REND. SEM. MAT. UNIV. PADOVA, Vol. 120 (2008)

Sheaves on Subanalytic Sites

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ABSTRACT - In [7] the authors introduced the notion of ind-sheaves and defined the six Grothendieck operations in this framework. As a byproduct, they obtained subanalytic sheaves and the six Grothendieck operations on them.
The aim of this paper is to give a direct construction of the six Grothendieck operations in the framework of subanalytic sites avoiding the heavy machinery of ind-sheaves. As an application, we show how to recover the subanalytic sheaves $\mathcal{O}^t$ and $\mathcal{O}^w$ of temperate and Whitney holomorphic functions respectively.

Introduction.

Let X be a real analytic manifold and k a field. The spaces of functions which are not defined by local properties, such as tempered distributions, tempered and Whitney $\mathcal{C}^\infty$ functions, etc., are very useful in the study of systems of linear partial differential equations (Laplace transform, tempered holomorphic solutions of $\mathcal{D}$-modules etc.). Although these spaces do not define sheaves on X, they define sheaves on a site associated to X, the subanalytic site X_{sa}, where one just considers open subanalytic sets and locally finite coverings.

In [7], Kashiwara and Schapira, motivated by the construction of the microlocalization functor, treated a more general theory, namely that of ind-sheaves. They defined the category $\mathrm{I}(k_X)$ of ind-sheaves on X as the category of ind-objects of the category $\mathrm{Mod}^c(k_X)$ of sheaves with compact support and they developped the six Grothendieck operations in this framework. When restricting to $\mathbb{R}$-constructible sheaves, they showed the equivalence between the category $\mathrm{I}_{\mathbb{R}\text{-c}}(k_X) = \mathrm{Ind}(\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X))$ of ind-$\mathbb{R}$-constructible sheaves on X and the category $\mathrm{Mod}(k_{X_{sa}})$ of sheaves on the subanalytic site associated to X.

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In this way, tempered distributions, tempered and Whitney $\mathcal{C}^\infty$ functions, etc., are obtained as a byproduct of the whole theory of ind-sheaves. It turns out to be useful to have a more straightforward introduction of these sheaves.

Our aim in this paper is to give a direct, self-contained and elementary construction of the six Grothendieck operations on $\mathrm{Mod}(k_{X_{sa}})$, without using the more sophisticated and much more difficult theory of ind-sheaves. Indeed, contrary to the category $\mathrm{I}(k_X)$, the category $\mathrm{Mod}(k_{X_{sa}})$ is a Grothendieck category.

We will start by recalling some results of [7], the definition of a subanalytic site, the natural functor of sites $\rho : X \to X_{sa}$, and the functors ρ_*, ρ^{-1} and $\rho_!$ relating the categories of "classical" and subanalytic sheaves. We also recall a very useful description of subanalytic sheaves as inductive limits of $\mathbb{R}$-constructible sheaves.

Then we go into the study of subanalytic sheaves, without using the notion of ind-sheaf. Let $f : X \to Y$ be a morphism of real analytic manifolds. The functors $\mathcal{H}om$, $\otimes$, f_* and f^{-1} are well defined since X_{sa} is a site. We introduce the proper direct image functor $f_{!!}$ and we study the relations between the above operations and the functors ρ_*, ρ^{-1} and $\rho_!$. In the derived category we obtain an exceptional inverse image, denoted by $f^!$, right adjoint to $Rf_{!!}$. This is obtained via Brown representability thanks to the existence of quasi-injective objects. We study quasi-injective objects without references to ind-sheaves, we show that they are acyclic with respect to the above functors and we prove that the quasi-injective dimension of $\mathrm{Mod}(k_{X_{sa}})$ is finite. We end this work giving some examples of subanalytic sheaves.

In more details, the contents of this paper are as follows.

In Section 1 we construct the operations in $\mathrm{Mod}(k_{X_{sa}})$. In § 1.1, we start by recalling the definitions of the functors ρ_*, ρ^{-1} and $\rho_!$ of [7] and their properties (a more detailed study of the functor $\rho_!$ is done in § 1.6). In § 1.2, we recall the internal operations and the functors of direct and inverse image (which are well defined on any site) and their relations with ρ_*, ρ^{-1} and $\rho_!$. We also define (in § 1.4) the proper direct image functor $f_{!!}$, where the notation $f_{!!}$ follows from the fact that $f_{!!} \circ \rho_* \not\simeq \rho_* \circ f_!$ in general. We study its properties and the relations with the other operations. While the functors f^{-1} and $\otimes$ are exact, the functors $\mathcal{H}om, f_*$ and $f_{!!}$ are left exact, and we introduce the subcategory of quasi-injective objects which is injective with respect to these functors. This is done in § 1.5.

In Section 2 we consider the derived category of $\mathrm{Mod}(k_{X_{sa}})$. In § 2.1, we consider the subcategory $D^b_{\mathbb{R}\text{-c}}(k_{X_{sa}})$ consisting of bounded complexes with $\mathbb{R}$-constructible cohomology and we prove the equivalence of derived categories $D^b_{\mathbb{R}\text{-c}}(k_X) \simeq D^b_{\mathbb{R}\text{-c}}(k_{X_{sa}})$. Then, in § 2.2, we study the derived functors of $\mathcal{H}om, f_*$ and $f_{!!}$ and we obtain the usual formulas (projection formula, base change formula, Künneth formula, etc.) in the framework of subanalytic sites. We also prove (in § 2.3) some vanishing theorems for subanalytic sheaves, in particular we prove that the quasi-injective dimension of $\mathrm{Mod}(k_{X_{sa}})$ is finite. Using the Brown representability theorem we prove the existence of a right adjoint to the functor $Rf_{!!}$, denoted by $f^!$. This is done in § 2.4. We calculate the functor $f^!$ by decomposing f as the composite of a closed embedding and a submersion.

In Section 3 we give some examples of subanalytic sheaves. We start by recalling the definition of sheaves of $\rho_!\mathcal{D}_X$-modules, where $\mathcal{D}_X$ denotes the sheaf of finite order differential operators on a complex analytic manifold X. Then in § 3.3 we show how to recover the sheaves of $\rho_!\mathcal{D}_X$-modules $\mathcal{O}^t_X$ and $\mathcal{O}^{\mathrm{w}}_X$ of temperate and Whitney holomorphic functions of [7] respectively. We prove the relations between the above sheaves and the functors of moderate and formal cohomology of [4] and [6].

Acknowledgments. We would like to thank Prof. Pierre Schapira who encouraged us to develop a theory of subanalytic sheaves independent of that of ind-sheaves, and for his many useful remarks.

1. Sheaves on subanalytic sites.

In the following X will be a real analytic manifold and k a field. References are made to [8] and [14] for an introduction to sheaves on Grothendieck topologies, to [5] for a complete exposition on classical sheaves and $\mathbb{R}$-constructible sheaves and to [1] and [10] for the theory of subanalytic sets (see also the Appendix for a short overview on some properties of subanalytic subsets).

1.1 – *The subanalytic site. Notations and review.*

We introduce the subanalytic site. The results of § 1.1 have already been proved in [7], for sake of completeness we reproduce here the proofs.

Denote by $\mathrm{Op}(X_{sa})$ the category of subanalytic subsets of X. One endows $\mathrm{Op}(X_{sa})$ with the following topology: $S \subset \mathrm{Op}(X_{sa})$ is a covering of

$U \in \mathrm{Op}(X_{sa})$ if for any compact subset K of X there exists a finite subset S_0 of S such that $K \cap \bigcup_{V \in S_0} V = K \cap U$. We will call X_{sa} the subanalytic site, and for $U \in \mathrm{Op}(X_{sa})$ we denote by $U_{X_{sa}}$ the category $\mathrm{Op}(X_{sa}) \cap U$ with the topology induced by X_{sa}. There is a natural morphism of sites $U_{sa} \to U_{X_{sa}}$.

REMARK 1.1.1. *We use the notation $U_{X_{sa}}$ to stress the difference from U_{sa}, the subanalytic site associated to U. For example, let $X = \mathbb{R}^2$ and $U = \mathbb{R}^2 \setminus \{0\}$. Let $V_n = \{x \in \mathbb{R}^2, \ \ |x| > \frac{1}{n}\}$. Then $\{V_n\}_{n \in \mathbb{N}} \in \mathrm{Cov}(U_{sa})$ but $\{V_n\}_{n \in \mathbb{N}} \notin \mathrm{Cov}(U_{X_{sa}})$.*

Let $\mathrm{Mod}(k_{X_{sa}})$ denote the category of sheaves on X_{sa}. Then $\mathrm{Mod}(k_{X_{sa}})$ is a Grothendieck category, i.e. it admits a generator and small inductive limits, and small filtrant inductive limits are exact. In particular as a Grothendieck category, $\mathrm{Mod}(k_{X_{sa}})$ has enough injective objects.

REMARK 1.1.2. *Denote by $\mathrm{Op}^c(X_{sa})$ the category of relatively compact subanalytic open subsets of X. One denotes by X_{sa}^c the category $\mathrm{Op}^c(X_{sa})$ with the topology induced by X_{sa}. The forgetful functor gives an equivalence of categories $\mathrm{Mod}(k_{X_{sa}}) \xrightarrow{\sim} \mathrm{Mod}(k_{X_{sa}^c})$.*

PROPOSITION 1.1.3. *Let $\{F_i\}_{i \in I}$ be a filtrant inductive system in $\mathrm{Mod}(k_{X_{sa}})$ and let $U \in \mathrm{Op}^c(X_{sa})$. Then*

$$\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i).$$

PROOF. By Remark 1.1.2 it is enough to prove the assertion in the category $\mathrm{Mod}(k_{X_{sa}^c})$. Denote by "$\varinjlim_i$" F_i the presheaf $V \mapsto \varinjlim_i \Gamma(V; F_i)$ on X_{sa}^c. Let $U \in \mathrm{Op}^c(X_{sa})$ and let S be a finite covering of U. Since $\varinjlim$ commutes with finite projective limits we obtain the isomorphism ("$\varinjlim_i$" $F_i)(S) \xrightarrow{\sim} \varinjlim_i F_i(S)$ and $F_i(U) \xrightarrow{\sim} F_i(S)$ since $F_i \in \mathrm{Mod}(k_{X_{sa}^c})$ for each i. Moreover the family of finite coverings of U is cofinal in $\mathrm{Cov}(U)$. Hence "$\varinjlim_i$" $F_i \xrightarrow{\sim}$ ("$\varinjlim_i$" $F_i)^+$. Applying once again the functor $(\,\cdot\,)^+$ we get

$$\text{“}\varinjlim_i\text{”} F_i \simeq (\text{“}\varinjlim_i\text{”} F_i)^+ \simeq (\text{“}\varinjlim_i\text{”} F_i)^{++} \simeq \varinjlim_i F_i.$$

Hence applying the functor $\Gamma(U; \cdot)$ we obtain the isomorphism $\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i)$ for each $U \in \mathrm{Op}^c(X_{sa})$. □

There is an easy way to construct sheaves on a subanalytic site

PROPOSITION 1.1.4. *Let F be a presheaf on X^c_{sa} and assume that*

(i) $F(\emptyset) = 0$,

(ii) *For any $U, V \in \mathrm{Op}^c(X_{sa})$ the sequence $0 \to F(U \cup V) \to \to F(U) \oplus F(V) \to F(U \cap V)$ is exact.*

Then $F \in \mathrm{Mod}(k_{X^c_{sa}}) \simeq \mathrm{Mod}(k_{X_{sa}})$.

PROOF. Let $U \in \mathrm{Op}^c(X_{sa})$ and let $\{U_j\}_{j=1}^n$ be a finite covering of U. Set for short $U_{ij} = U_i \cap U_j$. We have to show the exactness of the sequence

$$0 \to F(U) \to \oplus_{1 \leq k \leq n} F(U_k) \to \oplus_{1 \leq i < j \leq n} F(U_{ij}),$$

where the second morphism sends $(s_k)_{1 \leq k \leq n}$ to $(t_{ij})_{1 \leq i < j \leq n}$ by $t_{ij} = = s_i|_{U_{ij}} - s_j|_{U_{ij}}$. We shall argue by induction on n. For $n = 1$ the result is trivial, and $n = 2$ is the hypothesis. Suppose that the assertion is true for $j \leq n - 1$ and set $U' = \bigcup_{1 \leq k < n} U_k$. By the induction hypothesis the following commutative diagram is exact

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(U) & \longrightarrow & F(U') \oplus F(U_n) & \longrightarrow & F(U' \cap U_n) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \bigoplus_{i<n} F(U_i) \oplus F(U_n) & \longrightarrow & \bigoplus_{i<n} F(U_{in}) \\
 & & & & \downarrow & & \\
 & & & & \bigoplus_{i<j<n} F(U_{ij}). & &
\end{array}$$

Then the result follows. □

Let $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ be the abelian category of $\mathbb{R}$-constructible sheaves on X, and consider its subcategory $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ consisting of sheaves whose support is compact.

We denote by

$$\rho : X \to X_{sa}$$

the natural morphism of sites. We have functors

$$\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X) \subset \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X) \subset \mathrm{Mod}(k_X) \underset{\rho^{-1}}{\overset{\rho_*}{\rightleftarrows}} \mathrm{Mod}(k_{X_{sa}}). \tag{1.1}$$

We will still denote by ρ_* the restriction of ρ_* to $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$.

Remark 1.1.5. *By Proposition* 1.1.3 *for each* $F \in \mathrm{Mod}(k_X)$ *and* $V \in \mathrm{Op}^c(X_{sa})$ *one has*

$$\Gamma(V;\rho_*F) \simeq \varinjlim_U \Gamma(V;\rho_*F_U) \simeq \Gamma(V;\varinjlim_U \rho_*F_U), \tag{1.2}$$

where U ranges through the family of relatively compact open subanalytic subsets of X. The first isomorphism follows since V is relatively compact and $\Gamma(V,\rho_*F_U) \simeq \Gamma(V,\rho_*F)$ if $V \subset U$. The isomorphism (1.2) *implies* $\varinjlim_U \rho_*F_U \xrightarrow{\sim} \rho_*F$.

Remark 1.1.6. *The functor ρ_* does not commute with filtrant inductive limits. For example consider the family $\{V_n\}_{n\in\mathbb{N}}$ of Remark* 1.1.1. *We have $\rho_* \varinjlim_n k_{V_n} \simeq \rho_* k_{\mathbb{R}^2\setminus\{0\}}$, while for each $U \in \mathrm{Op}^c(\mathbb{R}^2_{sa})$ with $0 \in \partial U$ we have $\Gamma(U;\varinjlim_n \rho_*k_{V_n}) \simeq \varinjlim_n \Gamma(U;\rho_*k_{V_n}) = 0$.*

Proposition 1.1.7. *Let U be an open subanalytic subset of X and consider the constant sheaf $k_{U_{X_{sa}}} \in \mathrm{Mod}(k_{X_{sa}})$. We have $k_{U_{X_{sa}}} \simeq \rho_*k_U$.*

Proof. Let F be the presheaf on X_{sa} defined by $F(V) = k$ if $V \subset U$, $F(V) = 0$ otherwise. This is a separated presheaf and $k_{U_{X_{sa}}} = F^{++}$. Moreover there is an injective arrow $F(V) \hookrightarrow \rho_*k_U(V)$ for each $V \in \mathrm{Op}(X_{sa})$. Hence $F^{++} \hookrightarrow \rho_*k_U$ since the functor $(\,\cdot\,)^{++}$ is exact. Let $\mathcal{T}$ be the family of $W \in \mathrm{Op}(X_{sa})$ connected and such that W does not contain any connected component of U. Then $\mathcal{T}$ forms a basis for the topology of X_{sa} since the connected components of U are locally finite. For each $W \in \mathcal{T}$ we have $F(W) \simeq \rho_*k_U(W) \simeq k$ if $W \subset U$ and $F(W) = 0$ otherwise. Then $F^{++} \simeq \rho_*k_U$. □

Proposition 1.1.8. *The restriction of ρ_* to $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ is exact.*

Proof. (i) Let us consider an epimorphism $G \twoheadrightarrow F$ in $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$, we have to prove that $\psi : \rho_*G \to \rho_*F$ is an epimorphism. Let

$U \in \mathrm{Op}^c(X_{sa})$ and let $0 \neq s \in \Gamma(U; \rho_* F) \simeq \mathrm{Hom}_{k_X}(k_U, F)$. Set $G' = G \times_F k_U$. Then $G' \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ and moreover $G' \twoheadrightarrow k_U$. There exists a finite $\{U_i\}_{i \in I} \subset \mathrm{Op}^c(X_{sa})$ with U_i connected for each i such that $\oplus_i k_{U_i} \twoheadrightarrow G'$. The composition $k_{U_i} \to G' \to k_U$ is given by the multiplication by $a_i \in k$. Set $I_0 = \{k_{U_i};\ a_i \neq 0\}$, we may assume $a_i = 1$. We get a diagram

$$\begin{array}{ccccc} \oplus_{i \in I_0} k_{U_i} & \longrightarrow & G' & \longrightarrow & G \\ & \searrow & \downarrow & & \downarrow \\ & & k_U & \xrightarrow{s} & F. \end{array}$$

The composition $k_{U_i} \to G' \to G$ defines $t_i \in \mathrm{Hom}_{k_X}(k_{U_i}, G) = \Gamma(U_i; \rho_* G)$. Hence for each $s \in \Gamma(U; \rho_* F)$ there exists a finite covering $\{U_i\}$ of U and $t_i \in \Gamma(U_i; \rho_* G)$ such that $\psi(t_i) = s|_{U_i}$. This means that ψ is surjective.

(ii) Let $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. By Remark 1.1.5 $\rho_* F \simeq \varinjlim_U \rho_* F_U$, where U ranges through the family $\mathrm{Op}^c(X_{sa})$. The result follows since ρ_* is exact on $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ and filtrant $\varinjlim$ are exact. □

Proposition 1.1.9. *The restriction of* ρ_* *to* $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ *is fully faithful.*

Proof. It is enough to prove that $\rho^{-1}\rho_* F \simeq F$ for each $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. Since both functors are exact on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ we may reduce to the case $F = k_U$ with $U \in \mathrm{Op}(X_{sa})$ and the result follows from Proposition 1.1.7. □

Notations 1.1.10. *Since the functor* ρ_* *is fully faithful and exact on the category* $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$*, we can identify* $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ *with its image in* $\mathrm{Mod}(k_{X_{sa}})$*. When there is no risk of confusion we will write* F *instead of* $\rho_* F$*, for* $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$*.*

The following theorem gives a fundamental characterization of subanalytic sheaves and it will be used systematically in the following Sections.

Theorem 1.1.11. *(i) Let* $G \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ *and let* $\{F_i\}$ *be a filtrant inductive system in* $\mathrm{Mod}(k_{X_{sa}})$*. Then we have an isomorphism*

$$\varinjlim_i \mathrm{Hom}_{k_{X_{sa}}}(\rho_* G, F_i) \xrightarrow{\sim} \mathrm{Hom}_{k_{X_{sa}}}(\rho_* G, \varinjlim_i F_i).$$

(ii) Let $F \in \mathrm{Mod}(k_{X_{sa}})$. There exists a small filtrant inductive system $\{F_i\}_{i \in I}$ in $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ such that $F \simeq \varinjlim_i \rho_* F_i$.

Proof. (i) There exists an exact sequence $G_1 \to G_0 \to G \to 0$ with G_1, G_0 finite direct sums of constant sheaves k_U with $U \in \mathrm{Op}^c(X_{sa})$. Since ρ_* is exact on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and commutes with finite sums, by Proposition 1.1.7 we are reduced to prove the isomorphism $\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i)$. Then the result follows from Proposition 1.1.3.

(ii) Let $F \in \mathrm{Mod}(k_{X_{sa}})$, and define

$$I_0 := \{(U,s);\ U \in \mathrm{Op}^c(X_{sa}),\ s \in \Gamma(U;F)\}$$

$$G_0 := \oplus_{(U,s) \in I_0}\, \rho_* k_U$$

The morphism $\rho_* k_U \to F$, where the section $1 \in \Gamma(U; k_U)$ is sent to $s \in \Gamma(U;F)$ defines un epimorphism $\varphi : G_0 \to F$. Replacing F by $\ker \varphi$ we construct a sheaf $G_1 = \oplus_{(V,t) \in I_1} \rho_* k_V$ and an epimorphism $G_1 \twoheadrightarrow \ker \varphi$. Hence we get an exact sequence $G_1 \to G_0 \to F \to 0$. For $J_0 \subset I_0$ set for short $G_{J_0} = \oplus_{(U,s) \in J_0} \rho_* k_U$ and define similarly G_{J_1}. Set

$$J = \{(J_1, J_0);\ J_k \subset I_k,\ J_k \text{ is finite and } \operatorname{im} \varphi|_{G_{J_1}} \subset G_{J_0}\}.$$

The category J is filtrant and $F \simeq \varinjlim_{(J_1,J_0) \in J} \operatorname{coker}(G_{J_1} \to G_{J_0})$. □

Proposition 1.1.12. *Let $F \in \mathrm{Mod}(k_{X_{sa}})$, and let $U \in \mathrm{Op}(X)$. Then*

$$\Gamma(U;\, \rho^{-1} F) \simeq \varprojlim_{V \subset\subset U, V \in \mathrm{Op}^c(X_{sa})} \Gamma(V; F)$$

Proof. By Theorem 1.1.11 we may assume $F = \varinjlim_i \rho_* F_i$, with $F_i \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$. Then $\rho^{-1} F \simeq \varinjlim_i \rho^{-1} \rho_* F_i \simeq \varinjlim_i F_i$. We have the chain of isomorphisms

$$\begin{aligned}
\Gamma(U;\, \rho^{-1}F) &\simeq \varprojlim_{V \subset\subset U} \Gamma(\overline{V};\, \rho^{-1}F) &&\simeq \varprojlim_{V \subset\subset U} \Gamma(\overline{V}; \varinjlim_i F_i) \\
&\simeq \varprojlim_{V \subset\subset U} \varinjlim_i \Gamma(\overline{V}; F_i) &&\simeq \varprojlim_{V \subset\subset U} \varinjlim_i \Gamma(V; F_i) \\
&\simeq \varprojlim_{V \subset\subset U} \varinjlim_i \Gamma(V; \rho_* F_i) &&\simeq \varprojlim_{V \subset\subset U} \Gamma(V; F),
\end{aligned}$$

where $V \in \mathrm{Op}^c(X_{sa})$. The third isomorphism follows since $\overline{V}$ is compact and the last isomorphism follows from Proposition 1.1.3. □

PROPOSITION 1.1.13. *One has* $\rho^{-1} \circ \rho_* \xrightarrow{\sim} \mathrm{id}$, *in particular the functor* ρ_* *is fully faithful.*

PROOF. Let $F \in \mathrm{Mod}(k_X)$. Every $x \in X$ has a fundamental neighborhood system consisting of open subanalytic subsets. Hence we have the chain of isomorphisms

$$(\rho^{-1}\rho_* F)_x \simeq \varinjlim_{x\in U} \rho^{-1}\rho_* F(U) \simeq \varinjlim_{x\in U} \rho_* F(U) \simeq \varinjlim_{x\in U} F(U) \simeq F_x,$$

where U ranges through the family of open subanalytic neighborhoods of x. The second isomorphism follows since, by Proposition 1.1.12

$$\varinjlim_{x\in U} \rho^{-1}\rho_* F(U) \simeq \varinjlim_{x\in U} \varprojlim_{V\subset\subset U} \rho_* F(V) \simeq \varinjlim_{x\in U} \rho_* F(U).$$

□

Now we will describe a left adjoint to the functor ρ^{-1}.

PROPOSITION 1.1.14. *The functor* ρ^{-1} *admits a left adjoint, denoted by* $\rho_!$. *It satisfies*

(i) *for* $F \in \mathrm{Mod}(k_X)$ *and* $U \in \mathrm{Op}(X_{sa})$, $\rho_! F$ *is the sheaf associated to the presheaf* $U \mapsto \Gamma(\overline{U}; F)$,

(ii) *for* $U \in \mathrm{Op}(X)$ *one has* $\rho_! k_U \simeq \varinjlim_{V\subset\subset U, V\in \mathrm{Op}^c(X_{sa})} k_V$.

PROOF. Let $\widetilde{F} \in \mathrm{Psh}(k_{X_{sa}})$ be the presheaf $U \mapsto \Gamma(\overline{U}; F)$, and let $G \in \mathrm{Mod}(k_{X_{sa}})$. We will construct morphisms

$$\mathrm{Hom}_{\mathrm{Psh}(k_{X_{sa}})}(\widetilde{F}, G) \underset{\vartheta}{\overset{\xi}{\rightleftarrows}} \mathrm{Hom}_{k_X}(F, \rho^{-1}G).$$

To define ξ, let $\varphi : \widetilde{F} \to G$ and $U \in \mathrm{Op}(X)$. Then the morphism $\xi(\varphi)(U) : F(U) \to \rho^{-1}G(U)$ is defined as follows

$$F(U) \simeq \varprojlim_{V\subset\subset U, V\in \mathrm{Op}^c(X_{sa})} F(\overline{V}) \xrightarrow{\varphi} \varprojlim_{V\subset\subset U, V\in \mathrm{Op}^c(X_{sa})} G(V) \simeq \rho^{-1}G(U).$$

On the other hand, let $\psi : F \to \rho^{-1}G$ and $U \in \mathrm{Op}^c(X_{sa})$. Then the morphism $\vartheta(\psi)(U) : \widetilde{F}(U) \to G(U)$ is defined as follows

$$\widetilde{F}(U) \simeq \varinjlim_{U\subset\subset V\in \mathrm{Op}^c(X_{sa})} F(V) \xrightarrow{\psi} \varinjlim_{U\subset\subset V\in \mathrm{Op}^c(X_{sa})} \rho^{-1}G(V) \to G(U).$$

By construction one can check that the morphisms ξ and ϑ are inverse to each others. Then (i) follows from the chain of isomorphisms

$$\mathrm{Hom}_{k_{X_{sa}}}(\widetilde{F}^{++}, G) \simeq \mathrm{Hom}_{\mathrm{Psh}(k_{X_{sa}})}(\widetilde{F}, G) \simeq \mathrm{Hom}_{k_{X_{sa}}}(F, \rho^{-1}G).$$

To show (ii), consider the following sequence of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{k_{X_{sa}}}(\rho_! k_U, F) &\simeq \mathrm{Hom}_{k_X}(k_U, \rho^{-1}F) \\ &\simeq \varprojlim_{V \subset\subset U, V \in \mathrm{Op}^c(X_{sa})} \mathrm{Hom}_{k_{X_{sa}}}(k_V, F) \\ &\simeq \mathrm{Hom}_{k_{X_{sa}}}(\varinjlim_{V \subset\subset U, V \in \mathrm{Op}^c(X_{sa})} k_V, F),\end{aligned}$$

where the second isomorphism follows from Proposition 1.1.12. □

Proposition 1.1.15. *The functor $\rho_!$ is exact and commutes with $\varinjlim$ and $\otimes$.*

Proof. It follows by adjunction that $\rho_!$ is right exact and commutes with $\varinjlim$, so let us show that it is also left exact. With the notations of Proposition 1.1.14, let $F \in \mathrm{Mod}(k_X)$, and let $\widetilde{F} \in \mathrm{Psh}(k_{X_{sa}})$ be the presheaf $U \mapsto \Gamma(\overline{U}; F)$. Then $\rho_! F \simeq \widetilde{F}^{++}$, and the functors $F \mapsto \widetilde{F}$ and $G \mapsto G^{++}$ are left exact.

Let us show that $\rho_!$ commutes with $\otimes$. Let $F, G \in \mathrm{Mod}(k_X)$, the morphism

$$F(\overline{U}) \otimes G(\overline{U}) \to (F \otimes G)(\overline{U})$$

defines a morphism in $\mathrm{Mod}(k_{X_{sa}})$

$$\rho_! F \otimes \rho_! G \to \rho_!(F \otimes G)$$

by Proposition 1.1.14 (i). Since $\rho_!$ commutes with $\varinjlim$ we may suppose that $F = k_U$ and $G = k_V$ and the result follows from Proposition 1.1.14 (ii). □

Proposition 1.1.16. *The functor $\rho_!$ is fully faithful. In particular one has $\rho^{-1} \circ \rho_! \simeq \mathrm{id}$. Moreover, for $F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_{X_{sa}})$ one has*

$$\rho^{-1}\mathcal{H}om(\rho_! F, G) \simeq \mathcal{H}om(F, \rho^{-1}G).$$

Proof. For $F, G \in \mathrm{Mod}(k_X)$ we have by adjunction

$$\mathrm{Hom}_{k_X}(\rho^{-1}\rho_! F, G) \simeq \mathrm{Hom}_{k_X}(F, \rho^{-1}\rho_* G) \simeq \mathrm{Hom}_{k_X}(F, G).$$

This also implies that $\rho_!$ is fully faithful, in fact

$$\mathrm{Hom}_{k_{X_{sa}}}(\rho_! F, \rho_! G) \simeq \mathrm{Hom}_{k_X}(F,\, \rho^{-1}\rho_! G) \simeq \mathrm{Hom}_{k_X}(F, G).$$

Now let $K, F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_{X_{sa}})$, we have

$$\begin{aligned}
\mathrm{Hom}_{k_X}(K,\, \rho^{-1}\mathcal{H}om(\rho_! F, G)) &\simeq \mathrm{Hom}_{k_{X_{sa}}}(\rho_! K, \mathcal{H}om(\rho_! F, G)) \\
&\simeq \mathrm{Hom}_{k_{X_{sa}}}(\rho_! K \otimes \rho_! F, G) \\
&\simeq \mathrm{Hom}_{k_{X_{sa}}}(\rho_!(K \otimes F), G) \\
&\simeq \mathrm{Hom}_{k_X}(K \otimes F,\, \rho^{-1} G) \\
&\simeq \mathrm{Hom}_{k_X}(K, \mathcal{H}om(F,\, \rho^{-1} G))
\end{aligned}$$

and the result follows. □

1.2 – *Operations on the subanalytic site.*

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a real analytic map. This defines a morphism of sites $f : X_{sa} \to Y_{sa}$. We have a diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\
X_{sa} & \xrightarrow{f} & Y_{sa}.
\end{array}$$

The following functors are always well defined on a site

$$\begin{aligned}
\mathcal{H}om &: \mathrm{Mod}(k_{X_{sa}})^{op} \times \mathrm{Mod}(k_{X_{sa}}) \to \mathrm{Mod}(k_{X_{sa}}), \\
\otimes &: \mathrm{Mod}(k^{op}_{X_{sa}}) \times \mathrm{Mod}(k_{X_{sa}}) \to \mathrm{Mod}(k_{X_{sa}}), \\
f_* &: \mathrm{Mod}(k_{X_{sa}}) \to \mathrm{Mod}(k_{Y_{sa}}), \\
f^{-1} &: \mathrm{Mod}(k_{Y_{sa}}) \to \mathrm{Mod}(k_{X_{sa}}).
\end{aligned}$$

Let us summarize their properties:

- the functor $\mathcal{H}om$ is left exact and commutes with ρ_*,
- the functor $\otimes$ is exact and commutes with $\varinjlim$, ρ^{-1} and $\rho_!$,
- the functor f_* is left exact and commutes with ρ_* and $\varprojlim$,
- the functor f^{-1} is exact and commutes with $\varinjlim$, $\otimes$ and ρ^{-1},
- (f^{-1}, f_*) is a pair of adjoint functors.

Let Z be a subanalytic locally closed subset of X. As in classical sheaf theory we define

$$\begin{aligned} \Gamma_Z : \mathrm{Mod}(k_{X_{sa}}) &\to \mathrm{Mod}(k_{X_{sa}}) \\ F &\mapsto \mathcal{H}om(\rho_* k_Z, F) \\ (\,\cdot\,)_Z : \mathrm{Mod}(k_{X_{sa}}) &\to \mathrm{Mod}(k_{X_{sa}}) \\ F &\mapsto F \otimes \rho_* k_Z. \end{aligned}$$

We have

- the functor Γ_Z is left exact and commutes with ρ_* and $\varprojlim$,
- the functor $(\,\cdot\,)_Z$ is exact and commutes with $\varinjlim$, $\otimes$ and ρ^{-1},
- $((\cdot)_Z, \Gamma_Z)$ is a pair of adjoint functors.

1.3 – $\mathbb{R}$-*constructible sheaves on subanalytic sites.*

Let us consider the category $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. We prove that the subcategory of $\mathrm{Mod}(k_{X_{sa}})$ consisting of $\mathbb{R}$-constructible sheaves is stable under inverse image and tensor product.

PROPOSITION 1.3.1. *Let* $F, G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. *Then* $\rho_*(F \otimes G) \simeq \rho_* F \otimes \otimes \rho_* G$.

PROOF. We may reduce to the case $F = k_U$, $G = k_V$ with $U, V \in \mathrm{Op}(X_{sa})$. In this case $\rho_* k_{U\cap V} \simeq \rho_* k_U \otimes \rho_* k_V$ by Proposition 1.1.7. □

COROLLARY 1.3.2. *Let* $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$, *and let* Z *be a subanalytic locally closed subset of* X. *Then* $\rho_* F_Z \simeq (\rho_* F)_Z$.

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a real analytic map.

PROPOSITION 1.3.3. *Let* $f : X \to Y$ *be a real analytic map. Let* $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_Y)$. *Then* $\rho_* f^{-1} G \simeq f^{-1} \rho_* G$.

PROOF. Since the functor f^{-1} is exact, we may reduce to the case $G = k_V$, with $V \in \mathrm{Op}(Y_{sa})$. In this case we have $\rho_* f^{-1} k_V \simeq \rho_* k_{f^{-1}(V)} \simeq \simeq f^{-1} \rho_* k_V$, where the last isomorphism follows from Proposition 1.1.7. □

We apply the above results to calculate the functor $\mathcal{H}om$ in the category $\mathrm{Mod}(k_{X_{sa}})$.

PROPOSITION 1.3.4. *Let* $F = \varinjlim_i F_i$, *with* $F_i \in \mathrm{Mod}(k_{X_{sa}})$ *and let* $G = \varinjlim_j \rho_* G_j$ *with* $G_j \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. *One has*

$$\mathcal{H}om(G, F) \simeq \varprojlim_j \varinjlim_i \mathcal{H}om(\rho_* G_j, F_i).$$

PROOF. For each $U \in \mathrm{Op}^c(X_{sa})$ one has the isomorphisms

$$\begin{aligned}\Gamma(U, \mathcal{H}om(G, F)) &\simeq \mathrm{Hom}_{k_{X_{sa}}}(G_U, F) \\ &\simeq \varprojlim_j \varinjlim_i \mathrm{Hom}_{k_{X_{sa}}}(\rho_* G_{jU}, F_i) \\ &\simeq \varprojlim_j \varinjlim_i \Gamma(U; \mathcal{H}om(\rho_* G_j, F_i)) \\ &\simeq \Gamma(U; \varprojlim_j \varinjlim_i \mathcal{H}om(\rho_* G_j, F_i)).\end{aligned}$$

In the second isomorphism we used Corollary 1.3.2, and the last isomorphism follows from Proposition 1.1.3 and because $\Gamma(U; \cdot)$ commutes with $\varprojlim$. □

COROLLARY 1.3.5. *Let* $F = \varinjlim_i \rho_* F_i, G = \varinjlim_j \rho_* G_j$ *with* $F_i, G_j \in \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. *One has*

$$\mathcal{H}om(G, F) \simeq \varprojlim_j \varinjlim_i \rho_* \mathcal{H}om(G_j, F_i).$$

PROOF. It follows from the fact that $\mathcal{H}om$ commutes with ρ_* and from Proposition 1.3.4. □

COROLLARY 1.3.6. *Let* $F = \varinjlim_i \rho_* F_i$, *with* $F_i \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(X)$ *be a sheaf on* X_{sa}. *Let* Z *be a subanalytic locally closed subset of* X. *Then* $\Gamma_Z F \simeq \varinjlim_i \rho_* \Gamma_Z F_i$.

1.4 – *Proper direct image on* $\mathrm{Mod}(k_{X_{sa}})$.

In [7] the authors defined the functor $f_{!!}$ of proper direct image using ind-sheaves. Here we give a direct construction:

$$\begin{aligned}f_{!!} : \mathrm{Mod}(k_{X_{sa}}) &\to \mathrm{Mod}(k_{Y_{sa}}) \\ F &\mapsto \varinjlim_U f_* F_U \simeq \varinjlim_K f_* \Gamma_K F\end{aligned}$$

where U ranges through the family of relatively compact open subanalytic subsets of X and K ranges through the family of subanalytic compact subsets of X. One shall be aware that $\varinjlim$ is taken in the category $\mathrm{Mod}(k_{Y_{sa}})$. Let $V \in \mathrm{Op}^c(Y_{sa})$. Then $\Gamma(V;f_{!!}F) = \varinjlim_U \Gamma(f^{-1}(V);F_U) \simeq \varinjlim_K \Gamma(f^{-1}(V);\Gamma_K F)$, where U ranges through the family of relatively compact open subanalytic subsets of X and K ranges through the family of subanalytic compact subsets of X. If f is proper on $\mathrm{supp}(F)$ then $f_* \simeq f_{!!}$ and in this case $f_{!!} \circ \rho_* \simeq \rho_* \circ f_!$.

REMARK 1.4.1. *Remark that $f_{!!} \circ \rho_* \not\simeq \rho_* \circ f_!$ in general. Indeed let $V \in \mathrm{Op}^c(Y_{sa})$, then*

$$\Gamma(V;f_{!!}\rho_* F) = \varinjlim_K \Gamma(f^{-1}(V);\Gamma_K F),$$

$$\Gamma(V;\rho_* f_! F) = \varinjlim_Z \Gamma(f^{-1}(V);\Gamma_Z F),$$

where Z ranges through the family of closed subsets of $f^{-1}(V)$ such that $f|_Z : Z \to V$ is proper. Then

$$\Gamma(V;f_{!!}\rho_* F) = \{s \in \Gamma(f^{-1}(V);F);\ \overline{\mathrm{supp}\,(s)} \textit{ is compact in } X\},$$

$$\Gamma(V;\rho_* f_! F) = \{s \in \Gamma(f^{-1}(V);F);\ f : \mathrm{supp}\,(s) \to V \textit{ is proper}\}.$$

For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate, and let $V = (a,b) \in \mathrm{Op}^c(\mathbb{R}_{sa})$. Suppose that $\mathrm{supp}\,(s) = \{(x,y) \in (a,b) \times \mathbb{R}, y = \frac{1}{(x-a)(b-x)}\}$. Then $f : \mathrm{supp}\,(s) \to V$ is proper but $\overline{\mathrm{supp}\,(s)}$ is not compact.

PROPOSITION 1.4.2. *The functor $f_{!!}$ commutes with filtrant $\varinjlim$. Moreover $\rho^{-1} \circ f_{!!} \simeq f_! \circ \rho^{-1}$.*

PROOF. Let us show that $f_{!!}$ commutes with filtrant $\varinjlim$. Let $V \in \mathrm{Op}^c(Y_{sa})$ and let $\{F_i\}_i$ be a filtrant inductive system in $\mathrm{Mod}(k_{X_{sa}})$. Then

$$\begin{aligned}
\varinjlim_K \mathrm{Hom}_{k_{X_{sa}}}(k_{f^{-1}(V)}, \Gamma_K \varinjlim_i F_i) &\simeq \varinjlim_K \mathrm{Hom}_{k_{X_{sa}}}(k_{f^{-1}(V)\cap K}, \varinjlim_i F_i) \\
&\simeq \varinjlim_{i,K} \mathrm{Hom}_{k_{X_{sa}}}(k_{f^{-1}(V)\cap K}, F_i) \\
&\simeq \varinjlim_{i,K} \mathrm{Hom}_{k_{X_{sa}}}(k_{f^{-1}(V)}, \Gamma_K F_i) \\
&\simeq \varinjlim_i \mathrm{Hom}_{k_{Y_{sa}}}(k_V, f_{!!}F_i) \\
&\simeq \mathrm{Hom}_{k_{Y_{sa}}}(k_V, \varinjlim_i f_{!!}F_i),
\end{aligned}$$

where the second isomorphism follows from the fact that $k_{f^{-1}(V)\cap K} \in$
$\in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$.

Let us show $\rho^{-1} \circ f_{!!} \simeq f_! \circ \rho^{-1}$. Let $F = \varinjlim_i \rho_* F_i$. Since $f_{!!}$ commutes with filtrant $\varinjlim$ and F_i has compact support for each i we have $f_{!!}F = \varinjlim_i \rho_* f_! F_i$. We have the chain of isomorphisms

$$\begin{aligned} f_! \rho^{-1} \varinjlim_i \rho_* F_i &\simeq f_! \varinjlim_i \rho^{-1} \rho_* F_i \simeq f_! \varinjlim_i F_i \simeq \varinjlim_i f_! F_i \\ &\simeq \varinjlim_i \rho^{-1} \rho_* f_! F_i \simeq \rho^{-1} \varinjlim_i \rho_* f_! F_i. \end{aligned}$$

□

PROPOSITION 1.4.3. *The functor f_* commutes with ρ^{-1}.*

PROOF. Let $F \in \mathrm{Mod}(k_{X_{sa}})$. Then $f_* F \simeq \varprojlim_K f_* F_K$, where K ranges through the family of subanalytic compact subsets of X. We have the chain of isomorphisms

$$\begin{aligned} f_* \rho^{-1} F &\simeq \varprojlim_K f_* (\rho^{-1} F)_K \simeq \varprojlim_K f_{!!} (\rho^{-1} F)_K \simeq \varprojlim_K f_{!!} \rho^{-1} F_K \\ &\simeq \varprojlim_K \rho^{-1} f_! F_K \simeq \rho^{-1} \varprojlim_K f_! F_K \simeq \rho^{-1} \varprojlim_K f_* F_K \simeq \rho^{-1} f_* F, \end{aligned}$$

where the second and the sixth isomorphism follow from the fact that f is proper on a compact subset of X. □

COROLLARY 1.4.4. *The functor f^{-1} commutes with $\rho_!$.*

PROOF. It follows immediately by adjunction. □

PROPOSITION 1.4.5. *Let $F \in \mathrm{Mod}(k_{X_{sa}})$ and $G \in \mathrm{Mod}(k_{Y_{sa}})$. Then*

$$f_{!!} F \otimes G \simeq f_{!!}(F \otimes f^{-1} G).$$

PROOF. Let $F = \varinjlim_i \rho_* F_i$, $G = \varinjlim_j \rho_* G_j$. The functors $\otimes$, $f_{!!}$ and f^{-1} commute with $\varinjlim_i$. Moreover $\mathrm{supp}(F_i \otimes f^{-1} G_j)$ is compact for each i, j, hence f is proper on it. Then

$$\begin{aligned} f_{!!} \varinjlim_i \rho_* F_i \otimes \varinjlim_j \rho_* G_j &\simeq \varinjlim_{i,j} \rho_*(f_! F_i \otimes G_j) \\ &\simeq \varinjlim_{i,j} \rho_*(f_!(F_i \otimes f^{-1} G_j)) \\ &\simeq f_{!!}(\varinjlim_i \rho_* F_i \otimes f^{-1} \varinjlim_i \rho_* G_j). \end{aligned}$$

In the first isomorphism we used Proposition 1.3.1 and in the last one we used Propositions 1.3.1 and 1.3.3. □

Now let us consider a cartesian square

$$\begin{array}{ccc} X'_{sa} & \xrightarrow{f'} & Y'_{sa} \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X_{sa} & \xrightarrow{f} & Y_{sa} \end{array}$$

PROPOSITION 1.4.6. *Let* $F \in \mathrm{Mod}(k_{X_{sa}})$. *Then* $g^{-1} f_{!!} F \simeq f'_{!!} g'^{-1} F$.

PROOF. Let $F = \varinjlim_i \rho_* F_i$. All the functors in the above formula commute with $\varinjlim_i$. Moreover since $\mathrm{supp}(F_i)$ is compact, f' is proper on $\mathrm{supp}(g'^{-1} F_i)$ for each i. Then

$$g^{-1} f_{!!} \varinjlim_i \rho_* F_i \simeq \varinjlim_i \rho_* g^{-1} f_! F_i \simeq \varinjlim_i \rho_* f'_! g'^{-1} F_i \simeq f'_{!!} g'^{-1} \varinjlim_i \rho_* F_i,$$

where the first and the last isomorphisms follow from Proposition 1.3.3. □

The following isomorphism is the analogue for subanalytic sheaves of Corollary 4.3.15 of [7].

PROPOSITION 1.4.7. *Let* $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_Y)$ *and let* $F \in \mathrm{Mod}(k_{X_{sa}})$. *Then the natural morphism*

$$f_{!!} \mathcal{H}om(f^{-1} G, F) \to \mathcal{H}om(G, f_{!!} F)$$

is an isomorphism.

PROOF. Let us construct the morphism. By adjunction we have

$$f^{-1} G \otimes \mathcal{H}om(f^{-1} G, F) \to F,$$

hence, using the projection formula we get

$$G \otimes f_{!!}\mathcal{H}om(f^{-1}G,F) \simeq f_{!!}(f^{-1}G \otimes \mathcal{H}om(f^{-1}G,F)) \to f_{!!}F,$$

then by adjunction we obtain the desired morphism. Let us show that it is an isomorphism. We have the chain of isomorphisms

$$\begin{aligned} f_{!!}\mathcal{H}om(f^{-1}G,F) &\simeq \varinjlim_K f_*\Gamma_K\mathcal{H}om(f^{-1}G,F) \\ &\simeq \varinjlim_K f_*\mathcal{H}om(f^{-1}G,\Gamma_K F) \\ &\simeq \varinjlim_K \mathcal{H}om(G,f_*\Gamma_K F) \\ &\simeq \mathcal{H}om(G,\varinjlim_K f_*\Gamma_K F) \\ &\simeq \mathcal{H}om(G,f_{!!}F), \end{aligned}$$

where the fourth isomorphism follows from Proposition 1.3.4. □

1.5 – *Quasi-injective objects.*

Let us introduce a category which is useful in order to find acyclic objects with respect to the functors defined in the previous Sections. Although the definition is inspired to the definition of quasi-injective objects of [7] (or, more generally, to the definition of quasi-injective objects of a ind-category, see [8] for more details), the proofs of Theorems 1.5.4 and 1.5.16 are independent of the theory of ind-objects.

Definition 1.5.1. *An object* $F \in \mathrm{Mod}(k_{X_{sa}})$ *is quasi-injective if the functor* $\mathrm{Hom}_{k_{X_{sa}}}(\cdot,F)$ *is exact in* $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ *or, equivalently (see Theorem 8.7.2 of [8]) if for each* $U,V \in \mathrm{Op}^c(X_{sa})$ *with* $V \subset U$ *the restriction morphism* $\Gamma(U;F) \to \Gamma(V;F)$ *is surjective.*

It follows from the definition that injective sheaves belong to $\mathcal{J}_{X_{sa}}$. This implies that $\mathcal{J}_{X_{sa}}$ is cogenerating. Moreover the category $\mathcal{J}_{X_{sa}}$ is stable by filtrant $\varinjlim$ and $\prod$.

Proposition 1.5.2. *Let* $0 \to F' \to F \to F'' \to 0$ *be an exact sequence in* $\mathrm{Mod}(k_{X_{sa}})$ *and assume that* F' *is quasi-injective. Let* $U \in \mathrm{Op}^c(X_{sa})$.

Then the sequence

$$0 \to \Gamma(U;F') \to \Gamma(U;F) \to \Gamma(U;F'') \to 0$$

is exact.

PROOF. Let $s'' \in \Gamma(U;F'')$, and let $\{V_i\}_{i=1}^n$ be a finite covering of U such that there exists $s_i \in \Gamma(V_i;F)$ whose image is $s''|_{V_i}$. For $n \geq 2$ on $V_1 \cap V_2$ $s_1 - s_2$ defines a section of $\Gamma(V_1 \cap V_2; F')$ which extends to $s' \in \Gamma(U;F')$. Replace s_1 with $s_1 - s'$. We may suppose that $s_1 = s_2$ on $V_1 \cap V_2$. Then there exists $t \in \Gamma(V_1 \cup V_2)$ such that $t|_{V_i} = s_i$, $i = 1, 2$. Thus the induction proceeds. □

PROPOSITION 1.5.3. *Let* $F', F, F'' \in \mathrm{Mod}(k_{X_{sa}})$, *and consider the exact sequence*

$$0 \to F' \to F \to F'' \to 0.$$

Suppose that $F', F \in \mathcal{J}_{X_{sa}}$. *Then* $F'' \in \mathcal{J}_{X_{sa}}$.

PROOF. Let $U, V \in \mathrm{Op}^c(X_{sa})$ with $V \subset U$ and let us consider the diagram below

$$\begin{array}{ccc} \Gamma(U;F) & \longrightarrow & \Gamma(U;F'') \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ \Gamma(V;F) & \xrightarrow{\beta} & \Gamma(V;F''). \end{array}$$

The morphism α is surjective since F is quasi-injective and β is surjective by Proposition 1.5.2. Then γ is surjective. □

THEOREM 1.5.4. *The family of quasi-injective sheaves is injective with respect to the functor* $\mathrm{Hom}_{k_{X_{sa}}}(G, \cdot)$ *for each* $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$.

PROOF. (i) Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}(k_{X_{sa}})$ and assume that $F' \in \mathcal{J}_{X_{sa}}$. Let $G \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$. We have to show that the sequence

$$0 \to \mathrm{Hom}_{k_{X_{sa}}}(G, F') \to \mathrm{Hom}_{k_{X_{sa}}}(G, F) \to \mathrm{Hom}_{k_{X_{sa}}}(G, F'') \to 0$$

is exact. There is an epimorphism $\varphi : \oplus_{i \in I} k_{U_i} \to G$ where I is finite and $U_i \in \mathrm{Op}^c(X_{sa})$ for each $i \in I$.

The sequence $0 \to \ker\varphi \to \oplus_{i\in I} k_{U_i} \to G \to 0$ is exact. We set for short $G_1 = \ker\varphi$ and $G_2 = \oplus_{i \in I} k_{U_i}$. We get the following diagram where the first column is exact

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G, F') & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G, F) & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G, F'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_2, F') & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_2, F) & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_2, F'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_1, F') & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_1, F) & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_1, F'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The second row is exact by Proposition 1.5.2, hence the top row is exact by the snake lemma.

(ii) Let $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$, let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}(k_{X_{sa}})$ with $F' \in \mathcal{J}_{X_{sa}}$. Let $\{V_n\}_{n\in\mathbb{N}} \in \mathrm{Cov}(X_{sa})$ such that $V_n \subset\subset V_{n+1}$. By (i), all the sequences

$$0 \to \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F') \to \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F) \to \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F'') \to 0$$

are exact. Moreover since $F' \in \mathcal{J}_{X_{sa}}$ the morphism $\mathrm{Hom}_{k_{X_{sa}}}(G_{V_{n+1}}, F') \to$ $\to \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F')$ is surjective for all n. Then by the Mittag-Leffler property (see Proposition 1.12.3 of [5]) the sequence

$$0 \to \varprojlim_n \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F') \to \varprojlim_n \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F) \to \varprojlim_n \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F'') \to 0$$

is exact. Since $\varprojlim_n \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, \cdot) \simeq \mathrm{Hom}_{k_{X_{sa}}}(G, \cdot)$ the result follows. □

PROPOSITION 1.5.5. *Let $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. Then quasi-injective sheaves are injective with respect to the functor $\mathcal{H}om(G, \cdot)$.*

PROOF. Let $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. It is enough to check that for each $U \in \mathrm{Op}(X_{sa})$ and each exact sequence $0 \to F' \to F \to F'' \to 0$ with $F' \in \mathcal{J}_{X_{sa}}$, the sequence

$$0 \to \Gamma(U; \mathcal{H}om(G, F')) \to \Gamma(U; \mathcal{H}om(G, F)) \to \Gamma(U; \mathcal{H}om(G, F'')) \to 0$$

is exact. We have $\Gamma(U, \mathcal{H}om(G, \cdot)) \simeq \mathrm{Hom}_{k_{X_{sa}}}(G_U, \cdot)$, and quasi-injective objects are injective with respect to the functor $\mathrm{Hom}_{k_{X_{sa}}}(G_U, \cdot)$ for each $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$, and for each $U \in \mathrm{Op}(X_{sa})$. □

COROLLARY 1.5.6. *Quasi-injective sheaves are injective with respect to the functor Γ_Z for each locally closed subanalytic subset Z of X.*

COROLLARY 1.5.7. *Let $F \in \mathrm{Mod}(k_{X_{sa}})$ be quasi-injective. Then the functor $\mathcal{H}om(\cdot, F)$ is exact on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$.*

PROOF. Let $F \in \mathrm{Mod}(k_{X_{sa}})$ be quasi-injective. There is an isomorphism of functors $\Gamma(U; \mathcal{H}om(\cdot, F)) \simeq \mathrm{Hom}_{k_{X_{sa}}}((\cdot)_U, F)$ for each $U \in \mathrm{Op}(X_{sa})$. The functor $\mathrm{Hom}_{k_{X_{sa}}}((\cdot)_U, F)$ is exact on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and the result follows. □

PROPOSITION 1.5.8. *Let $F \in \mathrm{Mod}(k_{X_{sa}})$. Then F is quasi-injective if and only if $\mathcal{H}om(G, F)$ is quasi-injective for each $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$.*

PROOF. (i) Let F be quasi-injective, and let $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. We have $\mathrm{Hom}_{k_{X_{sa}}}(\cdot, \mathcal{H}om(G, F)) \simeq \mathrm{Hom}_{k_{X_{sa}}}(\cdot \otimes G, F)$, and $\mathrm{Hom}_{k_{X_{sa}}}(\cdot \otimes G, F)$ is exact on $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$.

(ii) Suppose that $\mathcal{H}om(G, F)$ is quasi-injective for each $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. The result follows by setting $G = k_X$. □

COROLLARY 1.5.9. *The functor Γ_Z sends quasi-injective objects to quasi-injective objects for each locally closed subanalytic subset Z of X.*

Let $f : X \to Y$ be a morphism of real analytic manifolds.

PROPOSITION 1.5.10. *Quasi-injective sheaves are injective with respect to the functor f_*. The functor f_* sends quasi-injective objects to quasi-injective objects.*

PROOF. (i) Let us consider $V \in \mathrm{Op}(Y_{sa})$. There is an isomorphism of functors $\Gamma(V; f_*(\,\cdot\,)) \simeq \Gamma(f^{-1}(V); \cdot)$. It follows from Proposition 1.5.4 that $\mathcal{J}_{X_{sa}}$ is injective with respect to the functor $\Gamma(f^{-1}(V); \cdot) \simeq$ $\simeq \mathrm{Hom}_{k_{X_{sa}}}(k_{f^{-1}(V)}, \cdot)$ for any $V \in \mathrm{Op}(Y_{sa})$.

(ii) Let $F \in \mathcal{J}_{X_{sa}}$. For each $G \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_Y)$ we have $\mathrm{Hom}_{k_{Y_{sa}}}(G, f_*F) \simeq$ $\mathrm{Hom}_{k_{X_{sa}}}(f^{-1}G, F)$. Since f^{-1} is exact and sends $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_Y)$ to $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$, Proposition 1.5.4 implies that the functor $\mathrm{Hom}_{k_{X_{sa}}}(f^{-1}(\,\cdot\,), F)$ is exact on $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_Y)$. □

PROPOSITION 1.5.11. *The family of quasi-injective sheaves is $f_{!!}$-injective. The functor $f_{!!}$ sends quasi-injective objects to quasi-injective objects.*

PROOF. (i) Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}(k_{X_{sa}})$ and assume that $F' \in \mathcal{J}_{X_{sa}}$. We have to check that the sequence $0 \to f_{!!}F' \to f_{!!}F \to f_{!!}F'' \to 0$ is exact. Since $F' \in \mathcal{J}_{X_{sa}}$, we have $\Gamma_K F' \in \mathcal{J}_{X_{sa}}$. Moreover $\mathcal{J}_{X_{sa}}$ is injective with respect to Γ_K and f_*. This implies that the sequence

$$0 \to f_*\Gamma_K F' \to f_*\Gamma_K F \to f_*\Gamma_K F'' \to 0$$

is exact. Applying the exact functor $\varinjlim_K$ we find that the sequence

$$0 \to \varinjlim_K f_*\Gamma_K F' \to \varinjlim_K f_*\Gamma_K F \to \varinjlim_K f_*\Gamma_K F'' \to 0$$

is exact.

(ii) Let K be a compact subanalytic subset of X. The functors Γ_K and f_* send quasi-injective objects to quasi-injective objects, then $f_*\Gamma_K F \in \mathcal{J}_{Y_{sa}}$. Since $\mathcal{J}_{Y_{sa}}$ is stable by filtrant $\varinjlim$, the result follows. □

Let S be a closed subanalytic subset of X and let $i_S : S \hookrightarrow X$ be the closed embedding. Let $F = \varinjlim_i \rho_* F_i \in \mathrm{Mod}(k_{X_{sa}})$ with $F_i \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$. We have $F_S \simeq \varinjlim_i \rho_* F_{iS} \simeq \varinjlim_i \rho_* i_{S*}\, i_S{}^{-1} F_i \simeq i_{S*}\, i_S{}^{-1} F$.

LEMMA 1.5.12. *Let S be a closed subanalytic subset of X and let $U \in \mathrm{Op}^c(X_{sa})$. Let $F \in \mathrm{Mod}(k_{X_{sa}})$. Then $\Gamma(U; F_S) \simeq \varinjlim_{V \supset S\cap U} \Gamma(V; F)$, with $V \in \mathrm{Op}^c(X_{sa})$.*

PROOF. Let $F \in \mathrm{Mod}(k_{X_{sa}})$. Then $F \simeq \varinjlim_i \rho_* F_i$ with $F_i \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$. We have the chain of isomorphisms

$$\begin{aligned} \Gamma(U; F_S) &\simeq \varinjlim_i \Gamma(U; F_{iS}) \\ &\xleftarrow{\sim} \varinjlim_{i, V\supset S\cap U} \Gamma(V; F_i) \\ &\simeq \varinjlim_{V\supset S\cap U} \Gamma(V; F), \end{aligned}$$

where V ranges through the family of relatively compact open subanalytic subsets of X containing $S \cap U$. The second isomorphism follows since F_i is $\mathbb{R}$-constructible for each i. □

REMARK 1.5.13. *The fact that $F_i \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ is locally constant on a subanalytic stratification of X plays an essential role. In fact the iso-*

morphism is not true in general in $\mathrm{Mod}(k_X)$*, since the family of relatively compact open subanalytic subsets of* X *containing* $S \cap U$ *is not cofinal to the family of open subsets of* X *containing* $S \cap U$.

PROPOSITION 1.5.14. *Let* S *be a closed subanalytic subset of* X *and let* $F \in \mathrm{Mod}(k_{X_{sa}})$ *be quasi-injective. Then* F_S *is quasi-injective.*

PROOF. Let $U, V \in \mathrm{Op}^c(X_{sa})$ with $V \subset U$. Since F is quasi-injective and inductive limits are right exact, the morphism $\varinjlim_{U' \supset S \cap U} \Gamma(U'; F) \to \varinjlim_{V' \supset S \cap V} \Gamma(V'; F)$ with $V', U' \in \mathrm{Op}^c(X_{sa})$, is surjective. Hence by Lemma 1.5.12 the morphism $\Gamma(U; F_S) \to \Gamma(V; F_S)$ is surjective and the result follows. $\square$

Recall that $F \in \mathrm{Mod}(k_X)$ is c-soft if the natural morphism $\Gamma(X; F) \to \Gamma(K, F)$ is surjective for each compact $K \subset X$. If F is c-soft and Z is a locally closed subset of X, then F_Z is c-soft. Moreover c-soft sheaves are $\Gamma(U; \cdot)$-injective for each $U \in \mathrm{Op}(X)$.

PROPOSITION 1.5.15. *Let* $F \in \mathrm{Mod}(k_{X_{sa}})$ *be quasi-injective. Then* $\rho^{-1}F$ *is c-soft.*

PROOF. Let K be a compact subset of X. Recall that if $U \in \mathrm{Op}(X)$ then $\Gamma(U; \rho^{-1}F) \simeq \varprojlim_{V \subset\subset U} \Gamma(V; F)$, where $V \in \mathrm{Op}(X_{sa})$. We have the chain of isomorphisms

$$\begin{aligned}
\Gamma(K; \rho^{-1}F) &\simeq \varinjlim_{U} \Gamma(U; \rho^{-1}F) \\
&\simeq \varinjlim_{U} \varprojlim_{V \subset\subset U} \Gamma(V; F) \\
&\simeq \varinjlim_{U} \Gamma(U; F)
\end{aligned}$$

where U ranges through the family of subanalytic relatively compact open subsets of X containing K and $V \in \mathrm{Op}(X_{sa})$.

Since F is quasi-injective and filtrant inductive limits are exact, the morphism $\Gamma(X; \rho^{-1}F) \simeq \Gamma(X; F) \to \varinjlim_{U} \Gamma(U; F) \simeq \Gamma(K; \rho^{-1}F)$, where U ranges through the family of subanalytic open subsets of X containing K, is surjective. $\square$

Let us consider the following subcategory of $\mathrm{Mod}(k_{X_{sa}})$:

$$\mathcal{P}_{X_{sa}} := \{G \in \mathrm{Mod}(k_{X_{sa}});\ G \text{ is } \mathrm{Hom}\,_{k_{X_{sa}}}(\cdot\,, F)\text{-acyclic for each } F \in \mathcal{J}_{X_{sa}}\}.$$

This category is generating, in fact if $\{G_j\}_j$ are $\mathbb{R}$-constructible sheaves $\oplus_j \rho_* G_j \in \mathcal{P}_{X_{sa}}$ by Theorem 1.5.4. Moreover $\mathcal{P}_{X_{sa}}$ is stable by $\cdot \otimes K$, where $K \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. In fact if $G \in \mathcal{P}_{X_{sa}}$ and $F \in \mathcal{J}_{X_{sa}}$ we have

$$\mathrm{Hom}_{k_{X_{sa}}}(G \otimes K, F) \simeq \mathrm{Hom}_{k_{X_{sa}}}(G, \mathcal{H}om(K, F))$$

and $\mathcal{H}om(K, F) \in \mathcal{J}_{X_{sa}}$ by Proposition 1.5.8.

THEOREM 1.5.16. *The category* $(\mathcal{P}^{op}_{X_{sa}}, \mathcal{J}_{X_{sa}})$ *is injective with respect to the functor* $\mathrm{Hom}_{k_{X_{sa}}}(\cdot, \cdot)$.

PROOF. (i) Let $G \in \mathcal{P}_{X_{sa}}$ and consider an exact sequence $0 \to F' \to$ $\to F \to F'' \to 0$ with $F' \in \mathcal{J}_{X_{sa}}$. We have to prove that the sequence

$$0 \to \mathrm{Hom}_{k_{X_{sa}}}(G, F') \to \mathrm{Hom}_{k_{X_{sa}}}(G, F) \to \mathrm{Hom}_{k_{X_{sa}}}(G, F'') \to 0$$

is exact. Since the functor $\mathrm{Hom}_{k_{X_{sa}}}(G, \cdot)$ is acyclic on quasi-injective sheaves we obtain the result.

(ii) Let $F \in \mathcal{J}_{X_{sa}}$, and let $0 \to G' \to G \to G'' \to 0$ be an exact sequence on $\mathcal{P}_{X_{sa}}$. Since the objects of $\mathcal{P}_{X_{sa}}$ are $\mathrm{Hom}_{k_{X_{sa}}}(\cdot, F)$-acyclic the sequence

$$0 \to \mathrm{Hom}_{k_{X_{sa}}}(G'', F) \to \mathrm{Hom}_{k_{X_{sa}}}(G, F) \to \mathrm{Hom}_{k_{X_{sa}}}(G', F) \to 0$$

is exact. □

COROLLARY 1.5.17. *The category* $(\mathcal{P}^{op}_{X_{sa}}, \mathcal{J}_{X_{sa}})$ *is injective with respect to the functor* $\mathcal{H}om(\cdot, \cdot)$.

PROOF. Let $G \in \mathcal{P}_{X_{sa}}$, and let $0 \to F' \to F \to F'' \to 0$ be an exact sequence with $F' \in \mathcal{J}_{X_{sa}}$. We shall show that for each $U \in \mathrm{Op}(X_{sa})$ the sequence

$$0 \to \Gamma(U; \mathcal{H}om(G, F')) \to \Gamma(U; \mathcal{H}om(G, F)) \to \Gamma(U; \mathcal{H}om(G, F'')) \to 0$$

is exact. This is equivalent to show that for each $U \in \mathrm{Op}(X_{sa})$ the sequence

$$0 \to \mathrm{Hom}_{k_{X_{sa}}}(G_U, F') \to \mathrm{Hom}_{k_{X_{sa}}}(G_U, F) \to \mathrm{Hom}_{k_{X_{sa}}}(G_U, F'') \to 0$$

is exact. This follows since $G_U \in \mathcal{P}_{X_{sa}}$. The proof of the exactness in $\mathcal{P}^{op}_{X_{sa}}$ is similar. □

1.6 – *The functor* $\rho_!$.

We have seen in § 1.1 that the functor $\rho^{-1} : \mathrm{Mod}(k_{X_{sa}}) \to \mathrm{Mod}(k_X)$ has a left adjoint $\rho_! : \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_{X_{sa}})$. The functor $\rho_!$ is fully faithful and exact. In particular, for $U \in \mathrm{Op}(X)$ one has $\rho_! k_U \simeq \varinjlim_{V \subset\subset U} \rho_* k_V$, where $V \in \mathrm{Op}(X_{sa})$.

Proposition 1.6.1. *Let S be a closed subset of X. Then $\rho_! k_S \simeq \varinjlim_{W \supset S} \rho_* k_{\overline{W}}$, where $W \in \mathrm{Op}(X_{sa})$.*

Proof. (i) Let $U = X \setminus S$. Since $\rho_!$ is exact we have an exact sequence

$$0 \to \rho_! k_U \to \rho_! k_X \to \rho_! k_S \to 0.$$

On the other hand, let $V \in \mathrm{Op}^c(X_{sa})$ and $V \subset\subset U$. We have an exact sequence $0 \to k_V \to k_X \to k_{X \setminus V} \to 0$. Since ρ_* is exact on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ the sequence $0 \to \rho_* k_V \to \rho_* k_X \to \rho_* k_{X\setminus V} \to 0$ is exact. Applying the exact $\varinjlim_{V \subset\subset U}$ we obtain an exact sequence

$$0 \to \varinjlim_{V \subset\subset U} \rho_* k_V \to \rho_* k_X \to \varinjlim_{V \subset\subset U} \rho_* k_{X \setminus V} \to 0.$$

We have $\varinjlim_{V \subset\subset U} \rho_* k_V \simeq \rho_! k_U$ and $\rho_* k_X \simeq \rho_! k_X$. Hence $\rho_! k_S \simeq \varinjlim_{V \subset\subset U} \rho_* k_{X\setminus V}$.

(ii) We shall show that for each $U' \in \mathrm{Op}^c(X_{sa})$ the natural morphism

$$\varinjlim_{V \subset\subset U} \Gamma(U'; k_{X\setminus V}) \to \varinjlim_{W \supset S} \Gamma(U'; k_{\overline{W}}) \tag{1.3}$$

is an isomorphism. We shall see that for each $W \in \mathrm{Op}(X_{sa})$ with $W \supset S$ there exists $W' \in \mathrm{Op}(X_{sa})$ such that $X \setminus \overline{W'} \subset\subset U$ and $\overline{W} \cap U' = \overline{W'} \cap U'$. Set $W' = W \cup (X \setminus \overline{U'})$. Since U' is relatively compact, $X \setminus \overline{W'} \subset\subset U$, and $\overline{W} \cap U' = \overline{W'} \cap U'$ by construction. Then

$$\varinjlim_{V \subset\subset U} \Gamma(U'; k_{X\setminus V}) \simeq \varinjlim_{(X\setminus\overline{W}) \subset\subset U} \Gamma(U'; k_{\overline{W}}) \simeq \varinjlim_{W \supset S} \Gamma(U'; k_{\overline{W}}).$$

□

Notations 1.6.2. *Let $Z = U \cap S$, where $U \in \mathrm{Op}(X)$ and let S be a closed subset of X. Let $F \in \mathrm{Mod}(k_{X_{sa}})$. We set for short ${}_Z F = F \otimes \rho_! k_Z$*

Lemma 1.6.3. *Let $F \in \mathrm{Mod}(k_{X_{sa}})$. Let $U \in \mathrm{Op}(X)$ and let S be a closed subset of X.*

(i) *One has* $_UF \simeq \varinjlim_{V \subset\subset U} F_V \simeq \varinjlim_{V \subset\subset U} \Gamma_{\overline{V}}F$, $V \in \mathrm{Op}^c(X_{sa})$.

(ii) *One has* $_SF \simeq \varinjlim_{W \supset S} F_{\overline{W}} \simeq \varinjlim_{W \supset S} \Gamma_W F$, $W \in \mathrm{Op}(X_{sa})$.

PROOF. (i) The first isomorphism is obvious. Let us show the second isomorphism. We have the chain of isomorphisms

$$\varinjlim_{V \subset\subset U} F_V \xleftarrow{\sim} \varinjlim_{V,V' \subset\subset U} (\Gamma_{\overline{V'}}F)_V \xrightarrow{\sim} \varinjlim_{V' \subset\subset U} \Gamma_{\overline{V'}}F,$$

where V, V' range through the family of subanalytic open subsets of X.
The proof of (ii) is similar. □

PROPOSITION 1.6.4. *Let Z be a locally closed subset of X. Let $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and $F \in \mathrm{Mod}(k_{X_{sa}})$. Then $_Z\mathcal{H}om(G,F) \simeq \mathcal{H}om(G, {_ZF})$.*

PROOF. (i) Let $U \in \mathrm{Op}(X)$. We have the chain of isomorphisms

$$\begin{aligned} {_U\mathcal{H}om(G,F)} &\simeq \varinjlim_{V \subset\subset U} \Gamma_{\overline{V}}\mathcal{H}om(G,F) \\ &\simeq \varinjlim_{V \subset\subset U} \mathcal{H}om(G, \Gamma_{\overline{V}}F) \\ &\simeq \mathcal{H}om(G, \varinjlim_{V \subset\subset U} \Gamma_{\overline{V}}F) \\ &\simeq \mathcal{H}om(G, {_UF}), \end{aligned}$$

where $V \in \mathrm{Op}(X_{sa})$. The third isomorphism follows from Proposition 1.3.4.
(ii) If S is a closed subset of X the proof is similar. □

PROPOSITION 1.6.5. *Let $F \in \mathrm{Mod}(k_{X_{sa}})$ be quasi-injective. Then $\rho_! K \otimes F$ is quasi-injective for each $K \in \mathrm{Mod}(k_X)$.*

PROOF. (i) Let us show the result when $K = k_Z$, for a locally closed subset Z of X. Let $G \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$. We have

$$\begin{aligned} \mathrm{Hom}_{k_{X_{sa}}}(G, {_ZF}) &\simeq \Gamma(X; \mathcal{H}om(G, {_ZF})) \\ &\simeq \Gamma(X; {_Z\mathcal{H}om(G,F)}) \\ &\simeq \Gamma(X; \rho^{-1}{_Z\mathcal{H}om(G,F)}) \\ &\simeq \Gamma(X; (\rho^{-1}\mathcal{H}om(G,F))_Z). \end{aligned}$$

Since F is quasi-injective, $\mathcal{H}om(G,F)$ is quasi-injective. Then by Proposition 1.5.15 the sheaf $(\rho^{-1}\mathcal{H}om(G,F))_Z$ is c-soft and it is injective with respect to the functor $\Gamma(X,\cdot)$. Hence the functor $\Gamma(X;\rho^{-1}\mathcal{H}om(\cdot,F)_Z)$ is exact on $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$.

(ii) Let $K \in \mathrm{Mod}(k_X)$. There exists an epimorphism $\oplus_{i\in I} k_{U_i} \twoheadrightarrow K$ with $U_i \in \mathrm{Op}(X_{sa})$ for each i. Let K_J be the image of $\oplus_{i\in J} k_{U_i}$, with $J \subset I$ finite. We have $K \simeq \varinjlim_J K_J$, hence $\rho_! K \simeq \varinjlim_J \rho_! K_J$ since $\rho_!$ commutes with $\varinjlim$. It is enough to prove the result for K_J. We argue by induction on the cardinal of J. Set $K = K_J$. If $|J| = 1$ then $K \simeq k_Z$ with Z locally closed subset of X and the result follows from (i).

Let us show $n-1 \Rightarrow n$. There is an epimorphism $\oplus_{i=1}^n k_{U_i} \twoheadrightarrow K$. Let K_1 be the image of $k_{U_1} \to K$ and let $K_2 = K/K_1$. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k_{U_1} & \longrightarrow & \oplus_{i=1}^n k_{U_i} & \longrightarrow & \oplus_{i=2}^n k_{U_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 & \longrightarrow & K & \longrightarrow & K_2 & \longrightarrow & 0,
\end{array}$$

where the vertical arrows are surjective, and the rows are exact. By the exactness of $\rho_!$ and $\otimes$ we obtain the exact sequence $0 \to \rho_! K_1 \otimes F \to$ $\to \rho_! K \otimes F \to \rho_! K_2 \otimes F \to 0$. By the inductive hypothesis $\rho_! K_1 \otimes F$ and $\rho_! K_2 \otimes F$ are quasi-injective, then $\rho_! K \otimes F$ is quasi-injective. □

PROPOSITION 1.6.6. *Let* $F \in \mathrm{Mod}(k_{X_{sa}})$, $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$ *and let* $K \in \mathrm{Mod}(k_X)$. *One has the isomorphism* $\mathcal{H}om(G,F) \otimes \rho_! K \simeq \mathcal{H}om(G, F \otimes \rho_! K)$.

PROOF. Both sides are left exact with respect to F. Hence we may assume that F is quasi-injective. Since quasi-injective sheaves are $\mathcal{H}om(G,\cdot)$-injective, both sides are exact with respect to K. Moreover as a consequence of Proposition 1.3.4 both sides commute with filtrant $\varinjlim$ with respect to K. We may reduce to the case $K = k_U$, with $U \in \mathrm{Op}(X_{sa})$. Then the result follows from Proposition 1.6.4. □

2. Derived category.

As usual, we denote $D(k_{X_{sa}})$ the derived category of $\mathrm{Mod}(k_{X_{sa}})$ and its full subcategory consisting of bounded (resp. bounded below, resp. bounded above) complexes is denoted by $D^b(k_{X_{sa}})$ (resp. $D^+(k_{X_{sa}})$, resp. $D^-(k_{X_{sa}})$).

2.1 – *The category* $D^b_{\mathbb{R}\text{-c}}(k_{X_{sa}})$.

As usual we denote by $D^b_{\mathbb{R}\text{-c}}(k_X)$ (resp. $D^b_{\mathbb{R}\text{-c}}(k_{X_{sa}})$) the full subcategory of $D^b(k_X)$ (resp. $D^b(k_{X_{sa}})$) consisting of objects with $\mathbb{R}$-constructible cohomology.

Recall that $\rho : X \to X_{sa}$ is the natural morphism of sites. It induces the functor $\rho_* : \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_{X_{sa}})$.

LEMMA 2.1.1. *Let $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. Then $R^j\rho_*F = 0$ for each $j \neq 0$.*

PROOF. The sheaf $R^j\rho_*F$ is the sheaf associated to the presheaf $V \to R^j\Gamma(V;F)$. We have to show that $R^j\Gamma(V;F) = 0$ for $j \neq 0$ on a family of generators of the topology of X_{sa}. This means that for each $V \in \mathrm{Op}^c(X_{sa})$ and for each $j \neq 0$, there exists I finite and $\{V_i\}_{i\in I} \in \mathrm{Cov}(V_{sa})$ such that $R^j\Gamma(V_i;R\rho_*F) \simeq R^j\Gamma(V_i;F) = 0$.

We use the notation of [5]. There exists a locally finite stratification $\{X_i\}_{i\in I}$ of X consisting of subanalytic subsets such that for all $j \in \mathbb{Z}$ and all $i \in I$ the sheaf $F|_{X_i}$ is locally constant. By the triangulation theorem there exist a simplicial complex (S,Δ) and a subanalytic homeomorphism $\psi : |S| \xrightarrow{\sim} X$ compatible with the stratification and such that V is a finite union of the images by ψ of open subsets $V(\sigma)$ of $|S|$, where $V(\sigma) = \bigcup_{\tau\in\Delta,\tau\supset\sigma}|\tau|$. By Proposition 8.1.4 of [5] we have $R^j\Gamma(\psi(V(\sigma));F) = 0$ for each σ and for each $j \neq 0$. The result follows because $V = \bigcup_{\psi(|\sigma|)\subset V}\psi(V(\sigma))$. □

Since $\mathbb{R}$-constructible sheaves are injective with respect to the functor ρ_*, the following diagram of derived categories is quasi-commutative.

$$
\begin{array}{ccc}
D^b_{\mathbb{R}\text{-c}}(k_X) & \underset{\rho^{-1}}{\overset{R\rho_*}{\rightleftarrows}} & D^b_{\mathbb{R}\text{-c}}(k_{X_{sa}}) \\
\uparrow{\scriptstyle\wr} & \nearrow_{\rho_*} & \\
D^b(\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)) & &
\end{array}
\tag{2.1}
$$

THEOREM 2.1.2. *One has the equivalence of categories*

$$D^b_{\mathbb{R}\text{-c}}(k_X) \simeq D^b(\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)) \simeq D^b_{\mathbb{R}\text{-c}}(k_{X_{sa}}).$$

PROOF. By dévissage, to prove the equivalence between $D^b(\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X))$ and $D^b_{\mathbb{R}\text{-c}}(k_{X_{sa}})$ it is enough to check that the functor ρ_* in (2.1) is fully faithful. We have $\rho^{-1} \circ \rho_* \simeq \mathrm{id}$ and the result follows.

The equivalence between $D^b(\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X))$ and $D^b_{\mathbb{R}\text{-c}}(k_X)$ was shown by Kashiwara in [4]. $\square$

2.2 – *Operations in the derived category.*

Let us study the operations in the derived category of $\mathrm{Mod}(k_{X_{sa}})$. Let $f : X \to Y$ be an analytic map. We will use quasi-injective objects and the results of § 1.5 to derive the formulas of Section 1.

Since $\mathrm{Mod}(k_{X_{sa}})$ has enough injectives, then the derived functors

$$R\mathcal{H}om : D^-(k_{X_{sa}})^{op} \times D^+(k_{X_{sa}}) \to D^+(k_{X_{sa}}),$$

$$Rf_* : D^+(k_{X_{sa}}) \to D^+(k_{Y_{sa}}),$$

$$Rf_{!!} : D^+(k_{X_{sa}}) \to D^+(k_{Y_{sa}}),$$

are well defined.

PROPOSITION 2.2.1. *Let $f : X \to Y$ be an analytic map. Then*

(i) *The functors Rf_* and $R\mathcal{H}om$ commute with $R\rho_*$.*

(ii) *The functors Rf_* and $Rf_{!!}$ commute with ρ^{-1}.*

(iii) *We have $R(g \circ f)_* \simeq Rg_* \circ Rf_*$ and $R(g \circ f)_{!!} \simeq Rg_{!!} \circ Rf_{!!}$.*

(iv) *The functor $R^k f_{!!} : \mathrm{Mod}(k_{X_{sa}}) \to \mathrm{Mod}(k_{Y_{sa}})$ commutes with filtrant inductive limits for each $k \in \mathbb{Z}$.*

(v) *If $F \in D^+(k_{X_{sa}})$ and f is proper on $\mathrm{supp}(F)$, then $Rf_{!!} \simeq Rf_*$.*

PROOF. (i) The functor ρ_* sends injective sheaves to injective sheaves, then Rf_* and $R\mathcal{H}om$ commute with $R\rho_*$.

(ii) Since ρ^{-1} has an exact left adjoint it sends injective sheaves to injective sheaves. Then Rf_* and $Rf_{!!}$ commute with ρ^{-1}.

(iii) The functor f_* (resp. $f_{!!}$) sends injective sheaves to injective (resp. quasi-injective) sheaves. Then $R(g \circ f)_* \simeq Rg_* \circ Rf_*$ and $R(g \circ f)_{!!} \simeq$ $\simeq Rg_{!!} \circ Rf_{!!}$.

(iv) Quasi-injective objects of $\mathrm{Mod}(k_{X_{sa}})$ are stable by filtrant $\varinjlim$, and the functor $f_{!!}$ commutes with such limits. Then $R^k f_{!!}$ commutes with filtrant $\varinjlim$ for each $k \in \mathbb{Z}$.

(v) We can find a representative F' of F in $K^+(\mathcal{J}_{X_{sa}})$ with f proper on $\mathrm{supp}(F')$. Then the result follows from the non derived case. □

PROPOSITION 2.2.2. *Let* $F = \varinjlim_i F_i$ *with* $F_i \in \mathrm{Mod}(k_{X_{sa}})$ *and let* $G \in D^b_{\mathbb{R}\text{-c}}(k_X)$*. One has* $R^k\mathcal{H}om(G, F) \simeq \varinjlim_i R^k\mathcal{H}om(G, F_i)$ *for each* $k \in \mathbb{Z}$*.*

PROOF. There exists (see [8], Corollary 9.6.7) an inductive system of injective resolutions $I_i^\bullet$ of F_i. Then $\varinjlim_i I_i^\bullet$ is a complex of quasi-injective objects quasi-isomorphic to F. Each object of $(\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)^{op}, \mathcal{J}_{X_{sa}})$ is $\mathcal{H}om(\cdot,\cdot)$-acyclic. Proposition 1.3.4 implies the isomorphism

$$\mathcal{H}om(G, \varinjlim_i I_i^\bullet) \simeq \varinjlim_i \mathcal{H}om(G, I_i^\bullet)$$

and the result follows. □

PROPOSITION 2.2.3. *Let* $F \in D^+(k_{X_{sa}})$*,* $G \in D^b_{\mathbb{R}\text{-c}}(k_X)$ *and let* $K \in D^+(k_X)$*. One has the isomorphism* $R\mathcal{H}om(G, F) \otimes \rho_! K \simeq R\mathcal{H}om(G, F \otimes \rho_! K)$*.*

PROOF. Let $I^\bullet$ be a quasi-injective resolution of F. By Proposition 1.6.5 we have that $I^\bullet \otimes \rho_! K$ is a complex of quasi-injective objects. Each object of $(\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)^{op}, \mathcal{J}_{X_{sa}})$ is $\mathcal{H}om(\cdot,\cdot)$-acyclic. Hence we are reduced to prove the isomorphism $\mathcal{H}om(G, I^\bullet) \otimes \rho_! K \simeq \mathcal{H}om(G, I^\bullet \otimes \rho_! K)$. The result follows from Proposition 1.6.6. □

PROPOSITION 2.2.4. *Let* $U \in \mathrm{Op}^c(X_{sa})$*. Let* $F \in \mathrm{Mod}(k_{X_{sa}})$ *be quasi-injective. Then* F_U *is* $\Gamma(V,\cdot)$*-acyclic for each* $V \in \mathrm{Op}(X_{sa})$*.*

PROOF. Since F_U has compact support, we may suppose that V is relatively compact. Let $S = X \setminus U$. Since F is quasi-injective and filtrant $\varinjlim$ are exact, the morphism $\Gamma(V;F) \to \varinjlim_{W \supset S\cap V} \Gamma(W;F) \xrightarrow{\sim} \Gamma(V;F_S)$ is surjective. Consider the exact sequence $0 \to F_U \to F \to F_S \to 0$. We get the exact sequence

$$0 \to \Gamma(V;F_U) \to \Gamma(V;F) \to \Gamma(V;F_S) \to 0.$$

By Proposition 1.5.14 F and F_S are quasi-injective, hence $\Gamma(V;\cdot)$-acyclic. This implies that F_U is $\Gamma(V;\cdot)$-acyclic. □

COROLLARY 2.2.5. *Let* $f : X \to Y$ *be a real analytic map and* *let* $U \in \mathrm{Op}^c(X_{sa})$ *Let* $F \in \mathrm{Mod}(k_{X_{sa}})$ *be quasi-injective. Then* F_U *is* $f_{!!}$*-acyclic.*

PROOF. Since F_U has compact support, $Rf_{!!}F_U \simeq Rf_* F_U$. The result follows because F_U is $\Gamma(f^{-1}(V);\cdot)$-acyclic for each $V \in \mathrm{Op}(Y_{sa})$. □

LEMMA 2.2.6. *Let* F *be quasi-injective object of* $\mathrm{Mod}(k_{X_{sa}})$ *and let* $G \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$*. Then* $F \otimes \rho_* G$ *is* $f_{!!}$*-acyclic.*

PROOF. Let $G \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$. Then G has a resolution

$$0 \to \oplus_{i_1 \in I_1} k_{U_{i_1}} \to \ldots \to \oplus_{i_n \in I_n} k_{U_{i_n}} \xrightarrow{\varphi} G \to 0,$$

where I_j is finite and $U_{i_j} \in \mathrm{Op}^c(X_{sa})$ for each $i_j \in I_j$, $j \in \{1, \ldots, n\}$. Let us argue by induction on the length n of the resolution.

If $n = 1$, then G is isomorphic to a finite sum $\oplus_i k_{U_i}$, with $U_i \in \mathrm{Op}^c(X_{sa})$, and the result follows from Corollary 2.2.5.

Let us show $n - 1 \Rightarrow n$. The sequence $0 \to \ker\varphi \to \oplus_{i_n \in I_n} k_{U_{i_n}} \to G \to 0$ is exact. The sheaf $\ker\varphi$ belongs to $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ and it has a resolution of length $n - 1$. Applying $F \otimes \rho_*(\cdot)$ we get the exact sequence

$$0 \to F \otimes \rho_* \ker\varphi \to \oplus_{i_n \in I_n} F_{U_{i_n}} \to F \otimes \rho_* G \to 0.$$

By the induction hypothesis $F \otimes \rho_* \ker\varphi$ is $f_{!!}$-acyclic. Moreover $\oplus_{i_n \in I_n} F_{U_{i_n}}$ is $f_{!!}$-acyclic, then $F \otimes \rho_* G$ is $f_{!!}$-acyclic. □

PROPOSITION 2.2.7. *Let* F *be quasi-injective object of* $\mathrm{Mod}(k_{X_{sa}})$ *and let* $G \in \mathrm{Mod}(k_{X_{sa}})$*. Then* $F \otimes G$ *is* $f_{!!}$*-acyclic.*

PROOF. Let $G \simeq \varinjlim_i \rho_* G_i$ with $G_i \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ for each i. Since the functors $\otimes$ and $R^k f_{!!}$ commute with filtrant $\varinjlim$ we have

$$R^k f_{!!}(F \otimes \varinjlim_i \rho_* G_i) \simeq \varinjlim_i R^k f_{!!}(F \otimes \rho_* G_i) = 0$$

if $k \neq 0$ by Lemma 2.2.6. □

PROPOSITION 2.2.8. *Let* $F \in D^+(k_{X_{sa}})$ *and* $G \in D^+(k_{Y_{sa}})$*. Then*

$$Rf_{!!}F \otimes G \simeq Rf_{!!}(F \otimes f^{-1}G).$$

PROOF. First assume that $F \in \mathrm{Mod}(k_{X_{sa}})$ is injective. By Proposition 2.2.7 $F \otimes f^{-1}G$ is $f_{!!}$-acyclic.

Now let $F \in D^+(k_{X_{sa}})$ and $G \in D^+(k_{Y_{sa}})$. Let F' be a complex of injective sheaves quasi-isomorphic to F. Then

$$Rf_{!!}F \otimes G \simeq f_{!!}F' \otimes G \simeq f_{!!}(F' \otimes f^{-1}G) \simeq Rf_{!!}(F \otimes f^{-1}G),$$

where the second isomorphism follows from Proposition 1.4.5. □

Now let us consider a cartesian square

$$\begin{array}{ccc} X'_{sa} & \xrightarrow{f'} & Y'_{sa} \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X_{sa} & \xrightarrow{f} & Y_{sa} \end{array}$$

PROPOSITION 2.2.9. *Let $F \in D^+(k_{X_{sa}})$. Then $g^{-1}Rf_{!!}F \simeq Rf'_{!!}g'^{-1}F$.*

PROOF. We have an isomorphism $f'_{!!}g'^{-1} \simeq g^{-1}f_{!!}$, and $R(g^{-1}f_{!!}) \simeq$ $\simeq g^{-1}Rf_{!!}$ since g^{-1} is exact. Then we obtain a morphism $g^{-1}Rf_{!!} \to Rf'_{!!}g'^{-1}$. It is enough to prove that for any $k \in \mathbb{Z}$ and for any $F \in \mathrm{Mod}(k_{X_{sa}})$ we have $g^{-1}R^kf_{!!}F \xrightarrow{\sim} R^kf'_{!!}g'^{-1}F$. Since both sides commute with filtrant $\varinjlim$, we may assume $F = \rho_*G$ with $G \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$. Moreover since $\mathrm{supp}(G)$ is compact, f' is proper on $\mathrm{supp}(g'^{-1}G)$. Then both sides commute with ρ_* and the result follows from the corresponding one for classical sheaves. □

As in classical sheaf theory, the Künneth formula follows from the projection formula and the base change formula.

PROPOSITION 2.2.10. *Consider a cartesian square*

$$\begin{array}{ccc} X'_{sa} & \xrightarrow{f'} & Y'_{sa} \\ \downarrow{\scriptstyle g'} & \searrow{\scriptstyle \delta} & \downarrow{\scriptstyle g} \\ X_{sa} & \xrightarrow{f} & Y_{sa} \end{array}$$

where $\delta = fg' = gf'$. There is a natural isomorphism

$$R\delta_{!!}(g'^{-1}F \otimes f'^{-1}G) \simeq Rf_{!!}F \otimes Rg_{!!}G$$

for $F \in D^+(k_{X_{sa}})$ and $G \in D^+(k_{Y'_{sa}})$.

PROOF. Using the projection formula and the base change formula we deduce

$$Rf'_{!!}(g'^{-1}F \otimes f'^{-1}G) \simeq Rf'_{!!}g'^{-1}F \otimes G \simeq g^{-1}Rf_{!!}F \otimes G.$$

Using the projection formula once again we find

$$Rg_{!!}Rf'_{!!}(g'^{-1}F \otimes f'^{-1}G) \simeq Rg_{!!}(g^{-1}Rf_{!!}F \otimes G) \simeq Rf_{!!}F \otimes Rg_{!!}G$$

and the result follows since $R\delta_{!!} \simeq Rg_{!!} \circ Rf'_{!!}$. □

The following isomorphism is the analogue for subanalytic sheaves of Lemma 5.2.8 of [7]

PROPOSITION 2.2.11. *Let $G \in D^b_{\mathbb{R}\text{-c}}(k_Y)$ and let $F \in D^+(k_{X_{sa}})$. Then the natural morphism*

$$Rf_{!!}R\mathcal{H}om(f^{-1}G, F) \to R\mathcal{H}om(G, Rf_{!!}F)$$

is an isomorphism.

PROOF. The morphism is obtained as in the non derived case. Let us show that it is an isomorphism. Let F' be a complex of injective sheaves quasi-isomorphic to F. Then

$$\begin{aligned} Rf_{!!}\mathcal{H}om(f^{-1}G, F) &\simeq f_{!!}\mathcal{H}om(f^{-1}G, F') \\ &\simeq \mathcal{H}om(G, f_{!!}F') \\ &\simeq R\mathcal{H}om(G, Rf_{!!}F), \end{aligned}$$

where the second isomorphism follows from Proposition 1.4.7. □

2.3 – *Vanishing theorems on* $\mathrm{Mod}(k_{X_{sa}})$.

In § 2.3 we prove some results on the vanishing of the cohomology of sheaves on a subanalytic site.

DEFINITION 2.3.1. *The quasi-injective dimension of the category* $\mathrm{Mod}(k_{X_{sa}})$ *is the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that for any $F \in \mathrm{Mod}(k_{X_{sa}})$ there exists an exact sequence*

$$0 \to F \to \mathrm{I}^0 \to \cdots \to I^n \to 0$$

with I^j quasi-injective for $0 \le j \le n$.

PROPOSITION 2.3.2. *The category* $\mathrm{Mod}(k_{X_{sa}})$ *has finite quasi-injective dimension.*

PROOF. Let $F \in \mathrm{Mod}(k_{X_{sa}})$. Then $F = \varinjlim\limits_i \rho_* F_i$, with $F_i \in \mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$. There exists (see [8], Corollary 9.6.7) an inductive system of injective resolutions $I_i^\bullet$ of F_i. By Proposition 3.3.11 of [5], the category $\mathrm{Mod}(k_X)$ has finite homological dimension. Then we may assume that $I_i^\bullet$ has length $N_0 < \infty$ for each i. Since F_i is ρ_*-injective for each i, $\rho_* I_i^\bullet$ is an injective resolution of $\rho_* F_i$ of length N_0. Taking the inductive limit we find that $\varinjlim\limits_i \rho_* I_i^\bullet$ is a resolution of F of length N_0, and $\varinjlim\limits_i \rho_* I_i^j \in \mathcal{J}_{X_{sa}}$ for each j. □

COROLLARY 2.3.3. *Let* $f : X \to Y$ *be a real analytic map, and let* $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. *The functors* f_*, $f_{!!}$ *and* $\mathcal{H}om(F, \cdot)$ *have finite cohomological dimension.*

PROPOSITION 2.3.4. *Let* $F \in \mathrm{Mod}(k_X)$ *and let* $G \in \mathrm{Mod}(k_{X_{sa}})$. *There exists a finite* $j_0 \in \mathbb{N}$ *such that*

$$R^j\mathcal{H}om(\rho_! F, G) = 0 \ \text{for}\ j > j_0.$$

PROOF. Let $U \in \mathrm{Op}(X_{sa})$. We have the chain of isomorphisms

$$\begin{aligned} R\Gamma(U; R\mathcal{H}om(\rho_! F, G)) &\simeq \mathrm{RHom}_{k_{X_{sa}}}(\rho_! F, R\Gamma_U G) \\ &\simeq \mathrm{RHom}_{k_X}(F,\ \rho^{-1} R\Gamma_U G). \end{aligned}$$

The functor Γ_U has finite cohomological dimension, and the homological dimension of the category $\mathrm{Mod}(k_X)$ is finite. Hence we can find a finite $j_0 \in \mathbb{N}$ such that $R^j\Gamma(U; R\mathcal{H}om(\rho_! F, G))$ vanishes for $j > j_0$ and for each $U \in \mathrm{Op}(X_{sa})$. This shows the result. □

REMARK 2.3.5. *We have seen that the functor* $\mathcal{H}om(F, \cdot)$ *has finite cohomological dimension when* F *is* $\mathbb{R}$*-constructible and when* $F = \rho_! G$ *with* $G \in \mathrm{Mod}(k_X)$. *We do not know if the cohomological dimension is finite for any* $F \in \mathrm{Mod}(k_{X_{sa}})$. *Indeed we do not know if the homological dimension of* $\mathrm{Mod}(k_{X_{sa}})$ *is finite or not.*

2.4 – *Duality.*

In the following we find a right adjoint to the functor $Rf_{!!}$, denoted by $f^!$. The construction is not inspired to the classical one as in [7], but it is done via Brown representability (we refer to [8] for an exposition on this subject). We calculate $f^!$ by decomposing f as the composite of a closed embedding and a submersion.

The subcategory $\mathcal{J}_{X_{sa}}$ of quasi-injective objects and the functor $f_{!!}$ have the following properties:

$$(2.2)\qquad \begin{cases} \text{(i) } \mathcal{J}_{X_{sa}} \text{ is cogenerating,} \\ \text{(ii) } \mathrm{Mod}(k_{X_{sa}}) \text{ has finite quasi-injective dimension,} \\ \text{(iii) } \mathcal{J}_{X_{sa}} \text{ is } f_{!!}\text{-injective,} \\ \text{(iv) } \mathcal{J}_{X_{sa}} \text{ is closed by small } \oplus, \\ \text{(v) } f_{!!} \text{ commutes with small } \oplus. \end{cases}$$

As a consequence of the Brown representability theorem (see [8], Corollary 14.3.7 for details) we find a right adjoint to the functor $Rf_{!!}$. Remark that the functor $Rf_{!!}$ extends to a functor $Rf_{!!} : D(k_{X_{sa}}) \to D(k_{Y_{sa}})$.

THEOREM 2.4.1. *The functor $Rf_{!!} : D(k_{X_{sa}}) \to D(k_{Y_{sa}})$ admits a right adjoint. We denote by $f^! : D(k_{Y_{sa}}) \to D(k_{X_{sa}})$ the adjoint functor.*

COROLLARY 2.4.2. *Let $G \in D^+(k_{Y_{sa}})$. Then $f^!G \in D^+(k_{X_{sa}})$.*

PROOF. We may reduce to the case that $G \in D^{\geq 0}(k_{Y_{sa}})$. Let N_0 be the quasi-injective dimension of $\mathrm{Mod}(k_{X_{sa}})$ and let $F \in D^{\leq -N_0-1}(k_{X_{sa}})$. Then $Rf_{!!}F \in D^{\leq -1}(k_{Y_{sa}})$ and

$$\mathrm{Hom}_{D(k_{Y_{sa}})}(Rf_{!!}F, G) \simeq \mathrm{Hom}_{D(k_{X_{sa}})}(F, f^!G) = 0.$$

Hence for each $F \in D^{\leq -N_0-1}(k_{X_{sa}})$ we have $\mathrm{Hom}_{D(k_{X_{sa}})}(F, f^!G) = 0$. Set for short $a = -N_0 - 1$, if $F = \tau^{\leq a} f^! G$ we have

$$\mathrm{Hom}_{D(k_{X_{sa}})}(\tau^{\leq a} f^!G, f^!G) \simeq \mathrm{Hom}_{D^{\leq a}(k_{X_{sa}})}(\tau^{\leq a} f^!G, \tau^{\leq a} f^!G) = 0.$$

This implies $\tau^{\leq a} f^!G = 0$, hence $f^!G \in D^+(k_{X_{sa}})$. $\square$

REMARK 2.4.3. *As in classical sheaf theory one can prove that for $F \in D^b(k_{X_{sa}})$ and $G \in D^+(k_{Y_{sa}})$ one has $Rf_* R\mathcal{H}om(F, f^!G) \simeq R\mathcal{H}om(Rf_{!!}F, G)$.*

REMARK 2.4.4. *As in classical sheaf theory, one can prove by adjunction the dual projection formula and the dual base change formula.*

PROPOSITION 2.4.5. *The functor $f^!$ commutes with $R\rho_*$, and the functor $H^kf^! : \mathrm{Mod}(k_{Y_{sa}}) \to \mathrm{Mod}(k_{X_{sa}})$ commutes with filtrant $\varinjlim$.*

PROOF. Since $Rf_{!!}$ commutes with ρ^{-1}, then $f^!$ commutes with $R\rho_*$ by adjunction.

Let us show that $H^kf^!$ commutes with $\varinjlim$. Let $\{F_i\}_i$ be a filtrant inductive system in $\mathrm{Mod}(k_{Y_{sa}})$. Remark that $\varinjlim_i H^kf^!F_i$ (resp. $H^kf^! \varinjlim_i F_i$) is the sheaf associated to the presheaf $U \mapsto \varinjlim_i R^k\Gamma(U; f^!F_i)$ (resp. $U \mapsto R^k\Gamma(U; f^! \varinjlim_i F_i)$), for $U \in \mathrm{Op}^c(X_{sa})$.

We will show the isomorphism $R^k\Gamma(U; f^! \varinjlim_i F_i) \xleftarrow{\sim} \varinjlim_i R^k\Gamma(U; f^!F_i)$ for each $U \in \mathrm{Op}^c(X_{sa})$. By adjunction it is enough to prove the isomorphism $R^k\mathrm{Hom}_{k_{Y_{sa}}}(Rf_!k_U, \varinjlim_i F_i) \simeq \varinjlim_i R^k\mathrm{Hom}_{k_{Y_{sa}}}(Rf_!k_U, F_i)$ (remark that we have $Rf_{!!}k_U \simeq Rf_!k_U \simeq Rf_*k_U$ since $U \in \mathrm{Op}^c(X_{sa})$).

Let $\mathcal{J}_{Y_{sa}}$ be the family of quasi-injective objects of $\mathrm{Mod}(k_{Y_{sa}})$. Each object of $(\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_Y)^{op}, \mathcal{J}_{Y_{sa}})$ is $\mathrm{Hom}_{k_{Y_{sa}}}(\cdot,\cdot)$-acyclic. Moreover $\mathcal{J}_{Y_{sa}}$ is stable by filtrant inductive limits. There exists (see [8], Corollary 9.6.7) an inductive system of injective resolutions $I_i^\bullet$ of F_i. Then $\varinjlim_i I_i^\bullet$ is a quasi-injective resolution of $\varinjlim_i F_i$. We have

$$\mathrm{Hom}_{K^+(k_{Y_{sa}})}(Rf_!k_U, \varinjlim_i I_i^\bullet) \simeq \varinjlim_i \mathrm{Hom}_{K^+(k_{Y_{sa}})}(Rf_!k_U, I_i^\bullet)$$

and the result follows. □

COROLLARY 2.4.6. *Let $F \in D^b(k_{Y_{sa}})$. Then $f^!F \in D^b(k_{X_{sa}})$.*

PROOF. We may reduce to the case $F \in \mathrm{Mod}(k_{Y_{sa}})$. Then $F \simeq \varinjlim_i \rho_*F_i$ with $F_i \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_Y)$ for each i. By Proposition 2.4.5 we have

$$H^kf^!F \simeq H^kf^! \varinjlim_i \rho_*F_i \simeq \varinjlim_i \rho_*H^kf^!F_i,$$

and $H^kf^!F_i = 0$ if $k > j_0$ for a fixed $j_0 \in \mathbb{N}$ and for each i. □

The following result is the analogue for subanalytic sheaves of Proposition 5.3.9 of [7].

PROPOSITION 2.4.7. *Let $F \in D^+(k_{Y_{sa}})$ and let $G \in D^+(k_Y)$. Then one has the isomorphism $f^!(F \otimes \rho_! G) \simeq f^! F \otimes \rho_! f^{-1} G$.*

PROOF. We have the chain of morphisms

$$Rf_{!!}(f^! F \otimes \rho_! f^{-1} G) \simeq Rf_{!!} f^! F \otimes \rho_! G \to F \otimes \rho_! G,$$

by adjunction we obtain the desired morphism. To prove that it is an isomorphism it is enough to show $R^k\Gamma(U; f^!(F \otimes \rho_! G)) \simeq R^k\Gamma(U; f^! F \otimes \rho_! f^{-1} G)$ for each $U \in \mathrm{Op}^c(X_{sa})$ and each $k \in \mathbb{Z}$. We have the chain of isomorphisms

$$\begin{aligned} R^k\Gamma(U; f^!(F \otimes \rho_! G)) &\simeq R^k\mathrm{Hom}_{k_{Y_{sa}}}(Rf_{!!} k_U, F \otimes \rho_! G) \\ &\simeq R^k\mathrm{Hom}_{k_{Y_{sa}}}(k_Y, R\mathcal{H}om(Rf_{!!} k_U, F \otimes \rho_! G)) \\ &\simeq R^k\mathrm{Hom}_{k_{Y_{sa}}}(k_Y, R\mathcal{H}om(Rf_{!!} k_U, F) \otimes \rho_! G) \\ &\simeq R^k\mathrm{Hom}_{k_{Y_{sa}}}(k_Y, Rf_{!!} R\mathcal{H}om(k_U, f^! F) \otimes \rho_! G) \\ &\simeq R^k\mathrm{Hom}_{k_{Y_{sa}}}(k_Y, Rf_{!!}(R\mathcal{H}om(k_U, f^! F) \otimes f^{-1}\rho_! G)) \\ &\simeq R^k\mathrm{Hom}_{k_{Y_{sa}}}(k_Y, Rf_{!!}(R\mathcal{H}om(k_U, f^! F \otimes \rho_! f^{-1} G))) \\ &\simeq R^k\mathrm{Hom}_{k_{X_{sa}}}(k_U, f^! F \otimes \rho_! f^{-1} G). \end{aligned}$$

Here the fourth and the last isomorphism follow from the fact that since k_U has compact support, then $R\mathcal{H}om(k_U, K)$ has compact support for any $K \in D^+(k_{X_{sa}})$ and $Rf_{!!} R\mathcal{H}om(k_U, K) \simeq Rf_* R\mathcal{H}om(k_U, K)$. □

PROPOSITION 2.4.8. *Let $F \in D^+(k_{Y_{sa}})$, and let $f : X \to Y$ be a closed embedding. Then $f^! F \simeq f^{-1} R\mathcal{H}om(k_X, F)$ and* id $\xrightarrow{\sim} f^! Rf_{!!}$.

PROOF. Since f is proper, then $Rf_* \simeq Rf_{!!}$. We have the isomorphisms $Rf_* f^! F \simeq Rf_* R\mathcal{H}om(k_X, f^! F) \simeq R\mathcal{H}om(k_X, F)$. Since $f^{-1} Rf_* f^! F \simeq f^! F$, then $f^! F \simeq f^{-1} R\mathcal{H}om(k_X, F)$.

Let $F' \in D^+(k_{X_{sa}})$. We have the isomorphisms

$$f^! Rf_* F' \simeq f^{-1} R\mathcal{H}om(k_X, Rf_* F') \simeq f^{-1} Rf_* R\mathcal{H}om(k_X, F') \simeq f^{-1} Rf_* F',$$

and $f^{-1} Rf_* F' \simeq F'$ since f is a closed embedding. □

Recall that f is a topological submersion (of fiber dimension n) if locally on X, f is isomorphic to the projection $Y \times \mathbb{R}^n \to Y$.

PROPOSITION 2.4.9. *Assume that f is a topological submersion. Then for $F \in D^+(k_{Y_{sa}})$ one has the isomorphism $f^{-1}F \otimes f^! k_Y \xrightarrow{\sim} f^! F$.*

PROOF. We have the chain of morphisms

$$Rf_{!!}(f^{-1}F \otimes f^! k_Y) \simeq F \otimes Rf_{!!} f^! k_Y \to F \otimes k_Y \simeq F,$$

by adjunction we obtain the desired morphism.

Let us show that it is an isomorphism. We may reduce to the case $F \simeq \varinjlim_i \rho_* F_i \in \mathrm{Mod}(k_{Y_{sa}})$. We have the chain of isomorphisms

$$\begin{aligned} H^k(f^{-1} \varinjlim_i \rho_* F_i \otimes f^! \rho_* k_Y) &\simeq \varinjlim_i \rho_* H^k(f^{-1} F_i \otimes f^! k_Y) \\ &\simeq \varinjlim_i \rho_* H^k f^! F_i \\ &\simeq H^k f^! \varinjlim_i \rho_* F_i. \end{aligned}$$

□

Using these results we can calculate explicitly the functor $f^!$. Let $f : X \to Y$ be an analytic map. We decompose it as the composite of a closed embedding and a submersion. In fact

$$f : X \overset{j}{\hookrightarrow} X \times Y \overset{p}{\to} Y$$

where p is the projection and j is the graph embedding $j(x) = (x, f(x))$. Let $F \in D^+(k_{Y_{sa}})$. Applying Propositions 2.4.8 and 2.4.9 we get

$$f^! F \simeq j^{-1} R\mathcal{H}om(k_{j(X)}, p^{-1}F \otimes p^! k_Y).$$

COROLLARY 2.4.10. *Assume that f is a topological submersion. Then:*
(i) *the functor $f^!$ commutes with ρ^{-1},*
(ii) *the functor $Rf_{!!}$ commutes with $\rho_!$.*

PROOF. (i) One has the chain of isomorphisms

$$\begin{aligned} \rho^{-1}(f^{-1}F \otimes f^! \rho_* k_Y) &\simeq \rho^{-1} f^{-1} F \otimes \rho^{-1} f^! \rho_* k_Y \\ &\simeq f^{-1} \rho^{-1} F \otimes \rho^{-1} f^! \rho_* k_Y \\ &\simeq f^{-1} \rho^{-1} F \otimes \rho^{-1} \rho_* f^! k_Y \\ &\simeq f^{-1} \rho^{-1} F \otimes f^! k_Y. \end{aligned}$$

The result follows from Proposition 2.4.9.

(ii) The result follows by adjunction. □

PROPOSITION 2.4.11. *Assume that f is a topological submersion and moreover that $Rf_! f^! k_Y \simeq k_Y$. Then for $F \in D^+(k_{Y_{sa}})$ the morphism $Rf_* f^{-1}F \to F$ is an isomorphism.*

PROOF. First let us show that $Rf_{!!} f^! k_Y \simeq k_Y$. We have the chain of isomorphisms

$$\begin{aligned} Rf_{!!} f^! \rho_* k_Y &\simeq Rf_{!!} \rho_* f^! k_Y \\ &\simeq Rf_{!!} \rho_! f^! k_Y \\ &\simeq \rho_! Rf_! f^! k_Y \\ &\simeq \rho_! k_Y \\ &\simeq \rho_* k_Y, \end{aligned}$$

where the second isomorphism follows because $f^! k_Y$ is locally constant and the third from Corollary 2.4.10 (ii). It follows from Proposition 2.4.9 that $f^{-1}F \simeq R\mathcal{H}om(f^! k_Y, f^! F)$. Then we have the chain of isomorphisms

$$\begin{aligned} Rf_* f^{-1} F &\simeq Rf_* R\mathcal{H}om(f^! k_Y, f^! F) \\ &\simeq R\mathcal{H}om(Rf_{!!} f^! k_Y, F) \\ &\simeq F. \end{aligned}$$

□

3. Examples of applications.

In this Section we give some example of subanalytic sheaves. Let X be a real analytic manifold, and let X_{sa} be the associated subanalytic site. We first introduce sheaves of $\mathcal{R}$-modules, where $\mathcal{R}$ is a sheaf of k-algebras on X_{sa}. Let $\mathcal{D}_X$ be the sheaf of finite order differential operators on X. We define the $\rho_! \mathcal{D}_X$-modules $\mathcal{O}_X^t$ and $\mathcal{O}_X^{\mathrm{w}}$ of tempered and Whitney holomorphic functions respectively. References are made to [8] for an exposition on sheaves of rings on a Grothendieck topology.

3.1 – *Modules over a $k_{X_{sa}}$-algebra.*

A sheaf of $k_{X_{sa}}$-algebras (or a $k_{X_{sa}}$-algebra, for short) is an object $\mathcal{R} \in \mathrm{Mod}(k_{X_{sa}})$ such that $\Gamma(U;\mathcal{R})$ is a k-algebra for each $U \in \mathrm{Op}(X_{sa})$ and the restriction maps are morphisms of algebras. Let us denote by $\mathcal{R}^{op}$ the opposite $k_{X_{sa}}$-algebra.

Let $\mathcal{R}$ be a $k_{X_{sa}}$-algebra and denote by $\mathrm{Mod}(\mathcal{R})$ the category of sheaves of (left) $\mathcal{R}$-modules. The category $\mathrm{Mod}(\mathcal{R})$ is a Grothendieck category and the family $\{\mathcal{R}_U\}_{U \in \mathrm{Op}^c(X_{sa})}$ is a small system of generators. Moreover the forgetful functor $\mathit{for} : \mathrm{Mod}(\mathcal{R}) \to \mathrm{Mod}(k_{X_{sa}})$ is exact.

In this Section we shall extend some results on $k_{X_{sa}}$-modules, by replacing $k_{X_{sa}}$ with $\mathcal{R}$. Since the formalism is similar to that we developed previously we shall not give proofs. The functors

$$\mathcal{H}om_{\mathcal{R}} : \mathrm{Mod}(\mathcal{R})^{op} \times \mathrm{Mod}(\mathcal{R}) \to \mathrm{Mod}(k_{X_{sa}}),$$

$$\otimes_{\mathcal{R}} : \mathrm{Mod}(\mathcal{R}^{op}) \times \mathrm{Mod}(\mathcal{R}) \to \mathrm{Mod}(k_{X_{sa}})$$

are well defined. Let us summarize their properties:

- the functor $\mathcal{H}om_{\mathcal{R}}$ is left exact,
- the functor $\otimes_{\mathcal{R}}$ is right exact and commutes with $\varinjlim$.

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a morphism of real analytic manifolds. Let $\mathcal{R}$ be a $k_{Y_{sa}}$-algebra. The functors f^{-1}, f_* and $f_{!!}$ induce functors

$$f^{-1} : \mathrm{Mod}(\mathcal{R}) \to \mathrm{Mod}(f^{-1}\mathcal{R}),$$

$$f_* : \mathrm{Mod}(f^{-1}\mathcal{R}) \to \mathrm{Mod}(\mathcal{R}),$$

$$f_{!!} : \mathrm{Mod}(f^{-1}\mathcal{R}) \to \mathrm{Mod}(\mathcal{R}).$$

Let us summarize their properties:

- the functor f^{-1} is exact and commutes with $\varinjlim$ and $\otimes_{\mathcal{R}}$,
- the functor f_* is left exact and commutes with $\varprojlim$,
- (f^{-1}, f_*) is a pair of adjoint functors,
- the functor $f_{!!}$ is left exact and commutes with filtrant $\varinjlim$.

Now we consider the derived category of sheaves of $\mathcal{R}$-modules.

DEFINITION 3.1.1. *An object $F \in \mathrm{Mod}(\mathcal{R})$ is flat if the functor $\mathrm{Mod}(\mathcal{R}^{op}) \ni G \to G \otimes_{\mathcal{R}} F$ is exact.*

Small direct sums and filtrant inductive limits of flat $\mathcal{R}$-modules are flat. Since the generators of $\mathrm{Mod}(\mathcal{R})$ are flat, then the subcategory of $\mathrm{Mod}(\mathcal{R})$ consisting of flat modules is generating. Thanks to flat objects we can find a left derived functor $\otimes^L_{\mathcal{R}}$ of the tensor product $\otimes_{\mathcal{R}}$.

DEFINITION 3.1.2. *An object $F \in \mathrm{Mod}(\mathcal{R})$ is quasi-injective if its image via the forgetful functor for* $: \mathrm{Mod}(\mathcal{R}) \to \mathrm{Mod}(k_{X_{sa}})$ *is quasi-injective.*

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a real analytic map. Let $\mathcal{R}$ be a $k_{Y_{sa}}$-algebra. As in § 1.5 one can prove that quasi-injective objects are injective with respect to the functors f_* and $f_{!!}$. The functors Rf_* and $Rf_{!!}$ are well defined and projection formula, base change formula and Künneth formula remain valid for $\mathcal{R}$-modules. Moreover hypothesis (2.2) are satisfied and we have

THEOREM 3.1.3. *The functor $Rf_{!!} : D^+(f^{-1}\mathcal{R}) \to D^+(\mathcal{R})$ admits a right adjoint. We denote by $f^! : D^+(\mathcal{R}) \to D^+(f^{-1}\mathcal{R})$ the adjoint functor.*

3.2 – *Sheaves of $\rho_!\mathcal{R}$-modules.*

We will consider the case where the ring is $\rho_!\mathcal{R}$, where $\mathcal{R}$ is a sheaf of k_X-algebras. We will also assume the following hypothesis:

$$\mathcal{R} \ \text{ has finite flat dimension.}$$

The functor $\rho_!$ induces an exact functor $\mathrm{Mod}(\mathcal{R}) \to \mathrm{Mod}(\rho_!\mathcal{R})$ which is left adjoint to $\rho^{-1} : \mathrm{Mod}(\rho_!\mathcal{R}) \to \mathrm{Mod}(\mathcal{R})$. We will still denote by $\rho_!$ that functor. The functor $\rho_* : \mathrm{Mod}(\mathcal{R}) \to \mathrm{Mod}(\rho_!\mathcal{R})$ is well defined too, in fact the morphism $\xi_F \in \mathrm{Hom}_{k_X}(\mathcal{R}, \mathcal{E}nd(F))$ defines a morphism in $\mathrm{Hom}_{k_{X_{sa}}}(\rho_!\mathcal{R}, \mathcal{E}nd(\rho_*F))$. That follows from the chain of isomorphism

$$\begin{aligned}\mathrm{Hom}_{k_{X_{sa}}}(\rho_!\mathcal{R}, \mathcal{E}nd(\rho_*F)) &\simeq \mathrm{Hom}_{k_{X_{sa}}}(\rho_!\mathcal{R}, \rho_*\mathcal{E}nd(F))\\ &\simeq \mathrm{Hom}_{k_X}(\rho^{-1}\rho_!\mathcal{R}, \mathcal{E}nd(F))\\ &\simeq \mathrm{Hom}_{k_X}(\mathcal{R}, \mathcal{E}nd(F)).\end{aligned}$$

We briefly summarize the properties of these functors:

- ρ^{-1} commutes with $\otimes^L_{\mathcal{R}}$, f^{-1} and $Rf_{!!}$,
- $R\rho_*$ commutes with $R\mathcal{H}om_{\mathcal{R}}$ and Rf_*,
- $\rho_!$ commutes with $\otimes^L_{\mathcal{R}}$ and f^{-1}.

Finally we recall the following result (which has been proved in [7])

PROPOSITION 3.2.1. *Denote by* $\widetilde{\mathcal{R}}$ *the presheaf* $U \to \Gamma(\overline{U};\mathcal{R})$, *where* $U \in \mathrm{Op}^c(X_{sa})$. *Suppose that* F *is a presheaf of* $\widetilde{\mathcal{R}}$*-modules and denote by* F^{++} *the sheaf associated to* F. *Then* $F^{++} \in \mathrm{Mod}(\rho_!\mathcal{R})$.

PROOF. Let $U \in \mathrm{Op}(X_{sa})$, and let $r \in \Gamma(\overline{U};\mathcal{R})$. Then r defines a morphism $\Gamma(\overline{V};\mathcal{R}) \otimes \Gamma(V;F) \to \Gamma(V;F)$ for each subanalytic $V \subset U$, hence an endomorphism of $(F^{++})|_{U_{X_{sa}}} \simeq (F|_{U_{X_{sa}}})^{++}$. This morphism defines a morphism of presheaves $\widetilde{\mathcal{R}} \to \mathcal{E}nd(F^{++})$ and $\widetilde{\mathcal{R}}^{++} \simeq \rho_!\mathcal{R}$ by Proposition 1.1.14. Then $F^{++} \in \mathrm{Mod}(\rho_!\mathcal{R})$. □

3.3 – *Some examples of subanalytic sheaves.*

From now on, the base field is $\mathbb{C}$. Let M be a real analytic manifold. One denotes by $\mathcal{C}^\infty_M$ and $\mathcal{D}b_M$ the sheaves of $\mathcal{C}^\infty$ functions and Schwartz's distributions respectively, and by $\mathcal{D}_M$ the sheaf of finite order differential operators with analytic coefficients. As usual, given a sheaf F on M, we set $D'F = R\mathcal{H}om(F, \mathbb{C}_M)$.

In [4] the author defined the functor

$$T\mathcal{H}om(\cdot, \mathcal{D}b_M) : \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)^{op} \to \mathrm{Mod}(\mathcal{D}_M)$$

in the following way: let U be a subanalytic open subset of M and $Z = M \setminus U$. Then the sheaf $T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_M)$ is defined by the exact sequence

$$0 \to \Gamma_Z\mathcal{D}b_M \to \mathcal{D}b_M \to T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_M) \to 0.$$

This functor is exact and extends as a functor in the derived category, from $D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ to $D^b(\mathcal{D}_M)$. Moreover the sheaf $T\mathcal{H}om(F, \mathcal{D}b_M)$ is soft for any $\mathbb{R}$-constructible sheaf F.

DEFINITION 3.3.1. *One denotes by* $\mathcal{D}b^t_M$ *the presheaf of tempered distributions on* M_{sa} *defined as follows:*

$$U \mapsto \Gamma(M;\mathcal{D}b_M)/\Gamma_{M\setminus U}(M;\mathcal{D}b_M).$$

As a consequence of the Łojasievicz's inequalities [11], for $U, V \in \mathrm{Op}(M_{sa})$ the sequence

$$0 \to \mathcal{D}b^t_M(U \cup V) \to \mathcal{D}b^t_M(U) \oplus \mathcal{D}b^t_M(V) \to \mathcal{D}b^t_M(U \cap V) \to 0$$

is exact. Then $\mathcal{D}b^t_M$ is a sheaf on M_{sa}. Moreover it follows by definition that $\mathcal{D}b^t_M$ is quasi-injective.

DEFINITION 3.3.2. *Let Z be a closed subset of M. We denote by* $\mathcal{I}^\infty_{M,Z}$ *the sheaf of* $\mathcal{C}^\infty$ *functions on M vanishing up to infinite order on Z.*

DEFINITION 3.3.3. *A Whitney function on a closed subset Z of M is an indexed family* $F = (F^k)_{k\in\mathbb{N}^n}$ *consisting of continuous functions on Z such that* $\forall m \in \mathbb{N}$, $\forall k \in \mathbb{N}^n$, $|k| \leq m$, $\forall x \in Z$, $\forall \varepsilon > 0$ *there exists a neighborhood U of x such that* $\forall y, z \in U \cap Z$

$$\left| F^k(z) - \sum_{|j+k|\leq m} \frac{(z-y)^j}{j!} F^{j+k}(y) \right| \leq \varepsilon d(y,z)^{m-|k|}.$$

We denote by $W^\infty_{M,Z}$ *the space of Whitney* $\mathcal{C}^\infty$ *functions on Z. We denote by* $\mathcal{W}^\infty_{M,Z}$ *the sheaf* $U \mapsto W^\infty_{U,U\cap Z}$.

In [6] the authors defined the functor

$$\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M : \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \to \mathrm{Mod}(\mathcal{D}_M)$$

in the following way: let U be a subanalytic open subset of M and $Z = M \setminus U$. Then $\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M = \mathcal{I}^\infty_{M,Z}$, and $\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M = \mathcal{W}^\infty_{M,Z}$. This functor is exact and extends as a functor in the derived category, from $D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ to $D^b(\mathcal{D}_M)$. Moreover the sheaf $F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M$ is soft for any $\mathbb{R}$-constructible sheaf F.

DEFINITION 3.3.4. *One denotes by* $\mathcal{C}^{\infty,\mathrm{w}}_M$ *the presheaf of Whitney* $\mathcal{C}^\infty$ *functions on* M_{sa} *defined as follows:*

$$U \mapsto \Gamma(M; \mathcal{H}om(\mathbb{C}_U, \mathbb{C}_M) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M).$$

As a consequence of a result of [12], for $U, V \in \mathrm{Op}(M_{sa})$ the sequence

$$0 \to \mathcal{C}^{\infty,\mathrm{w}}_M(U \cup V) \to \mathcal{C}^{\infty,\mathrm{w}}_M(U) \oplus \mathcal{C}^{\infty,\mathrm{w}}_M(V) \to \mathcal{C}^{\infty,\mathrm{w}}_M(U \cap V)$$

is exact. Then $\mathcal{C}^{\infty,\mathrm{w}}_M$ is a sheaf on M_{sa}.

REMARK 3.3.5. *Let us consider a locally cohomologically trivial (l.c.t.) subanalytic open subset, i.e.* $U \in \mathrm{Op}(M_{sa})$ *satisfying* $D'\mathbb{C}_U \simeq \mathbb{C}_{\overline{U}}$ *and* $D'\mathbb{C}_{\overline{U}} \simeq \mathbb{C}_U$. *Thanks to the triangulation theorem one can prove that l.c.t. open subanalytic subsets form a basis for the topology of* M_{sa}, *and given a* *l.c.t.* $U \in \mathrm{Op}(M_{sa})$ *we have*

$$\begin{aligned}\Gamma(U; \mathcal{C}_M^{\infty,\mathrm{w}}) &= \Gamma(M; \mathcal{H}om(\mathbb{C}_U, \mathbb{C}_M) \overset{\mathrm{w}}{\otimes} \mathcal{C}_M^\infty) \\ &\simeq \Gamma(M; \mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}_M^\infty) \\ &= W^\infty_{M,\overline{U}}.\end{aligned}$$

Remark that $\Gamma(U; \mathcal{D}b_M^t)$ and $\Gamma(U, \mathcal{C}_M^{\infty,\mathrm{w}})$ are $\Gamma(\overline{U}; \mathcal{D}_M)$-modules for each $U \in \mathrm{Op}(M_{sa})$, hence applying Proposition 3.2.1 the sheaves $\mathcal{D}b_M^t$ and $\mathcal{C}_M^{\infty,\mathrm{w}}$ belong to $\mathrm{Mod}(\rho_! \mathcal{D}_M)$.

We have the following result

PROPOSITION 3.3.6. *For each* $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$ *one has the isomorphisms*

$$\begin{aligned}\rho^{-1}\mathcal{H}om(F, \mathcal{D}b_M^t) &\simeq T\mathcal{H}om(F, \mathcal{D}b_M), \\ \rho^{-1}R\mathcal{H}om(F, \mathcal{C}_M^{\infty,\mathrm{w}}) &\simeq D'F \overset{\mathrm{w}}{\otimes} \mathcal{C}_M^\infty.\end{aligned}$$

PROOF. We may reduce to the case $F = \mathbb{C}_U$ with $U \in \mathrm{Op}^c(M_{sa})$. Let $V \in \mathrm{Op}^c(M_{sa})$.

By definition of $T\mathcal{H}om$ we have $\Gamma(V; T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_M)) \simeq$ $\simeq \Gamma(U \cap V; \mathcal{D}b_V^t)$. Let us consider a subanalytic $W \subset\subset V$. The natural morphism $\Gamma(U \cap V; \mathcal{D}b_V^t) \to \Gamma(U \cap W; \mathcal{D}b_M^t)$ defines the morphism

$$\varphi : \Gamma(U \cap V; \mathcal{D}b_V^t) \to \varprojlim_{W \subset\subset V} \Gamma(U \cap W; \mathcal{D}b_M^t) \simeq \Gamma(V;\, \rho^{-1}\Gamma_U \mathcal{D}b_M^t).$$

Since the family $\{W \in \mathrm{Op}^c(M_{sa});\ W \subset\subset V\}$ is a covering of V and $T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_M)$ is a sheaf φ is an isomorphism.

To prove the second isomorphism we shall first prove the isomorphism

$$\rho^{-1}\mathcal{H}om(F, \mathcal{C}_M^{\infty,\mathrm{w}}) \simeq \mathcal{H}om(F, \mathbb{C}_M) \overset{\mathrm{w}}{\otimes} \mathcal{C}_M^\infty \tag{3.1}$$

for $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. We may reduce to the case $F = \mathbb{C}_U$ with U l.c.t. and subanalytic. Let $V \in \mathrm{Op}^c(M_{sa})$ such that V and $U \cap V$ are l.c.t. and let us consider the family $\mathcal{T} = \{W \in \mathrm{Op}^c(M_{sa})$ l.c.t.; $W \subset\subset V,\ W \cap U$ l.c.t.$\}$. The natural morphism $\psi : \Gamma(V; \mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}_M^\infty) \simeq W^\infty_{V, V \cap \overline{U}} \to W^\infty_{M, \overline{U \cap W}} \simeq$ $\Gamma(M; \mathbb{C}_{\overline{U \cap W}} \overset{\mathrm{w}}{\otimes} \mathcal{C}_M^\infty)$ defines the morphism

$$\psi : \Gamma(V; \mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}_M^\infty) \to \varprojlim_{W \in \mathcal{T}} \Gamma(M; \mathbb{C}_{\overline{U \cap W}} \overset{\mathrm{w}}{\otimes} \mathcal{C}_M^\infty) \simeq \Gamma(V;\, \rho^{-1}\Gamma_U \mathcal{C}_M^{\infty,\mathrm{w}}),$$

where the second isomorphism follows since the family $\mathcal{T}$ is cofinal in $\{W \in \mathrm{Op}^c(M_{sa});\ W \subset\subset V\}$. Moreover we have the isomorphism

$$\varprojlim_{W \in \mathcal{T}} \Gamma(M; \mathbb{C}_{\overline{U \cap W}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M) \simeq \varprojlim_{W \in \mathcal{T}} \Gamma(W; \mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M).$$

Since the family $\mathcal{T}$ is a covering of V and $\mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M$ is a sheaf ψ is an isomorphism. Hence we get the desired isomorphism.

Now let $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. We have the chain of morphisms

$$\begin{aligned} D'F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M &\simeq \text{“}\varinjlim_{F' \to F}\text{”}\, \mathcal{H}om(F', \mathbb{C}_M) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M \\ &\simeq \text{“}\varinjlim_{F' \to F}\text{”}\, \rho^{-1}\mathcal{H}om(F', \mathcal{C}^{\infty,\mathrm{w}}_M) \\ &\simeq \rho^{-1}R\mathcal{H}om(F, \mathcal{C}^{\infty,\mathrm{w}}_M), \end{aligned}$$

where $F' \to F$ ranges to the family of qis. By Theorem 2.1.2 we may suppose $F' \in K^b(\mathrm{Mod}_{\mathbb{R}\text{-c}}(k_M))$, then we can restrict to $F' = \mathbb{C}_U$, $U \in \mathrm{Op}^c(M_{sa})$ l.c.t. and the second isomorphism follows from (3.1). The third isomorphism follows since ρ^{-1} is exact. □

Remark 3.3.7. *As a consequence of Proposition* 3.3.6, *given* $U \in \mathrm{Op}(M_{sa})$ *we have* $R\Gamma(U; \mathcal{C}^{\infty,\mathrm{w}}_M) \simeq R\Gamma(M; D'\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M)$. *In particular when* U *is l.c.t.* $R\Gamma(U; \mathcal{C}^{\infty,\mathrm{w}}_M)$ *is concentrated in degree zero since* $\mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M$ *is soft.*

Now let X be a complex manifold, $X_{\mathbb{R}}$ the underlying real analytic manifold and $\overline{X}$ the complex conjugate manifold. One denotes by $\mathcal{O}^t_X$ and $\mathcal{O}^{\mathrm{w}}_X$ the sheaves of tempered and Whitney holomorphic functions respectively which are defined as follows:

$$\begin{aligned} \mathcal{O}^t_X &:= R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X_{\mathbb{R}}}) \\ \mathcal{O}^{\mathrm{w}}_X &:= R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\mathrm{w}}_{X_{\mathbb{R}}}). \end{aligned}$$

By definition, $\mathcal{O}^t_X$ and $\mathcal{O}^{\mathrm{w}}_X$ belong to $D^b(\rho_!\mathcal{D}_X)$. The relation with the functors of temperate and formal cohomology is given by the following result

Proposition 3.3.8. *For each* $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ *one has the isomorphisms*

$$\begin{aligned} \rho^{-1}R\mathcal{H}om(F, \mathcal{O}^t_X) &\simeq T\mathcal{H}om(F, \mathcal{O}_X), \\ \rho^{-1}R\mathcal{H}om(F, \mathcal{O}^{\mathrm{w}}_X) &\simeq D'F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X. \end{aligned}$$

PROOF. We have the chain of isomorphisms

$$\begin{aligned}\rho^{-1}R\mathcal{H}om(F,\mathcal{O}^t_X) &\simeq \rho^{-1}R\mathcal{H}om(F,R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{O}_{\overline{X}},\mathcal{D}b^t_{X_\mathbb{R}}))\\ &\simeq \rho^{-1}R\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{O}_{\overline{X}},R\mathcal{H}om(F,\mathcal{D}b^t_{X_\mathbb{R}}))\\ &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_{\overline{X}},\rho^{-1}R\mathcal{H}om(F,\mathcal{D}b^t_{X_\mathbb{R}}))\\ &\simeq R\mathcal{H}om(\mathcal{O}_{\overline{X}},T\mathcal{H}om(F,\mathcal{D}b^t_{X_\mathbb{R}}))\\ &\simeq T\mathcal{H}om(F,\mathcal{O}_X).\end{aligned}$$

The proof of $\rho^{-1}R\mathcal{H}om(F,\mathcal{O}^{\mathrm{w}}_X)\simeq D'F\overset{\mathrm{w}}{\otimes}\mathcal{O}_X$ is similar. □

A. Appendix.

A.1 – *Review on subanalytic sets.*

We recall briefly some properties of subanalytic subsets. References are made to [1] and [10]. Let X be a real analytic manifold.

DEFINITION A.1.1. *Let A be a subset of X.*

(i) *A is said to be semi-analytic if it is locally analytic, i.e. each $x\in A$ has a neighborhood U_x such that $X\cap U_x=\cup_{i\in I}\cap_{j\in J}X_{ij}$, where I,J are finite sets and either $X_{ij}=\{y\in U_x; f_{ij}>0\}$ or $X_{ij}=\{y\in U_x; f_{ij}=0\}$ for some analytic function f_{ij}.*

(ii) *A is said to be subanalytic if it is locally a projection of a relatively compact semi-analytic subset, i.e. each $x\in A$ has a neighborhood U such that there exists a real analytic manifold Y and a relatively compact semi-analytic subset $A'\subset X\times Y$ satisfying $X\cap U=\pi(A')$, where $\pi:X\times Y\to X$ denotes the projection.*

(iii) *Let Y be a real analytic manifold. A continuous map $f:X\to Y$ is subanalytic if its graph is subanalytic in $X\times Y$.*

Let us recall some result on subanalytic subsets.

PROPOSITION A.1.2. *Let A,B be subanalytic subsets of X. Then $A\cup B$, $A\cap B$, $\overline{A}$, ∂A and $A\setminus B$ are subanalytic.*

PROPOSITION A.1.3. *Let A be a subanalytic subsets of X. Then the connected components of A are locally finite.*

PROPOSITION A.1.4. *Let $f: X \to Y$ be a subanalytic map. Let A be a relatively compact subanalytic subset of X. Then $f(A)$ is subanalytic.*

DEFINITION A.1.5. *A simplicial complex (K, Δ) is the data consisting of a set K and a set Δ of subsets of K satisfying the following axioms:*

S1 *any $\sigma \in \Delta$ is a finite and non-empty subset of K,*

S2 *if τ is a non-empty subset of an element σ of Δ, then τ belongs to Δ,*

S3 *for any $p \in K$, $\{p\}$ belongs to Δ,*

S4 *for any $p \in K$, the set $\{\sigma \in \Delta; p \in \sigma\}$ is finite.*

If (K, Δ) is a simplicial complex, an element of K is called a vertex. Let $\mathbb{R}^K$ be the set of maps from K to $\mathbb{R}$ equipped with the product topology. To $\sigma \in \Delta$ one associate $|\sigma| \subset \mathbb{R}^K$ as follows:

$$|\sigma| = \Big\{x \in \mathbb{R}^K;\ x(p) = 0 \text{ for } p \notin \sigma,\ x(p) > 0 \text{ for } p \in \sigma \text{ and } \sum_p x(p) = 1\Big\}.$$

As usual we set:

$$|K| = \bigcup_{\sigma \in \Delta} |\sigma|,$$

$$U(\sigma) = \bigcup_{\tau \in \Delta, \tau \supset \sigma} |\tau|,$$

and for $x \in |K|$:

$$U(x) = U(\sigma(x)),$$

where $\sigma(x)$ is the unique simplex such that $x \in |\sigma|$.

THEOREM A.1.6. *Let $X = \bigsqcup_{i \in I} X_i$ be a locally finite partition of X consisting of subanalytic subsets. Then there exists a simplicial complex (K, Δ) and a subanalytic homeomorphism $\psi : |K| \xrightarrow{\sim} X$ such that*

(i) *for any $\sigma \in \Delta$, $\psi(|\sigma|)$ is a subanalytic submanifold of X,*

(ii) *for any $\sigma \in \Delta$ there exists $i \in I$ such that $\psi(|\sigma|) \subset X_i$.*

A.2 – *Sheaves on Grothendieck topologies.*

We recall the definition of a Grothendieck topology. We will not treat the most general case, for which we refer to [8]. We will follow the presentation of [14] and [7].

Let $\mathcal{C}$ be a category admitting finite products and fiber products, and given $U \in \mathcal{C}$, denote by $\mathcal{C}_U$ the category of arrows $V \to U$. Given a morphism $V \to U$ and $S \subset \mathrm{Ob}(\mathcal{C}_U)$, one denotes by $V \times_U S \subset \mathrm{Ob}(\mathcal{C}_V)$ the subset defined by $\{V \times_U W \to V;\ W \in S\}$.

DEFINITION A.2.1. *If, $S_1, S_2 \subset \mathrm{Ob}(\mathcal{C}_U)$, one says that S_1 is a refinement of S_2 ($S_1 \preceq S_2$ for short) if any $V_1 \to U$ in S_1 factorizes as $V_1 \to V_2 \to U$ with $V_2 \to U \in S_2$.*

DEFINITION A.2.2. *A Grothendieck topology on $\mathcal{C}$ associates to each $U \in \mathcal{C}$ a family $\mathrm{Cov}(U) \subset \mathrm{Ob}(\mathcal{C}_U)$ satisfying the following axioms:*

GT1 $\{U \xrightarrow{id} U\} \in \mathrm{Cov}(U)$,

GT2 if $\mathrm{Cov}(U) \ni S_1 \preceq S_2 \subset \mathrm{Ob}(\mathcal{C}_U)$, *then* $S_2 \in \mathrm{Cov}(U)$,

GT3 if $S \in \mathrm{Cov}(U)$, *then for each* $V \to U$, $V \times_U S \in \mathrm{Cov}(V)$,

GT4 if $S_1, S_2 \subset \mathrm{Ob}(\mathcal{C}_U)$, $S_1 \in \mathrm{Cov}(U)$ *and* $V \times_U S_2 \in \mathrm{Cov}(V)$, *then* $S_2 \in \mathrm{Cov}(U)$.

An object $S \in \mathrm{Cov}(U)$ is called a covering of U.

DEFINITION A.2.3. *A site X is a category $\mathcal{C}_X$ endowed with a Grothendieck topology.*

Let $\mathcal{C}_X$ and $\mathcal{C}_Y$ be two categories admitting finite products and fiber products. A functor of sites $f : X \to Y$ is a functor $f^t : \mathcal{C}_Y \to \mathcal{C}_X$ which commutes with fiber products and such that if $U \in \mathcal{C}_Y$ and $S \in \mathrm{Cov}(U)$, then $f^t(S) \in \mathrm{Cov}(f^t(U))$.

Now let k be a field.

DEFINITION A.2.4. *Let X be a site. A presheaf of k-modules on X is a functor $\mathcal{C}_X^{op} \to \mathrm{Mod}(k)$.*

One denotes by $\mathrm{Psh}(k_X)$ the abelian category of presheaves of k-modules on X. Let $F \in \mathrm{Psh}(k_X)$, let $U \in \mathcal{C}_X$ and consider $V \to U \in \mathcal{C}_U$. The restriction morphism $F(U) \to F(V)$ is denoted by $s \mapsto s|_U$.

Let F be a presheaf of k-modules on X and let $S \subset \mathrm{Ob}(\mathcal{C}_U)$. One defines

$$F(S) = \ker\Big(\prod_{V \in S} F(V) \rightrightarrows \prod_{V', V'' \in S} F(V' \times_U V'')\Big)$$

Definition A.2.5. *A presheaf F of k-modules on X is a separated presheaf (resp. a sheaf) if for each $U \in \mathcal{C}_X$ and each $S \in \mathrm{Cov}(U)$ the morphism $F(U) \to F(S)$ is a monomorphism (resp. an isomorphism).*

One denotes by $\mathrm{Mod}(k_X)$ the category of sheaves of k-modules on X. We set for short Hom_{k_X} instead of $\mathrm{Hom}_{\mathrm{Mod}(k_X)}$.

We recall the construction of a sheaf associated to a presheaf. The relation "$\preceq$" defines a preorder on $\mathrm{Cov}(U)$, $U \in \mathcal{C}_X$. Let $F \in \mathrm{Psh}(k_X)$, one defines the functor $(\cdot)^+ : \mathrm{Psh}(k_X) \to \mathrm{Psh}(k_X)$ in the following way. For each $U \in \mathcal{C}_X$

$$F^+(U) = \varinjlim_{S \in \mathrm{Cov}(U)} F(S)$$

Theorem A.2.6. (i) *The functor $(\cdot)^+ : \mathrm{Psh}(k_X) \to \mathrm{Psh}(k_X)$ is left exact,*

(ii) *if $F \in \mathrm{Psh}(k_X)$, then F^+ is separated,*

(iii) *if $F \in \mathrm{Psh}(k_X)$ is separated, then $F^+ \in \mathrm{Mod}(k_X)$,*

(iv) *the functor $(\,\cdot\,)^{++} : \mathrm{Psh}(k_X) \to \mathrm{Mod}(k_X)$ is exact,*

(v) *let $F \in \mathrm{Psh}(k_X)$ and $G \in \mathrm{Mod}(k_X)$, one has the adjunction formula:*

$$\mathrm{Hom}_{\mathrm{Psh}(k_X)}(F, \imath G) \simeq \mathrm{Hom}_{k_X}(F^{++}, G),$$

where $\imath$ denotes the embedding functor.

Let $F \in \mathrm{Psh}(k_X)$, the sheaf F^{++} is called the sheaf associated to F.

Proposition A.2.7. *Let $F, G \in \mathrm{Mod}(k_X)$. A morphism $\varphi \in \mathrm{Hom}_{k_X}(F, G)$ is an epimorphism if and only if for each $U \in \mathcal{C}_X$ there exists $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ such that for each $s \in G(U)$ there exists $t_i \in F(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ for each i.*

Let $f : X \to Y$ be a morphism of sites. Let $F \in \mathrm{Psh}(k_X)$ and $G \in$ $\in \mathrm{Psh}(k_Y)$. One defines the functors

$$f_* : \mathrm{Psh}(k_X) \to \mathrm{Psh}(k_Y) \tag{A.1}$$

$$f^{\leftarrow} : \mathrm{Psh}(k_Y) \to \mathrm{Psh}(k_X) \tag{A.2}$$

in the following way: let $U \in \mathcal{C}_X$ and $V \in \mathcal{C}_Y$, then

$$f_* F(V) = F(f^t(V))$$

$$f^{\leftarrow} F(U) = \varinjlim_{U \to f^t(W)} G(W),$$

where $W \in \mathcal{C}_Y$.

DEFINITION A.2.8. *Let $f : X \to Y$ be a functor of sites*

(i) *the functor of direct image $f_* : \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_Y)$ is the functor induced by* (A.1),

(ii) *the functor of inverse image $f^{-1} : \mathrm{Mod}(k_Y) \to \mathrm{Mod}(k_X)$ is defined by $f^{-1} = (f^{\leftarrow}(\,\cdot\,))^{++}$.*

PROPOSITION A.2.9. (i) *The functor f_* is left exact and commutes with $\varprojlim$,*

(ii) *the functor f^{-1} is exact and commutes with $\varinjlim$,*

(iii) *(f^{-1}, f_*) is a pair of adjoint functors.*

DEFINITION A.2.10. *Let X be a site and let $F, G \in \mathrm{Mod}(k_X)$.*

(i) *One denotes by $\mathcal{H}om(F, G)$ the sheaf $U \mapsto \mathrm{Hom}_{k_U}(F|_U, G|_U)$,*

(ii) *one denotes by $F \otimes G$ the sheaf associated to the presheaf $U \mapsto F(U) \otimes G(U)$.*

PROPOSITION A.2.11. Let $F \in \mathrm{Mod}(k_X)$, $G, G' \in \mathrm{Mod}(k_Y)$.

(i) $\mathcal{H}om_{k_Y}(G, f_* F) \simeq f_* \mathcal{H}om_{k_X}(f^{-1}G, F)$,

(ii) $f^{-1}(G \otimes G') \simeq f^{-1}G \otimes f^{-1}G'$.

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Manoscritto pervenuto in redazione il 14 maggio 2007.