = JH$ and J satisfies the same basic conditions as I. Replacing R by H, I by J we have that R is a simple ring with minimal right ideals, moreover R is equals to its own socle and $IR = I$. Recall that, by Remark 2, we have

$$f(x_1, \ldots, x_n) = \sum_i t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)x_i.$$

Since R is a simple ring with minimal right ideals, it is a right-semisimple ring. Of course it is also a left-semisimple ring, so that R is a semisimple noetherian ring. Therefore any right (left) ideal of R is finitely generated. Moreover, since R is regular there exists $e^2 = e \in IR$ such that

$eR = wR + \sum_{i=1}^{n+2} b_i R$ such that, $ew = w, eb_i = b_i, i = 1, \ldots, n+2$. In particular for all $r_1, .., r_n \in R$ and for all $i = 1, .., n$ we have

$$(af(er_1, \ldots, er_i(1-e), \ldots, er_n))^m = (at_i(er_1, \ldots, er_n)er_i(1-e))^m = 0.$$

Hence for all $i = 1, .., n$

$$((1-e)at_i(er_1, \ldots, er_n)er_i)^{m+1} = 0$$

that is $(1-e)at_i(er_1, \ldots, er_n)eR$ is a nil right ideal of bounded index. Thus, by a well known result which is attribuited to Levitzki (for a proof see [6] and pp. 1-2 in [9]), $(1-e)aet_i(er_1, \ldots, er_n)er_i = 0$ for any $r_1, \ldots, r_n \in R$. By the primeness of R it follows that $(1-e)aet_i(er_1, \ldots, er_n)e = 0$ for all $i = 1, .., n$. Thus the left ideal Re satisfies the generalized identity $(1-e)aet_i(x_1, \ldots, x_n)$. By main theorem in [5] we have that $(1-e)ae(Re)t_i(Re, \ldots, Re) = (0)$. Therefore, again by the primeness of R, we get that either $(1-e)ae = 0$ or $t_i(x_1, \ldots, x_n)x_i$ is an identity for eR for all $i = 1, .., n$.

Firstly we assume that $(1-e)ae = 0$. Then $(eaef(ex_1e, \ldots, ex_ne)^m$ is an identity for R. In view of Fact 2, we have that either $eae = 0$ or $f(x_1, \ldots, x_n)$ is central for eRe. Second possibility gives that $[f(er_1, \ldots, er_n), er_{n+1}]er_{n+2} = 0$. In particular

$$0 = [f(eb_1, \ldots, eb_n), eb_{n+1}]eb_{n+2} = [f(b_1, \ldots, b_n), b_{n+1}]b_{n+2} \neq 0,$$

a contradiction. If $eae = 0$, then $0 = ae = aew = aw$, a contradiction again.

Finally we assume that $t_i(er_1, \ldots, er_n)er_i = 0$ for any $r_1, \ldots, r_n \in R$ and for all $i = 1, .., n$. Then, in particular, we get the contradiction

$$0 \neq [f(eb_1, \ldots, eb_n), eb_{n+1}]eb_{n+2} = \Big[\sum_i t_i(eb_1, \ldots, eb_n)eb_i, b_{n+1}\Big]b_{n+2} = 0.$$

□

LEMMA 2. *Let R be a prime ring, $a, b \in R$, $m \geq 1$ a fixed integer, I a non-zero right ideal of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over K such that $(af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b)^m = 0$, for all $r_1, \ldots, r_n \in I$.*

Then either $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I, or $(a+b)I = 0$ and $[b, I]I = 0$.

PROOF. If $b \in C$ then the proof is finished by Lemma 1. So we may assume that $b \notin C$. First aim is to show that R is a GPI-ring. Let $u \in I$ such

that $\{au, u\}$ are linearly C-independent. Then

$$(af(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^m \tag{2}$$

is a nontrivial generalized polynomial identity for R, as $(af(ux_1, \ldots, ux_n))^m$ occurs nontrivially in (2). So R is a GPI-ring. Let now $au = \alpha u$ for some $\alpha \in C$. Then R satisfies

$$(\alpha f(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^m \tag{3}$$

If $\{bu, u\}$ are C-independent, then $(f(ux_1, \ldots, ux_n)b)^m$ occurs nontrivially in (3). Then R is a GPI-ring. If $bu = \beta u$, for some $\beta \in C$, then the coefficients which occur in (3) are only $\{u, b\}$. If $u = \mu b$ for some $\mu \in C$, then, since $b \notin C$, R is GPI-ring. On the other hand, if $\{u, b\}$ are linearly C-independent, then (3) is again a non-trivial generalized polynomial identity for R. Therefore, again by [19], RC has a non-zero socle H with non-zero right ideal $J = IH$. As above H is a simple ring with minimal right ideals, moreover we have that $J = JH$ and J satisfies the same basic conditions as I in view of [16]. Replacing R by H, I by J we have that R is simple and equals to its own socle and $IR = I$. Notice that if $(a + b)I = 0$ then, from the assumption $(af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b)^m = 0$ for all $r_1, \ldots, r_n \in I$, it follows that $[b, f(r_1, \ldots, r_n)]^m = 0$ for all $r_1, \ldots, r_n \in I$ and by [2] we have that either $[b, I]I = 0$ or there exists an idempotent element $e \in H$ such that $IC = eRC$ and $f(x_1, .., x_n)$ is central valued on $eRCe$. In this last case we have $[f(er_1e, \ldots, er_ne), er_{n+1}e]e = 0$, for all $r_1, .., r_{n+1} \in R$, that is $[f(er_1, \ldots, er_n), er_{n+1}]e = 0$. In other words I satisfies the polynomial identity $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ and we are done.

Now we assume that the conclusion of the lemma doesn't hold. Then there exist $w, b_1, \ldots b_{n+2} \in I$ such that

$$(a + b)w \neq 0 \quad \text{and} \quad [f(b_1, \ldots, b_n), b_{n+1}]b_{n+2} \neq 0.$$

As in Lemma 1, since R is a simple regular ring, we write $\sum_{i=1}^{n+2} b_iR + wR = eR$ for some idempotent element $e \in R$. Moreover $e \in IR$ and $eb_i = b_i$ for $i = 1, \ldots, n + 2$ and $ew = w$. By the hypothesis we have that, for all $i = 1, .., n$, R satisfies

$$(af(ex_1, \ldots, ex_i(1 - e), \ldots ex_n) + f(ex_1, \ldots, ex_i(1 - e), \ldots ex_n)b)^m$$

and by calculations it follows that R satisfies

$$(at_i(ex_1, \ldots, ex_n)ex_i(1 - e) + t_i(ex_1, \ldots, ex_n)ex_i(1 - e)b)^m \tag{4}$$

for all $i = 1, .., n$. Left multiplying (4) by $(1 - e)$, we get

$$0 = (1 - e)(at_i(er_1, \ldots, er_n)er_i(1 - e))^m \qquad \forall r_1, .., r_n \in R.$$

It means that, for all $r_1, .., r_n \in R$ and for all $i = 1, .., n$,

$$((1-e)(at_i(er_1, \dots, er_n)eR)^{m+1} = (0).$$

By the above cited result of Levitzki (see again [6] and pp.1-2 in [9]) we have that R satisfies $(1-e)aet_i(ex_1, \dots, ex_n)ex_i$ for all $i = 1, .., n$.

Here we repeat the same argument in Lemma 1: by the primeness of R it follows that R satisfies $(1-e)aet_i(ex_1, \dots, ex_n)e$ for all $i = 1, .., n$. Thus the left ideal Re satisfies the generalized identity $(1-e)aet_i(x_1, \dots, x_n)$. By main theorem in [5] we have that $(1-e)ae(Re)t_i(Re, \dots, Re) = (0)$. It follows that either $(1-e)ae = 0$ or $t_i(x_1, \dots, x_n)x_i$ is an identity for eR for all $i = 1, .., n$.

If second case holds, then in particular we get the contradiction

$$0 \neq [f(eb_1, \dots, eb_n), eb_{n+1}]eb_{n+2} =$$

$$\Big[\sum_i t_i(eb_1, \dots, eb_{i-1}, eb_{i+1}, \dots, eb_n)eb_i, b_{n+1}\Big]b_{n+2} = 0.$$

So we have $ae = eae$. Now right multiplying (4) by e we get

$$(t_i(er_1, \dots, er_n)er_i(1-e)b)^m e = 0$$

for all $r_1, .., r_n \in R$ and $i = 1, .., n$, that is

$$((1-e)bet_i(er_1, \dots, er_n)eR)^{m+1} = (0).$$

Once again by Levitzki's theorem we have that R satisfies the generalized identity $(1-e)bet_i(ex_1, \dots, ex_n)ex_i$ for all $i = 1, .., n$.

Thus the left ideal Re satisfies the generalized identity $(1-e)bet_i(x_1, \dots, x_n)$. By main theorem in [5] we have that $(1-e)be(Re)t_i(Re, \dots, Re) = (0)$. It follows that either $(1-e)be = 0$ or $t_i(x_1, \dots, x_n)x_i$ is an identity for eR for all $i = 1, .., n$.

As above, in this last case we get the contradiction

$$0 \neq [f(eb_1, \dots, eb_n), eb_{n+1}]eb_{n+2} =$$

$$= \Big[\sum_i t_i(eb_1, \dots, eb_{i-1}, eb_{i+1}, \dots, eb_n)eb_i, b_{n+1}\Big]b_{n+2} = 0.$$

Hence it follows that $(1-e)be = 0$. Therefore the finite dimensional simple central algebra $eRCe$ satisfies $(af(x_1, \dots, x_n) + f(x_1, \dots, x_n)b)^m$. Then by Fact 1, either $eae = -ebe \in C$ or $f(ex_1e, \dots, ex_ne)$ is central. In the last case we have $[f(er_1, \dots, er_n), er_{n+1}]er_{n+2} = [f(r_1, \dots, r_n), r_{n+1}]r_{n+2} \neq 0$, a contradiction. Finally if $ae = eae = -ebe = -be$, then $0 = (a+b)e = (a+b)ew = (a+b)w \neq 0$, which is again a contradiction. □

THEOREM 1. *Let K be a commutative ring with unity, R a prime K-algebra with extended centroid C and g a non-zero generalized derivation of R. Suppose that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over K and I is a non-zero right ideal of R and $m \geq 1$, a fixed integer.*

If $(g(f(r_1, \ldots, r_n)))^m = 0$, for all $r_1, \ldots, r_n \in I$, then one of the following holds:

(i) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ *is an identity for I;*

(ii) $g(a) = ax$ *for all $x \in R$, where $a \in U$ such that $aI = 0$;*

(iii) $g(x) = ax - [q, x]$ *for all $x \in R$, where $a, q \in U$ such that $aI = 0$ and $[q, I]I = 0$.*

PROOF. As remarked above, every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$, for some $a \in U$ and derivation d on U. Therefore, for $u \in I$, U satisfies the following differential identity

$$(af(ux_1, \ldots, ux_n) + d(f(ux_1, \ldots, ux_n)))^m = 0$$

Suppose that $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is not identity for I.

If $d = 0$, then $(af(x_1, \ldots, x_n))^m = 0$ for all $x_1, \ldots, x_n \in I$. Hence by the initial hypothesis and Lemma 1 we get $g(x) = ax$ for all $x \in R$ and $aI = 0$. So the conclusion (ii) holds.

We may assume that $d \neq 0$. Thus we have that I satisfies

$$\Big(af(x_1, \ldots, x_n) + f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n)\Big)^m.$$

In light of Kharchenko's theory [12], we divide the proof into two case:

CASE 1. If d is inner derivation induced by the element $q \in U$, that is $d(x) = qx - xq$ for all $x \in U$. Thus $g(x) = ax + d(x) = (a + q)x - xq$ and U satisfies

$$\big((a + q)f(ux_1, \ldots, ux_n) - f(ux_1, \ldots, ux_n)q\big)^m.$$

Since $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is not identity for I, then by Lemma 2 we have that $0 = (a + q - q)I = aI$ and $[q, I]I = 0$.

CASE 2. Let now d an outer derivation of U. Since I and IU satisfies the same differential identities,

$$\big(af(x_1, \ldots, x_n) + d(f(x_1, \ldots, x_n))\big)^m$$

is an identity for IU, that is, for any $u \in I$,

$$\big(af(ux_1, \ldots, ux_n) + d(f(ux_1, \ldots, ux_n))\big)^m$$

is an identity for U. Thus U satisfies the following

$$\Big(af(ux_1,\dots,ux_n)+f^d(ux_1,\dots,ux_n)+\sum_i f(ux_1,\dots,d(u)x_i+ud(x_i),\dots,ux_n)\Big)^m.$$

Since d is an outer derivation, by Kharchenko's results in [12], U satisfies the identity

$$\Big(af(ux_1,\dots,ux_n)+f^d(ux_1,\dots,ux_n)+\sum_i f(ux_1,\dots,d(u)x_i+uy_i,\dots,ux_n)\Big)^m.$$

In particular taking $x_i=0$ it is easy to see that U satisfies the blended component $f(ux_1,\dots uy_i,\dots,ux_n)^m$, that is I satisfies $f(x_1,\dots,y_i,\dots,x_n)^m$. Since I satisfies a non-trivial polynomial identity, then there exists an idempotent $e\in socle(RC)$ such that $I=eRC$ (see Proposition in [15]). Hence R satisfies $f(ex_1,\dots,ex_n)^m$ that is R satisfies the identity $ef(ex_1,\dots,ex_n)^m$.

Then by main theorem in [5] we have that R satisfies $ef(ex_1,\dots,ex_n)ex_{n+1}$ that is $f(x_1,\dots,y_i,\dots,x_n)x_{n+1}$ is satisfied by $I=eRC$, a contradiction. □

4. The power central case.

At first we need to fix the following:

LEMMA 3. *Let R be a prime ring, I a non-zero right ideal of R and $m\geq 1$ a fixed integer. If $f(x_1,\dots,x_n)$ is a multilinear polynomial over K such that $f(r_1,\dots,r_n)^m\in Z(R)$ for all $r_1,\dots,r_n\in I$, then either $f(x_1,\dots,x_n)x_{n+1}$ is an identity for I or $f(x_1,\dots,x_n)$ is power central valued on R.*

PROOF. Suppose that $f(r_1,\dots,r_n)^m\in Z(R)$ for any $r_1,\dots,r_n\in I$. Since I satisfies a non-trivial polynomial identity, then there exists an idempotent $e\in socle(RC)$ such that $I=eRC$ (see Proposition in [15]). Thus, since $[f(r_1,\dots,r_n)^m,r]=0$ for all $r_1,\dots,r_n\in I$ and $r\in R$, we have that for any $r_1,\dots,r_{n+1}\in R$

$$0=[f(er_1,\dots,er_n)^m,r_{n+1}(1-e)]=f(er_1,\dots,er_n)^m r_{n+1}(1-e).$$

Since R is prime, either $(1-e)=0$ or $f(er_1,\dots,er_n)^m=0$. In the first case $e=1$ and $I=RC$. Thus $f(x_1,\dots,x_n)$ is power central valued on RC and so also on R. On the other hand, if $f(er_1,\dots,er_n)^m=0$, then $ef(er_1,\dots,er_n)^m=0$ and by main theorem in [5] $ef(er_1,\dots,er_n)eR=0$, for

all $r_1, \ldots, r_n \in R$. Hence $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for $I = eRC$ and we are done. □

THEOREM 2. *Let K be a commutative ring with unity, R a prime K-algebra, with extended centroid C, g a non-zero generalized derivation of R. Suppose that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over K, I is a non-zero right ideal of R and $m \geq 1$, a fixed integer such that $g(f(r_1, \ldots, r_n))^m \in Z(R)$ for all $r_1, \ldots, r_n \in I$. If there exist $a_1, \ldots, a_n \in I$ such that $g(f(a_1, \ldots, a_n))^m \neq 0$, then one of the following holds:*

(i) *$f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I;*
(ii) *$f(x_1, \ldots, x_n)$ is central valued on R;*
(iii) *$g(x) = ax$ for $a \in C$ and $f(x_1, \ldots, x_n)$ is power central valued on R;*
(iv) *R satisfies s_4.*

PROOF. Write again $g(x) = ax + d(x)$, for suitable $a \in U$ and d a derivation of R. Since $(af(x_1, \ldots, x_n) + d(f(x_1, \ldots, x_n)))^m$ is a central differential identity for I, by Theorem 1 in [3] R is a PI-ring and so RC is a finite dimensional central simple C-algebra. By Wedderburn-Artin theorem $RC \cong M_k(D)$ for some $k \geq 1$ and D a finite-dimensional central division C-algebra. By Theorem 2 in [16] $(af(x_1, \ldots, x_n) + d(f(x_1, \ldots, x_n)))^m \in C$ for all $x_1, \ldots, x_n \in IC$. Without loss of generality we may replace R with RC and assume that $R = M_k(D)$. Let $s_1, \ldots, s_n$ be arbitrary elements of I. Then R satisfies the following central differential identity in the variables X and Y:

$$\Big[\Big(af(s_1X, s_2, \ldots, s_n) + f^d(s_1X, s_2, \ldots, s_n) + f(d(s_1)X + s_1d(X), s_2, \ldots, s_n) + \\ + \sum_{i \geq 2} f(s_1X, \ldots, d(s_i), \ldots, s_n)\Big)^m, Y\Big].$$

If d is an outer derivation, fix $d(X) = y$, $X = 0$ and obtain $f(s_1r, s_2, \ldots, s_n)^m \in C$, for all $s_1, \ldots, s_n \in I$ and $r \in R$. By Lemma 3 either $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I or $f(x_1, \ldots, x_n)$ is power central valued on R. In the first case we are done.

Hence we assume that $f(x_1, \ldots, x_n)^m$ is central valued on R. In particular, if $f(r_1, \ldots, r_n)^m = 0$ for all $r_1, \ldots, r_n \in I$ then we obtain as a reduction of Lemma 3 that $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I. In the either case there exists $b_1, \ldots, b_n \in I$ such that $0 \neq f(b_1, \ldots, b_n)^m \in C$, that is I contains an invertible element of R, and so $I = R$. Hence we conclude by Wang's theorem in [21].

Suppose now that d is an inner derivation, say $d(x) = [q, x] = qx - xq$. Let F be a maximal subfield of D, so that $M_k(D) \otimes_C F \cong M_t(F)$ where $t = k \times [F : C]$. Hence the derivation d can be extended to $M_k(D) \otimes_C F$ and $(af(x_1, \ldots, x_n) + d(f(x_1, \ldots, x_n)))^m \in Z(M_t(F))$, for any $x_1, \ldots, x_n \in I \otimes F$ (Lemma 2 in [14] and Proposition in [18]). Therefore we may assume that $R \cong M_t(F)$. Notice that if $t = 1, 2$ then we have respectively that either $R \cong F$ or $R \cong M_2(F)$. In any case R satisfies the standard identity s_4 and we are done.

Here we assume that $t \geq 3$. Denote e_{ij} the usual matrix unit with 1 in the (i,j)-entry and zero elsewhere. Since there exists a set of matrix units that contains the idempotent generator of a given minimal right ideal, we observe that any minimal right ideal is part of a direct sum of minimal right ideals adding to R. In light of this and applying Proposition 5 on page 52 in [11], we may assume that any minimal right ideal of R is a direct sum of minimal right ideals, each of the form $e_{ii}R$.

Moreover we know that the right ideal I has a number of uniquely determinated simple components: they are minimal right ideals of R and I is their direct sum. In light of this argument, we may write $I = eR$ for some $e = \sum_{i=1}^{l} e_{ii}$ and $l \in \{1, 2, ..., t\}$, that is $I = eR = (e_{11}R + \ldots + e_{ll}R)$, where $t \geq 3$ and $l \leq t$.

Denote $q = \sum_{r,s} q_{rs}e_{rs}$, $a = \sum_{r,s} a_{rs}e_{rs}$, for $q_{rs}, a_{rs} \in F$.

If $f(x_1, \ldots, x_n)$ is not an identity for I, then by Lemma 3 in [2], for any $i \leq l, j > l$, the element e_{ij} falls in the additive subgroup of RC generated by all valuations of $f(x_1, \ldots, x_n)$ in I. Since the matrix $(ae_{ij} + qe_{ij} - e_{ij}q)^m$ has rank ≤ 2, then it is not central. Therefore $(ae_{ij} + qe_{ij} - e_{ij}q)^m = 0$ and right multiplying by e_{ii} we have $(-e_{ij}q)^m e_{ii} = 0$. This means that $q_{ji} = 0$ for any $i \leq l$ and for any $j > l$, that is $qI \subseteq I$. Moreover from $e_{jj}(ae_{ij} + qe_{ij} - e_{ij}q)^m = 0$ we also get $e_{jj}((a + q)e_{ij})^m = 0$, that is $a_{ji} = 0$, and $aI \subseteq I$. All this implies that $g(I) \subseteq I$. Since $0 \neq g(f(a_1, \ldots, a_n))^m \in \in I \cap F$, it is invertible and $I = R$. Also in this case we conclude thanks to Theorem 4 in [21].

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