

## A Group of Automorphisms of the Rooted Dyadic Tree and Associated Gelfand Pairs

DANIELE D'ANGELI (\*) - ALFREDO DONNO (\*)

ABSTRACT - In this paper we study the group  $G$  of automorphisms of the rooted dyadic tree generated by translation by 1 and multiplication by an odd integer  $q$ , showing that  $G$  admits a self-similar presentation and it is isomorphic to the Baumslag-Solitar group  $BS_q$ . Moreover, we show that the action of  $G$  on each level of the tree gives rise to a Gelfand pair.

### 1. Introduction.

Let  $T$  be the infinite binary rooted tree. Denote by  $L_n$  the  $n$ -th level of  $T$ , which consists of  $2^n$  vertices. The root of  $T$  can be identified with the group  $\mathbb{Z}$ , each vertex, say at level  $L_n$ , can be regarded as a coset of  $2^n\mathbb{Z}$  in  $\mathbb{Z}$ . Finally, the boundary  $\partial T$  corresponds to the ring of dyadic integers  $\mathbb{Z}_2$  (for more details see [F]).

Let  $G$  be the group of automorphisms of  $T$  generated by the translation by 1 and by the multiplication by an odd integer  $q$  for each vertex in  $T$ . Denote by  $a$  and  $b$  such automorphisms, respectively. The action of  $G$  on  $T$  is *self-similar*: we will directly prove that these automorphisms admit the following form

$$a = (1, a)\varepsilon, \quad b = (b, ba^h),$$

with  $q = 2h + 1$ , according to the fact that every self-similar group can be embedded in a wreath product. By using the self-similarity, we deduce that  $G$  is isomorphic to the Baumslag-Solitar group  $BS_q = \langle s, t : t^{-1}st = s^q \rangle$ , introduced in [BS]. We mention that self-similar groups are closely con-

(\*) Indirizzo degli A.: Section de Mathématiques, Université de Genève, 2-4, Rue du Lièvre, Case Postale 64, 1211 Genève 4, Suisse.

E-mail: daniele.dangeli@unige.ch    alfredo.donno@unige.ch

nected with automata. For example, Bartholdi and Šunik in [BS] obtained the same result in the more general context of a family of solvable groups generated by finite automata.

Let  $G_n$  be the finite homomorphic image of  $G$  acting faithfully on  $L_n$ . Also denote by  $K_n \leq G_n$  the parabolic subgroup which stabilizes a fixed vertex in  $L_n$ . This way,  $L_n$  will be identified with the corresponding homogeneous space  $G_n/K_n$ . We then prove that  $(G_n, K_n)$  is a Gelfand pair for every  $n \geq 1$ . In particular, we show, by direct computations involving characters, that the decomposition of the corresponding permutation representation into irreducible  $G$ -representations is multiplicity-free. Actually the result can also be obtained from the general theory of representations of semidirect products developed in [CST]. Finally, a formula for the relative spherical functions is given.

The idea for this work has been inspired by Professor A. Figà-Talamanca whom we thank for his interest and continuous encouragement.

## 2. Preliminaries.

Let  $T$  be the binary infinite rooted homogeneous tree, that is the tree in which each vertex has two children. For every  $n \geq 1$ , denote by  $L_n$  the  $n$ -th level of  $T$ , formed by  $2^n$  vertices. Identify each vertex of  $L_n$  with a word  $x_0 \dots x_{n-1}$  of length  $n$  in the alphabet  $X = \{0, 1\}$ . Let  $X^*$  be the set of all words of finite length in the alphabet  $X$ . The root will be represented by the empty word  $\emptyset$ . We will denote by  $T_x$  the subtree of  $T$  rooted at the vertex  $x$  and isomorphic to  $T$ .

The set  $L_n$  can be endowed with an ultrametric distance  $d$ , defined in the following way: if  $x = x_0 \dots x_{n-1}$  and  $y = y_0 \dots y_{n-1}$ , then

$$d(x, y) = n - \max\{i : x_k = y_k, \quad \forall k \leq i\}.$$

We observe that  $d = d'/2$ , where  $d'$  denotes the usual geodesic distance on  $T$ .

In this way  $(L_n, d)$  becomes an ultrametric space, in particular a metric space, on which the automorphisms group  $Aut(T)$  acts isometrically. Note that the diameter of  $(L_n, d)$  is exactly  $n$ .

Each automorphism  $g \in Aut(T)$  can be represented by its *labelling*. The labelling of  $g \in Aut(T)$  can be realized as follows: given a vertex  $x = x_0 \dots x_{n-1} \in T$ , we associate with  $x$  a permutation  $g_x \in S_2$  ( $S_2$  denotes the symmetric group on 2 elements) that gives the action of  $g$  on the two

children of  $x$ . Formally, the action of  $g$  on the vertex labelled with the word  $x = x_0 \dots x_{n-1}$  is

$$x^g = x_0^{g_0} x_1^{g_{x_0}} \dots x_{n-1}^{g_{x_0 \dots x_{n-2}}}.$$

It will be used also  $g(x)$  to indicate the element  $x^g$ . Let  $G \leq \text{Aut}(T)$ . We define the *stabilizer* of  $x \in T$  as  $\text{Stab}_G(x) = \{g \in G : g(x) = x\}$  and the stabilizer of the  $n$ -th level as  $\text{Stab}_G(n) = \bigcap_{x \in L_n} \text{Stab}_G(x)$ . Observe that  $\text{Stab}_G(n)$  is a normal subgroup of  $G$  of finite index for all  $n \geq 1$ .

An automorphism  $g \in \text{Stab}_G(1)$  can be identified with the elements  $g_i, i = 0, 1$  that describe the action of  $g$  on the respective subtrees  $T_i$ . So we get the following embedding

$$\varphi : \text{Stab}_G(1) \longrightarrow \text{Aut}(T) \times \text{Aut}(T)$$

that associates to  $g$  the pair  $(g_0, g_1)$ .

The following definitions can be found in [BGS].

- $G$  is *spherically transitive* if the action of  $G$  on  $L_n$  is transitive for every  $n$ .

In this case the subgroups  $\text{Stab}_G(x), x \in L_n$  are conjugate each other.

- $G$  is *fractal* if  $\text{Stab}_G(x)|_{T_x} \cong G$ , for each  $x \in T$ . This is equivalent to require that the map  $\varphi : \text{Stab}_G(1) \longrightarrow G \times G$  is a subdirect embedding, that is, surjective on each factor.

Clearly, a fractal group is spherically transitive if and only if it is transitive on the first level of  $T$ .

- $G$  is *self-similar* if for every  $g \in G, x \in X$ , there exists  $g_x \in G, x' \in X$  such that  $g(xw) = x'g_x(w)$  for all  $w \in X^*$ .

A self-similar group  $G$  can be embedded in the wreath product  $G \wr S_2 = G^2 \rtimes S_2$ , so that an element  $g \in G$  is represented as  $g = (g_0, g_1)\sigma$ , where  $g_i \in G$  describe the action on the subtree  $T_i$ , and  $\sigma \in S_2$  corresponds to the action on  $L_1$ . With this notation, the product between two elements  $g = (g_0, g_1)\sigma$  and  $h = (h_0, h_1)\tau$  is

$$gh = (g_0, g_1)(h_0, h_1)^\sigma \sigma \tau = (g_0 h_{\sigma^{-1}(0)}, g_1 h_{\sigma^{-1}(1)}) \sigma \tau.$$

Observe that in the product  $gh$  the automorphism  $g$  acts before the automorphism  $h$ , to have coherence with the wreath recursion. Consider again the binary tree but from another point of view. Observe that the

infinite sequence  $x_0x_1x_2\dots$  can be regarded as the dyadic number  $x_02^0 + x_12 + x_22^2 + \dots$ .

Let us identify the root of the tree with the ring  $\mathbb{Z}$ , the vertices of the first level with  $2\mathbb{Z}$  and  $2\mathbb{Z} + 1$  respectively. Again, the sons of  $2\mathbb{Z}$  are  $4\mathbb{Z}$  and  $4\mathbb{Z} + 2$ , whereas the sons of  $2\mathbb{Z} + 1$  are  $4\mathbb{Z} + 1$  and  $4\mathbb{Z} + 3$ . Actually, the  $2^n$  vertices of the  $n$ -th level are, for all  $n \geq 1$ , the cosets of  $2^n\mathbb{Z}$  in  $\mathbb{Z}$ .

In this way the boundary  $\partial T$  of the tree is identified with the dyadic closure of  $\mathbb{Z}$ , which will be denoted by  $\mathbb{Z}_2$ .

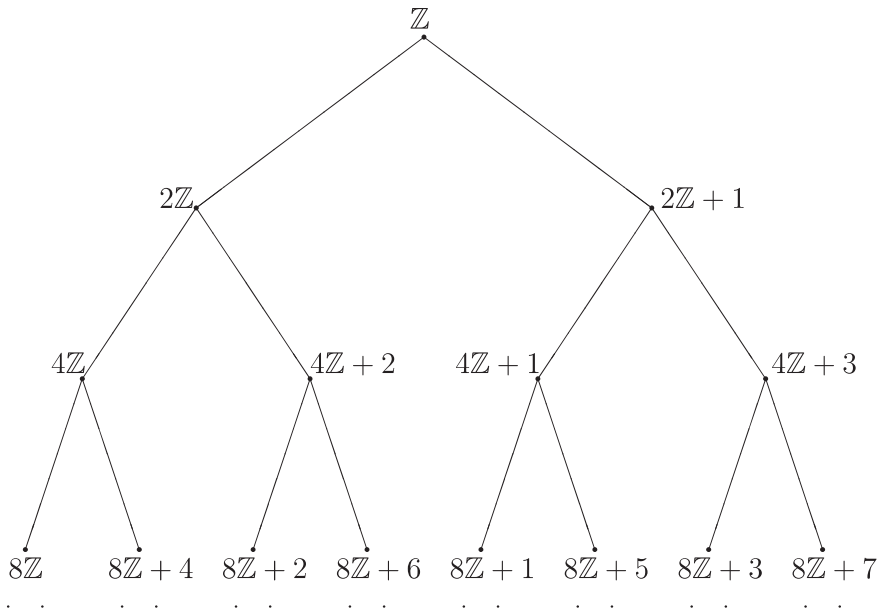


Fig. 1. – The dyadic tree.

### 3. Some automorphisms of the dyadic tree.

#### 3.1 – The automorphism $a$ .

Under this identification of the boundary of  $T$  with the ring  $\mathbb{Z}_2$  of dyadic integers, we introduce the automorphism  $a$  of  $T$  given by the translation by 1. Actually, this is the automorphism that generates the Adding Machine (see, for instance, [BGN]). If we consider the action of  $a$  on the level  $L_n$ , then the order of  $a$  is  $2^n$ . Therefore  $a$  admits the following *self-similar*

form:

$$(1) \quad a = (1, a)\varepsilon,$$

where  $\varepsilon$  is the nontrivial permutation of  $S_2$ .

The labelling of  $a$  is given in the following figure.

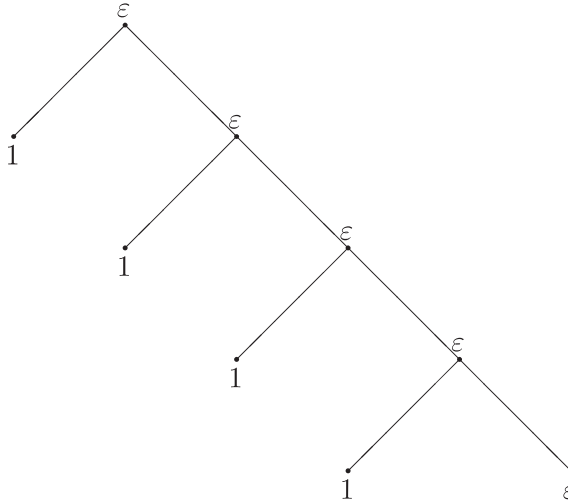


Fig. 2. – Labelling of  $a$ .

It is clear that the action of  $a$  fixes the root and so it preserves each level  $L_n$ , on which it acts transitively for every  $n \geq 1$ .

Let us verify, by induction on  $n$ , that the action of the automorphism  $a$  given in (1) is the translation by 1. On the first level the action of  $a$  is obviously transitive, since it exchanges the vertices corresponding to the cosets  $2\mathbb{Z}$  and  $2\mathbb{Z} + 1$ , that means to perform the sum of 1 modulo  $2\mathbb{Z}$ .

Assume the claim true for  $n - 1$  and let us prove it for  $n$ . First consider an element of the form  $2^n\mathbb{Z} + k$ , with  $k$  even. From the labelling of  $a$  it follows that the first letter 0 is changed in 1 (that corresponds to add 1), then  $a$  acts as the identity. We have:

$$\begin{aligned} 2^n\mathbb{Z} + k &\xrightarrow{\varepsilon} 2^n\mathbb{Z} + k + 1 \xrightarrow{-1} 2^n\mathbb{Z} + k \xrightarrow{:2} 2^{n-1}\mathbb{Z} + \frac{k}{2} \\ &\xrightarrow{id} 2^{n-1}\mathbb{Z} + \frac{k}{2} \xrightarrow{:2} 2^n\mathbb{Z} + k \xrightarrow{+1} 2^n\mathbb{Z} + k + 1. \end{aligned}$$

In fact, after applying  $\varepsilon$ , the element  $2^n\mathbb{Z} + k$  is in the subtree  $T_1$ . The translation by  $-1$  and the following division by 2 are made to identify the subtree  $T_1$  with the whole tree  $T$ . Now,  $a$  acts trivially. Finally, multi-

plication by 2 and translation by 1 give the image of the element  $2^n\mathbb{Z} + k$  in  $T_1$ , which is  $2^n\mathbb{Z} + k + 1$ , as desired.

A similar argument can be developed when  $a$  acts on an element of the form  $2^n\mathbb{Z} + k$ , with  $k$  odd. First we have the action of  $\varepsilon$ , which is the sum by  $-1$ , then again the action of  $a$ . We have:

$$\begin{aligned} 2^n\mathbb{Z} + k &\xrightarrow{\varepsilon} 2^n\mathbb{Z} + k - 1 \xrightarrow{:2} 2^{n-1}\mathbb{Z} + \frac{k-1}{2} \\ &\xrightarrow{a} 2^{n-1}\mathbb{Z} + \frac{k-1}{2} + 1 = 2^{n-1}\mathbb{Z} + \frac{k+1}{2} \xrightarrow{:2} 2^n\mathbb{Z} + k + 1. \end{aligned}$$

In this case, after applying  $\varepsilon$ , the element  $2^n\mathbb{Z} + k$  is in the subtree  $T_0$ . The division by 2 is made to identify  $T_0$  with the whole tree  $T$ . Now  $a$  acts again as  $a$ . By induction, this action is the translation by 1. Finally, multiplication by 2 gives the image of the element  $2^n\mathbb{Z} + k$  in  $T_0$ , which is  $2^n\mathbb{Z} + k + 1$ .

### 3.2 – The automorphism $b$ .

Let us introduce the automorphism  $b$  of  $T$  given by multiplication by an odd number  $q = 2h + 1$  different from 1. Since  $q$  is odd,  $b$  preserves each level and in particular it fixes the vertex  $2^n\mathbb{Z}$  for every  $n \geq 1$ . Moreover  $q$  has dyadic norm 1, so it is invertible in the ring of dyadic integers, but its inverse does not belong to  $\mathbb{Z}$  (except in the case  $q = -1$ ). So, it is not possible to identify the inverse of this automorphism with the multiplication by an integer. This is clear by considering that the order of the multiplication by  $q$  on  $L_n$  is not the same for all  $n \geq 1$ .

On  $L_1$  multiplication by  $q$  is trivial.

On  $L_2$  we have  $b^2 = 1$  for each  $q$  odd.

On  $L_3$  we still have  $b^2 = 1$  for each  $q$  odd. This is true because

$$8 \mid (q^2 - 1) \quad \text{for each } q \text{ odd.}$$

To compute the order of the multiplication by  $q$  on each level  $L_n$ , with  $n \geq 3$ , we have to study the equation  $q^{2^t} \equiv 1(2^n)$ . Consider the following decomposition:

$$q^{2^t} - 1 = (q^{2^{t-1}} + 1)(q^{2^{t-2}} + 1)(q^{2^{t-3}} + 1) \cdots (q^2 + 1)(q + 1)(q - 1).$$

Since  $q^2 \equiv 1(4)$  for every  $q$  odd, we get that each of the factors above, except the last two, are divisible by 2 but are not divisible by higher powers of 2. Moreover, the fact that  $8 \mid (q^2 - 1)$  for each  $q$  odd, implies that for all

$n \geq 3$  we have  $2^n \mid q^{2^{n-2}} - 1$ , and so  $q^{2^{n-2}} \equiv 1(2^n)$  for all  $n \geq 3$ . Actually, there exists some  $q$  whose order, at the level  $L_n$ , is strictly less than  $2^{n-2}$ . In fact, from the previous argument it follows that if

$$q^2 \equiv 1(2^k) \quad \text{but} \quad q^2 \not\equiv 1(2^{k+1}),$$

then, at the level  $L_n$ , the order of multiplication by  $q$  is exactly  $2^{n-k+1}$ , for all  $n \geq k$ . Also, we have  $k \geq 3$ . To describe the action of  $b$  on  $L_n$ ,  $n < k$ , we distinguish two cases.

If  $q \equiv 1(4)$ , then  $2^{k-1} \mid q - 1$  and so  $b = 1$  on each level  $L_n$ ,  $n < k$ . If  $q \equiv -1(4)$ , then  $2^{k-1} \mid q + 1$  and so  $b^2 = 1$ .

As an example, let us consider the case  $q = 3$ . Now  $q^2 - 1$  is divided by 8 but not by 16 and this implies that the order of the multiplication by 3 is  $2^{n-2}$  on  $L_n$  for each  $n \geq 3$ . In  $L_1$  one has  $b = 1$ , in  $L_2$   $b^2 = 1$ .

We can now investigate the order of the orbits of vertices of  $L_n$  under the action of  $b$ .

Let us consider the sphere of center  $2^n\mathbb{Z}$  and radius  $r$  with respect to the ultrametric distance already defined. It contains  $2^{r-1}$  vertices for  $r \geq 1$ . The first vertex from the left belonging to this sphere is the element  $2^n\mathbb{Z} + 2^{n-r}$ . Assume that  $2^l$  is the period of this element under the action of  $b$ , so that we get

$$2^{n-r}q^{2^l} \equiv 2^{n-r}(2^n).$$

This implies  $q^{2^l} \equiv 1(2^r)$ . Therefore, if the order of  $b$  on  $L_n$  is  $2^{n-k+1}$  for every  $n \geq k$ , it must be

$$l = r - k + 1,$$

with  $r \geq k$ . So, the orbit of the element  $2^n\mathbb{Z} + 2^{n-r}$  is exactly of length  $2^{r-k+1}$ .

It is easy to verify that the remaining vertices of  $L_n$  belonging to the sphere of radius  $r$  have the form

$$2^n\mathbb{Z} + 2^{n-r} + t2^{n-r+1}, \quad t = 1, \dots, 2^{r-1} - 1.$$

We want to prove that they have the same order  $2^{r-k+1}$ . Consider the equation

$$(2^{n-r} + t2^{n-r+1})q^{2^l} \equiv 2^{n-r} + t2^{n-r+1}(2^n).$$

Dividing by  $2^{n-r}$ , we get

$$q^{2^l} + 2tq^{2^l} \equiv 1 + 2t(2^r),$$

from which

$$(q^{2^l} - 1)(2t + 1) \equiv 0(2^r).$$

Since  $2t + 1$  is odd, we have again the equation  $q^{2^l} \equiv 1(2^r)$  and so  $l = r - k + 1$ , if  $q$  has order  $2^{r-k+1}$  modulo  $2^r$ .

Hence, the sphere of radius  $r$  decomposes, under the multiplication by  $q$ , in  $2^{k-2}$  orbits of length  $2^{r-k+1}$ , for all  $r \geq k$ . For  $r < k$ , from what observed, the sphere of radius  $r$  decomposes into orbits of length 1 or 2.

**PROPOSITION 3.1.** *The automorphism  $b$  admits the self-similar representation*

$$b = (b, ba^h),$$

where  $q = 2h + 1$ .

**PROOF.** Let us prove it by induction on  $n$ .

For  $n = 1$ , multiplication by  $q$  coincides with the identity, as in the self-similar form of  $b$ .

Assume the result true for  $n - 1$  and let us show it for  $n$ . First consider an element of the form  $2^n\mathbb{Z} + k$ , with  $k$  even. We have:

$$2^n\mathbb{Z} + k \xrightarrow{:2} 2^{n-1}\mathbb{Z} + \frac{k}{2} \xrightarrow{b} 2^{n-1}\mathbb{Z} + \frac{qk}{2} \xrightarrow{:2} 2^n\mathbb{Z} + qk.$$

In fact the element  $2^n\mathbb{Z} + k$  belongs to  $T_0$ . Division by 2 is made to identify the subtree  $T_0$  with the whole tree  $T$ . Now,  $b$  acts again as  $b$  and this action is, by induction, the multiplication by  $q$ . Finally, multiplication by 2 gives the image of  $2^n\mathbb{Z} + k$  in  $T_0$ , that is  $2^n\mathbb{Z} + qk$ , as desired.

A similar argument can be developed when  $b$  acts on an element of the form  $2^n\mathbb{Z} + k$ , with  $k$  odd. In this case, we have:

$$\begin{aligned} 2^n\mathbb{Z} + k &\xrightarrow{-1} 2^n\mathbb{Z} + k - 1 \xrightarrow{:2} 2^{n-1}\mathbb{Z} + \frac{k-1}{2} \xrightarrow{b} 2^{n-1}\mathbb{Z} + \frac{q(k-1)}{2} \\ &\xrightarrow{a^h} 2^{n-1}\mathbb{Z} + \frac{q(k-1)}{2} + h = 2^{n-1}\mathbb{Z} + \frac{q(k-1)}{2} + \frac{q-1}{2} = 2^{n-1}\mathbb{Z} + \frac{qk-1}{2} \\ &\xrightarrow{:2} 2^n\mathbb{Z} + qk - 1 \xrightarrow{+1} 2^n\mathbb{Z} + qk. \end{aligned}$$

In this case, the element  $2^n\mathbb{Z} + k$  is in the subtree  $T_1$ . The translation by  $-1$  and the following division by 2 are made to identify  $T_1$  with the whole



tree  $T$ . Now,  $b$  acts as  $ba^h$ . By induction, the action of  $b$  is multiplication by  $q$  and the following action of  $a^h$  is the translation by  $h$ . The multiplication by 2 and the sum of 1 give the final image of the element  $2^n\mathbb{Z} + k$  in  $T_1$ , that is  $2^n\mathbb{Z} + qk$ .  $\square$

### 3.3 – The group $G_q$ .

It is possible to verify that the automorphism group  $G = G_q$  of the dyadic tree generated by the translation by 1 and by the multiplication by an odd integer  $q = 2h + 1$  is a homomorphic image of the solvable Baumslag-Solitar group

$$BS_q = \langle s, t : t^{-1}st = s^q \rangle.$$

In fact, this relation holds for every level  $L_n$ , as we are going to prove.

We have

$$a = (1, a)\varepsilon, \quad b = (b, ba^h), \quad b^{-1} = (b^{-1}, a^{-h}b^{-1})$$

and so

$$b^{-1}ab = (a^h, a^{-h}b^{-1}ab)\varepsilon.$$

For  $n = 1$  the relation is satisfied for any  $q$  odd, because both the elements  $b^{-1}ab$  and  $a^q$  exchange the vertices in  $L_1$ .

Suppose, now, that the result is true for  $n - 1$  and let us show it for  $n$ . We have

$$\begin{aligned} b^{-1}ab &= (a^h, a^{-h}b^{-1}ab)\varepsilon = (a^h, a^{-h}a^q)\varepsilon = (a^h, a^{-h+2h+1})\varepsilon = \\ &= (a^h, a^{h+1})\varepsilon = a^{2h+1} = a^q, \end{aligned}$$

where the second equality follows by induction, and so the relation is proved.

Actually, the two groups coincide. In fact, for  $q \neq -1$ , the automorphisms  $a$  and  $b$  that we defined have infinite order. On the other hand, the following lemma holds.

**LEMMA 3.2.** *Let  $BS_q = \langle s, t : t^{-1}st = s^q \rangle$  the Baumslag-Solitar group. Then the order of at least one of  $s$  or  $t$  in any proper homomorphic image of  $BS_q$  must be finite.*

PROOF. Consider the factor group  $BS_q/N$ , where  $N$  is any proper normal subgroup of  $BS_q$ . If  $s^k$  or  $t^h$  belong to  $N$  for some  $k, h$  not trivial, then the claim is true. Using the following generalized relations

$$t^{-k}st^k = s^{q^k}, \quad t^{-1}s^kt = s^{kq}, \quad t^{-k}s^mt^k = s^{mq^k}$$

and suitable conjugations it is possible to write any word with occurrences of both  $s$  and  $t$  as a word of the form  $s^mt^k$ . Suppose that such an element belongs to  $N$ , this implies that  $t^{-k}s^mt^{2k} = s^{mq^k}t^k \in N$ . So the element  $s^mt^k(s^{mq^k}t^k)^{-1} = s^{m(1+q^k)}$  is in  $N$  and this completes the proof.  $\square$

So we get the following

**THEOREM 3.3.** *The automorphisms group  $G = G_q$  of the dyadic tree generated by the automorphism  $a$  which is the sum of 1 and the automorphism  $b$  which is the multiplication by  $q$  is isomorphic to the solvable Baumslag-Solitar group:*

$$BS_q = \langle a, b : b^{-1}ab = a^q \rangle.$$

Note that the group  $G$  is a self-similar group whose action on the tree  $T$  is spherically transitive. Since

$$a^2 = (a, a), \quad b = (b, ba^h), \quad ba^{-2h} = (ba^{-h}, b),$$

$G$  is a fractal group.

**REMARK 3.4.** The case  $q = -1$  yields the infinite dihedral group. In fact, the group we get is

$$G = \langle a, b : b^{-1}ab = a^{-1} \rangle,$$

and this is exactly the presentation of the infinite dihedral group (see, for example, [BGN]). It is obvious that in this case we have  $b^2 = 1$  and the generators for  $G$  are

$$a = (1, a)\varepsilon, \quad b = (b, ba^{-1}).$$

**REMARK 3.5.** In [BŠ] L. Bartholdi and Z. Šunik obtained these groups of automorphisms as groups generated by the automaton  $S_{m,n}$  in the more general case of the multiplication by  $m$  in the ring  $\mathbb{Z}_n$  of  $n$ -adic integers, with  $(m, n) = 1$ .

## 4. Gelfand pairs

### 4.1 – Some definitions

We present now some basic elements of the theory of finite Gelfand pairs that will be applied in what follows (see, for example, [CST1], [CST2] and [D] for many applications).

Let  $G$  be a finite group and let  $K \leq G$  be a subgroup, denote  $X = G/K = \{gK : g \in G\}$  the associated homogeneous space. In this way  $G$  acts transitively on  $X$  and  $K$  is the stabilizer of the element  $x_0 \equiv K \in X$ . The space  $L(G) = \{f : G \rightarrow \mathbb{C}\}$  is an algebra with respect to the convolution

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(gh)f_2(h^{-1}),$$

for all  $f_1, f_2 \in L(G)$  and  $g \in G$ . The subspace of  $L(G)$  consisting of the functions  $K$ -invariant to the right (i.e.  $f(gk) = f(g)$  for all  $g \in G, k \in K$ ) is a subalgebra that can be identified with  $L(X) = \{f : X \rightarrow \mathbb{C}\}$ .

This subalgebra can be endowed with a Hilbert space structure by the scalar product  $\langle f_1, f_2 \rangle = \sum_{x \in X} f_1(x)\overline{f_2(x)}$ , for all  $f_1, f_2 \in L(X)$ . Analogously, the subspace of  $L(X)$  consisting of the functions  $K$ -invariant is a subalgebra that we identify with  $L(K \backslash G / K) = \{f : G \rightarrow \mathbb{C} : f(kgk') = f(g), \text{ for all } k, k' \in K, g \in G\}$ . The group  $G$  acts on  $L(X)$  in the following way:  $gf(x) = f^g(x) = f(g^{-1}x)$ .

We will call  $(G, K)$  a *Gelfand pair* if the algebra  $L(K \backslash G / K)$  is commutative. The following are equivalent:

- (1)  $(G, K)$  is a Gelfand pair;
- (2) the decomposition of the space  $L(X)$  into irreducible submodules under the action of  $G$  is multiplicity-free, i.e. each irreducible submodule occurs with multiplicity 1;
- (3) given an irreducible representation  $V$  of  $G$ , the dimension of the subspace of  $K$ -invariant vectors  $V^K = \{v \in V : kv = v \ \forall k \in K\}$  is less or equal to 1, and it is 1 if and only if  $V \leq L(X)$ .

The reader is referred to [CST1] for details.

A particular example of Gelfand pair is given by the *symmetric Gelfand pairs*. A finite group  $G$  and a subgroup  $K \leq G$  constitute a symmetric Gelfand pair if for every  $g \in G$  the condition  $g^{-1} \in KgK$  is satisfied. In fact one can directly verify that this condition implies that the algebra

$L(K \backslash G / K)$  is commutative. Moreover, it is easy to verify that the condition  $g^{-1} \in KgK$  for every  $g \in G$  is equivalent to require that, for all  $x, y \in X$ , the pairs  $(x, y)$  and  $(y, x)$  are in the same orbit under the diagonal action of  $G$  on  $X \times X$ .

For example, if we consider the finite dihedral group  $D_n = C_n \rtimes C_2$ , it is easy to verify that  $(D_n, C_2)$  is a symmetric Gelfand pair. The condition  $g^{-1} \in C_2gC_2$ , for every  $g \in D_n$ , is trivially satisfied for this group.

**EXAMPLE 4.1.** Suppose that the finite group  $G$  acts on a finite metric space  $(X, d)$  isometrically and with a 2-points homogeneous action, i.e. in such a way that for all  $x, y, x', y' \in X$  such that  $d(x, y) = d(x', y')$  there exists  $g \in G$  such that  $gx = x'$  and  $gy = y'$ . Let  $K \leq G$  be the stabilizer of an element  $x_0 \in X$ . It follows from the previous argument that  $(G, K)$  is a symmetric Gelfand pair.

We can observe that in this case the  $K$ -orbits (which can be identified with the double cosets of  $K$  in  $G$ ) are the spheres

$$A_j = \{x \in X : d(x_0, x) = j\}.$$

Hence, a function  $f \in L(X)$  is  $K$ -invariant if and only if it is constant on the spheres  $A_j$ .

If  $(G, K)$  is a Gelfand pair and  $L(X) = \bigoplus_{i=0}^n V_i$  is a decomposition of  $L(X)$  into irreducible submodules, then for each  $i = 0, 1, \dots, n$  there exists a unique (up to normalization) bi- $K$ -invariant function  $\phi_i$  whose  $G$ -translates generate the  $V_i$ 's. In particular, we will require that these functions take value exactly 1 on the element  $x_0 \in X$  stabilized by  $K$ . The functions  $\phi_i$ ,  $i = 0, 1, \dots, n$  are called *spherical functions* and they form a basis for the algebra  $L(K \backslash G / K)$ . So, the number of  $K$ -orbits under the action of  $G$  on  $X$  is equal to the number of spherical functions. A different basis for the algebra  $L(K \backslash G / K)$  is given by the characteristic functions of the  $K$ -orbits.

A spherical function  $\phi$  can be also defined as a bi- $K$ -invariant function on  $G$  satisfying the following properties:

- (1)  $\phi * f = ((\phi * f)(1_G))\phi$  for every  $f \in L(K \backslash G / K)$ ;
- (2)  $\phi(1_G) = 1$ .

As an example, the function  $\phi_0 \equiv 1$  is a spherical function: this corresponds to the fact that the trivial representation always occurs in the decomposition of the space  $L(X)$  into irreducible submodules.

4.2 – Gelfand pairs associated with  $G$ .

From now on, we will apply the theory of Gelfand pairs to the groups we introduced in Section 3.

Let  $q = 2h + 1$ . Then, the multiplication by  $q$  has order  $2^{n-k+1}$  for each level  $L_n, n \geq k$ . With  $n$  fixed, the group  $G_n = G/Stab_G(n)$  which acts on the  $n$ -th level of  $T$ , is given by

$$G_n = C_{2^n} \times C_{2^{n-k+1}},$$

where  $\langle a \rangle = C_{2^n}$  and  $\langle b \rangle = C_{2^{n-k+1}}$ , since  $a^{2^n}, b^{2^{n-k+1}} \in Stab_G(n)$ , and the action is defined as  $b^{-1}ab = a^q$ . Observe that  $C_{2^n}$  acts transitively on  $L_n$ . Moreover  $C_{2^{n-k+1}} = Stab_{G_n}(2^n\mathbb{Z})$  for all  $n \geq k$ . This parabolic subgroup will be denoted by  $K_n$ .

We want to show that  $(G_n, K_n)$  are Gelfand pairs. In order to prove it we use the characterization of Gelfand pairs about the decomposition of the space  $L(L_n)$  into irreducible submodules, by showing that this decomposition is *multiplicity-free*.

We distinguish two cases.

CASE  $q \equiv 1(4)$ .

For every  $n < k$  we have  $G_n = C_{2^n}, K_n = \{1\}$  and so in the decomposition of the space  $L(L_n)$  all the irreducible representations (of dimension 1) of  $C_{2^n}$  occur with multiplicity 1. The pairs  $(C_{2^n}, 1)$  are not symmetric (except in the case  $n = 1$ ) Gelfand pairs.

Let now  $n \geq k$ . We have

$$G_n = C_{2^n} \times C_{2^{n-k+1}}.$$

The space  $L(L_n)$  decomposes under the action of  $C_{2^n}$  as

$$(2) \quad L(L_n) = \bigoplus_{j=0}^{2^n-1} V_j,$$

where the representation  $V_j$  is associated with the character  $\chi_j$  defined by  $\chi_j(a) = \omega^j$ , with  $\omega = e^{\frac{2\pi i}{2^n}}$ . We want to study how the automorphism  $b$  acts on the  $V_j$ 's. By using the relations holding in the group we get

$$a(bV_j) = ba^qV_j = \omega^{qj}bV_j$$

and so  $bV_j = V_{qj}$ . We want to understand which of these subspaces are invariant under this action. The equation  $qj \equiv j(2^n)$  gives  $j(q - 1) \equiv 0(2^n)$ , that implies  $j \equiv 0(2^{n-k+1})$ . The  $V_j$ 's such that  $j$  satisfies this property are exactly  $2^{k-1}$  and they correspond to the vertices of the spheres of radius  $r$ ,

with  $r < k$ . They are in fact irreducible submodules of dimension 1 in the representation of  $G_n$  on  $L(L_n)$ . More generally the  $V_j$ 's such that  $j \equiv 0(2^{n-r})$  but  $j \not\equiv 0(2^{n-r+1})$  correspond to the vertices of the sphere of radius  $r$  in  $L_n$  and in fact they are  $2^{r-1}$  in number. They are the  $V_j$ 's with  $j = 2^{n-r} + t2^{n-r+1}$ ,  $t = 0, \dots, 2^{r-1} - 1$ . Since we know that the order of  $b$  on the sphere of radius  $r$  is  $2^{r-k+1}$ , we deduce that these  $V_j$ 's will form  $2^{k-2}$  irreducible submodules for  $G_n$  of dimension  $2^{r-k+1}$ . As an example, the orbit

$$V_{2^{n-r}} \xrightarrow{b} V_{q2^{n-r}} \xrightarrow{b} V_{q^2 2^{n-r}} \xrightarrow{b} \dots \xrightarrow{b} V_{q^{2^{r-k+1}-1} 2^{n-r}}$$

generates the irreducible representation for  $G_n$

$$V_{2^{n-r}} \oplus V_{q2^{n-r}} \oplus V_{q^2 2^{n-r}} \oplus \dots \oplus V_{q^{2^{r-k+1}-1} 2^{n-r}}.$$

The matrices corresponding to this representation are

$$a \mapsto \begin{pmatrix} \omega^{2^{n-r}} & 0 & \dots & 0 \\ 0 & \omega q^{2^{n-r}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \omega q^{2^{r-k+1}-1} 2^{n-r} \end{pmatrix},$$

$$b \mapsto \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & 0 & & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let us call  $\psi$  the character of this representation. We have

$$\psi(a) = \chi_{2^{n-r}}(a) + \chi_{q2^{n-r}}(a) + \dots + \chi_{q^{2^{r-k+1}-1} 2^{n-r}}(a) = 0,$$

since the  $2^n$ -th roots of 1 which occur in the sum are pairwise opposite. Moreover  $\psi(b) = 0$ . Denote by  $\psi_d$ ,  $d = 1, \dots, 2^{k-2}$  the characters of the  $2^{k-2}$  irreducible representations associated with the sphere of radius  $r$ . In order to prove that they are pairwise non isomorphic, we present a suitable element of  $G_n$  on which the characters  $\psi_d$  take different values, for each  $d$ .

Let us consider the element  $a^{2^{r-k+1}} \in G_n$ . We observe that

$$(\omega^{2^{n-r}})^{2^{r-k+1}} = \omega^{2^{n-k+1}} = e^{\frac{2\pi i 2^{n-k+1}}{2^n}}.$$

Moreover we have  $2^{n-k+1} \equiv q^{2^{n-k+1}}(2^n)$ , since  $1 \equiv q(2^{k-1})$ . The same computation holds for the following powers of  $q$ . Hence, all the powers of  $\omega$  which occur in the diagonal of the matrix associated with  $a^{r-k+1}$  correspond to the same angle  $\frac{2\pi 2^{n-k+1}}{2^n} = \frac{2\pi}{2^{k-1}}$ .

Let us consider now any element  $2^{n-r} + t2^{n-r+1}$  in the sphere of radius  $r$ . We have

$$(\omega^{2^{n-r} + t2^{n-r+1}})^{2^{r-k+1}} = \omega^{2^{n-k+1} + t2^{n-k+2}}$$

and

$$q(2^{n-r} + t2^{n-r+1})2^{r-k+1} \equiv 2^{n-k+1} + t2^{n-k+2}(2^n),$$

as one can easily prove by using the property  $q \equiv 1(2^{k-1})$ . We want to study under which condition the elements  $2^{n-r} + t2^{n-r+1}$  and  $2^{n-r} + s2^{n-r+1}$  belong to the same orbit under the multiplication by  $q$ . We get the equation

$$2^{n-k+1} + t2^{n-k+2} \equiv 2^{n-k+1} + s2^{n-k+2}(2^n),$$

which gives  $t \equiv s(2^{k-2})$ .

So, the elements

$$V_{2^{n-r}}, V_{2^{n-r}+2^{n-r+1}}, V_{2^{n-r}+2 \cdot 2^{n-r+1}}, \dots, V_{2^{n-r}+(2^{k-2}-1)2^{n-r+1}}$$

belong to the  $2^{k-2}$  different orbits associated to the sphere of radius  $r$ . The angles corresponding to the respective powers of  $\omega$  are

$$\frac{2\pi}{2^{k-1}} + 2u \frac{2\pi}{2^{k-1}}, \quad u = 0, 1, \dots, 2^{k-2} - 1.$$

For  $u = 2^{k-2} - 1$  we get the angle  $-\frac{2\pi}{2^{k-1}}$  and so the angles associated with these orbits are pairwise different.

We have shown that the characters  $\psi_d$  take different values on the same element  $a^{2^{r-k+1}}$  and so the associated representations are not isomorphic, as required.

CASE  $q \equiv -1(4)$ .

For  $n = 1$  the group  $G_1$  is the cyclic group  $C_2$  and we get the symmetric Gelfand pair  $(C_2, 1)$ .

For  $2 \leq n \leq k - 1$  we have  $b^2 = 1$  and so

$$G_n = C_{2^n} \rtimes C_2,$$

which is the dihedral group of order  $2^{n+1}$ . The pairs  $(G_n, K_n)$  are symmetric Gelfand pairs (as we observed above).

Let  $n \geq k$ . Let the  $V_j$ 's be the spaces given in (2), with the relation  $bV_j = V_{qj}$ . In this case the subspaces which are invariant under the action of  $b$  are  $V_0$  and  $V_{2^{n-1}}$ , as one can get from the equation  $qj \equiv j(2^n)$  using that

$q \equiv 1(2)$  but  $q \not\equiv 1(4)$ . The spheres of radius  $1 < r < k$  decompose into  $2^{r-2}$  irreducible submodules of dimension 2 and each of them is the direct sum  $V_j \oplus V_{-j}$ , with  $j \equiv 0(2^{n-r})$  but  $j \not\equiv 0(2^{n-r+1})$  and they are non isomorphic.

Let  $r \geq k$ . We distinguish two subcases.

Let  $q = 4h - 1$  with  $h = 2x + 1$ . In this case necessarily  $k = 3$ , since  $(q - 1)(q + 1) = 8(2x + 1)(4x + 1)$ , and so the sphere of radius  $r$  decomposes exactly into two orbits under the action of  $b$ . Analogously to the case  $q \equiv 1(4)$ , we want to write a certain power of  $a$  on which the characters of these two representations take different values to deduce that they are non isomorphic.

Let us consider the element  $a^{2^{r-3}}$ . We observe that

$$(\omega^{2^{n-r}})^{2^{r-3}} = \omega^{2^{n-3}} = e^{\frac{2\pi i 2^{n-3}}{2^n}} = e^{\frac{i\pi}{4}}.$$

Moreover

$$(\omega q^{2^{n-r}})^{2^{r-3}} = \omega q^{2^{n-3}} = e^{\frac{2\pi i q 2^{n-3}}{2^n}} = e^{\frac{i q \pi}{4}}.$$

Now  $\frac{q\pi}{4} = \frac{4h\pi}{4} - \frac{\pi}{4} \equiv \pi - \frac{\pi}{4} (2\pi)$ , since  $h$  is odd. The same holds for the following powers of  $q$ .

Viceversa, the element  $\omega^{-2^{n-r}}$  raised to the power  $2^{r-3}$  gives the angle  $-\frac{\pi}{4}$ . As before, one can show that the element  $\omega^{-q 2^{n-r}}$  raised to the same power  $2^{r-3}$  gives the angle  $\pi + \frac{\pi}{4}$ . The same argument can be developed for the following powers of  $q$ . It is clear that the characters associated with these two representations take opposite nonzero values on the element  $a^{2^{r-3}} \in G_n$  and this completes the proof in this first case.

Now let  $q = 4h - 1$ , with  $h = 2x$ . Observe that

$$q^2 - 1 = (4h - 2)(4h) = 8h(2h - 1)$$

and so  $2^{k-3} \mid h$ . In this case we consider the element  $a^{2^{r-k}}$ . We have

$$(\omega^{2^{n-r}})^{2^{r-k}} = \omega^{2^{n-k}} = e^{\frac{2\pi i 2^{n-k}}{2^n}} = e^{\frac{2\pi i}{2^k}}.$$

Moreover

$$(\omega q^{2^{n-r}})^{2^{r-k}} = \omega q^{2^{n-k}} = e^{\frac{2\pi i q 2^{n-k}}{2^n}} = e^{\frac{2\pi q i}{2^k}}.$$

Now observe that

$$\frac{2\pi q}{2^k} = \frac{2\pi(4h - 1)}{2^k} = \frac{2\pi 4 \cdot 2^{k-3} m}{2^k} - \frac{2\pi}{2^k} = m\pi - \frac{2\pi}{2^k} \equiv \pi - \frac{2\pi}{2^k} (2\pi).$$



The last equality is true since the expression of  $h$  implies that  $m$  is odd. A similar argument can be developed for the following powers of  $q$ . So the angles  $\frac{2\pi}{2^k}$  and  $\pi - \frac{2\pi}{2^k}$  alternate in the orbit containing  $V_{2^{n-r}}$ .

Consider now any element  $2^{n-r} + t2^{n-r+1}$ , with  $t = 0, \dots, 2^{r-1} - 1$ . Its  $2^{r-k}$ -th power gives an angle equal to  $\frac{2\pi}{2^k}$  if the equation

$$2^{n-k} \equiv 2^{n-k} + t2^{n-k+1}(2^n),$$

which implies  $t \equiv 0(2^{k-1})$ , is satisfied.

On the other hand the  $2^{r-k}$ -th power gives an angle equal to  $\pi - \frac{2\pi}{2^k}$  if the equation

$$2^{n-k} + 2^{n-k} + t2^{n-k+1} \equiv 2^{n-1}(2^n),$$

which gives  $1 + t \equiv 2^{k-2}(2^{k-1})$ , is satisfied.

The values of  $t$  modulo  $2^{r-1}$  that satisfy these two equations are  $\frac{2 \cdot 2^{r-1}}{2^{k-1}} = 2^{r-k+1}$ , as many as the order of an orbit. So the elements whose  $2^{r-k}$ th power gives  $\frac{2\pi}{2^k}$  or  $\pi - \frac{2\pi}{2^k}$  are exactly those which satisfy the two equations.

Consider now the two equations

$$2^{n-k} + t2^{n-k+1} \equiv 2^{n-k} + s2^{n-k+1}(2^n)$$

and

$$2^{n-k} + t2^{n-k+1} + 2^{n-k} + s2^{n-k+1}(2^n) \equiv 2^{n-1}(2^n).$$

Their solutions are, respectively,

$$t \equiv s(2^{k-1}) \quad \text{and} \quad t \equiv 2^{k-2} - 1 - s(2^{k-1}).$$

So, if we fix a value of  $t$  modulo  $2^{r-1}$  not satisfying the first two equations, it is possible to find, by using these two new equations,  $2^{r-k+1}$  values which will form a new orbit, whose corresponding angle will be different from  $\frac{2\pi}{2^k}$  and  $\pi - \frac{2\pi}{2^k}$ .

This way we get  $2^{k-2}$  orbits in the decomposition of the sphere of radius  $r$ , all of them corresponding to different angles and so the corresponding  $2^{k-2}$  submodules have characters which take different values on  $a^{2^{r-k}}$  and so they are non isomorphic.

Consider now the general case of any odd  $q$  to show that the Gelfand pairs considered in this paper are non symmetric for every  $n \geq k$ . We

recall that the pair  $(G_n, K_n)$  is symmetric if the condition  $g^{-1} \in K_n g K_n$  is satisfied for every  $g \in G_n$ .

Let us choose  $g = a$  and consider  $K_n = \langle b : b^{2^{n-k+1}} = 1 \rangle$ . We can restrict our attention on the elements of the form  $b^{-i} a b^i$ . In fact the elements of the form  $b^i a b^i$  with  $i \neq -j$  give elements with occurrences of both  $a$  and  $b$  which are clearly different from  $a^{-1}$ . Since  $b^{-i} a b^i = a^{q^i}$ , we have to study the equation

$$(3) \quad q^i \equiv -1(2^n),$$

with  $i = 0, \dots, 2^{n-k+1} - 1$ . It is clear that in the case  $i = 0$  the equation is satisfied only in the trivial case  $n = 1$ , when  $G_n = C_2$ . Also in the case already developed  $q = -1$  this equation is satisfied by  $i = 1$ .

Consider now the equation (3). If  $2^n$  divides  $q^i + 1$ , then  $2^{n+1}$  divides  $(q^i + 1)(q^i - 1) = q^{2i} - 1$ . So one has  $q^{2i} \equiv 1(2^{n+1})$ . The smallest nonzero exponent satisfying this equation is, as we know,  $2^{n-k+2}$ . But it must be  $2i \leq 2^{n-k+2} - 2$ . This completes the proof. Collecting all the results obtained in this section we have the following

**THEOREM 4.2.** *Let  $q$  be an odd integer such that  $q^2 \equiv 1(2^k)$  but  $q^2 \not\equiv 1(2^{k+1})$ , then, for every  $n \geq k$ ,  $G_n = G/\text{Stab}_G(n) = C_{2^n} \times C_{2^{n-k+1}}$  and  $K_n = \text{Stab}_{G_n}(2^n \mathbb{Z}) = C_{2^{n-k+1}}$ . The associated pairs  $(G_n, K_n)$  are non symmetric Gelfand pairs.*

*For  $n < k$  and  $q \equiv 1(4)$  we get  $G_n = C_{2^n}$  and  $K_n = \{1\}$ , which are non symmetric (except in the case  $n = 1$ ) Gelfand pairs.*

*For  $1 < n < k$  and  $q \equiv -1(4)$  we get  $G_n = D_{2^n}$  and  $K_n = C_2$ , which are symmetric Gelfand pairs. For  $n = 1$  we have the symmetric Gelfand pair  $(C_2, 1)$ .*

### 4.3 – Final remarks.

In [CST] the authors show in a different way that the pairs  $(G_n, K_n)$  introduced above are Gelfand pairs by using the general theory of the Gelfand pairs associated to the semidirect products.

Let  $G = N \rtimes H$  be a finite group and let  $K \leq N$  be a subgroup of  $N$  invariant under the action of  $H$  and such that  $(N, K)$  is a Gelfand pair. Set  $X = N/K$  and let  $L(X) = \bigoplus_{i=0}^n V_i$  be the multiplicity-free decomposition of  $L(X)$  into irreducible  $N$ -submodules and let  $\phi_i \in V_i$  be the corresponding spherical functions. The map  $nK \mapsto nKH$  is a bijection between  $N/K$  and

$G/KH$ . The action of  $G$  on  $X = G/KH$  can be defined as  $nh(n'KH) = nhn'h^{-1}KH$  and the induced action on  $L(X)$  is, as usual,  $f^g(x) = f(g^{-1}x)$ . It is easy to check that if  $V_i$  is any irreducible  $N$ -invariant submodule in  $L(X)$ , then  $hV_i$  is still  $N$ -invariant and irreducible, for every  $h \in H$ , in other words  $H$  permutes the  $V_i$ 's.

So the decomposition of  $L(X)$  under the action of  $G$  into irreducible submodules is

$$(4) \quad L(X) = \bigoplus_{j=0}^r W_j,$$

where  $W_j = \bigoplus_{i:V_i \in \Gamma_j} V_i$  and  $\Gamma_j$  denotes, for  $j = 0, \dots, r$ , the  $H$ -orbits on  $\{V_0, V_1, \dots, V_n\}$ . The  $W_j$ 's are pairwise non isomorphic because the restrictions to  $N$  of the representations  $W_j$  and  $W_{j'}$  decompose into non isomorphic submodules for  $j \neq j'$ .

Hence the decomposition (4) is multiplicity-free and this implies that  $(G, KH)$  is a Gelfand pair. The corresponding spherical functions can be easily computed from the  $\phi_i$ 's and they are given by

$$\Phi_j = \frac{1}{|\Gamma_j|} \sum_{i:V_i \in \Gamma_j} \phi_i = \frac{1}{|H|} \sum_{h \in H} h\phi_i.$$

In the case of the groups introduced above we have  $N = C_{2^n}$ ,  $K = \{1\}$  and  $H = C_{2^{n-k+1}}$ . The  $V_i$ 's are the submodules in which decomposes the space  $L(L_n)$  under the action of  $C_{2^n}$  whose spherical functions are exactly the characters of  $C_{2^n}$ . The  $W_j$ 's are the orbits of the spaces  $V_i$ 's under the automorphism  $b$  with  $bV_i = V_{qi}$ .

The previous argument guarantees that  $(C_{2^n} \times C_{2^{n-k+1}}, C_{2^{n-k+1}})$  is a Gelfand pair and yields the associated spherical functions.

In Section 4.2 we preferred to give an explicit calculation of the irreducible subspaces of  $L(L_n)$  under the action of  $G_n$  using only congruences modulo  $2^n$  and identifying the spaces  $V_j$ 's with the vertices of the  $n$ -th level of the dyadic tree.

### REFERENCES

- [BGN] L. BARTHOLDI - R. I. GRIGORCHUK - V. NEKRASHEVYCH, *From fractal groups to fractal sets*, Fractals in Graz (P. Grabner and W. Woess editors), Trends in Mathematics, Birkäuser Verlag, Basel (2003), pp. 25–118.
- [BGŠ] L. BARTHOLDI - R. I. GRIGORCHUK - Z. ŠUNIK, *Branch groups*, Handbook of Algebra, North-Holland, Amsterdam (2003), Vol. 3, pp. 989–1112.

- [BŠ] L. BARTHOLDI - Z. ŠUNIK, *Some solvable automaton groups*, in *Topological and asymptotic aspects of group theory*, Contemp. Math., **394** (2006), pp. 11–30.
- [BS] G. BAUMSLAG - D. SOLITAR, *Some two-generator, one relator non-Hopfian groups*, Bull. Amer. Math. Soc., **68** (1962), pp. 199–201.
- [CST] T. CECCHERINI-SILBERSTEIN - F. SCARABOTTI - F. TOLLI, *Trees, wreath products and finite Gelfand pairs*, Adv. Math., no. 206 (2006), pp. 503–537.
- [CST1] T. CECCHERINI-SILBERSTEIN - F. SCARABOTTI - F. TOLLI, *Finite Gelfand pairs and their applications to Probability and Statistics*, J. Math. Sci. (New York), **141**, no. 2 (2007), pp. 1182–1229.
- [CST2] T. CECCHERINI-SILBERSTEIN - F. SCARABOTTI - F. TOLLI, *Harmonic Analysis on Finite Groups: Representation Theory, Gelfand Pairs and Markov Chains*, Cambridge Studies in Advanced Mathematics 108, Cambridge University Press, 2008.
- [D] P. DIACONIS, *Group Representations in Probability and Statistics*, IMS Hayward, CA, 1988.
- [F] A. FIGÀ-TALAMANCA, *Note del Seminario di Analisi Armonica*, a.a. 1992, Università di Roma “La Sapienza”.

Manoscritto pervenuto in redazione il 7 luglio 2007.