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A Threefold with $p_g = 0$ and $P_2 = 2$

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ABSTRACT - We construct a nonsingular threefold X with $q_1 = q_2 = p_g = 0$ and $P_2 = 2$ whose m-canonical transformation $\varphi_{|mK_X|}$ has the following properties

i) $\varphi_{|mK_X|}$ has the generic fiber of dimension ≥ 1, for $2 \leq m \leq 5$;
ii) it is generically a tranformation $2:1$, for $6 \leq m \leq 8$ and $m = 10$;
iii) it is birational for $m = 9$ and $m \geq 11$.

So, we have a gap for $m = 10$ in the birationality of $\varphi_{|mK_X|}$.

Introduction.

In the classification of nonsingular varieties X of general type, the m-canonical tranformation $\varphi_{|mK_X|}$, where K_X is a canonical divisor on X, plays an important part. The main problem concerning $\varphi_{|mK_X|}$ regards its birationality. The property of $\varphi_{|mK_X|}$ to have the generic fiber given by a finite set of points is important too.

In the case where X is a threefold, Meng Chen has given several limitations for the birationality of $\varphi_{|mK_X|}$. In the particular case where X has the geometric genus $p_g \geq 2$, Chen ([*Che*$_2$], [*Che*$_3$]) proved that:

if $p_g \geq 4$, then $\varphi_{|mK_X|}$ is birational for $m \geq 5$;
if $p_g = 3$, then $\varphi_{|mK_X|}$ is birational for $m \geq 6$;
if $p_g = 2$, then $\varphi_{|mK_X|}$ is birational for $m \geq 8$.

Such limitations are optimal, as demonstrated by examples costructed by Chen himself [*Che*$_2$] if $p_g \geq 4$, by S. Chiaruttini - R. Gattazzo ([*CG*]) if $p_g = 3$, by S. Chiaruttini ([*Chi*]) and by C. Hacon, considering an example of M. Reid [*Re*], if $p_g = 2$ (see [*Che*$_3$]).

In the case of $p_g = 1$ and $p_g = 0$, we have only partial results and the

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problem of finding an optimal limitation for the birationality of $\varphi_{|mK_X|}$ remains ([Che_1]). If $p_g = 1$ and the bigenus of X is $P_2 = 2$, then a Chen-Zuo's limitation ([CZ]) states that $\varphi_{|mK_X|}$ is birational for $m \geq 11$. We costructed ([S_4]) a threefold X with $q_1 = q_2 = 0$ (where q_1 and q_2 are the first and second irregularities of X) $p_g = 1$ and $P_2 = 2$ such that $\varphi_{|mK_X|}$ is birational if and only if $m \geq 11$, (cf. also X_{22} in [Re], p. 359, and [F]); so the above limitation is optimal.

As for threefolds with $p_g = 0$, we tried to find examples of X with $q_1 = q_2 = 0$, $P_2 = 2$ and with the birationality of $\varphi_{|mK_X|}$ for m large. The results obtained were worse than expected as regards the birationality of $\varphi_{|mK_X|}$, while an interesting result emerged for the gaps in the birationality of $\varphi_{|mK_X|}$. Having obtained the birationality of $\varphi_{|mK_X|}$ if and only if $m \geq 11$ in the case of $p_g = 1$ and $P_2 = 2$, the expected result in the new case of $p_g = 0$ and $P_2 = 2$ is birationality if and only if $m > 11$. Instead, all our constructions of threefolds X with $q_1 = q_2 = p_g = 0$ and $P_2 = 2$ have the 9-canonical transformation $\varphi_{|9K_X|}$, which is birational, but some of them also have $\varphi_{|10K_X|}$, which is not birational, and $\varphi_{|mK_X|}$, which is birational if and only if $m = 9$ and $m \geq 11$.

So, the threefolds with this property have a gap in the birationality of $\varphi_{|mK_X|}$ for $m = 10$. This came as a surprise because the only cases of gaps in the birationality of $\varphi_{|mK_X|}$ that we found were in threefolds with $q_1 = q_2 = p_g = P_2 = P_3 = 0$ or $q_1 = q_2 = p_g = P_2 = 0$. Such examples with gaps are in [S_3], where an example is constructed with the same properties as the example X_{46} in Reid's list ([Re]), and in [Ro_2].

In the present paper, we construct a threefold X with the properties described – i.e. $\varphi_{|mK_X|}$ is birational if and only if $m = 9$ and $m \geq 11$, $q_1 = q_2 = 0$ and $p_g = 0$, $P_2 = 2$ – and with further plurigenera $P_3 = 2$, $P_4 = P_5 = 4$, $P_6 = P_7 = 8$, $P_8 = 13$, $P_9 = 15$, $P_{10} = 19$, $P_{11} = 22$.

We note that X is birationally distinct from the threefolds appearing in the lists of [Re], pp. 358-359 and [F], pp. 151-154, 169-170, because X has different plurigenera from those of the threefolds in said lists.

The example X is constructed as a desingularization of a degree six hypersurface $V \subset \mathbb{P}^4$ endowed with a singularity at each of the five vertices A_0, A_1, A_2, A_3 and A_4 of the fundamental pentahedron. The construction is similar to those in [S_4]. Precisely, we put a triple point with an infinitely-near double surface at A_0 on V, we put a triple point with an infinitely-near triple curve at A_1, A_2, A_3, and an ordinary 4-ple point at A_4. Other unimposed singularities appear on V, but they do not affect the birational invariants of X.

The ground field k is an algebraically closed field of characteristic zero, which we can assume to be the field of complex numbers.

1. Imposing singularities on a degree six hypersurface V in $\mathbb{P}^4$.

Let (x_0,x_1,x_2,x_3,x_4) be homogeneous coordinates in $\mathbb{P}^4$ and let us indicate as $f_6(X_0,X_1,X_2,X_3,X_4)$ a form (homogeneous polynomial) of degree 6, in the variables X_0,X_1,X_2,X_3,X_4, defining a hypersurface $V\subset\mathbb{P}^4$ of degree six. We impose a triple point on V at each of the four vertices $A_0=(1,0,0,0,0)$, $A_1=(0,1,0,0,0)$, $A_2=(0,1,0,0,0)$, $A_3=(0,0,0,1,0)$ and an ordinary 4-ple (quadruple) point at $A_4=(0,0,0,0,1)$ of the fundamental pentahedron $X_0X_1X_2X_3X_4=0$.

The equation for V, with the imposed singularities, is of the following type

$$V: f_6(X_0,X_1,X_2,X_3,X_4)$$

$$=X_0^3(a_{33000}X_1^3+\cdots)+X_1^3(a_{23100}X_0^2X_2+\cdots)+X_2^3(\cdots)+X_3^3(\cdots)+X_4^2(\cdots)$$

$$+a_{22200}X_0^2X_1^2X_2^2+a_{22110}X_0^2X_1^2X_2X_3+\cdots+a_{00222}X_2^2X_3^2X_4^2=0,$$

where $a_{ijkhl}\in k$ denotes the coefficient of the monomial $X_0^iX_1^jX_2^kX_3^hX_4^l$.

Moreover, we impose a double surface $\mathcal{S}_0$ infinitely near A_0 in the first neighbourhood. We impose the same double surface $\mathcal{S}_0$, which is locally isomorphic to a plane as in [S_2]. In addition, we impose a triple curve $\mathcal{C}_i$ infinitely near A_i, $i=1,2,3$ in the first neighbourhood. $\mathcal{C}_i$ is locally isomorphic to a straight line as in [S_1].

As an example, we provide a few details on the realization of the singularity at A_0 on V. This will also enable a better understanding in the sequel of the computation of the m-canonical adjoints to V and of the m-genus P_m of a desingularization X of V, $\sigma: X\longrightarrow V$ (cf. section 5). Let us consider the affine open set $U_0\ni A_0$ in $\mathbb{P}^4$ given by $X_0\neq 0$ of affine coordinates $\left(x=\dfrac{X_1}{X_0}, y=\dfrac{X_2}{X_0}, z=\dfrac{X_3}{X_0}, t=\dfrac{X_4}{X_0}\right)$. The affine equation of $V\cap U_0$ is given by $f_6(1,x,y,z,t)=0$.

The affine coordinates of A_0 are $(0,0,0,0)$, so the blow-up of $\mathbb{P}^4$ at the point A_0 is locally given by the formulas:

$$\mathcal{B}_{x_1}:\begin{cases}x=x_1\\ y=x_1y_1\\ z=x_1z_1\\ t=x_1t_1\end{cases};\quad \mathcal{B}_{y_2}:\begin{cases}x=x_2y_2\\ y=y_2\\ z=y_2z_2\\ t=y_2t_2\end{cases};\quad \mathcal{B}_{z_3}:\begin{cases}x=x_3z_3\\ y=y_3z_3\\ z=z_3\\ t=z_3t_3\end{cases};\quad \mathcal{B}_{t_4}:\begin{cases}x=x_4t_4\\ y=y_4t_4\\ z=z_4t_4\\ t=t_4\end{cases}$$

and we consider $\mathcal{B}_{t_4}$. The strict (or proper) transform V' of V with respect to the local blow-up $\mathcal{B}_{t_4}$ has an affine equation given by

- $V' : \dfrac{1}{t_4^3} f_6(1, x_4t_4, y_4t_4, z_4t_4, t_4) = a_{31200}\, x_4 y_4^2 + \cdots + a_{00222}\, y_4^2 z_4^2 t_4^3 = 0.$

On this threefold V' we impose the plane $\mathcal{S}_0 \cap U_0$ given affinely by $\begin{cases} x_4 = 0 \\ t_4 = 0 \end{cases}$ as a singular plane of multiplicity two (i.e. as a double plane). The conditions on the coefficients a_{ijkhl}, such that V has the double plane $\mathcal{S}_0 \cap U_0$ infinitely near A_0, are given by

$$\begin{array}{llll}
a_{31200} = 0 & a_{30210} = 0 & a_{30012} = 0 & a_{20220} = 0 \\
a_{31110} = 0 & a_{30201} = 0 & a_{30003} = 0 & a_{20211} = 0 \\
a_{31101} = 0 & a_{30120} = 0 & a_{20310} = 0 & a_{20202} = 0 \\
a_{31020} = 0 & a_{30111} = 0 & a_{20301} = 0 & a_{20121} = 0 \\
a_{31011} = 0 & a_{30102} = 0 & a_{20130} = 0 & a_{20112} = 0 \\
a_{31002} = 0 & a_{30030} = 0 & a_{20031} = 0 & a_{20022} = 0. \\
a_{30300} = 0 & a_{30021} = 0 & &
\end{array}$$

In much the same way as above and precisely as in $[S_1]$, we impose a triple curve $\mathcal{C}_i$ infinitely near A_i and in the first neighbourhood, which is locally isomorphic to a straight line, for $i = 1, 2, 3$. Further information on the above singularities can be found in $[S_4]$.

We give the final equation for our hypersurface V after imposing all the above-mentioned singularities. We have chosen several coefficients as equal to zero because they are inessential for the computation of the birational invariants of a desingularization $\sigma : X \longrightarrow V$ of V. The shortest equation with the essential coefficients is

$$\begin{aligned}
V :& f_6(X_0, X_1, X_2, X_3, X_4) \\
&= a_{33000}\, X_0^3 X_1^3 + a_{32100}\, X_0^3 X_1^2 X_2 + a_{32001}\, X_0^3 X_1^2 X_4 + a_{23010}\, X_0^2 X_1^3 X_3 + a_{13020}\, X_0 X_1^3 X_3^2 \\
&+ a_{10302}\, X_0 X_2^3 X_4^2 + a_{03030}\, X_1^3 X_3^3 + a_{02031}\, X_1^2 X_3^3 X_4 + a_{01032}\, X_1 X_3^3 X_4^2 + a_{22200}\, X_0^2 X_1^2 X_2^2 \\
&+ a_{22020}\, X_0^2 X_1^2 X_3^2 + a_{22002}\, X_0^2 X_1^2 X_4^2 + a_{21210}\, X_0^2 X_1 X_2^2 X_3 + a_{21201}\, X_0^2 X_1 X_2^2 X_4 + \\
&+ a_{21102}\, X_0^2 X_1 X_2 X_4^2 + a_{21021}\, X_0^2 X_1 X_3^2 X_4 + a_{21012}\, X_0^2 X_2 X_3 X_4^2 + a_{12012}\, X_0 X_1^2 X_3 X_4^2 \\
&+ a_{02022}\, X_1^2 X_3^2 X_4^2 + a_{00222}\, X_2^2 X_3^2 X_4^2 = 0.
\end{aligned}$$

From here on, V denotes this last hypersurface defined by the above form $f_6(X_0, X_1, X_2, X_3, X_4)$ for a generic choice of the parameters a_{ijkhl}. As a reminder of this generic choice, we sometimes call V: the generic V.

2. Imposed and unimposed singularities of *V* : the actual singularities.

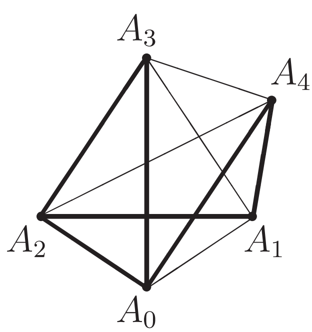
We consider the hypersurface V given at the end of section 1.

New unimposed singularities appear on the (generic) V close to the singularities imposed on V; they are actual or infinitely-near singularities. We call a singularity on V *actual* to distinguish it from those which are infinitely near. We call a singularity of V *unimposed* if it does not appear in the list of singulatities in section 1.

There are six unimposed actual double (straight) lines on V given by A_0A_2, A_0A_3, A_0A_4, A_1A_2, A_1A_4, A_2A_3 and the unimposed double plane cubic $\begin{cases} X_1 = 0 \\ X_2 = 0 \\ a_{01032} X_3^2 X_4 + a_{21021} X_0^2 X_3 + a_{21012} X_0^2 X_4 = 0. \end{cases}$

The generic V has no other actual singularities. It follows that the generic V is reduced, irreducible and normal.

The cubic lies on the plane $\begin{cases} X_1 = 0 \\ X_2 = 0 \end{cases}$, which is simple on V. The picture of the six double lines is as follows, where the double lines are drawn in bold type.



3. The infinitely-near singularities of *V*.

In section 2, we described the actual singularities on V; in the present section, we briefly describe the infinitely-near singularities. Here again, new infinitely-near singularities appear on the generic V alongside the infinitely-near singularities imposed on V. They are only double singular curves and isolated double points, so none of the unimposed singularities (be they actual or otherwise) affect the birational invariants of a desingularization $\sigma : X \longrightarrow V$ of V, such as the irregularities and the plurigenera of X. This means that, in calculating these invariants, we can assume that there are only the imposed singularities on V.

We compute said birational invariants of X using the theory of adjoints and pluricanonical adjoints developed in [S_1]. We can apply this theory because the singularities on the hypersurface V satisfy the hypotheses of [S_1], i.e. it must be possible to resolve the singularities on V with local blow-ups along linear affine subspaces; moreover, the degree six hypersurfaces in $\mathbb{P}^4$ must have singularities of codimension ≥ 2 (i.e. the hypersurfaces must be normal).

Such hypotheses on the singularities are satisfied by either actual or infinitely-near singularities of V. In particular, V is normal (section 2). To be precise, all the singularities of V are resolved with local blow-ups either along straight lines, that are double on V and on strict transforms of V, or along planes containing double curves and points. These planes are simple on V and on strict transforms of V, e.g. the simple plane $\begin{cases} X_1 = 0 \\ X_2 = 0 \end{cases}$, containing the cubic curve on V in section 2.

Having said as much, we only give details on the imposed infinitely-near singularities of V that are needed in the sequel.

From section 1, we already have the information that we need about the triple point A_0 and the double surface $\mathcal{S}_0$ infinitely near A_0.

Next, we consider the triple point A_1 on V and the blow-up at A_1. Let us consider the affine open set $U_1 \ni A_1$ in $\mathbb{P}^4$ given by $X_1 \neq 0$ of affine coordinates $\left(x = \dfrac{X_0}{X_1},\ y = \dfrac{X_2}{X_1},\ z = \dfrac{X_3}{X_1},\ t = \dfrac{X_4}{X_1}\right)$. The affine equations of $V \cap U_1$ are given by $f_6(x, 1, y, z, t) = 0$. The affine coordinates of A_1 are $(0, 0, 0, 0)$.

We can assume that the blow-up at A_1 is the first to be performed, so we can use the local blows-up $\mathcal{B}_{x_1}, \mathcal{B}_{y_2}, \mathcal{B}_{z_3}, \mathcal{B}_{t_4}$ in section 1.

The strict transform of $V \cap U_1$, with respect to $\mathcal{B}_{t_4}$, is given by

- $$V'_{t_4} : \frac{1}{t_4^3} f_6(x_4 t_4, 1, y_4 t_4, z_4 t_4, t_4) = a_{33000}\, x_4^3 + \cdots + a_{03030} z_4^3 + \cdots + a_{12012}\, x_4 z_4 t_4 + \cdots = 0.$$

We are interested in the triple curve infinitely near A_1. So, we focus locally on the triple line on V'_{t_4} belonging to the exceptional divisor $t_4 = 0$ of the local blow-up $\mathcal{B}_{t_4}$. This triple line is given by $\begin{cases} x_4 = 0 \\ z_4 = 0 \\ t_4 = 0 \end{cases}$.

Let us go on to consider the triple point A_2 on V, the blow-up at A_2 and the affine open set $U_2 \ni A_2$ in $\mathbb{P}^4$ given by $X_2 \neq 0$ of affine coordinates

$\left(x = \dfrac{X_0}{X_2}, \; y = \dfrac{X_1}{X_2}, \; z = \dfrac{X_3}{X_2}, \; t = \dfrac{X_4}{X_2}\right)$. The affine equations of $V \cap U_2$ are given by $f_6(x, y, 1, z, t) = 0$. The affine coordinates of A_2 are $(0, 0, 0, 0)$.

Here again, we can assume that the blow-up at A_2 is the first to be performed, so we can use the local blow-ups $\mathcal{B}_{x_1}, \mathcal{B}_{y_2}, \mathcal{B}_{z_3}, \mathcal{B}_{t_4}$ in section 1.

The strict transform of $V \cap U_2$, with respect to $\mathcal{B}_{y_2}$, is given by

- $V'_{y_2} : \dfrac{1}{y_2^3} f_6(x_2 y_2, y_2, 1, y_2 z_2, y_2 t_2)$

$$= a_{10302}\, x_2 t_2^2 + \cdots + a_{22200}\, x_2^2 y_2 + \cdots + a_{00222}\, y_2 z_2^2 t_2^2 = 0.$$

We are interested in the triple curve infinitely near A_2, so we focus locally on the triple line on V'_{y_2} belonging to the exceptional divisor $y_2 = 0$ of the local blow-up $\mathcal{B}_{y_2}$. This triple line is given by $\begin{cases} x_2 = 0 \\ y_2 = 0 \\ t_2 = 0 \end{cases}$.

Finally, let us consider the triple point A_3 on V, the blow-up at A_3 and the affine open set $U_3 \ni A_3$ in $\mathbb{P}^4$ given by $X_3 \neq 0$ of affine coordinates $\left(x = \dfrac{X_0}{X_3}, \; y = \dfrac{X_1}{X_3}, \; z = \dfrac{X_2}{X_3}, \; t = \dfrac{X_4}{X_3}\right)$. The affine equations of $V \cap U_3$ are given by $f_6(x, y, z, 1, t) = 0$. The affine coordinates of A_3 are $(0, 0, 0, 0)$.

We can again assume that the blow-up at A_3 is the first to be performed, so we can use the local blow-ups $\mathcal{B}_{x_1}, \mathcal{B}_{y_2}, \mathcal{B}_{z_3}, \mathcal{B}_{t_4}$ in section 1.

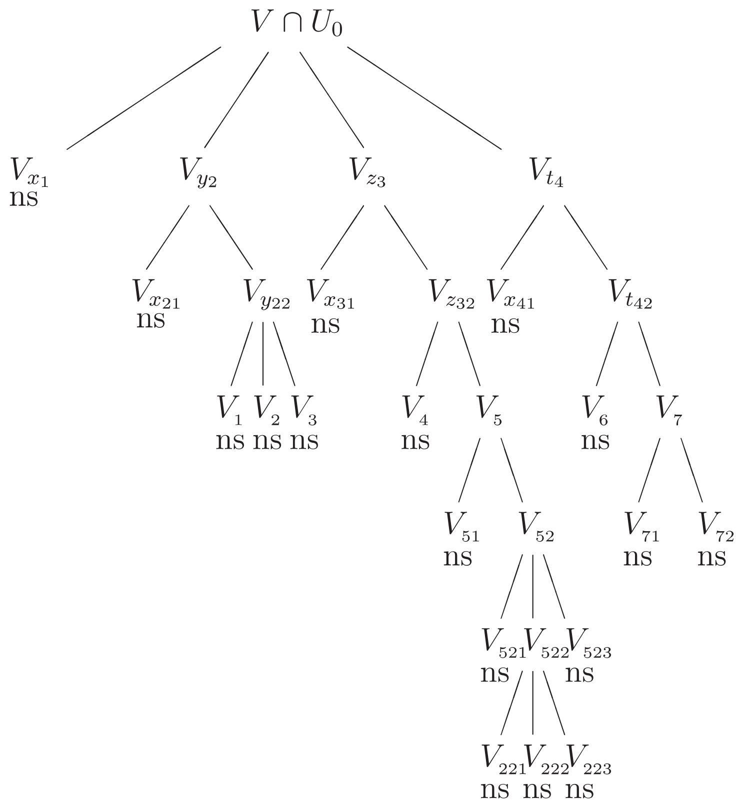
The strict transform of $V \cap U_3$, with respect to $\mathcal{B}_{x_1}$, is given by

- $V'_{x_1} : \dfrac{1}{x_1^3} f_6(x_1, x_1 y_1, x_1 z_1, 1, x_1 t_1)$

$$= a_{03030}\, y_1^3 + \cdots + a_{22020}\, x_1 y_1^2 + \cdots + a_{21021}\, x_1 y_1 t_1 + \cdots = 0.$$

We are interested in the triple curve infinitely near A_3, so we focus locally on the triple line on V'_{x_1} belonging to the exceptional divisor $x_1 = 0$ of the local blow-up $\mathcal{B}_{x_1}$. This triple line is given by $\begin{cases} x_1 = 0 \\ y_1 = 0 \\ t_1 = 0 \end{cases}$.

To end this section, we add one more item of information, drawing the picture of the tree of local blow-ups resolving the singularity at A_0 and those infinitely near.



where “ns” means “‘nonsingular”.

4. The m-canonical adjoints to $V \subset \mathbb{P}^4$.

Let

$$\mathbb{P}_r \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_3} \mathbb{P}_2 \xrightarrow{\pi_2} \mathbb{P}_1 \xrightarrow{\pi_1} \mathbb{P}_0 = \mathbb{P}^4$$

be a sequence of blow-ups solving the singularities of V.

If we call $V_i \subset \mathbb{P}_i$ the *strict transform* of V_{i-1} with respect to π_i, then the above sequence gives us

$$X = V_r \xrightarrow{\pi'_r} \cdots \xrightarrow{\pi'_3} V_2 \xrightarrow{\pi'_2} V_1 \xrightarrow{\pi'_1} V_0 = V,$$

where $\pi'_i = \pi_{i|V_i} : V_i \longrightarrow V_{i-1}$ and $\sigma_{|_X} : X \longrightarrow V$, $\sigma = \pi_r \circ \cdots \circ \pi_1$, is a desingularization of $V \subset \mathbb{P}^4$.

Let us assume that π_i is a blow-up along a subvariety Y_{i-1} of $\mathbb{P}_{i-1}$, of dimension j_{i-1}, which can be either a singular or a nonsingular subvariety of $V_{i-1} \subset \mathbb{P}_{i-1}$ (i.e. Y_{i-1} is a locus of singular or simple points of V_{i-1}). Let m_{i-1} be the multiplicity of the variety Y_{i-1} on V_{i-1}.

Let us set $n_{i-1} = -3 + j_{i-1} + m_{i-1}$, for $i = 1, ..., r$ and $\deg(V) = d$.

A hypersurface $\Phi_{m(d-5)}$ of degree $m(d-5)$, $m \geq 1$, in $\mathbb{P}^4$ is an *m-canonical adjoint* to V (with respect to the sequence of blow-ups $\pi_1, ..., \pi_r$) if the restriction to X of the divisor

$$D_m = \pi_r^*\{\pi_{r-1}^*[\cdots \pi_1^*(\Phi_{m(d-5)}) - mn_0E_1 \cdots] - mn_{r-2}E_{r-1}\} - mn_{r-1}E_r$$

is effective, i.e. $D_{m|_X} \geq 0$, where $E_i = \pi^{-1}(Y_{i-1})$ is the exceptional divisor of π_i and $\pi_i^* : Div(\mathbb{P}_{i-1}) \longrightarrow Div(\mathbb{P}_i)$ is the homomorphism of the Cartier (or locally principal) divisor groups (cf. [S_1], sections 1,2).

An m-canonical adjoint $\Phi_{m(d-5)}$ is an *global* m-canonical adjoint to V (with respect to $\pi_1, ..., \pi_r$) if the divisor D_m is effective on $\mathbb{P}_r$, i.e. $D_m \geq 0$ (loc. cit.).

Note that, if $\Phi_{m(d-5)}$ is an m-canonical adjoint to V, then $D_{m|_X} \equiv mK$, where '$\equiv$' denotes linear equivalence and K denotes a canonical divisor on X.

In our above example, an order can be established in the sequence of blow-ups, e.g. let us assume that π_1 is the blow-up at the triple point A_0, π_2 is the blow-up along the double surface $\mathcal{S}_0$ infinitely near A_0, π_3 is the blow-up at the triple point A_1, π_4 is the blow-up along the triple curve $\mathcal{C}_1$ infinitely near A_1, π_5 is the blow-up at the triple point A_2, π_6 is the blow-up along the triple curve $\mathcal{C}_2$ infinitely near A_2, π_7 is the blow-up at the triple point A_3, π_8 is the blow-up along the triple curve $\mathcal{C}_3$ infinitely near A_3 and the blow-up π_9 is the one at the 4-ple point A_4.

The example V has degree $d = 6$ and D_m, relative to our X, is given by:

$$(*) \qquad D_m = \pi_r^* \cdots \pi_3^*\{\pi_2^*[\pi_1^*(\Phi_m)] - mE_2\} - mE_4 - mE_6 - mE_8 - mE_9\,,$$

where E_i is the exceptional divisor of the blow-up π_i and, to be more specific, E_1 is the exceptional divisor of the blow-up π_1 at the triple point A_0, E_2 is the exceptional divisor of the blow-up π_2 along $\mathcal{C}_1$, ... and E_9 is the exceptional divisor of the blow-up π_9 at the 4-ple point A_4.

No other exceptional divisors are subtracted in D_m because, as we said before, the unimposed singularities are either actual or infinitely-near double singular curves or isolated double points on our (generic) V. Put more precisely, the exceptional divisors of the blow-ups along the double curves appear with coefficient $n_i = 0$ in the above expression of D_m and the exeptional divisors of the blow-ups along simple planes appear again with coefficient $n_j = 0$. Since we have resolved all the unimposed singularities with blow-ups either along double curves or along simple planes, only the exeptional divisors E_2, E_4, E_6, E_8 and E_9 appear in D_m. Note, moreover, that the exceptional divisor of a blow-up at a triple point also appears with coefficient $n_h = 0$ in D_m.

5. The plurigenera of a desingularization X of V.

Let us consider the equation of $V : f_6(X_0, X_1, X_2, X_3, X_4) = 0$ at the end of section 1 and arrange the form f_6 according to the powers of X_4.

$$(**) \quad f_6 = \varphi_4(X_0, X_1, X_2, X_3)X_4^2 + \varphi_5(X_0, X_1, X_2, X_3)X_4 + \varphi_6(X_0, X_1, X_2, X_3),$$

where $\varphi_i(X_0, X_1, X_2, X_3)$ is a form of degree i in X_0, X_1, X_2, X_3 and precisely

$$\varphi_4(X_0, X_1, X_2, X_3) = a_{10302}\, X_0X_2^3 + a_{01032}\, X_1X_3^3 + a_{22002}\, X_0^2X_1^2 + \cdots + a_{00222}\, X_2^2X_3^2.$$

Next, let us consider the hypersurface Φ_m, appearing in (*) section 4 and assume that its equation is $F_m(X_0, X_1, X_2, X_3, X_4) = 0$, of degree m. Arranging the form F_m according to the powers of X_4, we can write

$$(***) \quad F_m(X_0, X_1, X_2, X_3, X_4) = \psi_s(X_0, X_1, X_2, X_3)X_4^{m-s} + \psi_{s+1}(X_0, X_1, X_2, X_3)X_4^{m-s-1} + \cdots + \psi_m(X_0, X_1, X_2, X_3),$$

where $\psi_j(X_0, X_1, X_2, X_3)$ is a form of degree j in X_0, X_1, X_2, X_3 and s is an integer satisfying $0 \leq s \leq m$.

Under the sole hypothesis that V has a 4-ple point at A_4 the following lemma holds.

LEMMA 1. *With the above notations, if Φ_m is an m-canonical adjoint* (be it global or not), *then, modulo $V : f_6 = 0$, we can assume that $s \geq m - 1$ in* (***); i.e. *if $\Phi_m : F_m = 0$ is an m-canonical adjoint, then we can assume that*

$$F_m = \psi_{m-1}(X_0, X_1, X_2, X_3)X_4 + \psi_m(X_0, X_1, X_2, X_3).$$

Moreover, we have the equality

$$\psi_{m-1}(X_0, X_1, X_2, X_3) = A_{m-5}(X_0, X_1, X_2, X_3)\varphi_4(X_0, X_1, X_2, X_3),$$

where $A_{m-5}(X_0, X_1, X_2, X_3)$ is a form of degree $m - 5$ in X_0, X_1, X_2, X_3 and $\varphi_4(X_0, X_1, X_2, X_3)$ is defined above in (**).

The idea for the proof of the above lemma came from M. C. Ronconi [CR], [Ro_1]. A detailed proof can be found in [S_4] (Lemma 1, section 5).

REMARK 1. In Lemma 1, we have $F_m = A_{m-5}\varphi_4 X_4 + \psi_m$. We see that, if $A_{m-5} = 0$, then F_m defines a global m-canonical adjoint Φ_m to V, whereas if $A_{m-5} \neq 0$, then Φ_m is a "non-global" m-canonical adjoint to V. The non-global m-canonical adjoints to V are important for establishing the birationality of the m-canonical transformation $\varphi_{|mK_X|}$ (see next section).

The following lemma is proved in [S_4], Lemma 2, section 12, where the singularities at three fundamental points on a degree six hypersurface V' differ from those on V in the present case. More precisely, V' has three triple points with an infinitely-near double plane, whereas V has three triple points with an infinitely-near triple curve. But the proof remains the same in both cases.

LEMMA 2. *The m-canonical adjoint to V given by*

$$\Phi_m : A_{m-5}(X_0, X_1, X_2, X_3)\varphi_4(X_0, X_1, X_2, X_3)X_4 + \psi_m(X_0, X_1, X_3, X_4) = 0,$$

has the following property

$$D_{m|_X} \geq 0 \Longleftrightarrow D_m + E_9 \geq 0,$$

where $D_m = \pi_r^* \cdots \pi_3^*\{\pi_2^*[\pi_1^*(\Phi_m)] - mE_2\} - mE_4 - mE_6 - mE_8 - mE_9$, *is defined in* (*), *section* 4.

REMARK 2. Roughly speaking, the result in Lemmas 1 and 2, that permits us an easy computation of the m-genus P_m ($\forall m$) of a desingularization $\sigma : X \longrightarrow V$ of V, is the following. Our degree six hypersurface V has a 4-ple point, so from Lemma 1 we can assume that the m-canonical adjoint Φ_m is defined by a form of the type $F_m = A_{m-5}\varphi_4 X_4 + \psi_m$, where the variable X_4 appears to the power 1. In order to compute the linear conditions given by the other singularities to the hypersurfaces Φ_m so that they are m-canonical adjoints to V, i.e. to obtain $D_{m|_X} \geq 0$, we find that we do not need to restrict D_m to X and, after imposing $D_{m|_X} \geq 0$, we only need to have $D_m + E_9 \geq 0$. This follows from the fact that F_m contains the variable X_4 to the power 1, whereas the form f_6 defining V contains the variable X_4 to the power 2, and also from the particular singularities obtained in our examples. We note that E_9 has to be added to D_m, otherwise D_m may not be effective (when $A_{m-5} \neq 0$, see Remark 1). So it is very easy to compute the conditions on F_m such that $D_m + E_9$ is effective and, since P_m = number of linearly independent forms contained in F_m (cf. [S_1]), the computation of P_m, $\forall m$, is very easy too.

Now, we are ready to compute the plurigenera of a desingularization $\sigma : X \longrightarrow V$ of V. Let us write

$$A_{m-5}(X_0, X_1, X_2, X_3)X_4 = \left(\sum_{i+j+k+h=m-5} a_{ijkh} X_0^i X_1^j X_2^k X_3^h \right) X_4,$$

$$\psi_m(X_0, X_1, X_2, X_3) = \sum_{i'+j'+h'+l'=m} b_{ijkh} X_0^{i'} X_1^{j'} X_2^{k'} X_3^{h'},$$

where $a_{ijkh}, b_{ijkh} \in k$.

•• *First let us consider the two blows-up π_1 and π_2.* We know that the blow-up π_1 of $\mathbb{P}^4$ at A_0 is given by $\mathcal{B}_{x_1}, \mathcal{B}_{y_2}, \mathcal{B}_{z_3}, \mathcal{B}_{t_4}$ (cf. section 1). Let us consider the affine open set $U_0 = \{X_0 \neq 0\}$ as in section 1.

The total transform of $\Phi_m \cap U_0$ with respect to $\mathcal{B}_{t_4}$ is given by

$$\mathcal{B}_{t_4}^*(\Phi_m \cap U_0) : A_{m-5}(1, x_4t_4, y_4t_4, z_4t_4)\varphi_4(1, x_4t_4, y_4t_4, z_4t_4)t_4 +$$

$$\psi_m(1, x_4t_4, y_4t_4, z_4t_4, t_4) = 0.$$

The double surface S_0 infinitely near A_0 in affine coordinates (x_4, y_4, z_4, t_4) is given by $\begin{cases} x_4 = 0 \\ t_4 = 0 \end{cases}$ (cf. section 1).

The blow-up π_2 along $\mathcal{S}_0$ is locally given by the formulas:

$$\mathcal{B}_{x_{41}} : \begin{cases} x_4 = x_{41} \\ y_4 = y_{41} \\ z_4 = z_{41} \\ t_4 = x_{41}t_{41} \end{cases} ; \quad \mathcal{B}_{t_{42}} : \begin{cases} x_4 = x_{42}t_{42} \\ y_4 = y_{42} \\ z_4 = z_{42} \\ t_4 = t_{42} \end{cases} .$$

The total transform of $\mathcal{B}_{t_4}^*(\Phi_m \cap U_0)$ with respect to $\mathcal{B}_{x_{41}}$ is given by

$$\mathcal{B}_{x_{41}}^*[\mathcal{B}_{t_4}^*(\Phi_m \cap U_0)] :$$

$$A_{m-5}(1, x_{41}^2t_{41}, x_{41}y_{41}t_{41}, x_{41}z_{41}t_{41})\varphi_4(1, x_{41}^2t_{41}, x_{41}y_{41}t_{41}, x_{41}z_{41}t_{41})x_{41}t_{41} +$$

$$\psi_m(1, x_{41}^2t_{41}, x_{41}y_{41}t_{41}, x_{41}z_{41}t_{41}, x_{41}t_{41}) = 0.$$

With the above notations, this total transform is given by

$$\mathcal{B}^*_{x_{41}}[\mathcal{B}^*_{t_4}(\Phi_m \cap U_0)] :$$

$$\left(\sum_{i+j+k+h=m-5} a_{ijkh}\, x_{41}^{2j+k+h}\, y_{41}^k z_{41}^h\right)\varphi_4(1, x_{41}^2 t_{41}, x_{41}\, y_{41} t_{41}, x_{41} z_{41} t_{41}) x_{41} t_{41}$$
$$+ \sum_{i'+j'+h'+l'=m} b_{ijkh}\, x_{41}^{2j'+k'+h'}\, y_{41}^{k'} z_{41}^{h'} = 0.$$

The following claims hold true; they are corollaries to Lemma 1 and 2 and consequences of the desingularization of V.

Claim 1. The composition of the two local blows-up $\mathcal{B}_{x_{41}} \circ \mathcal{B}_{t_4}$ coincides, up to isomorphisms, with the desingularization $\sigma_{|_X}$ on the affine open set $V_{x_{41}}$, because $V_{x_{41}}$ is nonsingular (see the tree of blow-ups at the end of section 3). In fact, $V_{x_{41}}$ is isomorphic to an open set on X and the two above morphisms can be identified on $V_{x_{41}}$.

Claim 2. Since Φ_m is an m-canonical adjoint to V, by definition we have $D_{m|_X} \geq 0$; so, from Lemma 2, we can say that: $D_m + E_9 \geq 0$.

Claim 3. From Claims 1 and 2, we deduce (up to isomorphisms) that

$$\mathcal{B}^*_{x_{41}}[\mathcal{B}^*_{t_4}(\Phi_m \cap U_0)] - mE_2 + E_9 \geq 0.$$

This last inequality is equivalent to the following equality of polynomials

$$\left(\sum_{i+j+k+h=m-5} a_{ijkh}\, x_{41}^{2j+k+h}\, y_{41}^k z_{41}^h\right)\varphi_4(1, x_{41}^2 t_{41}, x_{41}\, y_{41} t_{41}, x_{41} z_{41} t_{41}) x_{41} t_{41}$$
$$+ \sum_{i'+j'+h'+l'=m} b_{ijkh}\, x_{41}^{2j'+k'+h'}\, y_{41}^{k'} z_{41}^{h'} = x_{41}^m(\ldots)$$

Claim 4. Since $\varphi_4(1, x_{41}^2 t_{41}, x_{41}\, y_{41} t_{41}, x_{41} z_{41} t_{41}) = x_{41}^3(\ldots)$, the latter equality of polynomials is equivalent to the inequalities

$$\begin{cases} 2j + k + h + 3 + 1 \geq m \\ 2j' + k' + h' \geq m \end{cases}, \quad \text{i.e.} \quad \begin{cases} j \geq i + 1 \\ j' \geq i' \end{cases}.$$

•• *Next, let us consider the two blows-up π_3 and π_4.* As in section 3, we can assume that the first blow-up that we perform is π_3 at A_1, so we can use the local blows-up $\mathcal{B}_{x_1}, \mathcal{B}_{y_2}, \mathcal{B}_{z_3}, \mathcal{B}_{t_4}$ in section 1.

As in the above case of π_1 and π_2, here too for π_3 and π_4, we find that the total transform of $\Phi_m \cap U_1$ with respect to $\mathcal{B}_{t_4}$ is given by

$$\mathcal{B}^*_{t_4}(\Phi_m \cap U_1) : A_{m-5}(x_4t_4, 1, y_4t_4, z_4t_4)\varphi_4(x_4t_4, 1, y_4t_4, z_4t_4)t_4 \\ + \psi_m(x_4t_4, 1, y_4t_4, z_4t_4, t_4) = 0.$$

The triple curve C_1 infinitely near A_1 in affine coordinates (x_4, y_4, z_4, t_4) is given by (section 3) $\begin{cases} x_4 = 0 \\ z_4 = 0 \\ t_4 = 0 \end{cases}$.

The blow-up π_4 along $\mathcal{C}_1$ is locally given by the formulas:

$$\mathcal{B}_{x_{41}} : \begin{cases} x_4 = x_{41} \\ y_4 = y_{41} \\ z_4 = x_{41}z_{41} \\ t_4 = x_{41}t_{41} \end{cases} ; \quad \mathcal{B}_{z_{42}} : \begin{cases} x_4 = x_{42}z_{42} \\ y_4 = y_{42} \\ z_4 = z_{42} \\ t_4 = z_{42}t_{42} \end{cases} ; \quad \mathcal{B}_{t_{43}} : \begin{cases} x_4 = x_{43}t_{43} \\ y_4 = y_{43} \\ z_4 = z_{43}t_{43} \\ t_4 = t_{43} \end{cases} .$$

The total transform of $\mathcal{B}^*_{t_4}(\Phi_m \cap U_1)$ with respect to $\mathcal{B}_{x_{41}}$ is given by

$$\mathcal{B}^*_{x_{41}}[\mathcal{B}^*_{t_4}(\Phi_m \cap U_1)] :$$

$$\left(\sum_{i+j+k+h=m-5} a_{ijkh}\, x_{41}^{2i+k+2h}\, y_{41}^k z_{41}^h \right) \varphi_4(x_{41}^2 t_{41}, 1, x_{41}\, y_{41}t_{41}, x_{41}^2 z_{41}t_{41}) x_{41}t_{41} \\ + \sum_{i'+j'+h'+l'=m} b_{ijkh}\, x_{41}^{2i'+k'+2h'}\, y_{41}^{k'} z_{41}^{h'} = 0.$$

From the analogous four claims written above and from the equality

$$\varphi_4(x_{41}^2 t_{41}, 1, x_{41}\, y_{41}t_{41}, x_{41}^2 z_{41}t_{41}) = x_{41}^4(\ldots),$$

we obtain the inequalities

$$\begin{cases} 2i + k + 2h + 4 + 1 \geq m \\ 2i' + k' + 2h' \geq m \end{cases}, \quad \text{i.e.} \quad \begin{cases} i + h \geq j \\ i' + h' \geq j' \end{cases}.$$

•• *Let us move on now to consider the two blows-up π_5 and π_6.* Once again, we can assume that the blow-up π_3 at A_2 is performed first, so we can again use the local blows-up $\mathcal{B}_{x_1}, \mathcal{B}_{y_2}, \mathcal{B}_{z_3}, \mathcal{B}_{t_4}$ in section 1.

As in the above cases, here for π_5 and π_6 we obtain that the total transform of $\Phi_m \cap U_2$, with respect to $\mathcal{B}_{y_2}$, is given by

$$\mathcal{B}^*_{y_2}(\Phi_m \cap U_2) : A_{m-5}(x_2y_2, y_2, 1, y_2z_2)\varphi_4(x_2y_2, y_2, 1, y_2z_2)y_2t_2 \\ + \psi_m(x_2y_2, y_2, 1, y_2z_2, y_2t_2) = 0.$$

The triple curve C_2 infinitely near A_2 in affine coordinates (x_2, y_2, z_2, t_2) is given by (section 3) $\begin{cases} x_2 = 0 \\ y_2 = 0 \\ t_2 = 0 \end{cases}.$

The blow-up π_6 along $\mathcal{C}_2$ is locally given by the formulas:

$$\mathcal{B}_{x_{21}} : \begin{cases} x_2 = x_{21} \\ y_2 = x_{21} y_{21} \\ z_2 = z_{21} \\ t_2 = x_{21} t_{21} \end{cases}; \quad \mathcal{B}_{y_{22}} : \begin{cases} x_2 = x_{22} y_{22} \\ y_2 = y_{22} \\ z_2 = z_{22} \\ t_2 = y_{22} t_{22} \end{cases}; \quad \mathcal{B}_{t_{23}} : \begin{cases} x_2 = x_{23} t_{23} \\ y_2 = y_{23} t_{23} \\ z_2 = z_{23} \\ t_2 = t_{23} \end{cases}.$$

The total transform of $\mathcal{B}^*_{y_2}(\Phi_m \cap U_2)$ with respect to $\mathcal{B}_{x_{21}}$ is given by

$$\mathcal{B}^*_{x_{21}}[\mathcal{B}^*_{y_2}(\Phi_m \cap U_2)] :$$

$$\left(\sum_{i+j+k+h=m-5} a_{ijkh}\, x_{21}^{2i+j+h}\, y_{21}^{j} z_{21}^{h} \right) \varphi_4(x_{21}^2 y_{21}, x_{21} y_{21}, 1, x_{21} z_{21}) x_{21}^2 y_{21} t_{21}$$
$$+ \sum_{i'+j'+h'+l'=m} b_{ijkh}\, x_{21}^{2i'+j'+h'}\, y_{21}^{j'} z_{21}^{h'} = 0.$$

From the same four claims written above, and from the equality

$$\varphi_4(x_{21}^2 y_{21}, x_{21} y_{21}, 1, x_{21} z_{21}) = x_{21}^2(...),$$

we obtain the inequalities

$$\begin{cases} 2i + j + h + 2 + 2 \geq m \\ 2i' + j' + h' \geq m \end{cases}, \quad \text{i.e.} \quad \begin{cases} i \geq k + 1 \\ i' \geq k' \end{cases}.$$

•• *Finally, considering the two blows-up* π_7 *and* π_8, as in the case of π_5 and π_6, we obtain the inequalities

$$\begin{cases} i + 2j + k + 2 + 2 \geq m \\ i' + 2j' + k' \geq m \end{cases}, \quad \text{i.e.} \quad \begin{cases} j \geq h + 1 \\ j' \geq h' \end{cases}.$$

Joining the above inequalities, we obtain

$$(**) \qquad \begin{cases} i + h \geq j \geq i + 1 \geq k + 2, & j \geq h + 1 \\ i' + h' \geq j' \geq i' \geq k', & j' \geq h' \end{cases}.$$

From the inequalities in the first line of (**), we deduce $j \geq 2$, $i \geq 1$, $h \geq 1$. Bearing in mind that $i + j + k + h = m - 5$,

i) there are no values of i, j, k, h satisfying (**) and corresponding to m, for $m \leq 8$;

ii) the values $[i = 1, j = 2, k = 0, h = 1]$ correspond to $m = 9$;

iii) there are no values of i, j, k, h satisfying (**) and corresponding to $m = 10$;

iv) the two sets of values $[i = 2,\ j = 3,\ k = 0,\ h = 1]$ and $[i = 1, j = 3, k = 0, h = 2]$ satisfy (**) and correspond to $m = 11$, and so on; there are values of i, j, k, h satisfying (**) that correspond to any value of $m \geq 12$.

As for the inequalities in the second line of (**), and given that $i' + j' + k' + h' = m$,

1) there are no values of i', j', k', h' satisfying (**) and corresponding to $m = 1$;
2) the two sets of values $[i' = j' = 1,\ k' = h' = 0]$ and $[j' = h' = 1, i' = k' = 0]$ satisfy (**) and correspond to $m = 2$;
3) the two sets of values $[i' = j' = k' = 1,\ h' = 0]$ and $[i' = j' = h' = 1, k' = 0]$ satisfy (**) and correspond to $m = 3$;
4) there are 4 sets of values satisfying (**) and corresponding to $m = 4$, there are also 4 sets of values satisfying (**) and corresponding to $m = 5$, 8 sets satisfying (**) and corresponding to $m = 6$ and 8 sets satisfying (**) and corresponding to $m = 7$.
5) The following sets $[i' = j' = 3,\ k' = h' = 0]$, $[i' = j' = h' = 2,\ k' = 0]$, $[i' = j' = 2, k' = h' = 1]$, $[i' = 2, j' = 3, k' = 0, h' = 1]$ are 4 of the 8 sets of values satisfying (**) that correspond to $m = 6$.

The following sets $[i' = j' = 3,\ k' = 1,\ h' = 0]$, $[i' = h' = 2,\ j' = 3, k' = 0]$, $[i' = 1, j' = 3, k' = 1, h' = 2]$, $[i' = 1, j' = h' = 3, k' = 0]$ are 4 of the 8 sets of values satisfying (**) that correspond to $m = 7$.

Consequences. Let us just recall that we have written the equation of an m-canonical adjoint Φ_m as follows:

$$\Phi_m : A_{m-5}(X_0, X_1, X_2, X_3)\varphi_4(X_0, X_1, X_2, X_3)X_4 + \psi_m(X_0, X_1, X_3, X_4) = 0,$$

where

$$A_{m-5}(X_0, X_1, X_2, X_3)X_4 = \Bigg(\sum_{i+j+k+h=m-5} a_{ijkh}\, X_0^i X_1^j X_2^k X_3^h\Bigg) X_4$$

and

$$\psi_m(X_0, X_1, X_2, X_3) = \sum_{i'+j'+h'+l'=m} b_{ijkh}\, X_0^{i'}\, X_1^{j'}\, X_2^{k'}\, X_3^{h'}.$$

From i),...,vi), we deduce that the form A_{m-5} is zero if and only if $m \leq 8$ and $m = 10$.

Since the m-genus P_m of a desingularization X of V is the number of the linearly independent forms defining m-canonical adjoints to V (cf. [S_1]), from 1), ..., 4), we deduce the following results regarding the plurigenera of a desingularization X of V.

From 1), we can establish that there are no 1-canonical adjoints (also called canonical adjoints) to V; this implies that the geometric genus of X is $p_g = 0$.

From 2), we find that $\Phi_2 : \psi_2(X_0, X_1, X_3, X_4) = X_1(\lambda_1 X_0 + \lambda_2 X_3) = 0$, where $\lambda_i \in k$; this implies that the bigenus of X is $P_2 = 2$.

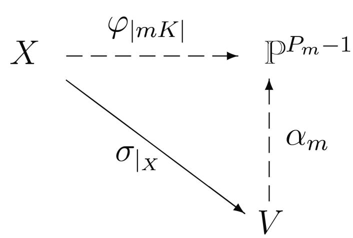
From 3), we learn that $\Phi_3 : \psi_3(X_0, X_1, X_3, X_4) = X_0 X_1(\mu_1 X_2 + \mu_2 X_3) = 0$, $\mu_i \in k$; this implies that the trigenus of X is $P_3 = 2$.

From 4), we obtain that $P_4 = P_5 = 4$, $P_6 = 8$ and $P_7 = 8$.

In addition, X has the plurigenera $P_8 = 13, P_9 = 15, P_{10} = 19, P_{11} = 22$.

6. The m-canonical transformation $\varphi_{|mK_X|}$, $m \geq 2$.

Let us use $\alpha_m : V - - \rightarrow \mathbb{P}^{P_m - 1}$ to indicate the rational transformation associated with the linear system of m-canonical adjoints Φ_m to V. The following triangle



is commutative.

Let us consider the linear system of m-canonical adjoints Φ_m. From i) and 1), ..., 4) and the Consequences, we can see that if $2 \leq m \leq 5$, then Φ_m is given by $\psi_m(X_0, X_1, X_3, X_4) = 0$; moreover, the rational transformation α_m has the generic fiber of dimension ≥ 1. From the commutativity of the above triangle, $\varphi_{|mK_X|}$ also has the generic fiber of dimension ≥ 1.

From i) and 5) and the Consequences, we know that Φ_m, for $m = 6, 7$, is again given by $\psi_m(X_0, X_1, X_3, X_4) = 0$, and that the rational transformation α_m, as well as $\varphi_{|mK_X|}$, is generically $2:1$. As a consequence of this and of the fact that $P_2 \neq 0$, $\varphi_{|mK_X|}$ is either generically $2:1$ or birational (to its image) for $m \geq 8$. It is not difficult to prove that $\varphi_{|6K_X|}$ and $\varphi_{|7K_X|}$ are generically $2:1$, since all we have to do is consider the rational transformation defined by the 4 sets of values given in 5) (in both cases $m = 6, 7$).

Next, we note that a necessary condition for the birationality of $\varphi_{|mK_X|}$ is that $A_{m-5} \neq 0$ in the equation $A_{m-5}\varphi_4 X_4 + \psi_m = 0$ of Φ_m; in other words, Φ_m must be a non-global canonical adjoint to V (cf. Remark 1, section 5).

To be more precise, let us consider $\Phi_m : A_{m-5}\varphi_4 X_4 + \psi_m = 0$ and assume that the rational transformation $\alpha'_m : V - - \rightarrow \mathbb{P}^{P_m - 1}$ defined by the linear system $\psi_m = 0$ of global m-canonical adjoints to V (see Remark 1, section 5) is generically $2:1$, then $\varphi_{|mK_X|}$ is birational if and only if $A_{m-5} \neq 0$. This is immediately proved by the presence of the addendum $A_{m-5}X_4$, which contains X_4 to the power 1; indeed, this addendum separates the two distinct points on $V : \varphi_4 X_4^2 + \varphi_5 X_4 + \varphi_6 = 0$ that are mapped to one point.

As a corollary of this latter fact, in the light of i),...iv) and the Cosequences, $\varphi_{|mK_X|}$ is birational if and only if $m = 9$ and $m \geq 11$. So, for $m = 10$, there is a gap in the birationality of $\varphi_{|mK_X|}$.

This concludes our examination of $\varphi_{|mK_X|}$, for $m \geq 2$.

7. Computing the irregularities of X.

This brings us to the demonstration that $q_i = \dim_k H^i(X, \mathcal{O}_X) = 0$, for $i = 1, 2$. We know that $q_1 = \dim_k H^1(X, \mathcal{O}_X) = q(S_r) = \dim_k H^1(S_r, \mathcal{O}_{S_r})$, where $S_r \subset X$ is the strict transform of a generic hyperplane section S of V (cf. [S_1], section 4, for instance). S has several isolated (actual or infinitely-near) double points and no other singularities. This follows from the fact that, outside the points A_0, A_1, A_2, A_3 and A_4, the hypersurface V only has actual or infinitely-near double curves and isolated double points. So, $q_1 = 0$.

To prove that $q_2 = 0$, we use the formula (36) in section 4 of [S_1], which states that:

$$q_2 = p_g(X) + p_g(S_r) - \dim_k (W_2),$$

where W_2 is the vector space of the degree 2 forms defining global adjoints Φ_2 to V, i.e. defining hyperquadrics Φ_2 such that

$$\pi_r^* \dots \pi_2^*[\pi_1^*(\Phi_2)] - E_2 - E_4 - E_6 - E_8 - E_9 \geq 0,$$

(cf. the expression of D_m in (*), section 4). So the above hyperquadrics Φ_2

are those passing through the points A_0, A_1, A_2, A_3 and A_4. Thus, $\dim_k(W_2) = 15 - 5 = 10$. It follows from $p_g(S_r) = 10$ and $p_g(X) = 0$ (cf. Consequences at the end of section 5) that $q_2 = 0$.

REFERENCES

[*Che*$_1$] M. CHEN, *On the* $\mathbb{Q}$*-divisor method and its application*, J. Pure and Appl. Algebra, **191** (2004), pp. 143–156.

[*Che*$_2$] M. CHEN, *Canonical stability of 3-folds of general type with* $p_g \geq 3$, Internat. J. Math., **14**, No. 5 (2003), pp. 515–528.

[*Che*$_3$] M. CHEN, *A sharp lower bound for the volume of 3-folds of general type*, preprint, arXiv:math.AG/0407397 (2005).

[*Chi*] S. CHIARUTTINI, *A new Example of Birationality of pluricanonical Maps*, preprint.

[*CG*] S. CHIARUTTINI - R. GATTAZZO, *Examples of birationality of pluricanonical maps*, Rend. Sem. Mat. Univ. Padova, **107** (2002), pp. 81–94.

[*CZ*] M. CHEN - K. ZUO, *Complex projective threefolds with non-negative canonical Euler-Poincare' characteristic*, Commun. in An. and Geom., Vol. **16**, No. 1 (2008), pp. 159–182.

[*F*] A. R. IANO - FLETCHER, *Working with weighted complete intersections*, Explicit birational geometry of 3-folds, London Math. Soc., Lecture Note Ser. 281, Cambridge Univ. Press, Cambridge (2000), pp. 101–173.

[*Re*] M. REID, *Young Person's Guide to Canonical Singularity*, Proc. Algebraic Geometry, Bowdoin 1985, Vol. **46**, A.M.S. (1987), pp. 345–414.

[*Ro*$_1$] M. C. RONCONI, *A threefold of general type with* $q_1 = q_2 = p_g = P_2 = 0$, Proc. Monodromy Conference, Steklov Institute, Moscow, June 25-30, 2001, Acta Appl. Mathematicae, **75** (2003), pp. 133–150.

[*Ro*$_2$] M. C. RONCONI, *Examples of birationality of* $\Phi_{|mK|}$ *for large m with and without gaps*, to appear.

[*S*$_1$] E. STAGNARO, *Adjoints and pluricanonical adjoints to an algebraic hypersurface*, Annali di Mat. Pura ed Appl., **180** (2001), pp. 147–201.

[*S*$_2$] E. STAGNARO, *Pluricanonical maps of a threefold of general type*, Proc. Greco Conference, Catania, "Le Matematiche", Vol. **LV** - Fasc. II (2000), pp. 533–543.

[*S*$_3$] E. STAGNARO, *Gaps in the birationality of pluricanonical transformations*, Accademia Ligure di Sc. e Lettere, Collana di Studi e Ricerche, Genova (2004), pp. 5–53.

[*S*$_4$] E. STAGNARO, *Threefolds with Kodaira dimension 0 or 3*, Bollettino U.M.I. (8) **10-B** (2007), pp. 1149–1182.

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