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On the Torsion of Brieskorn Modules of Homogeneous Polynomials.

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ABSTRACT - Let $f \in \mathbb{C}[X_1, ..., X_n]$ be a homogeneous polynomial and $B(f)$ be the corresponding Brieskorn module. We describe the torsion of the Brieskorn module $B(f)$ for $n = 2$ and show that any torsion element has order 1. For $n > 2$, we find some examples in which the torsion order is strictly greater than 1.

1. The Milnor algebra and the Brieskorn module.

Let $f \in R = \mathbb{C}[x_1, ..., x_n]$ be a homogeneous polynomial of degree $d > 1$. Then the Koszul complex of the partial derivatives $f_j = \dfrac{\partial f}{\partial x_j}; j = 1, ..., n$ in R can be identified to the complex

$$0 \longrightarrow \Omega^0 \xrightarrow{df\wedge} \Omega^1 \xrightarrow{df\wedge} ... \xrightarrow{df\wedge} \Omega^{n-1} \xrightarrow{df\wedge} \Omega^n \longrightarrow 0, \tag{1.1}$$

where Ω^j denotes the regular differential forms of degree j on $\mathbb{C}^n$.

Let J_f be the Jacobian ideal spanned by the partial derivatives f_j, $j = 1, ..., n$, in R and $M(f) = R/J_f$ be the Milnor algebra of f. One has the following obvious isomorphism of graded vector spaces

$$M(f)(-n) = \frac{\Omega^n}{df \wedge \Omega^{n-1}}. \tag{1.2}$$

Here, for any graded $\mathbb{C}[t]$-module M, the shifted module $M(m)$ is defined by setting $M(m)_s = M_{m+s}$ for all $s \in \mathbb{Z}$.

We define the (algebraic) Brieskorn module as the quotient

$$B(f) = \frac{\Omega^n}{df \wedge d(\Omega^{n-2})} \tag{1.3}$$

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in analogy with the (analytic) local situation considered in [3], see also [11], as well as the submodule

$$C(f) = \frac{df \wedge \Omega^{n-1}}{df \wedge d(\Omega^{n-2})}. \tag{1.4}$$

These modules are modules over the ring $\mathbb{C}[t]$ and the multiplication by t is given by multiplying by the polynomial f. Sometimes $B(f)$ is denoted by $G_f^{(0)}$ and $C(f)$ by $G_f^{(-1)}$, see [2], [6].

One has the following basic relation between the Milnor algebra and the Brieskorn module, see [7], Prop. 1.6.

PROPOSITION 1.1.

$$df \wedge \Omega^{n-1} = df \wedge d(\Omega^{n-2}) + f \cdot \Omega^n.$$

In particular

$$B(f)/f.B(f) \simeq M(f)(-n).$$

Let $B(f)_{\mathrm{tors}}$ be the submodule of $\mathbb{C}[t]$-torsion elements in $B(f)$ and define the reduced Brieskorn module $\overline{B}(f) = B(f)/B(f)_{\mathrm{tors}}$.

REMARK 1.2. The reduced Brieskorn module $\overline{B}(f)$ is known to be a free $\mathbb{C}[t]$-module of rank $b_{n-1}(F)$, where $F = \{x \in \mathbb{C}^n | \, f(x) = 1\}$ is the affine Milnor fiber of f. Indeed, it follows from the section (1.8) in [7] that one has a canonical isomorphism $\oplus_{j=0,d-1}\overline{B}(f)_{qd+j} = H^{n-1}(F, \mathbb{C})$ for any $q \geq n$. If we assume the $\mathbb{C}[t]$-basis of $\overline{B}(f)$ to be formed by homogeneous elements (which is always possible), each basis element will contribute by 1 to the dimension of $\oplus_{j=0,d-1}\overline{B}(f)_{qd+j}$ for q large enough.

Moreover, it exists an integer $N > 0$ such that $t^N B(f)_{\mathrm{tors}} = 0$, see [7], Remark 1.7. The least N satisfying this condition is called the *torsion order* of f and is denoted by $N(f)$.

REMARK 1.3. As noted in [7], Remark 1.7, there is a slight difference between the Brieskorn module defined above and the Brieskorn module considered by Barlet and Saito in [1]. In fact, the latter one is defined to be the n-th cohomology group $H^n(\mathcal{A}_f^*)$, where $\mathcal{A}_f^j = \ker(df\wedge : \Omega^j \to \Omega^{j+1})$ and the differential $d_f : \mathcal{A}_f^j \to \mathcal{A}_f^{j+1}$ is induced by the exterior differential $d : \Omega^j \to \Omega^{j+1}$. Since $df \wedge d(\Omega^{n-2}) \subset d_f(\mathcal{A}_f^{n-1})$, one has an epimorphism

$$B(f) \to H^n(\mathcal{A}_f^*),$$

(note that the direction of this arrow is misstated in [7], Remark 1.7). Moreover, when $n = 2$, it follows from Proposition 3.7 in [1] that $H^2(\mathcal{A}_f^*)$ is torsion-free. Our results in section 2 show that the torsion module $B(f)_{\text{tors}}$ can even be not of finite type in this setting, and hence the above epimorphism is far away from an isomorphism in general.

DEFINITION 1.4. For $b \in B(f)$, we say that b is t-torsion of order $k_b \geq 1$ if $t^{k_b} \cdot b = 0$ and $t^{k_b - 1} \cdot b \neq 0$.

It is clear that such a torsion order k_b divides the torsion order $N(f)$ of f, since $t^{N(f)} \cdot b = 0$ and that $N(f)$ is the G.C.D. of all the t-torsion orders k_b for $b \in B(f)$ a torsion element.

1.5. *The case of an isolated singularity.*

Assume that $f \in \mathbb{C}[x_1, ..., x_n]$ is a homogeneous polynomial having an isolated singularity at the origin of $\mathbb{C}^n$. Then the dimension of the Milnor algebra $M(f)$ (as a $\mathbb{C}$-vector space) is the Milnor number of f at the origin, denoted by $\mu(f)$. One has in this case $\mu(f) = b_{n-1}(F)$, see for instance [5]. In this case, the structure of the module $B(f)$ is as simple as possible.

PROPOSITION 1.6. *The $\mathbb{C}[t]$-module $B(f)$ is free of rank $\mu(f)$.*

PROOF. Indeed, a homogeneous polynomial f having an isolated singularity at the origin induces a tame mapping $f : \mathbb{C}^n \to \mathbb{C}$, to which the results in [6], [12] and [13] apply. Indeed, the Gauss-Manin system G_f of f, which is an $A_1 = \mathbb{C}[t]\langle\partial_t\rangle$-module, contains a $\mathbb{C}[t]$-submodule $G_f^{(0)}$, which is known to be free of finite type for a tame polynomial, see for instance Remark 3.3 in [6]. As we have already mentionned above, $G_f^{(0)} = B(f)$. □

COROLLARY 1.7. *There is an isomorphism*

$$B(f)(n) = M(f) \otimes_{\mathbb{C}} \mathbb{C}[t]$$

of graded modules over the graded ring $\mathbb{C}[t]$. In particular, one has, at the level of the associated Poincaré series, the following equality.

$$P_{B(f)}(t) = P_{M(f)}(t) \cdot \frac{t^n}{1-t} = \frac{t^n \cdot (1-t^{d-1})^n}{(1-t)^{n+1}}.$$

EXAMPLE 1.8. For $n = 3$, take $f(x, y, z) = x^3 + y^3 + z^3$. Then a $\mathbb{C}$-basis of $M(f)$ is given by $1, x, y, z, xy, yz, xz, xyz$. The same 8 monomials form a

basis of $B(f)$ as a free $\mathbb{C}[t]$-module. In this case

$$P_{B(f)}(t) = \frac{t^3(1-t^2)^3}{(1-t)^4} = t^3 + 4t^4 + 7t^5 + 8t^6 + 8t^7 + 8t^8 + \cdots.$$

COROLLARY 1.9. *The $\mathbb{C}[t]$-module $B(f)$ is torsion free if and only if 0 is an isolated singularity of the homogeneous polynomial f.*

PROOF. If 0 is not an isolated singularity, then the Milnor algebra $M(f)$ is an infinite dimensional $\mathbb{C}$-vector space. The isomorphism in Proposition 1.1 implies that in this case $B(f)$ is not finitely generated over $\mathbb{C}[t]$. By Remark 1.2, it follows that the canonical projection

$$B(f) \to \overline{B}(f)$$

is not an isomorphism, hence $B(f)_{tors} \neq 0$. □

Proposition 1.1 implies that $f \cdot B(f) = C(f)$, which in turn yields the following.

COROLLARY 1.10. *Assume that 0 is not an isolated singularity of the homogeneous polynomial f. Then $N(f) = 1$ if and only if the $\mathbb{C}[t]$-module $C(f)$ is torsion free.*

PROOF. Let $b \in B(f)_{\text{tors}}$ be a non-zero element, which exists by our assumption. By Remark 1.2, it follows that b is t-torsion, say of order k. If $k > 1$, then

$$0 = t^k \cdot b = t^{k-1} \cdot (tb).$$

By Proposition 1.1, we know that $t \cdot b \in C(f)$. If $C(f)$ is torsion free, we get that $t \cdot b = 0$ in $C(f)$, i.e. $t \cdot b = 0$ in $B(f)$, a contradiction. Hence $k = 1$ for any $b \in B(f)_{tors}$, in other words $N(f) = 1$. □

2. The case $n = 2$.

In this section we suppose that $f \in \mathbb{C}[x, y]$ is a homogeneous polynomial of degree $d > 1$, which is not the power g^r of some other polynomial $g \in \mathbb{C}[x, y]$ for some $r > 1$. This condition is equivalent to asking the Milnor fiber F of f to be connected and such polynomials are sometimes called *primitive*. For more on this, see [8], final Remark, part (I).

PROPOSITION 2.1. *The submodule $C(f)$ is a free $\mathbb{C}[t]$-module of rank $b_1(F)$.*

PROOF. The fact the $C(f)$ is torsion free follows from Proposition 7. (ii) in [2]. Indeed, it is shown there that the localization morphism

$$\ell : C(f) \to C(f) \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}]$$

is injective, which is clearly equivalent to the fact that there are no t-torsion elements.

For the second claim, note that the composition of the inclusion $C(f) \to B(f)$ and the canonical projection $B(f) \to \overline{B}(f)$ gives rise to an embedding of $C(f)$ into $\overline{B}(f)$ whose image is exactly $t \cdot \overline{B}(f)$ (use again Proposition 1.1 and the fact that $C(f)$ is torsion free). Since $\mathbb{C}[t]$ is a principal ideal domain, the claim follows from the structure theorem of submodules of free modules of finite rank over such rings. □

Corollary 1.10 implies the following:

COROLLARY 2.2. *If $f \in \mathbb{C}[x, y]$ is a homogeneous polynomial with a non-isolated singularity at the origin, then $N(f) = 1$.*

The above Corollary can be restated by saying that

$$0 \to B(f)_{\text{tors}} \to B(f) \xrightarrow{t} C(f) \to 0$$

is an exact sequence of graded $\mathbb{C}[t]$-modules. We get thus an isomorphism of graded $\mathbb{C}[t]$-modules

$$\overline{B}(f)(-d) \simeq C(f).$$

EXAMPLE 2.3. Let $f = x^p y^q$ with $(p, q) = 1$. In order to compute the torsion of the Brieskorn module $B(f)$, we start by finding $\mathbb{C}$-vector space monomial bases for $\dfrac{B(f)}{C(f)} \simeq M(f)$ (up-to a shift in gradings) and $C(f)$. Note that the Jacobian ideal is given by

$$J_f = \langle x^{p-1} y^q, x^p y^{q-1} \rangle = x^{p-1} \cdot y^{q-1} \langle x, y \rangle.$$

It follows that $\dfrac{B(f)}{C(f)} \simeq \dfrac{S}{J_f}$ is an infinite dimensional $\mathbb{C}$-vector space with a monomial basis given by $x^a y^b$ with $a \leq p-2$ or $b \leq q-2$ or $(a, b) = (p-1, q-1)$.

We have to compute $df \wedge d\Omega^0 = \{df \wedge dg\}$, where g is a polynomial function on $\mathbb{C}^2$. Since we are working with homogeneous polynomials, it is enough to work with one monomial, say $(x^a y^b)$, at a time. We have

$$df \wedge d(x^a y^b) = (pb - qa)x^{a+p-1}y^{b+q-1}dx \wedge dy.$$

Since

$$C(f) = \frac{J_f \cdot \Omega^2}{df \wedge d\Omega^0} = \frac{x^{p-1}y^{q-1}\langle x, y\rangle \Omega^2}{df \wedge d\Omega^0},$$

a system of generators of the $\mathbb{C}$-vector space $C(f)$ is given by the classes of the elements $w \in x^{p-1}y^{q-1}\langle x, y\rangle \Omega^2$, which do not belong to $df \wedge d\Omega^0$. To find them, it is enough to look at monomial differential forms i.e. $x^\alpha y^\beta dx \wedge dy$ with $\alpha \geq p-1$, $\beta \geq q-1$ and $\alpha + \beta \geq p + q - 1$.

So, if $a + p - 1 = \alpha, b + q - 1 = \beta$ and $pb - qa \neq 0$, then

$$df \wedge d\left(\frac{x^a y^b}{pb - aq}\right) = x^\alpha y^\beta dx \wedge dy.$$

Hence, the only elements $x^\alpha y^\beta dx \wedge dy$ which are not in $df \wedge d\Omega^0$ are $x^{a+p-1}y^{b+q-1}dx \wedge dy$, where $pb = qa$, i.e. $a = kp, b = kq$ for some $k \geq 0$. It follows that $C(f)$, as a $\mathbb{C}$-vector space, has a basis given by $x^{(k+1)p-1}y^{(k+1)q-1}$ where $k \geq 0$, which can be written as

$$C(f) = \mathbb{C}[t] \cdot x^{2p-1}y^{2q-1}dx \wedge dy$$

i.e. $C(f)$ is a free $\mathbb{C}[t]$-module of rank 1.

Moreover, the corresponding monomial basis as a $\mathbb{C}$-vector space for the Brieskorn module $B(f)$ is given by $x^a y^b dx \wedge dy$ with $a \leq p-2$ or $b \leq q-2$ or $a = (k+1)p - 1, b = (k+1)q - 1$ for some $k \geq 0$.

With respect to this basis, $B(f)_{\text{tors}}$ is the linear span of $x^a y^b dx \wedge dy$ with $a \leq p-2$ or $b \leq q-2$. Indeed our computation above shows that

$$df \wedge d(x^{a+1}y^{b+1}) = (p(b+1) - q(a+1))f \cdot w$$

i.e. $tw = 0$ if the coefficient $p(b+1) - q(a+1)$ is non-zero.

It follows that

$$\overline{B}(f) = \mathbb{C}[t] \cdot [x^{p-1}y^{q-1}dx \wedge dy],$$

a free $\mathbb{C}[t]$-module of rank 1.

It remains to explain why $b_1(F) = 1$, where $F = \{(x, y) \in \mathbb{C}^2; x^p y^q = 1\}$. Consider the covering

$$\mathbb{Z}_q \hookrightarrow F \xrightarrow{\phi} \mathbb{C}^*$$

where ϕ maps (x, y) to x.

Now a covering yields an exact sequence (see for instance [10], page 376),

$$0 \hookrightarrow \pi_1(F,(x_0,y_0)) \xrightarrow{\phi_\#} \pi_1(\mathbb{C}^*,x_0) \to \pi_0(\mathbb{Z}_q,(x_0,y_0)) = \mathbb{Z}_q \to 0.$$

Hence $\pi_1(F) \simeq \mathbb{Z}$, which shows that the first Betti number $b_1(F)$ is 1.

3. Eigenvalues of the monodromy and torsion of Brieskorn modules.

Any homogeneous polynomial $f \in \mathbb{C}[x_1,...,x_n]$ induces a locally trivial fibration $f : \mathbb{C}^n \setminus f^{-1}(0) \to \mathbb{C}^*$, with fiber F and semisimple monodromy operators

$$T_f^k : H^k(F,\mathbb{C}) \to H^k(F,\mathbb{C})$$

for $k = 0,...,n-1$. The eigenvalues of T_f are d-roots of unity, where d is the degree of f, and for each such eigenvalue λ we denote by $H^*(F,\mathbb{C})_\lambda$ the corresponding eigenspace.

According to [7], see the discussion just before Remark 1.9, one has for $q < n$ an inclusion

$$t^{n-q} : \overline{B}(f)_{qd-j} \to \overline{B}(f)_{nd-j}$$

and, for $q \geq n$, an identification

$$\overline{B}(f)_{qd-j} = H^{n-1}(F,\mathbb{C})_\lambda,$$

where $\lambda = \exp\left(\dfrac{2\pi j\sqrt{-1}}{d}\right)$, with $j = 0,1,...,d-1$. Let $\omega_n = dx_1 \wedge ... \wedge dx_n$ and note that $[\omega_n] \in \overline{B}(f)_{qd-j}$ for some q if and only if $n-j$ is divisible by d. This yields the following, see [7], Corollary 1.10.

COROLLARY 3.1. *Assume that* $\lambda = \exp\left(\dfrac{2\pi n\sqrt{-1}}{d}\right)$ *is not an eigenvalue of the monodromy operator* T_f *acting on* $H^{n-1}(F,\mathbb{C})$. *Then* $[\omega_n]$ *is a non-zero torsion element in* $B(f)$.

Here are some examples of torsion elements in $B(f)$ obtained using this approach, and having torsion order $N(f) > 1$.

EXAMPLE 3.2. Let $f = x^3 + y^2z$, $n = 3$, $d = 3$, $j = 0$ (this defines a cuspidal cubic curve $\mathcal{C}$ in $\mathbb{P}^2$). In this case $\lambda = 1$ is not an eigenvalue of the monodromy acting on $H^2(F,\mathbb{C})$. Indeed, one has $H^2(F,\mathbb{C})_1 = H^2(U,\mathbb{C}) = 0$. Here $U = \mathbb{P}^2 \setminus \mathcal{C}$ and the last vanishing comes from

the following obvious equalities: $b_0(U) = 1$, $b_1(U) = 0$ and $\chi(U) = \chi(\mathbb{P}^2) - \chi(C) = 3 - 2 = 1$ as C is homeomorphic to a sphere S^2.

For $w = dx \wedge dy \wedge dz$, to find the value of k such that $t^k \cdot [w] = 0$ in $B(f)$ is equivalent to saying that $f^k \cdot w \in df \wedge d\Omega^1$. We have to check for which value of k, we have solutions of the equation:

$$f^k \cdot w = \left[3x^2\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + 2yz\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + y^2\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dx \wedge dy \wedge dz,$$

where $P, Q, R \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of degree $3k - 1$.

For $k = 1$, we get a system of non-homogeneous linear equations in which we have 15 unknowns and 9 equations. Then using the software Mat Lab we compute the rank of the matrix corresponding to the homogeneous system (containing 9 rows and 15 columns) and get 8. On other hand, the rank of the matrix associated to the non-homogeneous system (containing 9 rows and 16 columns) is 9, which show that this system has no solution.

For $k = 2$, we get another system of non-homogeneous linear equations containing 60 unknowns and 26 equations. Then using Mat Lab we compute the rank of the matrix corresponding to the homogeneous system (containing 26 rows and 60 columns) and get 26. This time the corresponding non-homogeneous system matrix (containing 26 rows and 61 columns) has also rank 26, which shows that this system of equations has a solution. An explicit solution for $k = 2$, is $P = \frac{32}{3}(x^3yz)$, $Q = xy^2z^2$ and $R = \frac{1}{3}(x^4y)$. Hence $[w]$ has t-torsion order 2 in $B(f)$.

The next example shows somehow what happens when $H^{n-1}(F, \mathbb{C})_\lambda \neq 0$.

Example 3.3. Let $f = x^2y^2 + xz^3 + yz^3$, where $n = 3$, $d = 4$ and $j = 3$. The corresponding curve $C : f = 0$ in $\mathbb{P}^2$ has two cusps as singularities. It follows from the study of the plane quartic curves, see [5], p. 130, that $\pi_1(U) = \mathbb{Z}_4$ and hence $H^1(F, \mathbb{C}) = 0$.

Next $\chi(U) = \chi(\mathbb{P}^2) - \chi(C) = 3 - (2 - 2 \cdot 3 + 2 \cdot 2) = 3$. The zeta-function of the monodromy operator T_f looks like

$$\det(t \cdot Id - T_f^0) \cdot \det(t \cdot Id - T_f^2) = (t^4 - 1)^3$$

see for instance [5], p. 108. Hence we can not apply Corollary 3.1 to this polynomial to infer that $[w] = [\omega_3]$ is a torsion element in $B(f)$.

However, we can try to find values of k, for which we have solutions of the equation:

$$f^k \cdot w = \left[(2xy^2 + z^3)\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + (2x^2y + z^3)\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + (3xz^2 + 3yz^2)\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] w$$

where $P, Q, R \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of degree $4k - 2$.

For $k = 1$ and $k = 2$ a similar computation of matrix ranks as above shows that the corresponding systems have no solution. Hence $[w]$ is either a non-torsion element, or it has t-torsion order greater or equal to 3 in $B(f)$.

REMARK 3.4. Note that in both examples above the element $t \cdot [w]$ is a non-zero torsion element in $C(f)$. Hence Proposition 2.1 fails for $n > 2$. It also shows that in these cases the module $C(f)$ is not torsion free, compare to Corollary 1.10.

The last example shows that even for rather complicated examples (here the zero set of f is a surface S with non-isolated singularities) one may still have 1 as the t-torsion order of $[\omega_n]$.

REMARK 3.5. Let $f = x^2z + y^3 + xyt$, $n = 4$, $d = 3$, $j = 1$ be the equation of a cubic surface S in $\mathbb{P}^3$. It follows from [4], Example 4.3, that $H^3(F, \mathbb{C}) = 0$. Hence we can apply Corollary 3.1 to this polynomial to infer that $[w] = [\omega_4]$ is a torsion element in $B(f)$.

We have to check for which values of k, we have solutions of the equation:

$$f^k \cdot w = \left[(2xz + yt)\left(\frac{\partial S}{\partial t} - \frac{\partial T}{\partial z} + \frac{\partial U}{\partial y}\right) + (3y^2 + xt)\left(-\frac{\partial Q}{\partial t} + \frac{\partial R}{\partial z} - \frac{\partial U}{\partial x}\right) + x^2\left(\frac{\partial P}{\partial t} - \frac{\partial R}{\partial y} + \frac{\partial T}{\partial x}\right) + xy\left(\frac{\partial P}{\partial z} - \frac{\partial Q}{\partial y} - \frac{\partial S}{\partial x}\right)\right] dx \wedge dy \wedge dz \wedge dt,$$

where $P, Q, R, S, T, U \in \mathbb{C}[x, y, z, t]$ are homogeneous polynomials of degree $3k - 1$.

For $k = 1$, we get a system of non-homogeneous linear equations in which we have 42 unknowns and 15 equations. Using MatLab we compute

the rank of the corresponding homogeneous system (containing 15 rows and 42 columns) and get 14. And the rank of non-homogeneous system (containing 15 rows and 43 columns) has the same rank 14, which show that this system has a solution.

Hence $t \cdot [w] = 0$ in $B(f)$.

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