Bar Billiards and Poncelet's Porism

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ABSTRACT - We prove that for a given oval C there exist ovals C_{in} and C_{out}, inside and outside of C, such that the pairs (C, C_{in}) and (C_{out}, C) satisfy the Poncelet's porism for almost any number of reflections in their bar billiards.

1. Introduction.

Let us consider two ovals C_1 and C_2, i.e., two smooth strictly convex closed plane curves. Let the oval C_2 lie inside the second oval C_1. From any point on C_1, draw a tangent to C_2 and extend it to C_1 in the opposite direction. From this point we draw another tangent, etc. For all tangents, the resulting Poncelet's transverse will be called a bar billiard since it is similar to a traditional game played years ago. In general in this game players scored points by knocking balls into the holes while avoiding toppling a skittle in the middle of the table. Here the role of the skittle is played by the oval C_2 and it must be "toppled" by the tangent line. The general behavior of such billiards is worth investigating but in this paper we will concentrate on the Poncelet's porism for this specific setting. Namely, we say that a bar billiard has a Poncelet's porism property if the following is true: if, on the oval C_1, there is one point of origin for which a Poncelet transverse is closed, then the transverse will also close for any other point of origin on the oval. One can find an extensive bibliography on Poncelet's porism in [4]; a nice introduction to theory of billiards is provided in [3].

In this paper we prove that for a given oval C, in a suitably small C^2-neighborhood there exist ovals C_{in} and C_{out} inside and outside of C, such that for a suitable large number of segments the pairs (C, C_{in}) and (C_{out}, C)

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satisfy Poncelet's porism in the above sense. In the course of the proof we bring into play the isoptics introduced and developed in [1] and [2].

2. Outer oval.

Let us begin with a simple algebraic lemma

LEMMA 2.1.   *Let $v, w \in \mathbb{R}^2$ and $\langle v, w \rangle$ and $[v, w]$ denote the dot product and the determinant of these vectors. Then*

(1) $v\langle v, w\rangle - w\langle w, v\rangle = [v, w] J(w)$
(2) $\langle v, w\rangle = [J(w), v],$

where J is the positive rotation of angle $\pi/2$.

Next we consider an operator $\dfrac{D}{D\alpha}$ of the form

$$\frac{Dp(t)}{D\alpha} = \frac{p(t + \alpha) - p(t)\cos\alpha}{\sin\alpha}$$

acting on real functions defined on all $\mathbb{R}$. From the elementary calculus it is clear that, for any C^1-function p,

$$\lim_{\alpha \to 0} \frac{Dp(t)}{D\alpha} = p'(t)$$

for any $t \in \mathbb{R}$.

In what follows we give a preview of certain facts concerning isoptics as the fundamental tool in our constructions.

DEFINITION 2.1. *A plane, closed, simple, positively oriented, curve of positive curvature is called an oval.*

We take a coordinate system with origin O in the interior of C. Let $p(t)$, $t \in [0, 2\pi]$, be the distance from O to the support line $l(t)$ of C perpendicular to the vector $e^{it} = \cos t + i \sin t$. It is well-known that $p(t)$ is of class C^1 and that the parametrization of C in terms of $p(t)$ is given by the formula

(2.1) $z(t) = p(t)e^{it} + p'(t)ie^{it},$

where $ie^{it} = -\sin t + i\cos t$. Note that the support function p can be extended to a periodic function on $\mathbb{R}$ with the period 2π.

DEFINITION 2.2. *Let C_α be a locus of vertices of a fixed angle $\pi - \alpha$ formed by two support lines of the oval C. The curve C_α will be called an α-isoptic of C.*

It is convenient to parametrize the α-isoptic C_α by the same angle t so that the equation of C_α takes the form

$$(2.2) \qquad\qquad z_\alpha(t) = p(t)e^{it} + \frac{Dp(t)}{D\alpha} ie^{it}.$$

Since the functions involved are at least of class C^1 we observe that the given parametrization of an isoptic is of class C^1. It can be shown that it is a regular curve. Note that

$$(2.3) \qquad z_\alpha(t) = z(t) + \lambda(t, \alpha)ie^{it} = z(t + \alpha) + \mu(t, \alpha)ie^{i(t+\alpha)}$$

$$(2.4) \qquad z'_\alpha(t) = -\lambda(t, \alpha)e^{it} + \rho(t, \alpha)ie^{it}$$

for suitable functions λ, μ and ρ. Moreover, we have

$$(2.5) \qquad\qquad \lambda(t, \alpha) = \frac{Dp(t)}{D\alpha} - p'(t),$$

$$(2.6) \qquad\qquad \rho(t, \alpha) = p(t) + \frac{Dp'(t)}{D\alpha}.$$

COROLLARY 2.1. *With the notations above we have*

    1) $\lim_{\alpha \to 0} z_\alpha(t) = z(t)$,

    2) $\lim_{\alpha \to 0} \lambda(t, \alpha) = 0$,

    3) $\lim_{\alpha \to 0} \rho(t, \alpha) = R(t)$, *where $R(t) = p(t) + p''(t)$ is the curvature radius of the oval C at t.*

LEMMA 2.2. *The function*

$$(2.7) \qquad\qquad k_\alpha(t) = \begin{cases} \kappa_\alpha(t), & 0 < \alpha < \pi \\ \kappa(t), & \alpha = 0 \end{cases}$$

is continuous from the right on α, where $\kappa_\alpha(t)$ is the curvature of the isoptic $z_\alpha(t)$ at t.

PROOF. It follows immediately from the equation of an isoptic that

$$(2.8) \qquad\qquad z'_\alpha(t) = -\lambda(t, \alpha)e^{it} + \rho(t, \alpha)ie^{it}$$

and

$$(2.9) \qquad |z'_\alpha(t)|^2 = \lambda^2(t, \alpha) + \rho^2(t, \alpha).$$

The latter formula ensures that the isoptics of ovals are always regular curves. By the above considerations we get

$$(2.10) \quad \lim_{\alpha \to 0} z'_\alpha(t) = \lim_{\alpha \to 0} \left[\left(p'(t) - \frac{Dp(t)}{D\alpha} \right) e^{it} + \left(p(t) + \frac{Dp'}{D\alpha} \right) i e^{it} \right]$$

$$= (p(t) + p''(t)) i e^{it} = z'(t)$$

and

$$(2.11) \quad \lim_{\alpha \to 0} |z'_\alpha(t)|^2 = \lim_{\alpha \to 0} \left[\left(p'(t) - \frac{Dp(t)}{D\alpha} \right)^2 + \left(p(t) + \frac{Dp'}{D\alpha} \right)^2 \right]$$

$$= (p(t) + p''(t))^2 = |z'(t)|^2.$$

The formula for curvature $\kappa_\alpha(t)$ ([2]) of an isoptic curve can be transformed to the following

$$(2.12) \quad \kappa_\alpha(t) = \frac{1}{\left(\lambda^2 + \rho^2 \right)^{\frac{3}{2}}} \left(2\lambda^2 + 2\rho^2 - R(t)\rho + \lambda R(t) \cot \alpha - \lambda \frac{\varrho(\tau + \alpha)}{\sin \alpha} \right)$$

$$= \frac{1}{\left(\lambda^2 + \rho^2 \right)^{\frac{3}{2}}} \left(2\lambda^2 + 2\rho^2 - R(t)\rho - \lambda \frac{DR(t)}{D\alpha} \right).$$

Hence,

$$(2.13) \qquad \lim_{\alpha \to 0} \kappa_\alpha(t) = \frac{1}{R(t)} = \kappa(t).$$

$\square$

COROLLARY 2.2. *There exists $\varepsilon > 0$ such that if $0 < \alpha < \varepsilon$ then $k_\alpha(t) > 0$ for any $t \in \mathbb{R}$.*

THEOREM 2.1. *For any oval C there exists an oval O containing oval C inside such that the corresponding bar billiard has the Poncelet's porism property.*

PROOF. Let $\alpha = \dfrac{2\pi}{n}$ be an angle with $0 < \alpha < \varepsilon$ for the above ε. Then we can consider a bar billiard such that the starting oval is C_2 and its α-isoptic is C_1. In this case, for any t Poncelet's transverses close after n reflections and form an n-gon with or without self intersections. $\square$

3. Inner oval.

In this section we consider the reverse problem: for a given oval C, does there exist an oval O inside C such that for a corresponding bar billiard Poncelet's porism holds?

First, we assume that the oval C is parametrized as in formula (2.1) and that the origin of coordinates is placed at the Steiner point of C. Furthermore, the coordinate system is chosen in such a way that the tangent line at $z(0)$ is perpendicular to the x-axis. Then, in particular, we have $z(0) = (a, 0)$ for some $a > 0$ and $p'(0) = 0$. If we introduce the following notation

$$(3.1) \qquad q(t) = (q_1(t), q_2(t)) = z(t) - z(t + \alpha)$$

then the line $l(t)$ through the points $z(t)$ and $z(t + \alpha)$ is given by

$$(3.2) \qquad l(t): \quad -yq_1(t) + xq_2(t) - [z(t), q(t)] = 0.$$

Hence, we easily calculate $h(t)$, its distance from the origin $O = (0, 0)$

$$(3.3) \qquad h(t) = -\frac{[z(t), q(t)]}{|q(t)|}.$$

Let $\varphi(t)$ be the angle contained between the positive direction of the axis Ox and the vector $q(t)$, and $x(t)$ be the angle between the positive direction of Ox and OO', the shortest segment joining O with the line (3.2). Evidently, we have the relation

$$(3.4) \qquad x(t) = \frac{\pi}{2} + \varphi(t).$$

Note that the following formulas are true for $\varphi(t)$

$$(3.5) \qquad \cos \varphi(t) = \frac{\langle z(0), q(t) \rangle}{|q(t)||z(0)|}$$

$$(3.6) \qquad \sin \varphi(t) = \frac{[z(0), q(t)]}{|q(t)||z(0)|}.$$

Differentiating both sides of formula (3.5) we obtain

$$(3.7) \qquad \varphi'(t) = \frac{[q(t), q'(t)]}{|q(t)|^2}$$

and, after some further manipulations we get

$$(3.8) \qquad \varphi'(t) = \frac{\lambda(t, \alpha)R(t + \alpha) - \mu(t, \alpha)R(t)}{\sin \alpha \, (\lambda^2(t, \alpha) + \rho^2(t, \alpha))} > 0,$$

where $\lambda(t, \alpha)$ and $\mu(t, \alpha)$ are the same as in formulas (2.3) and (2.4). Moreover from the geometric interpretation of these functions, we see that they are nonzero and that $\mu < 0$. Thus, it follows from elementary calculus that there exists an inverse function $\psi(x) = \varphi^{-1}\left(x - \dfrac{\pi}{2}\right) = t$. This function allows us to define a new support function

$$(3.9) \qquad\qquad P(x) = h(\psi(x)).$$

Note that $|OO'| = P(x)$ and recall that x is the angle between the positive direction of axis Ox and $\overrightarrow{OO'}$. This construction will be used to find an envelope z_{izo} of the family of lines (3.2) in the form

$$(3.10) \qquad\qquad z_{izo}(x) = P(x)e^{ix} + \frac{dP(x)}{dx}ie^{ix}.$$

After some computations we have

$$(3.11) \qquad \frac{dP(x)}{dx} = \frac{dh}{dt}(\psi(x))\,\frac{d\psi(x)}{dx} = -\frac{[z', q]\langle q, q\rangle + \langle q, z\rangle[q, q']}{|q|[q, q']}(\psi(x)).$$

In this framework t is a generic parameter and we may consider an equation of the envelope in the form

$$(3.12)
\begin{aligned}
z_{izo}(t) &\\
&= P\left(\frac{\pi}{2} + \varphi(t)\right)e^{i\left(\frac{\pi}{2}+\varphi(t)\right)} + \frac{dP}{dx}\left(\frac{\pi}{2} + \varphi(t)\right)ie^{i\left(\frac{\pi}{2}+\varphi(t)\right)} \\
&= P(t)E^{it} + P'_x(t)iE^{it},
\end{aligned}$$

where

$$(3.13) \qquad\qquad E^{it} = \left\{\frac{[z(0), q(t)]}{|q(t)||z(0)|}, \frac{\langle z(0), q(t)\rangle}{|q(t)||z(0)|}\right\},$$

$$(3.14) \qquad\qquad iE^{it} = -\left\{\frac{\langle z(0), q(t)\rangle}{|q(t)||z(0)|}, \frac{[z(0), q(t)]}{|q(t)||z(0)|}\right\}.$$

Ultimately, the above considerations lead to

$$(3.15) \qquad z_{izo}(t) = -\frac{[z(t), q(t)]}{|q(t)|}E^{it}$$
$$-\frac{[z'(t), q(t)]\langle q(t), q(t)\rangle + \langle q(t), z(t)\rangle[q(t), q'(t)]}{|q(t)|[q(t), q'(t)]}iE^{it},$$

where $z'(t) = \dfrac{dz}{dt}(t)$ and $q'(t) = \dfrac{dq}{dt}(t)$. Let us make the convention that in the reminder of this paper prime will denote differentiation with respect to t. By taking the derivative, we obtain

LEMMA 3.1.

$$(3.16) \qquad \left(E^{it}\right)' = iE^{it} \frac{[q(t), q'(t)]}{|q(t)|^2},$$

$$(3.17) \qquad \left(iE^{it}\right)' = -E^{it} \frac{[q(t), q'(t)]}{|q(t)|^2}.$$

THEOREM 3.1.   *The curve $z_{izo}(t)$ is the closed envelope of the family of lines $\{l(t) : \ t \in \mathbb{R}\}$.*

PROOF.   By lemma (3.1) it follows that

$$(3.18) \qquad z'_{izo}(t) = \left(P(t) \frac{[q(t), q'(t)]}{|q(t)|^2} + \left(\frac{dP}{dx}(t) \right)' \right) iE^{it}$$

and from (3.14) we obtain that

$$(3.19) \qquad iE^{it} = a \frac{q(t)}{|q(t)|}.$$

This means that the tangent vector to $z'_{izo}(t)$ is linearly dependent on $q(t)$. Note, that we don't claim that it is a nonzero vector. □

We do not yet know whether $z_{izo}(t) \to z(t)$ when $\alpha \to 0$. To this end we prove

LEMMA 3.2.

$$(3.20) \qquad \lim_{\alpha \to 0} z_{izo}(t) = z(t).$$

PROOF.   Write

$$(3.21) \qquad \frac{[z(0), q(t)]}{|q(t)||z(0)|} = \frac{(\rho \sin \alpha - \lambda \cos \alpha) \sin t - (\lambda \sin \alpha + \rho \cos \alpha) \cos t}{\sqrt{\lambda^2 + \rho^2}},$$

$$(3.22) \qquad \frac{\langle z(0), q(t) \rangle}{|q(t)||z(0)|} = \frac{(\rho \sin \alpha - \lambda \cos \alpha) \cos t + (\lambda \sin \alpha + \rho \cos \alpha) \cos t}{\sqrt{\lambda^2 + \rho^2}},$$

where $\lambda = \lambda(t, \alpha)$ and $\rho = \rho(t, \alpha)$. By passing to the limit, we obtain

$$(3.23) \qquad \lim_{\alpha \to 0} E^{it} = e^{it},$$

$$(3.24) \qquad \lim_{\alpha \to 0} iE^{it} = ie^{it}.$$

Similarly, since

$$(3.25) \qquad \frac{q(t)}{|q(t)|} = \frac{(\rho \sin\alpha - \lambda \cos\alpha)e^{it} - (\lambda \sin\alpha + \rho \cos\alpha)ie^{it}}{\sqrt{\lambda^2 + \rho^2}},$$

we get

$$(3.26) \qquad \lim_{\alpha \to 0} P(t) = -\lim_{\alpha \to 0} \frac{[z(t), q(t)]}{|q(t)|} = p(t).$$

When we analyze formula (3.25) we easily see that

$$(3.27) \qquad \lim_{\alpha \to 0} \frac{q(t)}{\sin\alpha} = -R(t)ie^{it},$$

$$(3.28) \qquad \lim_{\alpha \to 0} \frac{q'(t)}{\sin\alpha} = R(t)e^{it} - R'(t)ie^{it}$$

The above formulas and the formula (3.15) give

$$(3.29) \qquad \lim_{\alpha \to 0} \frac{dP}{dx}(t) = p'(t),$$

which implies that

$$(3.30) \qquad \lim_{\alpha \to 0} z_{izo}(t) = z(t).$$

$\square$

Our next task is to prove

$$(3.31) \qquad \lim_{\alpha \to 0} z'_{izo}(t) = z'(t).$$

Recall that

$$(3.32) \qquad \frac{dP}{dx}(t) = h'(t)\frac{|q(t)|^2}{[q(t), q'(t)]},$$

so that

$$(3.33) \qquad z'_{izo}(t) = \left(h(t)\frac{[q(t), q'(t)]}{|q(t)|^2} + \left(h'(t)\frac{|q(t)|^2}{[q(t), q'(t)]} \right)' \right) iE^{it}.$$

In order to determine $\lim_{\alpha \to 0} z'_{izo}(t)$ we need only two facts:

$$(3.34) \qquad \lim_{\alpha \to 0} \frac{[q(t), q'(t)]}{|q(t)|^2} = 1,$$

$$(3.35) \qquad \lim_{\alpha \to 0} \frac{q'(t)}{\sin\alpha} = 2R'(t)e^{it} + (R(t) - R''(t))ie^{it}.$$

Hence, after some tedious computations we get

(3.36) $$\lim_{\alpha \to 0} \left(h'(t) \frac{|q(t)|^2}{[q(t), q'(t)]} \right)' = p''(t),$$

that implies

LEMMA 3.3. *We have*

(3.37) $$\lim_{\alpha \to 0} z'_{izo}(t) = z'(t).$$

In particular, the envelope which can be called an inner isoptic is a regular curve if α is sufficiently small.

Note that for some values of α the inner isoptic can develop cusps.

Our last problem before formulating the final theorem is to find $\lim_{\alpha \to 0} z''_{izo}(t)$.

LEMMA 3.4. *Under the above notations*

(3.38) $$\lim_{\alpha \to 0} z''_{izo}(t) = R'(t)ie^{it} - R(t)e^{it} = z''(t).$$

In particular, the function

(3.39) $$k_{izo}(t) = \begin{cases} \kappa_{izo}(t), & 0 < \alpha < \pi, \\ \kappa(t), & \alpha = 0, \end{cases}$$

where $\kappa_{izo}(t)$ denotes the curvature of the inner isoptic $z_{izo}(t)$, is continuous from the right.

PROOF. Write

$$(3.40) \quad z''_{izo}(t) = \left\{ h'(t) \frac{[q(t), q'(t)]}{|q(t)|^2} + h(t) \left(\frac{[q(t), q'(t)]}{|q(t)|^2} \right)' \right.$$
$$\left. + \left(h'(t) \frac{|q(t)|^2}{[q(t), q'(t)]} \right)'' \right\} iE^{it}$$
$$- \left\{ h(t) \frac{[q(t), q'(t)]}{|q(t)|^2} + \left(h'(t) \frac{|q(t)|^2}{[q(t), q'(t)]} \right)' \right\} \frac{[q(t), q'(t)]}{|q(t)|^2} E^{it}.$$

Note that in the above representation the second term tends to $-R(t)e^{it}$ and $h'(t) \frac{[q(t), q'(t)]}{|q(t)|^2} \to p'(t)$. It remains to find two other limits in the first term.

This is a very long and cumbersome job, and after computations similar to those we have done earlier, we obtain

$$(3.41) \qquad \lim_{\alpha \to 0} \left(\frac{[q(t), q'(t)]}{|q(t)|^2} \right)' = 0,$$

$$(3.42) \qquad \lim_{\alpha \to 0} \left(h'(t) \frac{|q(t)|^2}{[q(t), q'(t)]} \right)'' = p'''(t).$$

Hence, the lemma holds. □

COROLLARY 3.1. *There exists $\varepsilon > 0$ such that if $0 < \alpha < \varepsilon$ then $k_{izo}(t) > 0$ for any $t \in \mathbb{R}$.*

THEOREM 3.2. *For any oval C there exists an oval O, contained in C such that corresponding bar billiard has Poncelet's porism property.*

PROOF. Let $\alpha = \dfrac{2\pi}{n}$ be an angle such that $0 < \alpha < \varepsilon$ for the above ε. Then we can consider a bar billiard with the starting oval C_1 and its inner isoptic $z_{izo}(t)$ taken for C_2. In this case, for any t, the Poncelet's transverses close after n reflections and form an n-gon with or without self intersections. □

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