

Sectional Invariants of Scrolls Over a Smooth Projective Variety

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ABSTRACT - Let X be a smooth complex projective variety of dimension n and let \mathcal{E} be an ample vector bundle of rank r on X . Then we calculate the i th sectional Euler number $e_i(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ and the i th sectional Betti number $b_i(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ for $i \geq 2n - 3$ or $i = 1$, and the i th sectional Hodge number of type $(j, i - j)$ $h_i^{j, i-j}(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ for $i \geq 2n - 1$ and $0 \leq j \leq i$, where $\mathbb{P}_X(\mathcal{E})$ is the projective space bundle associated with \mathcal{E} and $H(\mathcal{E})$ is its tautological line bundle. Moreover we define a new invariant $v(X, \mathcal{E})$ of (X, \mathcal{E}) for $r \geq n - 1$. This invariant is thought to be a generalization of curve genus. We will investigate some properties of this invariant.

1. Introduction.

Let X be a projective variety of dimension n defined over the field of complex numbers, and let L be an ample (resp. a nef and big) line bundle on X . Then (X, L) is called a *polarized (resp. quasi-polarized) variety*. If X is smooth, then we say that (X, L) is a polarized (resp. quasi-polarized) *manifold*. In order to study polarized varieties, it is important to use an invariant of (X, L) . There are the following three invariants of (X, L) which are well-known.

- The degree L^n .
- The sectional genus $g(L)$.
- The Δ -genus $\Delta(L)$.

By using these invariants, many authors studied polarized varieties. In particular, P. Ionescu classified polarized manifolds by the degree under the assumption that L is very ample with $L^n \leq 8$ ([13], [14], and [15]), and

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T. Fujita classified polarized varieties by the Δ -genus and the sectional genus ([3]).

In [5], in order to study polarized varieties more deeply, the author introduced the notion of the i th sectional geometric genus $g_i(X, L)$ of (X, L) for every integer i with $0 \leq i \leq n$. This is a generalization of the degree and the sectional genus of (X, L) . Namely $g_0(X, L) = L^n$ and $g_1(X, L) = g(L)$.

Here we recall the reason why this invariant is called the i th sectional geometric genus. Let (X, L) be a polarized manifold of dimension $n \geq 2$ with $\text{Bs}|L| = \emptyset$, where $\text{Bs}|L|$ is the base locus of the complete linear system $|L|$. Let i be an integer with $1 \leq i \leq n$. Let X_{n-i} be the transversal intersection of general $n - i$ members of $|L|$. In this case X_{n-i} is a smooth projective variety of dimension i . Then we can prove that $g_i(X, L) = h^i(\mathcal{O}_{X_{n-i}})$, that is, $g_i(X, L)$ is the geometric genus of X_{n-i} .

Hence we can expect that $g_i(X, L)$ has analogous properties of the geometric genus of i -dimensional varieties.

In [6] and [7], we introduced the notion of the i th sectional H -arithmetic genus $\chi_i^H(X, L)$ of (X, L) . By definition we can prove that if $\text{Bs}|L| = \emptyset$, then $\chi_i^H(X, L) = \chi(\mathcal{O}_{X_{n-i}})$, where X_{n-i} is the transversal intersection of general $n - i$ members of $|L|$. Namely $\chi_i^H(X, L)$ is the Euler-Poincaré characteristic of the structure sheaf of X_{n-i} . ($\chi(\mathcal{O}_{X_{n-i}})$ is called the arithmetic genus of X_{n-i} in the sense of Hirzebruch. (See [12, 15.5, Section 15, Chapter IV]. We also call $\chi(\mathcal{O}_{X_{n-i}})$ the H -arithmetic genus of X_{n-i} .)

In [8], we also introduced some i th sectional invariants of (X, L) , that is, the i th sectional Euler number $e_i(X, L)$, the i th sectional Betti number $b_i(X, L)$ and the i th sectional Hodge number $h_i^{j, i-j}(X, L)$ of type $(j, i - j)$ for every integer j with $0 \leq j \leq i$, and we investigated some properties of these. In particular we proved that polarized manifolds' version of the Hodge duality and the Hodge decomposition hold (see [8, Theorem 3.1]).

In this paper we consider the i th sectional Euler number and the i th sectional Betti number of scrolls over a smooth projective variety. In this paper we say that a polarized manifold (P, H) is a *scroll over a smooth projective variety* X if there exists an ample vector bundle \mathcal{E} on X such that $(P, H) \cong (\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle on $\mathbb{P}_X(\mathcal{E})$.

In section 3, we calculate $e_i(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ and $b_i(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ for $i \geq 2n - 3$ and $i = 1$ (see Theorems 3.1 and 3.2). We also calculate $h_i^{j, i-j}(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ for $i \geq 2n - 1$ and $0 \leq j \leq i$ (see Theorem 3.3). In particular, by using these results, we can calculate these invariants of $(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ completely for $n = 1$ or 2 (see Corollaries 3.1, 3.3 and 3.4).

In section 4, we will define a new invariant of generalized polarized manifolds. Here a *generalized polarized manifold* means the pair (X, \mathcal{E}) where X is a smooth projective variety and \mathcal{E} is an ample vector bundle on X . Let $r := \text{rank}(\mathcal{E})$. Here we state the history of invariants of (X, \mathcal{E}) . First in [2], Fujita introduced the c_1 -sectional genus and the $\mathcal{O}(1)$ -sectional genus of (X, \mathcal{E}) . Next, in [1], for the case where $r = n - 1$, Ballico defined an invariant of (X, \mathcal{E}) which is called the *curve genus* $cg(X, \mathcal{E})$ of (X, \mathcal{E}) (see Definition 2.3), and several authors (in particular Lanteri, Maeda, Somese, and so on) studied this invariant.

As a generalization of the curve genus, for any ample vector bundle \mathcal{E} with $r \leq n - 1$, Ishihara ([16, Definition 1.1]) defined an invariant $g(X, \mathcal{E})$, which is called the c_r -*sectional genus* of (X, \mathcal{E}) , and in [9] we investigated some properties about $g(X, \mathcal{E})$. We note that if $n - r = 1$, then $g(X, \mathcal{E})$ is the curve genus. This invariant means the following: If a general element of $H^0(\mathcal{E})$ has a zero locus Z which is smooth of expected dimension $n - r$, then $g(X, \mathcal{E}) = g(Z, \det \mathcal{E}|_Z)$, that is, $g(X, \mathcal{E})$ is the sectional genus of $(Z, \det \mathcal{E}|_Z)$. Recently Fusi and Lanteri generalized this invariant. See [11] for detail.

In this paper, we will introduce a new invariant $v(X, \mathcal{E})$ of generalized polarized manifolds (X, \mathcal{E}) with $r \geq n - 1$, which is defined by using a result in section 3 (see Definition 4.1). Here we note that $v(X, \mathcal{E})$ is equal to the curve genus if $r = n - 1$. We will investigate $v(X, \mathcal{E})$ and give some results about this invariant (see Theorems 4.1 and 4.2, and Proposition 4.1).

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Notation and Conventions.

We say that X is a *variety* if X is an integral separated scheme of finite type. In particular X is irreducible and reduced if X is a variety. Varieties are always assumed to be defined over the field of complex numbers. In this article, we shall study mainly a smooth projective variety. The words “line bundles” and “Cartier divisors” are used interchangeably. The tensor products of line bundles are denoted additively.

$\mathcal{O}(D)$: invertible sheaf associated with a Cartier divisor D on X .

\mathcal{O}_X : the structure sheaf of X .

$\chi(\mathcal{F})$: the Euler-Poincaré characteristic of a coherent sheaf \mathcal{F} .

$h^i(\mathcal{F}) := \dim H^i(X, \mathcal{F})$ for a coherent sheaf \mathcal{F} on X .

$h^i(D) := h^i(\mathcal{O}(D))$ for a Cartier divisor D .

$q(X) (= h^1(\mathcal{O}_X))$: the irregularity of X .

$b_i(X) := \dim H^i(X, \mathbb{C})$.

K_X : the canonical divisor of X .

\mathbb{P}^n : the projective space of dimension n .

\mathbb{Q}^n : a smooth quadric hypersurface in \mathbb{P}^{n+1} .

\sim (or $=$): linear equivalence.

$\det(\mathcal{E}) := \wedge^r \mathcal{E}$, where \mathcal{E} is a vector bundle of rank r on X .

$\mathbb{P}_X(\mathcal{E})$: the projective space bundle associated with a vector bundle \mathcal{E} on X .

$\mathcal{H}(\mathcal{E})$: the tautological line bundle on $\mathbb{P}_X(\mathcal{E})$.

\mathcal{E}^\vee : the dual of a vector bundle \mathcal{E} .

$c_i(\mathcal{E})$: the i -th Chern class of a vector bundle \mathcal{E} .

$c_i(X) := c_i(\mathcal{T}_X)$, where \mathcal{T}_X is the tangent bundle of a smooth projective variety X .

For a real number m and a non-negative integer n , let

$$[m]^n := \begin{cases} m(m+1) \cdots (m+n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

$$[m]_n := \begin{cases} m(m-1) \cdots (m-n+1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Then for n fixed, $[m]^n$ and $[m]_n$ are polynomials in m whose degree are n .

For any non-negative integer n ,

$$n! := \begin{cases} [n]_n & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Assume that m and n are integers with $n \geq 0$. Then we put

$$\binom{m}{n} := \frac{[m]_n}{n!}$$

We note that $\binom{m}{n} = 0$ if $0 \leq m < n$, and $\binom{m}{0} = 1$.

2. Preliminaries.

NOTATION 2.1. (1) Let Y be a smooth projective variety of dimension i , let \mathcal{T}_Y be the tangent bundle of Y , and let $\Omega_Y (= \Omega_Y^1)$ be the dual bundle of \mathcal{T}_Y and $\Omega_Y^j := \wedge^j \Omega_Y$. For every integer j with $0 \leq j \leq i$, we put

$$\begin{aligned} h_{i,j}(c_1(Y), \dots, c_i(Y)) &:= \chi(\Omega_Y^j) \\ &= \int_Y \text{ch}(\Omega_Y^j) \text{Td}(\mathcal{T}_Y). \end{aligned}$$

(Here $\text{ch}(\mathcal{O}_Y^j)$ (resp. $\text{Td}(\mathcal{T}_Y)$) denotes the Chern character of \mathcal{O}_Y^j (resp. the Todd class of \mathcal{T}_Y). See [10, Example 3.2.3 and Example 3.2.4].)

(2) Let (M, L) be a polarized manifold of dimension m . For every integers i and j with $0 \leq j \leq i \leq m$, we put

$$C_j^i(M, L) := \sum_{l=0}^j (-1)^l \binom{m-i+l-1}{l} c_{j-l}(M) L^l,$$

$$w_i^j(M, L) := h_{i,j}(C_1^i(M, L), \dots, C_i^i(M, L)) L^{n-i}.$$

(3) Let M be a smooth projective variety of dimension m . For every integers i and j with $0 \leq j \leq i \leq m$, we put

$$H_1(i, j) := \begin{cases} \sum_{s=0}^{i-j-1} (-1)^s h^s(\mathcal{O}_M^j) & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases}$$

$$H_2(i, j) := \begin{cases} \sum_{t=0}^{j-1} (-1)^{i-t} h^t(\mathcal{O}_M^{i-j}) & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

DEFINITION 2.1. (See [8, Definition 3.1].) Let (M, L) be a polarized manifold of dimension m , and let i and j be integers with $0 \leq i \leq m$ and $0 \leq j \leq i$.

(1) The i th sectional Euler number $e_i(M, L)$ of (M, L) is defined by the following:

$$e_i(M, L) := \sum_{l=0}^i (-1)^l \binom{m-i+l-1}{l} c_{i-l}(M) L^{m-i+l}.$$

(2) The i th sectional Betti number $b_i(M, L)$ of (M, L) is defined by the following:

$$b_i(M, L) := \begin{cases} e_0(M, L) & \text{if } i = 0, \\ (-1)^i \left(e_i(M, L) - \sum_{j=0}^{i-1} 2(-1)^j b_j(M) \right) & \text{if } 1 \leq i \leq m. \end{cases}$$

(3) The i th sectional Hodge number $h_i^{j, i-j}(M, L)$ of type $(j, i-j)$ of

(M, L) is defined by the following:

$$h_i^{j,i-j}(M, L) := (-1)^{i-j} \left\{ w_i^j(M, L) - H_1(i, j) - H_2(i, j) \right\}.$$

REMARK 2.1. (1) If $i = 0$, then

$$e_0(M, L) = b_0(M, L) = h_0^{0,0}(M, L) = L^m.$$

(2) If $i = 1$, then

$$e_1(M, L) = 2 - 2g(L),$$

$$b_1(M, L) = 2g(L),$$

$$h_1^{1,0}(M, L) = h_1^{0,1}(M, L) = g(L).$$

(3) If $i = m$, then

$$e_m(M, L) = e(M),$$

$$b_m(M, L) = b_m(M),$$

$$h_m^{j,m-j}(M, L) = h^{j,m-j}(M),$$

$$h_m^{m-j,j}(M, L) = h^{m-j,j}(M).$$

PROPOSITION 2.1. *Let (M, L) be a polarized manifold of dimension m .*

(1) *For every integer i with $0 \leq i \leq m$, the following hold:*

$$(1.1) \quad b_i(M, L) = \sum_{k=0}^i h_i^{k,i-k}(M, L).$$

$$(1.2) \quad h_i^{j,i-j}(M, L) = h^{i-j,j}(M, L).$$

$$(1.3) \quad h_i^{i,0}(M, L) = h_i^{0,i}(M, L) = g_i(M, L).$$

(2) *Assume that L is base point free. Then for every integers i and j with $1 \leq i \leq m$ and $0 \leq j \leq i$ the following hold.*

$$(2.1) \quad b_i(M, L) \geq b_i(M).$$

$$(2.2) \quad h_i^{j,i-j}(M, L) \geq h^{j,i-j}(M).$$

PROOF. See [8, Theorem 3.1 (3.1.1), (3.1.3), (3.1.4) and Proposition 3.3 (2) and (3)]. \square

PROPOSITION 2.2. For every integers a, k, l and r with $0 \leq l$,

$$\sum_{j=0}^l (-1)^j \binom{r+j-a}{j} \binom{r-k}{l-j} = (-1)^l \binom{k-a+l}{l}.$$

PROOF. See [8, Proposition 2.5]. \square

NOTATION 2.2. Let X be a smooth projective variety of dimension $n \geq 1$ and let \mathcal{E} be an ample vector bundle of rank r on X . We put $P := \mathbb{P}_X(\mathcal{E})$, $H := H(\mathcal{E})$ and $m := \dim P$. Then $m = n + r - 1$. In this paper we assume that $r \geq 2$.

REMARK 2.2. Let $X, \mathcal{E}, P, H, m, n$ and r be as in Notation 2.2.

(1) By [18, (2.1) Proposition], we have

$$b_j(P) = b_j(X) + b_{j-2}(X) + \cdots + b_{j-2r+2}(X).$$

(2) Let i be an integer with $i \leq m$. Then $n + r - 1 \geq i$ and we obtain $r \geq i - n + 1$.

(2.1) If $i \geq 2n - 2$ and $i - 1 \geq j$, then

$$\begin{aligned} j - 2r + 2 &\leq (i - 1) - 2(i - n + 1) + 2 \\ &= 2n - 1 - i \\ &\leq 1. \end{aligned}$$

So by (1) above, if $i \geq 2n - 2$ and $i - 1 \geq j$, then by (1) we have

$$b_j(P) = \begin{cases} \sum_{k=0}^l b_{j-2k}(X) & \text{if } j = 2l, \\ \sum_{k=0}^l b_{j-2k}(X) & \text{if } j = 2l + 1. \end{cases}$$

By the same argument as this, if $i \geq 2n - 1$, then we see that

$$b_i(P) = \begin{cases} \sum_{k=0}^l b_{i-2k}(X) & \text{if } i = 2l, \\ \sum_{k=0}^l b_{i-2k}(X) & \text{if } i = 2l + 1. \end{cases}$$

(2.2) Assume that $i \leq m - 1$. If $i = 2n - 3$ and $i - 1 \geq j$, then $j - 2r + 2 \leq 2$. If this equality holds, then $i = n + r - 1 = m$. But this contradicts the assumption. Hence $j - 2r + 2 \leq 1$, and by (1) above we have

$$b_j(P) = \begin{cases} \sum_{k=0}^l b_{j-2k}(X) & \text{if } j = 2l, \\ \sum_{k=0}^l b_{j-2k}(X) & \text{if } j = 2l + 1. \end{cases}$$

By the same argument as this, if $i \leq m - 1$, $n \geq 1$ (resp. $n \geq 2$) and $i = 2n - 2$ (resp. $i = 2n - 3$), then we see that $b_{2n-2}(P) = \sum_{k=0}^{n-1} b_{2n-2-2k}(X)$ (resp. $b_{2n-3}(P) = \sum_{k=0}^{n-2} b_{2n-3-2k}(X)$).

DEFINITION 2.2. Let X be a smooth projective variety of dimension n and let \mathcal{E} be a vector bundle of rank r on X .

(1) The *Chern polynomial* $c_t(\mathcal{E})$ is defined by the following:

$$c_t(\mathcal{E}) = 1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \cdots.$$

(2) For every integer j with $j \geq 0$, the *j th Segre class* $s_j(\mathcal{E})$ of \mathcal{E} is defined by the following equation: $c_t(\mathcal{E}^\vee)s_t(\mathcal{E}) = 1$, where $c_t(\mathcal{E}^\vee)$ is the Chern polynomial of \mathcal{E}^\vee and $s_t(\mathcal{E}) = \sum_{j \geq 0} s_j(\mathcal{E})t^j$.

REMARK 2.3. (1) Let X be a smooth projective variety and let \mathcal{F} be a vector bundle on X . Let $\tilde{s}_j(\mathcal{F})$ be the Segre class which is defined in [10, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.

(2) For every integer i with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

DEFINITION 2.3. Let X be a smooth projective variety of dimension n and let \mathcal{E} be an ample vector bundle of rank $n - 1$ on X . Then the *curve genus* $cg(X, \mathcal{E})$ of (X, \mathcal{E}) is defined as follows:

$$cg(X, \mathcal{E}) = 1 + \frac{1}{2}(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}).$$

3. Sectional invariants of scrolls over a smooth projective manifold.

THEOREM 3.1. *Let $X, \mathcal{E}, P, H, m, n$ and r be as in Notation 2.2. Then the following hold.*

(3.1.1) If $i \geq 2n - 1$, then $e_i(P, H) = (i - n + 1)c_n(X)$.

(3.1.2) If $n \geq 2$, then $e_{2n-2}(P, H) = (n - 1)c_n(X) + c_n(\mathcal{E})$.

(3.1.3) If $n \geq 3$, then $e_{2n-3}(P, H) = (n - 2)(c_n(X) - c_n(\mathcal{E})) - c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E}))$.

(3.1.4) $e_1(P, H) = -(n - 2)s_n(\mathcal{E}) - (c_1(\mathcal{E}) + K_X)s_{n-1}(\mathcal{E})$.

(3.1.5) $e_0(P, H) = s_n(\mathcal{E})$.

PROOF. By [10, Example 3.2.11] and Remark 2.3 (1), for every integer l with $0 \leq l \leq m$

$$c_l(P) = \sum_{j=0}^l \sum_{k=0}^j \binom{r-k}{j-k} c_k(p^* \mathcal{E}^\vee) H(\mathcal{E})^{j-k} c_{l-j}(p^* \mathcal{T}_X).$$

(Here $p : P \rightarrow X$ denotes the projection.) Hence

$$\begin{aligned} e_i(P, H) &= \sum_{j=0}^i (-1)^j \binom{m-i+j-1}{j} c_{i-j}(P) H^{m-i+j} \\ &= \sum_{j=0}^i (-1)^j \binom{m-i+j-1}{j} \\ &\quad \left\{ \sum_{s=0}^{i-j} \sum_{u=0}^s \binom{r-u}{s-u} c_u(p^* \mathcal{E}^\vee) H(\mathcal{E})^{m-i+j-u+s} c_{i-j-s}(p^* \mathcal{T}_X) \right\} \\ &= \sum_{\substack{0 \leq k, t \\ 0 \leq k+t \leq i}} \left(\sum_{j=0}^{i-t-k} (-1)^j \binom{r+j+n-i-2}{j} \binom{r-k}{i-t-k-j} \right) \\ &\quad c_k(p^* \mathcal{E}^\vee) c_t(p^* \mathcal{T}_X) H(\mathcal{E})^{m-k-t}. \end{aligned}$$

By Proposition 2.2 and [3, (3.7) in Chapter 0], we get

$$\begin{aligned} (1) e_i(P, H) &= \sum_{\substack{0 \leq k, t \\ 0 \leq k+t \leq i}} (-1)^{i-t-k} \binom{k+(n-i-2)+i-t-k}{i-t-k} c_k(p^* \mathcal{E}^\vee) c_t(p^* \mathcal{T}_X) H(\mathcal{E})^{m-k-t} \\ &= \sum_{\substack{0 \leq k, t \\ 0 \leq k+t \leq i}} (-1)^{i-t-k} \binom{n-t-2}{i-t-k} c_k(p^* \mathcal{E}^\vee) c_t(p^* \mathcal{T}_X) H(\mathcal{E})^{m-k-t} \\ &= \sum_{\substack{0 \leq k, t \\ 0 \leq k+t \leq i}} (-1)^{i-t-k} \binom{n-t-2}{i-t-k} c_k(\mathcal{E}^\vee) c_t(\mathcal{T}_X) s_{n-k-t}(\mathcal{E}). \end{aligned}$$

Here we put

$$E(i, k, t) := (-1)^{i-t-k} \binom{n-t-2}{i-t-k} c_k(\mathcal{E}^\vee) c_t(\mathcal{T}_X) s_{n-k-t}(\mathcal{E}).$$

In order to calculate $e_i(P, H)$, we have only to consider the case where $E(i, k, t) \neq 0$. We note that if $k+t > n$, then $c_k(\mathcal{E}^\vee) c_t(\mathcal{T}_X) s_{n-k-t}(\mathcal{E}) = 0$. So we may assume that $k+t \leq n$.

(a) The case where $i \geq 2n-1$.

First we note that if $(n, i) = (1, 1)$, then

$$e_i(P, H) = e_1(P, H) = 2 - b_1(P, H) = 2 - 2g_1(P, H) = 2 - 2q(X) = c_1(X).$$

So we assume that $(n, i) \neq (1, 1)$.

Here we note that $(n-t-2) - (i-t-k) = n-i-2+k \leq -n+k-1 \leq -1$

and $k+t \leq i$. Hence $\binom{n-t-2}{i-t-k} \neq 0$ if and only if one of the following holds.

(a.1) $i-t-k = 0$.

(a.2) $i-t-k > 0$ and $n-t-2 < 0$.

(a.1) If $i-t-k = 0$, then $2n-1 \leq i = t+k \leq n$. Hence $n = 1$ and $i = 1$. But this contradicts the assumption here.

(a.2) If $i-t-k > 0$ and $n-t-2 < 0$, then $t = n-1$ or n because $k+t \leq n$ and $k \geq 0$. Hence $(t, k) = (n-1, 0)$, $(n-1, 1)$ or $(n, 0)$.

If $(t, k) = (n-1, 0)$, then

$$\binom{n-t-2}{i-t-k} = \binom{-1}{i-n+1} = (-1)^{i-n+1}.$$

If $(t, k) = (n-1, 1)$, then

$$\binom{n-t-2}{i-t-k} = \binom{-1}{i-n} = (-1)^{i-n}.$$

If $(t, k) = (n, 0)$, then

$$\binom{n-t-2}{i-t-k} = \binom{-2}{i-n} = (-1)^{i-n}(i-n+1).$$

Hence if $(n, i) \neq (1, 1)$, then

$$\begin{aligned} e_i(P, H) &= (-1)^{2(i-n+1)}c_{n-1}(\mathcal{T}_X)s_1(\mathcal{E}) + (-1)^{2(i-n)}c_1(\mathcal{E}^\vee)c_{n-1}(\mathcal{T}_X) \\ &\quad + (-1)^{2(i-n)}(i-n+1)c_n(\mathcal{T}_X) \\ &= (i-n+1)c_n(X). \end{aligned}$$

Therefore in any case

$$e_i(P, H) = (i-n+1)c_n(X)$$

if $i \geq 2n-1$.

(b) The case where $i = 2n-2$. Here we note that $n \geq 2$ in this case.

Then $\binom{n-t-2}{i-t-k} = \binom{n-t-2}{2n-t-2-k}$. Here we note that $(n-t-2) \leq 2n-t-2-k$. Hence $\binom{n-t-2}{2n-t-2-k} \neq 0$ if and only if one of the following holds.

(b.1) $k = n$.

(b.2) $k < n$ and $n-t-2 < 0$.

(b.1) If $k = n$, then $t = 0$ because $t+k \leq n$ and $t \geq 0$, and

$\binom{n-t-2}{2n-t-2-k} = 1$. Here we note that $n-t-2 \geq 0$ in this case because $n \geq 2$ and $t = 0$. Hence

$$E(2n-2, n, 0) = (-1)^{2n-2-n}c_n(\mathcal{E}^\vee) = c_n(\mathcal{E}).$$

(b.2) If $k < n$ and $n-t-2 < 0$, then $t = n-1$ or n . Hence $(t, k) = (n-1, 0)$, $(n-1, 1)$ or $(n, 0)$. By the same argument as (a.2), we obtain

$$\begin{aligned} &E(2n-2, 0, n-1) + E(2n-2, 1, n-1) + E(2n-2, 0, n) \\ &= (-1)^{2(2n-2-n+1)}c_{n-1}(X)s_1(\mathcal{E}) + (-1)^{2(2n-2-n)}c_1(\mathcal{E}^\vee)c_{n-1}(X) \\ &\quad + (-1)^{2(2n-2-n)}(n-1)c_n(X) \\ &= (n-1)c_n(X). \end{aligned}$$

Hence we get

$$e_{2n-2}(P, H) = (n-1)c_n(X) + c_n(\mathcal{E}).$$

(c) Assume that $i = 2n - 3$. Here we note that $n \geq 3$ in this case. Then

$$\binom{n-t-2}{2n-t-k-3} \neq 0 \text{ if and only if one of the following holds.}$$

(c.1) $k = n$.

(c.2) $k = n - 1$.

(c.3) $k \leq n - 2$ and $t > n - 2$.

First we consider the case (c.1). Then $t = 0$ because $k + t \leq n$ and $t \geq 0$. So we have

$$\begin{aligned} E(2n-3, n, 0) &= (-1)^{2n-3-0-n} \binom{n-0-2}{2n-3-0-n} c_n(\mathcal{E}^\vee) \\ &= - \binom{n-2}{n-3} c_n(\mathcal{E}) \\ &= -(n-2)c_n(\mathcal{E}). \end{aligned}$$

Next we consider the case (c.2). Then $t = 0$ or 1 , and $\binom{n-t-2}{2n-t-k-3} = 1$. Hence we have $E(2n-3, n-1, 0) = -c_{n-1}(\mathcal{E})s_1(\mathcal{E}) = -c_{n-1}(\mathcal{E})c_1(\mathcal{E})$ and $E(2n-3, n-1, 1) = c_{n-1}(\mathcal{E})c_1(X)$.

Finally we consider the case (c.3). Then $(k, t) = (0, n-1), (1, n-1)$ or $(0, n)$. Hence by the same argument as above

$$E(2n-3, 0, n-1) + E(2n-3, 1, n-1) + E(2n-3, 0, n) = (n-2)c_n(X).$$

Therefore

$$\begin{aligned} e_{2n-3}(P, H) &= -(n-2)c_n(\mathcal{E}) - c_{n-1}(\mathcal{E})c_1(\mathcal{E}) + c_{n-1}(\mathcal{E})c_1(X) + (n-2)c_n(X) \\ &= (n-2)(c_n(X) - c_n(\mathcal{E})) + c_{n-1}(\mathcal{E})(c_1(X) - c_1(\mathcal{E})) \\ &= (n-2)(c_n(X) - c_n(\mathcal{E})) - c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})). \end{aligned}$$

(d) The case where $i = 1$. Then by (1) we get

$$\begin{aligned} e_1(P, H) &= \sum_{\substack{0 \leq k, t \\ 0 \leq k+t \leq 1}} (-1)^{1-t-k} \binom{n-t-2}{1-t-k} c_k(\mathcal{E}^\vee) c_t(\mathcal{T}_X) s_{n-k-t}(\mathcal{E}) \\ &= -(n-2)s_n(\mathcal{E}) + (c_1(\mathcal{E}^\vee) + c_1(\mathcal{T}_X))s_{n-1}(\mathcal{E}) \\ &= -(n-2)s_n(\mathcal{E}) - (c_1(\mathcal{E}) + K_X)s_{n-1}(\mathcal{E}). \end{aligned}$$

(e) The case where $i = 0$. Then by (1) we get $e_0(P, H) = s_n(\mathcal{E})$.

We get the assertion of Theorem 3.1. \square

By Theorem 3.1, we get the following for $n = 1$ and 2.

COROLLARY 3.1. *Let X, \mathcal{E}, P, H and n be as in Notation 2.2.*

(3.1.1) *Assume that $n = 1$. Then we get the following:*

$$e_i(P, H) = \begin{cases} i(2 - 2g(X)) & \text{if } i \geq 1, \\ \deg \mathcal{E} & \text{if } i = 0. \end{cases}$$

(3.1.2) *Assume that $n = 2$. Then we get the following:*

$$e_i(P, H) = \begin{cases} (i-1)c_2(X) & \text{if } i \geq 3, \\ c_2(X) + c_2(\mathcal{E}) & \text{if } i = 2, \\ -(c_1(\mathcal{E}) + K_X)c_1(\mathcal{E}) & \text{if } i = 1, \\ s_2(\mathcal{E}) & \text{if } i = 0. \end{cases}$$

THEOREM 3.2. *Let $X, \mathcal{E}, P, H, m, n$ and r be as in Notation 2.2. Then the following hold.*

(3.2.1) *If $m \geq i \geq 2n - 1$, then $b_i(P, H) = b_i(P)$.*

(3.2.2) *If $n \geq 2$ and $m > 2n - 2$, then $b_{2n-2}(P, H) = b_{2n-2}(P) + c_n(\mathcal{E}) - 1$.*

(3.2.3) *If $n \geq 3$ and $m > 2n - 3$, then*

$$b_{2n-3}(P, H) = b_{2n-3}(P) + (n-2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) + 2 - 2g(X).$$

(3.2.4) $b_1(P, H) = 2 + (n-2)s_n(\mathcal{E}) + (c_1(\mathcal{E}) + K_X)s_{n-1}(\mathcal{E})$.

(3.2.5) $b_0(P, H) = s_n(\mathcal{E})$.

PROOF. Since $e_i(P, H)$ has been calculated in Theorem 3.1, we need to study $\sum_{k=0}^{i-1} (-1)^k b_k(P)$. By using Remark 2.2, we calculate this.

(a) The case where $i \geq 2n - 1$. Since $b_m(P, H) = b_m(P)$, we assume that $i < m$.

(a.1) Assume that i is even. Then $i \geq 2n$. Here we note that $b_j(X) = 0$ if $j > 2n$. Then

$$\begin{aligned} & \sum_{k=0}^{i-1} (-1)^k b_k(P) \\ &= \sum_{k=0}^n \binom{i}{2-k} b_{2k}(X) - \sum_{k=1}^n \binom{i}{2-k+1} b_{2k-1}(X) \\ &= \sum_{k=0}^n (n-k+1) b_{2k}(X) - \sum_{k=1}^n (n-k+2) b_{2k-1}(X) + \frac{i-2n-2}{2} e(X). \end{aligned}$$

Here we note that by Remark 2.2 (2.1), we have $b_i(P) = \sum_{k=0}^n b_{2k}(X)$ because $i \geq 2n$ in this case. Since i is even, we have

$$\begin{aligned} & b_i(P, H) - b_i(P) \\ &= (-1)^i \left(e_i(P, H) - 2 \sum_{j=0}^{i-1} (-1)^j b_j(P) \right) - \sum_{k=0}^n b_{2k}(X) \\ &= (i-n+1) c_n(X) - 2 \sum_{k=0}^n (n-k+1) b_{2k}(X) \\ & \quad + 2 \sum_{k=1}^n (n-k+2) b_{2k-1}(X) - (i-2n-2) e(X) - \sum_{k=0}^n b_{2k}(X) \\ &= (n+3) e(X) + \sum_{k=0}^n (2k-2n-3) b_{2k}(X) + \sum_{k=1}^n (2n-2k+4) b_{2k-1}(X) \\ &= \sum_{k=0}^n (2k-n) b_{2k}(X) - \sum_{k=1}^n (2k-n-1) b_{2k-1}(X). \end{aligned}$$

Here we prove the following.

$$\text{CLAIM 3.1.} \quad (1) \sum_{k=0}^n (2k-n) b_{2k}(X) = 0.$$

$$(2) \sum_{k=1}^n (2k-n-1) b_{2k-1}(X) = 0.$$

PROOF. (1) Assume that $n = 2l$. Then we note that $(2k - n)b_{2k}(X) = 0$ if $k = l$. So by Poincaré duality

$$\begin{aligned} & \sum_{k=0}^n (2k - n)b_{2k}(X) \\ &= \sum_{k=0}^{l-1} (2k - n)b_{2k}(X) + \sum_{k=l+1}^n (2k - n)b_{2k}(X) \\ &= \sum_{k=0}^{l-1} (2k - n)b_{2k}(X) + \sum_{k=0}^{l-1} (n - 2k)b_{2n-2k}(X) \\ &= \sum_{k=0}^{l-1} (2k - n)b_{2k}(X) - \sum_{k=0}^{l-1} (2k - n)b_{2k}(X) \\ &= 0. \end{aligned}$$

If $n = 2l + 1$, then by Poincaré duality

$$\begin{aligned} & \sum_{k=0}^n (2k - n)b_{2k}(X) \\ &= \sum_{k=0}^l (2k - n)b_{2k}(X) + \sum_{k=l+1}^n (2k - n)b_{2k}(X) \\ &= \sum_{k=0}^l (2k - n)b_{2k}(X) + \sum_{k=0}^l (n - 2k)b_{2n-2k}(X) \\ &= \sum_{k=0}^l (2k - n)b_{2k}(X) - \sum_{k=0}^l (2k - n)b_{2k}(X) \\ &= 0. \end{aligned}$$

Hence we obtain the assertion of (1).

(2) This can be proved by the same argument as (1). \square

By Claim 3.1, $b_i(P, H) = b_i(P)$ if $i \geq 2n - 1$ and i is even.

(a.2) Assume that i is odd. Then

$$\begin{aligned}
 & \sum_{j=0}^{i-1} (-1)^j b_j(P) \\
 &= \sum_{k=0}^n \left(\frac{i+1}{2} - k \right) b_{2k}(X) - \sum_{k=1}^n \left(\frac{i-1}{2} - k + 1 \right) b_{2k-1}(X) \\
 &= \sum_{k=0}^n (n - k + 1) b_{2k}(X) - \sum_{k=1}^n (n - k + 1) b_{2k-1}(X) \\
 & \quad + \left(\frac{i+1}{2} - n - 1 \right) e(X).
 \end{aligned}$$

By Remark 2.2 (2.1), we have $b_i(P) = \sum_{k=1}^n b_{2k-1}(X)$ because $i \geq 2n - 1$ in this case. Hence by Theorem 3.1 and Claim 3.1, we have

$$\begin{aligned}
 & b_i(P, H) - b_i(P) \\
 &= - \left((i - n + 1)e(X) - 2 \sum_{j=0}^{i-1} (-1)^j b_j(P) \right) - b_i(P) \\
 &= -(n + 2)e(X) + \sum_{k=0}^n (2n - 2k + 2) b_{2k}(X) - \sum_{k=1}^n (2n - 2k + 3) b_{2k-1}(X) \\
 &= \sum_{k=0}^n (n - 2k) b_{2k}(X) - \sum_{k=1}^n (n - 2k + 1) b_{2k-1}(X) \\
 &= 0.
 \end{aligned}$$

Therefore $b_i(P, H) = b_i(P)$ if $i \geq 2n - 1$ and i is odd.

In any case, if $m \geq i \geq 2n - 1$, then $b_i(P, H) = b_i(P)$.

(b) The case where $i = 2n - 2$ and $2n - 2 < m$. Then by Remark 2.2 (2.1) we obtain

$$b_j(P) = \begin{cases} \sum_{k=0}^l b_{j-2k}(X) & \text{if } j = 2l, \\ \sum_{k=0}^l b_{j-2k}(X) & \text{if } j = 2l + 1 \end{cases}$$

for every integer j with $j < 2n - 2$. Hence

$$\sum_{j=0}^{2n-3} (-1)^j b_j(P) = \sum_{k=0}^{n-2} (n-k-1) b_{2k}(X) - \sum_{k=1}^{n-1} (n-k) b_{2k-1}(X).$$

By assumption, we have $n + r - 1 = m \geq i + 1 = 2n - 1$. So by Remark 2.2 (2.2) we obtain

$$b_{2n-2}(P) = \sum_{k=0}^{n-1} b_{2k}(X).$$

Therefore

$$\begin{aligned} & b_{2n-2}(P, H) - b_{2n-2}(P) \\ &= \left(e_{2n-2}(P, H) - 2 \sum_{j=0}^{2n-3} (-1)^j b_j(P) \right) - b_{2n-2}(P) \\ &= (n-1)e(X) + c_n(\mathcal{E}) - \sum_{k=0}^{n-2} (2n-2k-2)b_{2k}(X) \\ &\quad + \sum_{k=1}^{n-1} (2n-2k)b_{2k-1}(X) - \sum_{k=0}^{n-1} b_{2k}(X) \\ &= (n-1)e(X) + c_n(\mathcal{E}) - \sum_{k=0}^{n-2} (2n-2k-1)b_{2k}(X) \\ &\quad + \sum_{k=1}^{n-1} (2n-2k)b_{2k-1}(X) - b_{2n-2}(X) \\ &= (n-1)e(X) + c_n(\mathcal{E}) + e(X) - 1 - \sum_{k=0}^{n-1} (2n-2k)b_{2k}(X) \\ &\quad + \sum_{k=1}^{n-1} (2n-2k+1)b_{2k-1}(X) + b_{2n-1}(X) \\ &= ne(X) + c_n(\mathcal{E}) - 1 - \sum_{k=0}^{n-1} (2n-2k)b_{2k}(X) \\ &\quad + \sum_{k=1}^n (2n-2k+1)b_{2k-1}(X). \end{aligned}$$

Since

$$\sum_{k=0}^{n-1} (2n - 2k)b_{2k}(X) = \sum_{k=0}^n (2n - 2k)b_{2k}(X),$$

we obtain

$$\begin{aligned} & b_{2n-2}(P, H) - b_{2n-2}(P) \\ &= ne(X) + c_n(\mathcal{E}) - 1 - \sum_{k=0}^n (2n - 2k)b_{2k}(X) \\ &\quad + \sum_{k=1}^n (2n - 2k + 1)b_{2k-1}(X) \\ &= c_n(\mathcal{E}) - 1 + \sum_{k=0}^n (2k - n)b_{2k}(X) \\ &\quad - \sum_{k=1}^n (2k - n - 1)b_{2k-1}(X). \end{aligned}$$

By Claim 3.1, we get

$$b_{2n-2}(P, H) = b_{2n-2}(P) + c_n(\mathcal{E}) - 1.$$

(c) The case where $i = 2n - 3$ and $i < m$. Then by Remark 2.2 (2.2) we obtain

$$b_j(P) = \begin{cases} \sum_{k=0}^l b_{j-2k}(X) & \text{if } j = 2l, \\ \sum_{k=0}^l b_{j-2k}(X) & \text{if } j = 2l + 1 \end{cases}$$

for every integer j with $j < 2n - 3$. Hence

$$\sum_{j=0}^{2n-4} b_j(P) = \sum_{k=0}^{n-2} (n - k - 1)b_{2k}(X) - \sum_{k=1}^{n-2} (n - k - 1)b_{2k-1}(X).$$

By assumption we have $n + r - 1 = m \geq i + 1 = 2n - 2$. Hence by Remark 2.2 (2.2) we obtain

$$b_{2n-3}(P) = \sum_{k=1}^{n-1} b_{2k-1}(X).$$

Therefore

$$\begin{aligned}
& b_{2n-3}(P, H) - b_{2n-3}(P) \\
&= -e_{2n-3}(P, H) + 2 \sum_{j=0}^{2n-4} (-1)^j b_j(P) - b_{2n-3}(P) \\
&= -(n-2)(e(X) - c_n(\mathcal{E})) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) \\
&\quad + \sum_{k=0}^{n-2} (2n-2k-2)b_{2k}(X) - \sum_{k=1}^{n-2} (2n-2k-2)b_{2k-1}(X) \\
&\quad - \sum_{k=1}^{n-1} b_{2k-1}(X) \\
&= -(n-2)(e(X) - c_n(\mathcal{E})) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) \\
&\quad + \sum_{k=0}^{n-2} (2n-2k-2)b_{2k}(X) - \sum_{k=1}^{n-2} (2n-2k-1)b_{2k-1}(X) \\
&\quad - b_{2n-3}(X) \\
&= -(n-2)(e(X) - c_n(\mathcal{E})) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) \\
&\quad - 2e(X) + 2b_{2n-2}(X) + 2b_{2n}(X) - 3b_{2n-3}(X) - 2b_{2n-1}(X) \\
&\quad + \sum_{k=0}^{n-2} (2n-2k)b_{2k}(X) - \sum_{k=1}^{n-2} (2n-2k+1)b_{2k-1}(X) \\
&= -(n-2)(e(X) - c_n(\mathcal{E})) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) \\
&\quad + \sum_{k=0}^n (2n-2k)b_{2k}(X) - \sum_{k=1}^n (2n-2k+1)b_{2k-1}(X) \\
&\quad - 2e(X) + 2b_{2n}(X) - b_{2n-1}(X) \\
&= -(n-2)e(X) + (n-2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) \\
&\quad + \sum_{k=0}^n (n-2k)b_{2k}(X) - \sum_{k=1}^n (n-2k+1)b_{2k-1}(X) \\
&\quad + (n-2)e(X) + 2b_{2n}(X) - b_{2n-1}(X).
\end{aligned}$$

Since $b_{2n}(X) = 1$ and $b_{2n-1}(X) = b_1(X) = 2q(X)$, by Claim 3.1 we obtain

$$\begin{aligned} & b_{2n-3}(P, H) - b_{2n-3}(P) \\ &= (n-2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) + 2 - 2q(X). \end{aligned}$$

(d) The case where $i = 1$.

Then by Theorem 3.1 (3.1.4) and the definition of $b_1(P, H)$

$$\begin{aligned} b_1(P, H) &= -e_1(P, H) + 2b_0(P) \\ &= 2 + (n-2)s_n(\mathcal{E}) + (c_1(\mathcal{E}) + K_X)s_{n-1}(\mathcal{E}). \end{aligned}$$

(e) The case where $i = 0$.

Then by Theorem 3.1 (3.1.5) and the definition of $b_0(P, H)$,

$$b_0(P, H) = e_0(P, H) = s_n(\mathcal{E}).$$

Therefore we get the assertion. \square

COROLLARY 3.2. *Let X, \mathcal{E}, P, H, m and n be as in Notation 2.2. If $n \geq 2$ and $m > i \geq 2n - 2$, then $b_i(P, H) \geq b_i(P)$.*

PROOF. If $i \geq 2n - 1$, then by Theorem 3.2 (3.2.1) we get the assertion. Next we consider the case where $i = 2n - 2$. Since \mathcal{E} is ample, we have $c_n(\mathcal{E}) \geq 1$. Therefore by Theorem 3.2 (3.2.2), we see $b_{2n-2}(P, H) \geq b_{2n-2}(P)$. This completes the proof. \square

COROLLARY 3.3. *Let X, \mathcal{E}, P, H, m and n be as in Notation 2.2. (3.3.1) Assume that $n = 1$. Then we get the following:*

$$b_i(P, H) = \begin{cases} b_i(P) & \text{if } m \geq i \geq 1, \\ \deg \mathcal{E} & \text{if } i = 0. \end{cases}$$

(3.3.2) Assume that $n = 2$. Then we get the following:

$$b_i(P, H) = \begin{cases} b_i(P) & \text{if } m \geq i \geq 3, \\ b_2(P) + c_2(\mathcal{E}) - 1 & \text{if } i = 2, \\ c_1(\mathcal{E})(c_1(\mathcal{E}) + K_X) + 2 & \text{if } i = 1, \\ s_2(\mathcal{E}) & \text{if } i = 0. \end{cases}$$

PROOF. By Theorem 3.2, we get the assertion. (Here we note that if $n = 2$, then $2n - 2 = 2 < n + r - 1 = m$ because we assume $r \geq 2$ in this paper.) \square

REMARK 3.1. Since \mathcal{E} is ample, we see that if $n = 1$ or 2 , then $b_i(P, H) \geq 0$ for any i .

Here we calculate $h_i^{j,i-j}(P, H)$ for the case where $m \geq i \geq 2n - 1$.

THEOREM 3.3. *Let X, \mathcal{E}, P, H, m and n be as in Notation 2.2. If $m \geq i \geq 2n - 1$ and $0 \leq j \leq i$, then $h_i^{j,i-j}(P, H) = h^{j,i-j}(P)$.*

PROOF. First we note that we can take an ample line bundle A on X such that $\mathcal{E} \otimes A^{\otimes t}$ is ample and spanned for every positive integer t . Hence $H(\mathcal{E} \otimes A^{\otimes t})$ is ample and spanned. We also note that there exists an isomorphism $\phi : \mathbb{P}_X(\mathcal{E} \otimes A^{\otimes t}) \rightarrow P$ with $H(\mathcal{E} \otimes A^{\otimes t}) = \phi^*(H \otimes p^*(A^{\otimes t}))$, where $p : P \rightarrow X$ is the projection. Therefore $H \otimes p^*(A^{\otimes t})$ is also ample and spanned. By this ϕ , we identify $\mathbb{P}_X(\mathcal{E} \otimes A^{\otimes t})$ and P . By Theorem 3.2 we have

$$b_i(P, H(\mathcal{E} \otimes A^{\otimes t})) = b_i(P).$$

Hence

$$\begin{aligned} b_i(P, H \otimes p^*(A^{\otimes t})) &= b_i(P, H(\mathcal{E} \otimes A^{\otimes t})) \\ &= b_i(P). \end{aligned}$$

On the other hand by Proposition 2.1 (1.1) and (2) we have

$$h_i^{j,i-j}(P, H \otimes p^*(A^{\otimes t})) = h^{j,i-j}(P)$$

because $H \otimes p^*(A^{\otimes t})$ is ample and spanned. But since $F(t) := h_i^{j,i-j}(P, H \otimes p^*(A^{\otimes t})) - h^{j,i-j}(P)$ is a polynomial in t and $F(t) = 0$ for every positive integer t , we see that $F(0) = 0$, that is,

$$h_i^{j,i-j}(P, H) = h^{j,i-j}(P).$$

This completes the proof. \square

COROLLARY 3.4. *Let X, \mathcal{E}, P, H, m and n be as in Notation 2.2. (3.4.1) Assume that $n = 1$. Then we get the following:*

$$h_i^{j,i-j}(P, H) = \begin{cases} h^{j,i-j}(P) & \text{if } m \geq i \geq 1 \text{ and } 0 \leq j \leq i, \\ \deg \mathcal{E} & \text{if } i = 0 \text{ and } j = 0. \end{cases}$$

(3.4.2) Assume that $n = 2$. Then we get the following:

$$h_i^{j,i-j}(P, H) = \begin{cases} h^{j,i-j}(P) & \text{if } m \geq i \geq 3 \text{ and } 0 \leq j \leq i, \\ h^{0,2}(P) & \text{if } i = 2 \text{ and } j = 0, \\ h^{2,0}(P) & \text{if } i = 2 \text{ and } j = 2, \\ h^{1,1}(P) + c_2(\mathcal{E}) - 1 & \text{if } i = 2 \text{ and } j = 1, \\ \frac{1}{2}(c_1(\mathcal{E})(c_1(\mathcal{E}) + K_X)) + 1 & \text{if } i = 1 \text{ and } j = 0, 1, \\ s_2(\mathcal{E}) & \text{if } i = 0 \text{ and } j = 0. \end{cases}$$

PROOF. If $n = 1$, then by Corollary 3.3 and Theorem 3.3 we get the assertion.

Assume that $n = 2$. Then by Corollary 3.3 and Theorem 3.3 we get the assertion for the case where $m \geq i \geq 3$ and $0 \leq j \leq i$. If $(i, j) = (2, 0)$ (resp. $(2, 2)$), then by Proposition 2.1 (1.3) and [5, Example 2.10 (8)] we have $h_2^{0,2}(P, H) = g_2(P, H) = h^2(\mathcal{O}_P) = h^{0,2}(P)$ (resp. $h_2^{2,0}(P, H) = g_2(P, H) = h^2(\mathcal{O}_P) = h^{0,2}(P) = h^{2,0}(P)$). Moreover by Corollary 3.3 (3.3.2) and Proposition 2.1 (1.1) we get $h_2^{1,1}(P, H) = h^{1,1}(P) + c_2(\mathcal{E}) - 1$.

Assume that $i = 1$. Then by Proposition 2.1 (1.1) we have $b_1(P, H) = h_1^{1,0}(P, H) + h_1^{0,1}(P, H)$. Moreover by Proposition 2.1 (1.2) we have $h_1^{1,0}(P, H) = h_1^{0,1}(P, H)$. Therefore by Corollary 3.3 we get the assertion for the case $i = 1$.

By Corollary 3.3 we get the assertion for the case where $i = 0$. \square

REMARK 3.2. Since \mathcal{E} is ample, we see that if $n = 1$ or 2 , then $h_i^{j,i-j}(P, H) \geq 0$ for any i and j with $0 \leq j \leq i \leq m$.

4. A new invariant of generalized polarized manifolds.

Here we use Notation 2.2. Assume that $i = 2n - 3$, $n \geq 3$ and $i < m$. Then by Theorem 3.2 (3.2.3) we see that $(n - 2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E}))$ is even because $b_{2n-3}(P, H)$ and $b_{2n-3}(P)$ are even (see [8, Theorem 3.1 (3.1.2)]). We put

$$v(X, \mathcal{E}) := 1 + \frac{1}{2}((n - 2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E}))).$$

Here we note that $r \geq n - 1$ since $n + r - 1 = m \geq i + 1 = 2n - 2$.

If $r = n - 1$, then

$$v(X, \mathcal{E}) = 1 + \frac{1}{2}(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}).$$

So $v(X, \mathcal{E})$ is thought to be a generalization of the curve genus $cg(X, \mathcal{E})$ of (X, \mathcal{E}) (see Definition 2.3). Here we define $v(X, \mathcal{E})$ again.

DEFINITION 4.1. Let (X, \mathcal{E}) be a generalized polarized manifold of dimension $n \geq 3$. Assume that $r \geq n - 1$. Then the invariant $v(X, \mathcal{E})$ of (X, \mathcal{E}) is defined as follows.

$$v(X, \mathcal{E}) := 1 + \frac{1}{2}((n - 2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E}))).$$

Since $b_{2n-3}(P, H) - b_{2n-3}(P) = 2v(X, \mathcal{E}) - 2q(X)$ by Theorem 3.2 (3.2.3), we can propose the following conjecture.

CONJECTURE 4.1. Let (X, \mathcal{E}) be a generalized polarized manifold of dimension $n \geq 3$. Assume that $r \geq n - 1$. Then $v(X, \mathcal{E}) \geq q(X)$.

Here we study the non-negativity of $v(X, \mathcal{E})$.

THEOREM 4.1. Let (X, \mathcal{E}) be a generalized polarized manifold of dimension $n \geq 3$. Assume that $r \geq n - 1$. Then $v(X, \mathcal{E}) \geq 0$.

PROOF. If $r = n - 1$, then $v(X, \mathcal{E})$ is the curve genus and then $v(X, \mathcal{E}) \geq 0$ by [19, Theorem 1]. So we may assume that $r \geq n$.

If $r \geq n$ and $K_X + c_1(\mathcal{E})$ is nef, then $(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}) \geq 0$ because \mathcal{E} is ample. Furthermore $c_n(\mathcal{E}) \geq 1$. So we obtain $v(X, \mathcal{E}) \geq 2$.

If $r \geq n$ and $K_X + c_1(\mathcal{E})$ is not nef, then $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$ by [22, Theorem 1 and Theorem 2]. In this case $(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}) = -n$ and $c_n(\mathcal{E}) = 1$. Hence $v(X, \mathcal{E}) = 0$. Therefore we get the assertion. \square

THEOREM 4.2. Let (X, \mathcal{E}) be a generalized polarized manifold of dimension $n \geq 3$. Assume that $r \geq n - 1$.

(1) If $v(X, \mathcal{E}) = 0$, then (X, \mathcal{E}) is one of the following.

- (1.1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$.
- (1.2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-1})$.
- (1.3) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^n}(2))$.
- (1.4) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n-1})$.
- (1.5) $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{F})$ for some vector bundle \mathcal{F} of rank n on \mathbb{P}^1 , and $\mathcal{E} \cong \bigoplus_{j=1}^{n-1} (H(\mathcal{F}) + \pi^* \mathcal{O}_{\mathbb{P}^1}(b_j))$, where $\pi : X \rightarrow \mathbb{P}^1$ is the bundle projection.

(2) If $v(X, \mathcal{E}) = 1$, then (X, \mathcal{E}) is one of the following.

(2.1) $X \cong \mathbb{P}_C(\mathcal{F})$ for some vector bundle \mathcal{F} of rank n on a smooth elliptic curve C , and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus n-1}$ for any fiber F of the bundle projection $X \rightarrow C$.

(2.2) $K_X + \det(\mathcal{E}) = \mathcal{O}_X$ and $r = n - 1$.

PROOF. If $v(X, \mathcal{E}) \leq 1$, then by the proof of Theorem 4.1, one of the following cases occurs:

(4.2.1) $r = n - 1$.

(4.2.2) $r \geq n$ and $K_X + c_1(\mathcal{E})$ is not nef.

If $r = n - 1$ and $v(X, \mathcal{E}) = 0$ (resp. $v(X, \mathcal{E}) = 1$), then $0 = v(X, \mathcal{E}) = cg(X, \mathcal{E})$ (resp. $1 = v(X, \mathcal{E}) = cg(X, \mathcal{E})$) and by [19, Theorem 1 and Theorem 2] we get the above types.

If $r \geq n$ and $K_X + c_1(\mathcal{E})$ is not nef, then $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$ by [22, Theorem 1 and Theorem 2] and then $v(X, \mathcal{E}) = 0$. Hence we get the assertion. \square

REMARK 4.1. (1) If (X, \mathcal{E}) is the type (2.2) in Theorem 4.2, then a classification of (X, \mathcal{E}) has been obtained by [20].

(2) If $v(X, \mathcal{E}) \leq 1$, then we see that $v(X, \mathcal{E}) \geq q(X)$.

PROPOSITION 4.1. Let (X, \mathcal{E}) be a generalized polarized manifold of dimension $n \geq 3$. Assume that $r \geq n - 1$ and \mathcal{E} is spanned. Then $v(X, \mathcal{E}) \geq q(X)$.

PROOF. Set $(P, H) := (\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$. Then H is ample and spanned. So by Proposition 2.1 (2.1) we obtain

$$b_{2n-3}(P, H) \geq b_{2n-3}(P).$$

On the other hand

$$\begin{aligned} b_{2n-3}(P, H) - b_{2n-3}(P) \\ = 2v(X, \mathcal{E}) - 2q(X). \end{aligned}$$

Hence we get the assertion. \square

REMARK 4.2. (1) By Theorem 4.2 we can determine the type of (X, \mathcal{E}) such that $v(X, \mathcal{E}) = q(X)$ and $q(X) \leq 1$.

- (2) For every integer $q \geq 2$, there exists an example of (X, \mathcal{E}) with $v(X, \mathcal{E}) = q = q(X)$ (This was given by the referee.): Let C be a smooth projective curve with $g(C) = q \geq 2$. We note that there exist vector bundles \mathcal{F} and \mathcal{G} on C with $\text{rank } \mathcal{F} = n$ and $\text{rank } \mathcal{G} = n - 1$ such that $\mathcal{E} := H(\mathcal{F}) \otimes \pi^*(\mathcal{G})$ is an ample vector bundle of rank $n - 1$ on X , where $X := \mathbb{P}_C(\mathcal{F})$, $H(\mathcal{F})$ is the tautological line bundle of \mathcal{F} and $\pi : X \rightarrow C$ is the bundle projection. Then we can easily check that $v(X, \mathcal{E}) = cg(X, \mathcal{E}) = q \geq 2$.
- (3) We see that if \mathcal{E} is ample and spanned, $\text{rank } \mathcal{E} = n - 1$ and $v(X, \mathcal{E}) = q(X) \geq 2$, then (X, \mathcal{E}) is isomorphic to the type in (2) above. This has been proved by [17, Theorem].

Here we give the following conjecture which was pointed out by the referee:

CONJECTURE 4.2. *Let X be a smooth projective variety of dimension $n \geq 3$ and let \mathcal{E} be an ample vector bundle on X with $\text{rank } \mathcal{E} = r$. Assume that $r \geq n - 1$ and $q(X) \geq 2$. If \mathcal{E} is spanned and $v(X, \mathcal{E}) = q(X)$, then $r = n - 1$.*

REMARK 4.3. We consider the case where $r \geq n \geq 3$. (The following (a) and (b) were also pointed out by the referee.)

- (a) If $K_X + c_1(\mathcal{E})$ is not ample, then by [4, Main Theorem] (X, \mathcal{E}) is one of the following 6 types and we can calculate $v(X, \mathcal{E})$ and $q(X)$:
 - (a.1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1})$. In this case $(v(X, \mathcal{E}), q(X)) = (1 + (1/2) \cdot (n - 2)(n + 1), 0)$.
 - (a.2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$. In this case $(v(X, \mathcal{E}), q(X)) = (0, 0)$.
 - (a.3) $X \cong \mathbb{P}_C(\mathcal{F})$ for a vector bundle \mathcal{F} of rank n on a smooth curve C , and $\mathcal{E} \cong H(\mathcal{F}) \otimes \pi^*(\mathcal{G})$ for a vector bundle \mathcal{G} on C with $\text{rank } \mathcal{G} = n$, where $\pi : X \rightarrow C$ is the bundle projection. In this case $(v(X, \mathcal{E}), q(X)) = (g(C) + (n - 1)(g(C) - 1 + c_1(\mathcal{F}) + c_1(\mathcal{G})), g(C))$.
 - (a.4) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^n}(2))$. In this case $(v(X, \mathcal{E}), q(X)) = (n - 1, 0)$.
 - (a.5) $(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n})$. In this case $(v(X, \mathcal{E}), q(X)) = (1 + (1/2)(n - 2) \cdot (n + 1), 0)$.
 - (a.6) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n})$. In this case $(v(X, \mathcal{E}), q(X)) = (n - 1, 0)$.

Here we consider the case where (X, \mathcal{E}) is the type (a.3) above. Then $1 \leq c_n(\mathcal{E}) = c_1(\mathcal{F}) + c_1(\mathcal{G})$ because \mathcal{E} is ample. Hence $g(C) - 1 + c_1(\mathcal{F}) +$

$+c_1(\mathcal{G}) \geq 0$ and we get $v(X, \mathcal{E}) \geq g(C) = q(X)$. Moreover if $g(C) \geq 1$, then we see that $v(X, \mathcal{E}) \geq q(X) + (n-1) > q(X)$.

We also note that if \mathcal{E} is spanned by global sections, then $c_n(\mathcal{E}) \geq 2$ by [21, (3.4) Theorem] and we obtain $v(X, \mathcal{E}) \geq q(X) + (n-1) > q(X)$.

(b) Next we assume that $K_X + c_1(\mathcal{E})$ is ample. Then we can prove the following:

PROPOSITION 4.2. (1) *If $K_X + c_1(\mathcal{E})$ is ample, then $v(X, \mathcal{E}) \geq 1 + \frac{1}{2}(n-1)$.*

(2) *If $K_X + c_1(\mathcal{E})$ is ample and \mathcal{E} is spanned, then $v(X, \mathcal{E}) \geq n$.*

PROOF. (1) By assumption, we have $c_n(\mathcal{E}) \geq 1$ and $(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}) \geq 1$. Hence we get the assertion of (1).

(2) Since \mathcal{E} is spanned and $K_X + c_1(\mathcal{E})$ is ample, we see that $c_n(\mathcal{E}) \geq 2$ by [21, (3.4) Theorem]. Hence $(n-2)c_n(\mathcal{E}) + (K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}) \geq 2(n-2) + 2$ because the term on the left is even. Therefore we get the assertion of (2). \square

The above results suggest that for the case where $r \geq n$ and $v(X, \mathcal{E}) \neq q(X)$ there are some gaps for the value of $v(X, \mathcal{E}) - q(X)$, depending on n . We will investigate this in a future paper.

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