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Hölder Type Estimates for the $\overline{\partial}$-Equation in Strongly Pseudoconvex Domains.

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ABSTRACT - In this paper we generalize the Hölder space with a majorant function, define its order, and prove the existence and regularity for the solutions of the Cauchy-Riemann equation in the generalized Hölder space over a bounded strongly pseudoconvex domain.

1. Introduction and regular majorant.

If D is a bounded domain in $\mathbb{C}^n$, the Hölder space of order α, $\Lambda_\alpha(D)$ $(0 < \alpha < 1)$, is defined as the set of all functions g on D which satisfy for a constant $C = C_g > 0$ the condition

$$|g(z) - g(\zeta)| \leq C|z - \zeta|^\alpha, \quad z, \zeta \in D.$$

We first generalize this Hölder space following Dyakonov [Dya97] (also see Pavlović's book [Pav04]). For this purpose we introduce the notion of a *regular majorant*. Let ω be a continuous increasing function on $[0, \infty)$. We assume $\omega(0) = 0$, and suppose that $\omega(t)/t$ is non-increasing

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and satisfies the inequality

$$\int_0^\delta \frac{\omega(t)}{t}\,dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2}\,dt \le C\omega(\delta), \quad \text{for any} \quad 0 < \delta < 1, \tag{1.1}$$

for a suitable constant $C = C(\omega)$. Such a function ω is called a *regular majorant*. Given a regular majorant ω, the Hölder type space, $\Lambda(\omega, D)$ is defined as the family of all functions g on D such that

$$|g(z) - g(\zeta)| \le C\omega(|z - \zeta|), \quad z, \zeta \in D. \tag{1.2}$$

The norm $\|g\|_\omega$ of $g \in \Lambda(\omega, D)$ is given by $C_g + \|g\|_\infty$, where $C_g \ge 0$ is the smallest constant satisfying (1.2) and $\|g\|_\infty$ is the L^∞ norm in D. Note that with this norm $\Lambda(\omega, D)$ is a Banach space and $\Lambda(\omega, D) \subset L^\infty(D)$ (see the chapter 10 of Pavlović's book [Pav04]). We denote by $\Lambda_q(\omega, D)$ the set of the differential forms of type $(0, q)$ whose coefficients are in $\Lambda(\omega, D)$. We define the *order* of a regular majorant as follows:

DEFINITION 1.1. We say that a regular majorant ω has *order* α $(0 < \alpha < 1)$ if there exist an α and a positive real number t_0 such that

$$\begin{aligned} \alpha &= \sup\left\{\gamma : \frac{\omega(t)}{t^\gamma} \text{ is increasing } \forall t,\ 0 < t < t_0\right\} \\ &= \inf\left\{\gamma : \frac{\omega(t)}{t^\gamma} \text{ is decreasing } \forall t,\ 0 < t < t_0\right\}. \end{aligned}$$

If a regular majorant ω has *order* α, then we let $\omega = \omega_\alpha$ and call $\Lambda(\omega_\alpha, D)$ the Hölder type space of order α. By definition of the *order* of a regular majorant, it is uniquely determined, if it exists. Now we state our main result of this paper.

THEOREM 1.2. *Let $D \subset\subset \mathbb{C}^n$ $(n \ge 2)$ be a strongly pseudoconvex domain with C^4-boundary and $0 < \alpha < 1/2$. If a regular majorant ω_α has order α and $f \in \Lambda_q(\omega_\alpha, D)$ with $\bar\partial f = 0$ $(1 \le q \le n)$, then there is a solution $u \in \Lambda_{(q-1)}(t^{1/2}\omega_\alpha, D)$ of $\bar\partial u = f$ such that for some constant $C = C(\omega_\alpha)$*

$$\|u\|_{t^{1/2}\omega_\alpha} \le C\|f\|_{\omega_\alpha}. \tag{1.3}$$

The above inequality (1.3) generalizes the estimate by Henkin-Romanov [RH71] and Lieb-Range [LR80]

$$\|u\|_{t^{\alpha+1/2}} \le C\|f\|_{t^\alpha}.$$

For the proof of the Hölder type estimate (1.3), we need a variant of the Hardy-Littlewood Lemma [HL84].

LEMMA 1.3. *Let $D \subset\subset \mathbb{R}^n$ be a bounded domain with C^1 -boundary. If g is a $C^1(D)$-function and ω_γ is a regular majorant of order γ,* $0 < \gamma < 1$ *such that for some constant c_g depending on g,*

$$|dg(x)| \leq c_g \frac{\omega_\gamma(|\rho(x)|)}{|\rho(x)|}, \quad x \in D,$$

then we have

$$|g(x) - g(y)| \leq c_g \ \omega_\gamma(|x-y|).$$

As convention we use the notation $A \lesssim B$ or $A \gtrsim B$ if there are constants c_1, c_2, independent of the quantities under consideration, satisfying $A \leq c_1 B$ and $A \geq c_2 B$, respectively.

Before proving our theorem, we discuss some properties of a regular majorant and some examples.

EXAMPLE 1.4. (i) The most typical example is a function $\omega(t) = t^\alpha$ $(0 < \alpha < 1)$. Clearly, ω is a regular majorant and has order α.

(ii) A non-trivial example is the function, $\omega(t) = t^\alpha |\log t|^\beta$ on $[0, t_0]$ extended continuously for $t > t_0$ to be a regular majorant. Here $0 < \alpha < 1$, $-\infty < \beta < \infty$ and t_0 must be chosen sufficiently small so that the function ω should be a regular majorant (t_0 depends on α, β). Since $\lim_{t \searrow 0} t^\varepsilon |\log t|^\beta = 0$ for any $\varepsilon > 0$, it follows that $\omega(t) = t^\alpha |\log t|^\beta$ has order α for any choice of β.

(iii) Define the function $m(t) = 1/|\log t|^\beta$, $\beta > 0$ for $0 < t < t_0$ and $m(0) = 0$. Then $m(t)$ is continuous and increasing near 0, but it is not a regular majorant.

We end this section by describing useful properties of a regular majorant.

REMARK 1.5. (i) If ω, m are two regular majorants and have orders α, β respectively with $(0 < \alpha < \beta < 1)$, then letting ω_α, m_β, there exist $t_0 > 0$ and c such that $m_\beta(t) \leq c\omega_\alpha(t)$, $0 \leq t \leq t_0$. Hence we have the inclusion $\Lambda(m_\beta, D) \subset \Lambda(\omega_\alpha, D)$. Note that if two regular majorants, ω, m have the same order α, then generally there is no inclusion relation between $\Lambda(\omega, D)$ and $\Lambda(m, D)$.

(ii) In our Theorem 1.2, for a general regular majorant of order $1/2$, the estimate $\|u\|_{\omega_{1/2}} \lesssim \|f\|_\infty$ does not hold. In fact, the celebrated Henkin's theorem [RH71] holds only for the special regular majorant $\omega_{1/2}(t) = |t|^{1/2}$ and this number 1/2 is the sharp bound [Ran86]. But there is a regular majorant $m_{1/2}(t) = |t|^{1/2}|\log t|$ near the origin of order 1/2, which is strictly bigger than $|t|^{1/2}$.

REMARK 1.6. Let ω be a regular majorant of order α $(0 < \alpha < 1/2)$, say $\omega = \omega_\alpha$. Then $t^{1/2}\omega_\alpha$ is also a regular majorant of order $(\alpha + 1/2)$. In fact, $t^{1/2}\omega_\alpha$ is increasing and $(t^{1/2}\omega_\alpha)/t$ is non-increasing, since ω_α/t^γ, $\gamma > \alpha$ is decreasing. Here we use the fact that ω_α has order α. It remains to show that $t^{1/2}\omega_\alpha$ also satisfies (1.1). Since $(t^{1/2}\omega_\alpha)/t$ is non-increasing, we have for any δ, $(0 < \delta < 1)$,

$$\int_0^\delta \frac{s^{1/2}\omega_\alpha(s)}{s}\, ds \lesssim \delta^{1/2} \int_0^\delta \frac{\omega_\alpha(s)}{s}\, ds \lesssim \delta^{1/2}\omega_\alpha(\delta). \tag{1.4}$$

On the other hand, for a given $0 < \alpha < 1/2$, we can choose a sufficiently small ε such that $\alpha < 1/2 - \varepsilon$. It follows from the order of ω_α that $\omega_\alpha/t^{(1/2-\varepsilon)}$ is decreasing. Hence we obtain

$$\begin{aligned} \delta \int_\delta^\infty \frac{s^{1/2}\omega_\alpha(s)}{s^2}\, ds &= \delta \int_\delta^\infty \frac{\omega_\alpha(s)}{s^{1/2-\varepsilon}} \frac{1}{s^{1+\varepsilon}}\, ds \\ &\lesssim \delta \cdot \frac{\omega_\alpha(\delta)}{\delta^{1/2-\varepsilon}} \int_\delta^\infty \frac{1}{s^{1+\varepsilon}}\, ds \lesssim \delta^{1/2}\omega_\alpha(\delta). \end{aligned} \tag{1.5}$$

By (1.4) and (1.5), $t^{1/2}\omega_\alpha$ is a regular majorant of order $(\alpha + 1/2)$.

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2. Henkin's solution operator of the $\bar\partial$-equation.

In this section, we introduce the Henkin's solution operator [HL84] of the $\bar\partial$-equation and prove the integral estimates for the solution operator in a strongly pseudoconvex domain in $\mathbb{C}^n$. Let D be defined by a function ρ, i.e., $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$, where $\rho \in C^4$ and $\nabla\rho \neq 0$ on bD.

To construct the integral formula for solutions of the $\overline{\partial}$-equation in a strongly pseudoconvex domain, we need a support function (see [HL84]). For the global support function, we follow Fornaess construction [For76]. He showed that there exist a neighborhood U of $\overline{D}$ and a function $\phi(\cdot,\cdot) \in C^3(U \times U)$ such that for all $\zeta \in U$, $\phi(\zeta,\cdot)$ is holomorphic in U and $\phi(\zeta,z) = \langle \Phi, \zeta - z\rangle$, where we define $\Phi = \Phi(\zeta,z) = (\phi_1(\zeta,z),\ldots,\phi_n(\zeta,z))$ and $\langle \Phi, \zeta - z\rangle = \sum_{j=1}^{n} \phi_j(\zeta,z)(\zeta_j - z_j)$. In [For76], Fornaess also showed that $\phi_j \in C^3(U \times U)$ is holomorphic in z and there is a constant c such that for all $z \in \overline{D}$ and $\zeta \in D$ we have

$$2\mathrm{Re}\,\phi(\zeta,z) \geq \rho(\zeta) - \rho(z) + c|\zeta - z|^2 \tag{2.6}$$

and $d_\zeta\phi(\zeta,z)|_{z=\zeta} = \partial\rho(\zeta)$. Suppose that $f \in \Lambda_q(\omega_\alpha, D)$ $(1 \leq q \leq n)$ and $\overline{\partial} f = 0$. Then f is uniformly continuous in D. Using the above global support function ϕ, we define Henkin kernel $H(\zeta,z)$ as follows:

$$H(\zeta,z) = \frac{1}{(2\pi i)^n}\frac{\langle\bar{\zeta}-\bar{z},d\zeta\rangle}{|\zeta-z|^2}\wedge\frac{\langle\Phi,d\zeta\rangle}{\langle\Phi,\zeta-z\rangle}\wedge\sum_{k+\ell=n-2}\left(\frac{\langle\bar{\zeta}-\bar{z},d\zeta\rangle}{|\zeta-z|^2}\right)^k\wedge\left(\frac{\langle\bar{\partial}_{\zeta,z}\Phi,d\zeta\rangle}{|\zeta-z|^2}\right)^\ell,$$

where $\bar{\partial}_{\zeta,z}\Phi = \bar{\partial}_\zeta\Phi$ and $d\zeta = (d\zeta_1,\ldots,d\zeta_n)$. Note that Φ is holomorphic in z. We also define the Bochner-Martinelli kernel:

$$K(\zeta,z) = \frac{1}{(2\pi i)^n}\frac{\langle\bar{\zeta}-\bar{z},d\zeta\rangle}{|\zeta-z|^2}\wedge\left(\frac{\langle d\bar{\zeta}-d\bar{z},d\zeta\rangle}{|\zeta-z|^2}\right)^{n-1}.$$

For the construction of the above kernels, see [Ran86] or [CS01]. We have the Henkin's solution operator $\mathbb{S}f = \mathbb{K}f - \mathbb{H}f$ of the $\overline{\partial}$-equation, where

$$\mathbb{H}f(z) = \int_{\zeta\in bD} f(\zeta)\wedge H(\zeta,z), \quad \mathbb{K}f(z) = \int_{\zeta\in D} f(\zeta)\wedge K(\zeta,z).$$

We remark that the fact that the support function $\phi(\zeta,z)$ is holomorphic in z is very crucial in the construction of the solution operator $\mathbb{S}f$ of the $\overline{\partial}$-equation.

To prove the Hölder type estimate (1.3) of the main Theorem 1.2, we use Lemma 1.3. Hence, we have to estimate the differential of the Henkin solution operator, $d_z\,\mathbb{S}f$. Using the fact that $|\zeta - z|^2 \lesssim |\phi(\zeta,z)|$ for $(\zeta,z) \in bD \times \overline{D}$, straightforward computations give the kernel estimate (for the details, see [Ran86])

$$|d_zH(\zeta,z)| \lesssim \frac{1}{|\phi(\zeta,z)|^2|\zeta-z|^{2n-3}}, \quad (\zeta,z)\in bD\times\overline{D}.$$

REMARK 2.1. Without loss of generality, we assume that the differential of Henkin kernel, $d_z H(\zeta, z)$ has the following form:

$$d_z H(\zeta, z) = \frac{A(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^{2n-2}}, \tag{2.7}$$

where $A(\cdot, z)$ belongs to $C^1(\overline{D})$ and satisfies $|A(\zeta, z)| \lesssim |\zeta - z|$. Actually, $d_z H(\zeta, z)$ contains more terms whose singularity order is lower than that of (2.7) and so we can ignore other terms (refer to § 3. of chapter 4 in [Ran86]).

Generally, the Bochner-Martinelli integral, $\mathbb{K}f$, has a good regularity, so $\mathbb{K}$ is a bounded operator from L^∞-forms to Λ_α-forms for any $0 < \alpha < 1$. This kind of regularity still holds for a regular majorant of order α $(0 < \alpha < 1)$. Hence, we only prove the estimate for the differential of Henkin kernel, $d_z\, \mathbb{H}f$, which is the main part of this paper.

PROPOSITION 2.2. *For any α with $0 < \alpha < 1/2$, there exists a constant $C_\alpha > 0$ such that*

$$|d_z \mathbb{H} f(z)| \le C_\alpha \|f\|_{\omega_\alpha} \frac{\omega_\alpha(|\rho(z)|)}{|\rho(z)|^{1/2}} \quad \text{for} \quad z \in D. \tag{2.8}$$

PROOF. Since the singularities of the Henkin kernel are located in the diagonal $bD \times bD$, to show the inequality (2.8), it suffices to estimate the integral of (2.8) near boundary points. Fix a point $z \in D$ which is sufficiently close to the boundary of D and choose a ball $B(z, r)$ with $B(z, r) \cap bD \neq \emptyset$, in which we have a C^1 coordinates system $(t_1, \dots, t_{2n}) = t = t(\zeta, z)$ such that $t_1 = -\rho(\zeta), t_2 = \operatorname{Im} \phi(\zeta, z), t(z, z) = (-\rho(z), \dots, 0)$, and $|t(\zeta, z)| < 1$ for $\zeta \in B(z, r)$. (For the detail, see [HL84].) Moreover, this coordinate system t satisfies

$$|t| \lesssim |\zeta - z| \lesssim |t|, \quad \zeta \in B(z, r) \cap bD.$$

Also, note that the new coordinate system satisfies $t(\zeta, z) = (0, t')$ for $\zeta \in B(z, r) \cap bD$, where $t' = (t_2, \dots, t_{2n})$. By Remark 2.1, we have to show that

$$I(z) = \left| \int_{bD \cap B(z,r)} \frac{f(\zeta)\chi(\zeta)A(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^{2n-2}}\, dV(\zeta) \right| \lesssim \|f\|_{\omega_\alpha} \frac{\omega_\alpha(\delta(z))}{\delta(z)^{1/2}},$$

where χ is a compactly supported cut-off function in $B(z, r)$. For this kind of estimate of Hölder type, we choose $\zeta' \in B(z, r) \cap bD$ satisfying $t(\zeta', z) = (0, 0, t_3, \dots, t_{2n})$. This gives the obvious estimate, $I(z) \le I_1(z) +$

$+ I_2(z)$, where

$$I_1(z) = \left| \int\limits_{bD\cap B(z,r)} \frac{(f(\zeta) - f(\zeta'))\chi(\zeta)A(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^{2n-2}} \, dV(\zeta) \right|,$$

$$I_2(z) = \left| \int\limits_{bD\cap B(z,r)} \frac{f(\zeta')\chi(\zeta)A(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^{2n-2}} \, dV(\zeta) \right|.$$

It follows from the definition of $\|\cdot\|_{\omega_\alpha}$ and the inequality $|A(\zeta, z)| \lesssim |\zeta - z|$, that

$$I_1(z) \leq \|f\|_{\omega_\alpha} \int\limits_{bD\cap B(z,r)} \frac{\omega_\alpha(|\zeta - \zeta'|)}{|\phi(\zeta, z)|^2 |\zeta - z|^{2n-3}} \, dV(\zeta). \tag{2.9}$$

To estimate the integral of the right hand side of (2.9), we use the coordinate system t, the inequality (2.6), and introduce polar coordinates in $t'' = (t_3, \dots, t_{2n}) \in \mathbb{R}^{2n-2}$, and also set $r = |t''|$. Then we have

$$\begin{aligned} I_1(z) &\lesssim \|f\|_{\omega_\alpha} \int\limits_{|t'|<1} \frac{\omega_\alpha(|t_2|)}{(|t_2| + |t'|^2 + |\rho(z)|)^2 |t'|^{2n-3}} \, dV(t') \\ &\lesssim \|f\|_{\omega_\alpha} \int\limits_{|t_2|<1} \omega_\alpha(|t_2|) \left[\int\limits_0^1 \frac{r^{2n-3} dr}{(|t_2| + r^2 + |\rho(z)|)^2 r^{2n-3}} \right] dt_2 \\ &\lesssim \|f\|_{\omega_\alpha} \int\limits_0^1 \frac{\omega_\alpha(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2. \end{aligned}$$

We may assume that $0 < |\rho(z)| < 1$, since $z \in D$ is close to the boundary. We decompose the integral as follows:

$$\int\limits_0^1 \frac{\omega_\alpha(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2 = \int\limits_0^{|\rho(z)|} \frac{\omega_\alpha(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2 + \int\limits_{|\rho(z)|}^1 \frac{\omega_\alpha(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2.$$

Since ω_α is a regular majorant, by the first term of the left hand side of (1.1), we have

$$\int\limits_0^{|\rho(z)|} \frac{\omega_\alpha(t_2)}{(t_2 + |\rho(z)|)^{3/2}} \, dt_2 \lesssim \frac{1}{|\rho(z)|^{1/2}} \int\limits_0^{|\rho(z)|} \frac{\omega_\alpha(t_2)}{t_2} \, dt_2 \lesssim \frac{\omega_\alpha(|\rho(z)|)}{|\rho(z)|^{1/2}}.$$

Similarly, since $s^{1/2}\omega_\alpha(s)$ is also a regular majorant, by the second term of the left hand side of (1.1), it follows that

$$\int_{|\rho(z)|}^{1} \frac{\omega_\alpha(t_2)}{(t_2+|\rho(z)|)^{3/2}}\,dt_2 = \int_{|\rho(z)|}^{1} \frac{t_2^{1/2}\omega_\alpha(t_2)}{t_2^2}\frac{t_2^{3/2}}{(t_2+|\rho(z)|)^{3/2}}\,dt_2$$

$$\lesssim \int_{|\rho(z)|}^{1} \frac{t_2^{1/2}\omega_\alpha(t_2)}{t_2^2}\,dt_2$$

$$\lesssim \frac{|\rho(z)|^{1/2}\omega_\alpha(|\rho(z)|)}{|\rho(z)|} = \frac{\omega_\alpha(|\rho(z)|)}{|\rho(z)|^{1/2}}.$$

These inequalities imply $I_1(z)\lesssim \|f\|_{\omega_\alpha}\omega_\alpha(|\rho(z)|)/|\rho(z)|^{1/2}$.

For $I_2(z)$, we need a somewhat different method. The integration by parts allows one to lower the singularity order of the Henkin kernel. This kind of method was used in [Ran92].

We see that

$$\frac{1}{\phi^2} = -\left(\frac{\partial\phi}{\partial t_2}\right)^{-1}\frac{\partial}{\partial t_2}\left(\frac{1}{\phi}\right).$$

Therefore, by integration by parts, we have

$$\begin{aligned}(2.10)\qquad I_2(z) &\le \left|\int_{|t'|\le 1} -\left(\frac{\partial\phi}{\partial t_2}\right)^{-1}\frac{\partial}{\partial t_2}\left(\frac{1}{\phi}\right)\frac{f(0,0,t'')\chi(t')A(t',z)}{|t'|^{2n-2}}\,dt'\right| \\ &= \left|\int_{|t'|\le 1} f(0,0,t'')\,\frac{1}{\phi}\,\frac{\partial}{\partial t_2}\left[\left(\frac{\partial\phi}{\partial t_2}\right)^{-1}\frac{\chi(t')A(t',z)}{|t'|^{2n-2}}\right]dt'\right| \\ &= \left|\int_{|t'|\le 1} f(0,0,t'')\frac{1}{\phi}\left(\frac{\partial\phi}{\partial t_2}\right)^{-2}B(t',z)dt'\right|,\end{aligned}$$

where

$$B(t',z) = -\frac{\partial^2\phi}{\partial t_2^2}\frac{\chi(t')A(t',z)}{|t'|^{2n-2}} + \frac{\partial\phi}{\partial t_2}\frac{\partial}{\partial t_2}\left(\frac{\chi(t')A(t',z)}{|t'|^{2n-2}}\right).$$

In the second equality of (2.10), we use the fact that $f(0,0,t'')$ does not depend on t_2. Since $t_2 = \operatorname{Im}\phi$, we have $|\partial\phi/\partial t_2| \ge 1$. Therefore, we have

$$I_2(z)\lesssim \|f\|_\infty \int_{|t|'\le 1} \frac{dt'}{|\phi||t'|^{2n-2}}.$$

For the moment, we assume that for any $\varepsilon > 0$,

$$J(z) = \int\limits_{|t'|\le 1} \frac{dt'}{|\phi||t'|^{2n-2}} \lesssim |\rho(z)|^{-\varepsilon}, \tag{2.11}$$

which will be proved later as an independent lemma. Since (2.11) holds for arbitrary $\varepsilon > 0$, one can choose $\varepsilon > 0$ so that $0 < 1/2 - \varepsilon < \alpha$. Moreover, ω_α, $0 < \alpha < 1/2$, is a regular majorant and so $\omega_\alpha(t)/t^{1/2-\varepsilon}$ is increasing, or equivalently, $|\rho(z)|^{-\varepsilon} \lesssim \omega_\alpha(|\rho(z)|)/|\rho(z)|^{1/2}$. It follows that

$$I_2(z) \lesssim \|f\|_\infty J(z) \lesssim \|f\|_{\omega_\alpha} \frac{\omega_\alpha(|\rho(z)|)}{|\rho(z)|^{1/2}}.$$

These two estimates for $I_1(z)$ and $I_2(z)$ complete the proof. □

We end this section with the proof of (2.11).

LEMMA 2.3. *For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$J(z) \le C_\varepsilon |\rho(z)|^{-\varepsilon}.$$

PROOF. We have

$$J(z) \lesssim \int\limits_{|t'|<1} \frac{dt'}{(|t_2| + |\rho(z)| + |t'|^2)|t'|^{2n-2}}$$

$$\lesssim \int\limits_{|(t_2,t_3,t_4)|<1} \frac{dt_2 dt_3 dt_4}{(|t_2| + |\rho(z)|)(t_2^2 + t_3^2 + t_4^2)}.$$

Again, using polar coordinates in (t_3, t_4), say $x = |(t_3, t_4)|$, one obtains

$$J(z) \lesssim \int\limits_{|t_2|<1} \frac{1}{|t_2| + |\rho(z)|} \left(\int_0^1 \frac{x\,dx}{t_2^2 + x^2} \right) dt_2$$

$$\lesssim \int_0^1 \frac{|\log t_2|}{(t_2 + |\rho(z)|)}\, dt_2$$

$$\le C_\varepsilon \int_0^1 \frac{t_2^{-\varepsilon}}{(t_2 + |\rho(z)|)}\, dt_2.$$

By the change of variable $s = t_2/|\rho(z)|$, we have

$$J(z) \lesssim \int_0^1 \frac{|t_2|^{-\varepsilon}}{(|t_2| + |\rho(z)|)} \, dt_2$$

$$\lesssim |\rho(z)|^{-\varepsilon} \int_0^\infty \frac{ds}{(1+s)s^\varepsilon} \leq C_\varepsilon |\rho(z)|^{-\varepsilon}.$$

□

3. Proof of Theorem 1.2.

In this section, we complete the proof of our main Theorem 1.2 using Proposition 2.2 and Lemma 1.3.

The inequality (2.8) in Proposition 2.2 implies that

$$|d\mathbb{H}f(z)| \leq c_\alpha \|f\|_\infty \frac{|\rho(z)|^{1/2} \omega_\alpha(|\rho(z)|)}{|\rho(z)|}.$$

Therefore by Lemma 1.3 and regularities of the operator $\mathbb{H}f$ in the Hölder type spaces we can prove the inequality (1.3) of Theorem 1.2.

Finally, we include a brief sketch of the proof of Lemma 1.3.

PROOF. Because $\overline{D}$ is compact, by the local coordinate change argument, it suffices to show the following in the special domain $D(k) = \{ (x_1, x') \in \mathbb{R}^n : 0 < x_1 < k, \ |x'| < k \}$: if

$$|dg(x)| \leq c_g \frac{\omega_\gamma(x_1)}{x_1} \tag{3.12}$$

for $x, y \in D(k/2)$ with $|x - y| \leq k/2$, then we have

$$|g(x) - g(y)| \leq c \cdot c_g \ \omega_\gamma(|x-y|). \tag{3.13}$$

To show this, fix two points $x, y \in D(k/2)$ with $|x - y| \leq k/2$ and let $d = |x - y|$. Here we may assume that $k \leq 1/2$ and by symmetry we may also suppose $x_1 \leq y_1$.

First it follows from (3.12) that

$$|g(x_1, x') - g(x_1 + d, x')| \leq \int_{x_1}^{x_1+d} \left| \frac{\partial g}{\partial x_1}(t, x') \, dt \right| \tag{3.14}$$

$$\leq c_g \int_{x_1}^{x_1+d} \frac{\omega_\gamma(t)}{t} \, dt \leq c \cdot c_g \ \omega_\gamma(d)$$

In fact, if $0 < d \le x_1$, then

$$\int_{x_1}^{x_1+d} \frac{\omega_\gamma(t)}{t}\, dt \le d\, \frac{\omega_\gamma(x_1)}{x_1} \le \omega_\gamma(d),$$

since $\omega_\gamma(t)/t$ is decreasing. If $0 < x_1 \le d$, then

$$\int_{x_1}^{x_1+d} \frac{\omega_\gamma(t)}{t}\, dt \lesssim \int_0^d \frac{\omega_\gamma(t)}{t}\, dt \lesssim \omega_\gamma(d). \tag{3.15}$$

Since $\omega_\gamma(t)/t$ is decreasing, the first inequality of (3.15) holds and by (1.1) the second inequality of (3.15) is also true.

Next, by the Mean Value Theorem and (3.12), since $\omega_\gamma(t)/t$ is decreasing, we have

$$|g(x_1 + d, x') - g(y_1 + d, y')| \le c_g\, d\, \frac{\omega_\gamma(a_1)}{a_1} \le c_g\ \omega_\gamma(d) \tag{3.16}$$

for some a_1 in the line segment between $x_1 + d$ and $y_1 + d$. Since

$$\begin{aligned}|g(x) - g(y)| \le{}& |g(x_1, x') - g(x_1 + d, x')| \\ &+ |g(x_1 + d, x') - g(y_1 + d, y')| + |g(y_1 + d, y') - g(y_1, y')|,\end{aligned}$$

(3.13) follows from the estimates (3.14) and (3.16). □

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