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A Geometric Realization of $\boldsymbol{sl}(6, \mathbb{C})$

GIOVANNI GAIFFI (*) - MICHELE GRASSI (*)

ABSTRACT - Given an orientable weakly self-dual manifold X of rank two, we build a geometric realization of the Lie algebra $\boldsymbol{sl}(6, \mathbb{C})$ as a naturally defined algebra $\mathcal{L}_{\mathbb{C}}$ of endomorphisms of the space of differential forms of X. We provide an explicit description of Serre generators in terms of natural generators of $\mathcal{L}_{\mathbb{C}}$. This construction gives a bundle on X which is related to the search for a natural Gauge theory on X. We consider this paper as a first step in the study of a rich and interesting algebraic structure.

1. Introduction.

This paper is a step in a broader program, which aims at finding a geometric counterpart to the Mirror Symmetry phenomenon, and possibly a geometric language in which to formulate a physical theory interpolating between different σ-models. While we direct the reader to [G2], [G3] for more details, we list here only some aspects of this theory to put the present work into context.

In the Strominger-Yau-Zaslow approach to Mirror Symmetry one has that two mirror dual Calabi-Yaus should posses (in some limiting sense) semi-flat special lagrangian torus fibrations $f : M \to B, \hat{f} : \hat{M} \to B$ which have as fibres flat tori which are dual in the metric sense (see [SYZ], and [G2] for the terminology and the definitions). As it is widely known, the major drawback of this approach is that it is very difficult to build special lagrangian tori fibrations. Usually this construction can be carried out only when the dual Calabi-Yau manifolds are actually hyperkahler, and the special lagrangian tori can be viewed as complex submanifolds (with respect to a rotated complex structure), so that the methods of complex algebraic geometry can be put to work.

(*) Indirizzo degli A.: Dipartimento di Matematica "L. Tonelli", Largo Pontecorvo, 5 - 56127 Pisa.

E-mail: gaiffi@dm.unipi.it grassi@dm.unipi.it

When you do have the fibrations, then the idea is to construct the mirror map as a sort of Fourier-Mukai transform (see for example [BMP]). This Fourier-Mukai transform is a correspondence induced by pull-back and push forward from the space $X = M \times_B \hat{M}$. In the hyperkähler case this space is a complex manifold, while in the general case (for example for Mirror Symmetry for Calabi-Yau threefolds) it is just a real manifold of (real) dimension $3 \cdot dim_{\mathbb{C}}(M)$.

BACKGROUND. The notion of (Weakly) self-dual manifold (cf. [G2]) was conceived in the first place to isolate the geometric aspects of the X above which are needed to obtain Mirror Symmetry between M and $\hat{M}$. We reproduce here the definition for the reader, while referring to [G2] and [G3] for all the remarks, examples and observations:

DEFINITION 1.1. *A* weakly self-dual manifold *(WSD manifold for brevity) is given by a smooth manifold X, together with two smooth* 2*-forms ω_1, ω_2, a Riemannian metric and a third smooth* 2*-form ω_D (the dualizing form) on it, which satisfy the following conditions:*

1) *$d\omega_1 = d\omega_2 = d\omega_D = 0$ and the distribution $\omega_1^0 + \omega_2^0$ is integrable.*

2) *For all $p \in X$ there exists an orthogonal basis $dx_1, \dots, dx_m$, $dy_1^1, \dots, dy_m^1, dy_1^2, \dots, dy_m^2$, $dz_1, \dots, dz_c, dw_1, \dots, dw_c$ of T_p^*X such that the $dx_1, \dots, dx_m, dy_1^1, \dots, dy_m^1, dy_1^2, \dots, dy_m^2$ are orthonormal and*

$$(\omega_1)_p = \sum_{i=1}^{m} dx_i \wedge dy_i^1, \qquad (\omega_2)_p = \sum_{i=1}^{m} dx_i \wedge dy_i^2,$$

$$(\omega_D)_p = \sum_{i=1}^{m} dy_i^1 \wedge dy_i^2 + \sum_{i=1}^{c} dz_i \wedge dw_i.$$

Any orthogonal basis of T_pX dual to a basis of 1*-forms as above is said to be adapted to the structure, or standard. The number m is the rank of the structure.*

For a more intrinsic definition of WSD manifolds the reader should refer to [G2]. Here we have chosen the quickest way to introduce them.

When the forms $\omega_1, \omega_2, \omega_D$ are covariant constant with respect to the Levi-Civita connection, we speak of 2-Kähler manifolds. An example of these comes from mirror symmetry for abelian varieties.

REMARK 1.2. *The form ω_D is symplectic once restricted to $\omega_1^0 + \omega_2^0$. We have therefore that $\omega_D^{dim(X)-m} \neq 0$.*

DEFINITION 1.3. 1) *A WSD manifold is* nondegenerate *if* $dim(\omega_1^0 \cap \omega_2^0)_p = 0$ *at all points (equivalently if its dimension is* 3 *times the rank).*

2) *A WSD manifold is* self-dual *(SD manifold for brevity) if all the leaves of the distribution* $\omega_1^0 + \omega_2^0$ *have volume one (with respect to the volume form induced by the metric)*

Using Self dual manifolds, you can give a first naïve geometric definition of Mirror Symmetry as follows:

Two Calabi-Yau manifolds with B-field (M, B_M) *and* $(\hat{M}, B_{\hat{M}})$ *are mirror dual if there is a Self-dual manifold* X *together with surjections* $\pi : X \to M$ *and* $\hat{\pi} : X \to \hat{M}$ *such that:*

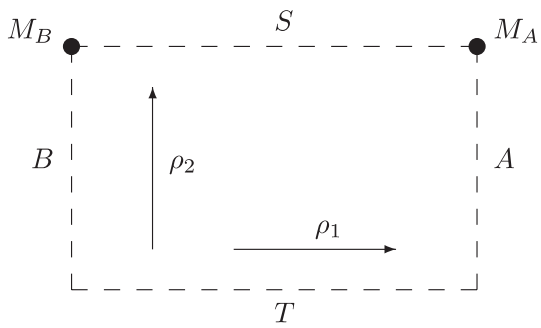
a) $\pi^*(\omega_M) = \omega_1$, $\hat{\pi}^*(\omega_{\hat{M}}) = \omega_2$.
b) The leaves of $\omega_1^\perp$ *are the fibres of* $\hat{\pi}$.
c) The leaves of $\omega_2^\perp$ *are the fibres of* π.
d) The induced B-fields on M *and* $\hat{M}$ *are the ones given.*

Here make their first appearence the B-fields B_M and $B_{\hat{M}}$, which are flat unitary gerbes on M and $\hat{M}$ respectively, and which are not relevant for the discussions of this paper. In [G2] it was shown that this picture works well in the case of elliptic curves, and for some other flat situations.

In the paper [G3] there is a toric construction of a two-dimensional family $\mathbb{X}^m_{k_1,k_2}$ of WSD manifolds of rank m and (real) dimension $3m+2$ (see Definition 3.11 on page 11 of that paper). The construction is inspired by Delzant's method of constructing toric projective manifolds (see [Gu]). As the real parameters k_1, k_2 vary, these WSD manifolds interpolate between physically significant asymptotic limits, as described in the following. To these manifolds one can apply the constructions of the present paper when $m = 2$. In this case the resulting degenerate WSD manifolds have dimension $6 + 2$.

PHYSICAL MOTIVATION. One of the reasons to introduce SD manifolds however was to get rid of special lagrangian fibrations, which are so difficult to construct, and to be able to attack the problem of Mirror Symmetry also when these fibrations are not expected to exist. In this more general context one expects that the Mirror Symmetry phaenomenon will not be obtained directly from fibrations of a SD manifold to the dual Calabi-Yaus, but via a more sophisticated procedure, which involves a Gromov-Hausdorff type of limit. In [G3] it was shown that for the family of anticanonical divisors in complex projective space one can build a (real) two-dimensional family of WSD manifolds, which degenerate in a nor-

malized Gromov-Hausdorff sense to the correct limits of the mirror dual Calabi-Yaus. The picture, taken from [G3], is the following:



where M_A and M_B are the large Kähler and large complex structure limits of M and $\hat{M}$ respectively. To be precise, the manifolds which come out of the costruction of [G3] are (degenerate) Weakly self-dual manifolds or rank m and dimension $3m+2$ for $m \geq 1$.

The point of view of [G3] is very different from the current one in the main literature on mathematical Mirror Symmetry: instead of considering the fibre product $M \times_B \hat{M}$ (when it exists) as a device for proving Mirror Symmetry for Calabi-Yaus, the limiting Calabi-Yaus of Mirror Symmetry are seen as very special limits of a family of Self-Dual manifolds, which are the main objects of study. This is actually more in line with what can be found in the physical literature, where the σ-models defining the string theories from which Mirror Symmetry originates are seen as just "phases" of a unique theory, which is not necessarily in the form of a σ-model but could very likely be similar to a quantized Gauge theory on an 11-dimensional manifold. To make this circle of ideas more concrete (and hence more verifiable) at the end of [G3] it is suggested that one should try to build a natural gauge theory on Self-dual manifolds: the hope is that once quantized this gauge theory might interpolate between the σ-models associated to the Calabi-Yau's, and as a byproduct prove Mirror Symmetry for them. In the present paper we perform a first step in this direction, constructing natural bundles on rank two WSD manifolds. In view of [G3], this is potentially relevant to Mirror Symmetry for quartic hypersurfaces in $\mathbb{P}^3$, i.e. K3 surfaces. Of course one can always put a gauge bundle on the Self-dual manifolds "artificially", but a natural bundle which depends only on the geometric structure is much more appealing. We ignore here the issue of which action to put on the theory, but it too should be a natural geometric one.

Finally, on [GG] we analyzed the situation for rank three WSD mani-

folds, and we found that in this case the corresponding natural bundle is formed by complex Lie superalgebras. We were able to find a geometrically motivated real form, and to split it into simple factors. The results of [GG] confirm the suspicion that on a WSD manifold of high enough rank there could be enough natural algebraic bundles of operators to build interesting gauge theories.

THE CONSTRUCTION OF $\mathcal{L}_{\mathbb{C}}$. From a physical point of view the case of Calabi-Yau threefolds (i.e. rank three WSD manifolds) or fourfolds (i.e. rank four WSD manifolds) would be the most interesting one to start with. However, its technical difficulty convinced us to start more modestly from the case of Calabi-Yau two-folds (i.e. K3 surfaces) which correspond to rank two Weakly Self-dual manifolds. We also considered only orientable nondegenerate Weakly Self-dual manifolds of rank two (hence of dimension 6): one can immediately verify that the relations among the resulting generators of the algebra $\mathcal{L}_{\mathbb{C}}$ remain unchanged with respect to the degenerate case.

This could be considered a proof of concept from a physicist's point of view, however Mirror Symmetry for K3's is in itself very interesting mathematically, so we hope that our results could have some useful geometric consequences. The rank three case is treated in our subsequent [GG], as mentioned in the previous section of this introduction. The main result of the present paper is the following (which is a geometric restatement of Theorem 5.11):

The Lie algebra $\boldsymbol{sl}(6,\mathbb{C})$ *acts via canonical operators (depending only on the geometric structure) on the smooth differential forms of any orientable WSD manifold of rank* 2.

This action generalizes naturally the action of $\boldsymbol{sl}(2,\mathbb{C})$ on smooth differential forms of any almost Kähler manifold, and is induced by a bundle action on the exterior power of the cotangent bundle.

Recall that a WSD manifold is a Riemannian manifold with three "compatible" closed differential forms. We will build a Lie algebra of pointwise operators on complex differential forms on X, as smooth sections of a bundle of Lie algebras of operators on the complexified cotangent bundle of X. To start, one can define the following operators:

DEFINITION 1.4. For $\phi \in \Omega^*_{\mathbb{C}} X$,

$$L_0(\phi) = \omega_D \wedge \phi, \qquad L_1(\phi) = -\omega_2 \wedge \phi, \qquad L_2(\phi) = \omega_1 \wedge \phi.$$

One can notice immediately the strong resemblance of the operators above with the Lefschetz operator of Kähler geometry. Indeed, one can elaborate on this similarity, and use the metric to define the adjoints $\Lambda_j = L_j^*$ (using a pointwise procedure, as in the almost Kähler case).

Simply using the L_j and the Λ_j, one can show that the algebra generated is isomorphic to $SL(4,\mathbb{C})$ ([G2]). However, there are other natural differential forms on a WSD manifold (which do not have a counterpart in the Kähler case), namely the volume forms of the distributions $\omega_1^\perp$, $\omega_2^\perp$, $\omega_D^\perp$ of vectors which contract to zero with the forms ω_1, ω_2 and ω_D respectively. If one calls V_0, V_1, V_2 the corresponding wedge operators, and A_0, A_1, A_2 their adjoints, the complexity of the calculations to describe the generated Lie algebra grows a lot. We called $\mathcal{L}$ the algebra generated by the L_j, V_j and their adjoints, and $\mathcal{L}_{\mathbb{C}}$ its complexification. To study $\mathcal{L}_{\mathbb{C}}$ we introduced an operator J, which is a complex structure on each of the two-dimensional distributions mentioned above and generates a group isomorphic to $\boldsymbol{SO}(2,\mathbb{R})$ (recall that we are in the "hyperkahler" case, corresponding to Mirror Symmetry for K3's, so an "extra" complex structure should not be surprising; moreover the holonomy of a WSD manifold in which all $\omega_1, \omega_2, \omega_D$ are invariant is actually always included in the group generated by J). One checks that all the operators introduced commute with it:

$$\forall j\ [L_j, J] = [\Lambda_j, J] = [V_j, J] = [A_j, J] = 0$$

and therefore one can try to decompose $\Lambda^* T^*_{\mathbb{C}} X$ with respect to J and then use Schur's Lemma to reduce to the study of the operators on the isotypical components. One should mention that in the (very) good cases (for instance 2-Kähler manifolds) the operators above are all covariant constant with respect to the metric connection, and define an action on the cohomology of X much in the same way as in the Kähler setting the operators L and Λ do (due to Hodge-type identities). We do not explore this aspect here, although it may be relevant to the (homological) mirror map construction.

Coming back to the construction, we point out the inclusion of the Lie algebra $\mathcal{L}_{\mathbb{C}}$ inside a copy of the Clifford algebra $\boldsymbol{Cl}_{6,6}$.

Using this Clifford algebra one can identify "degree two" or "quadratic" operators (in a way similar to the ones involved in the Spinor representations on standard Spin manifolds) and among these the $\boldsymbol{SO}(2,\mathbb{R})$-invariant ones. A posteriori, it turns out that the operators of $\mathcal{L}_{\mathbb{C}} \oplus \langle J \rangle$ are all the J-invariant operators of "degree two", and this strengthens the rationale in our selection of natural operators.

As a last step one finds that inside $\Lambda^* T^* X$ there is an $\boldsymbol{SO}(2,\mathbb{R})$-isotypical component of dimension 6, and by direct computation we prove that indeed

the operators restricted to this sub-representation determine a copy of $\boldsymbol{sl}(6,\mathbb{C})$ (with the defining representation). Using the bound on the dimension of $\mathcal{L}_{\mathbb{C}}$ obtained computing "quadratic" invariants, one then shows that the representation on this isotypical component is faithful. This provides as a byproduct a method for giving presentation of standard Serre generators of $\mathcal{L}_{\mathbb{C}}$, explicitely written in terms of the natural geometrical generators.

2. Basic operators.

In this section we fix a point p in the WSD manifold X. The WSD structure splits the cotangent space as $T^*_pX = W_0 \oplus W_1 \oplus W_2$ where the W_j are three mutually orthogonal canonical distributions defined as:

$$W_0 = \{\phi \in T^*_pX \mid \phi \wedge \omega_1^2 = \phi \wedge \omega_2^2 = 0\},$$
$$W_1 = \{\phi \in T^*_pX \mid \phi \wedge \omega_1^2 = \phi \wedge \omega_D^2 = 0\},$$
$$W_2 = \{\phi \in T^*_pX \mid \phi \wedge \omega_2^2 = \phi \wedge \omega_D^2 = 0\}.$$

The WSD structure also determines canonical pairwise linear identifications among W_0, W_1 and W_2, so that one can also write $T^*_pX = W_0 \otimes_{\mathbb{R}} \mathbb{R}^3$ or more simply

$$T^*_pX = W \otimes_{\mathbb{R}} \mathbb{R}^3$$

where $W = W_0 \cong W_1 \cong W_2$.

Let us now come back to the canonical operators L_j mentioned in the introduction:

DEFINITION 1.4 *For* $\phi \in \Omega^*_{\mathbb{C}}X$,

$$L_0(\phi) = \omega_D \wedge \phi, \qquad L_1(\phi) = -\omega_2 \wedge \phi, \qquad L_2(\phi) = \omega_1 \wedge \phi.$$

We now choose a (non-canonical) orthonormal basis γ_1, γ_2 for W_0, and this together with the standard identifications of the W_j determines an orthonormal basis for T^*_pX, which we write as $\{v_{ij} = \gamma_i \otimes e_j \mid i = 1,2,\ j = 0,1,2\}$. We remark that the v_{ij} are an *adapted coframe* for the WSD structure, and therefore we have the explicit expressions:

$$\omega_1 = v_{10} \wedge v_{11} + v_{20} \wedge v_{21},$$
$$\omega_2 = v_{10} \wedge v_{12} + v_{20} \wedge v_{22},$$
$$\omega_D = v_{11} \wedge v_{12} + v_{21} \wedge v_{22}.$$

A different choice of the γ_1, γ_2 would be related to the previous one by an element in $\boldsymbol{O}(2, \mathbb{R})$ or, taking into account the orientability of X mentioned in the Introduction, an element of $\boldsymbol{SO}(2, \mathbb{R})$. The Lie algebra of the group $\boldsymbol{SO}(2, \mathbb{R})$ expressing the change from one oriented adapted basis to another is generated (point by point) by the global operator J:

DEFINITION 2.1. *The operator $J \in End_{\mathbb{R}}(\Omega^*(X))$ is induced by its pointwise action on the $\Lambda^* T^*_p X$ for varying $p \in X$, defined in terms of the standard basis v_{ij} as*

$$J(v_{1j}) = v_{2j}, \qquad J(v_{2j}) = -v_{1j} \qquad \text{for } j \in \{0,1,2\}$$

and $J(v \wedge w) = J(v) \wedge w + v \wedge J(w)$ for $v, w \in \Lambda^* T^*_p X$.

REMARK 2.2. *As J commutes with itself, it is well defined, independently of the choice of an oriented adapted basis.*

Using the chosen (orthonormal) basis, one can define corresponding (non canonical) wedge and contraction operators:

DEFINITION 2.3. *Let $i \in \{1,2\}$ and $j \in \{0,1,2\}$. The operators E_{ij} and I_{ij} are respectively the wedge and the contraction operator with the form v_{ij} on $\bigwedge^* T^* X$ (defined using the given basis); we use the notation $\dfrac{\partial}{\partial v_{ij}}$ to indicate the element of $T_p X$ dual to $v_{ij} \in T^*_p X$:*

$$E_{ij}(\phi) = v_{ij} \wedge \phi, \qquad I_{ij}(\phi) = \frac{\partial}{\partial v_{ij}} \rightharpoonup \phi.$$

PROPOSITION 2.4. *The operators E_{ij}, I_{ij} satisfy the following relations:*

$$\begin{gathered}
\forall i,j,k,l \qquad E_{ij}E_{kl} = -E_{kl}E_{ij}, \quad I_{ij}I_{kl} = -I_{kl}I_{ij}, \\
\forall i,j \qquad E_{ij}I_{ij} + I_{ij}E_{ij} = Id, \\
\forall (i,j) \neq (k,l) \qquad E_{ij}I_{kl} = -I_{kl}E_{ij}, \\
\forall i,j \qquad E^*_{ij} = I_{ij}, \quad I^*_{ij} = E_{ij}
\end{gathered}$$

where $*$ is adjunction with respect to the metric.

PROOF. The proof is a simple direct verification, which we omit. □

It is then immediate to verify that:

PROPOSITION 2.5. *J can be expressed as*

$$J = \sum_{j=0}^{2} \left(E_{2j} I_{1j} - E_{1j} I_{2j}\right)$$

on the whole $\bigwedge^* T_p^* X$. From this expression and the previous proposition one obtains that $J^* = -J$, i.e. for every p the Lie algebra generated by J is a subalgebra of $\boldsymbol{o}(\bigwedge^* T_p^* X)$ isomorphic to $\boldsymbol{so}(2,\mathbb{R}) \cong \mathbb{R}$. Moreover, the exponential images inside $Aut_{\mathbb{R}}(\Omega^*(X))$of the operators of type tJ for $t \in \mathbb{R}$ form a group isomorphic to $\boldsymbol{SO}(2,\mathbb{R}) \cong S^1$, as this isomorphism holds for the (faithful) restriction of the group action to $T_p^* X$.

Using the (non canonical) operators E_{ij} we can obtain simple expressions for the pointwise action of the other canonical operators, the volume forms V_j:

DEFINITION 2.6. *For $\phi \in \bigwedge^* T_p^* X$,*

$$V_0(\phi) = E_{10}E_{20}(\phi), \qquad V_1(\phi) = E_{11}E_{21}(\phi), \qquad V_2(\phi) = E_{12}E_{22}(\phi).$$

Remember however that the operators V_j do not depend on the choice of a basis, as they are simply multiplication by the volume forms of the spaces W_j.

We use the v_{ij} also as a orthonormal basis for the complexified space $T_p^* \otimes_{\mathbb{R}} \mathbb{C}$ (with respect to the induced hermitian inner product). We indicate with the same symbols V_j the complexified operators acting on the spaces $\bigwedge^*_{\mathbb{C}} T_p^* X$.

The riemannian metric induces a Riemannian metric on $T_p^* X$ and on the space $\bigwedge^* T_p^* X$.

DEFINITION 2.7. For $j \in \{0,1,2\}$

$$\Lambda_j = L_j^*, \qquad A_j = V_j^*.$$

By construction the canonical operators L_j, V_j, Λ_j, A_j on $\bigwedge^* T_p^* X$ are the pointwise restrictions of corresponding global operators on smooth differential forms, which we indicate with the same symbols: for $j \in \{0,1,2\}$,

$$L_j, V_j, \Lambda_j, A_j : \Omega^*(X) \to \Omega^*(X).$$

Summing up:

DEFINITION 2.8. *The $*$-Lie algebra $\mathcal{L}$ is the $*$-Lie subalgebra of $End_{\mathbb{R}}(\Omega^*(X))$ generated by the operators*

$$\{L_j, V_j, \Lambda_j, A_j \mid \textit{for } \ j = 0,1,2\}.$$

The $*$ operator on $\mathcal{L}$ is induced by the adjoint with respect to the Riemannian metric. The $*$-Lie algebra $\mathcal{L}_{\mathbb{C}}$ is $\mathcal{L}\otimes\mathbb{C}$, and is in a natural way a $*$-Lie subalgebra of $End_{\mathbb{C}}\big(\Omega^*_{\mathbb{C}}(X)\big)$. The $*$ operator on $\mathcal{L}_{\mathbb{C}}$ is induced by the adjoint with respect to the induced Hermitian metric.

The canonical splitting $T^*_pX = W_0\oplus W_1\oplus W_2$ together with the canonical identifications $W_0\cong W_1\cong W_2$ induce an action of the symmetric group $\mathcal{S}_3$, which propagates to $\bigwedge^* T^*X$ and to its $\mathcal{C}^\infty$ sections. At every point, the action can be written explicitly in terms of the basis as

$$\sigma(v_{ij}) = v_{i\sigma(j)}.$$

The induced action on endomorphisms via conjugation, $\sigma(\phi) = \sigma\circ\phi\circ\sigma^{-1}$, preserves $\mathcal{L}_{\mathbb{C}}$. Indeed, one can check directly using the basis v_{ij} at every point that for $\sigma\in\mathcal{S}_3$

$$\sigma(V_j) = V_{\sigma(j)},\qquad \sigma(L_j) = \varepsilon(\sigma)L_{\sigma(j)}.$$

Since $\mathcal{S}_3$ acts on $\mathcal{L}_{\mathbb{C}}$ by conjugation with unitary operators, its action commutes with adjunction (the $*$ operator), and therefore

$$\sigma(A_j) = A_{\sigma(j)},\qquad \sigma(\Lambda_j) = \varepsilon(\sigma)\Lambda_{\sigma(j)}.$$

Moreover, one also has that $\sigma(J) = J$ which means that the action of $\mathcal{S}_3$ commutes with that of $\boldsymbol{so}(2,\mathbb{R})$.

3. The action of $\boldsymbol{so}(2,\mathbb{R})$.

When one deals with mirror simmetry for 2-Kähler manifolds (see the Introduction), the WSD manifolds which arise have the property that the forms ω_1,ω_2 and ω_D are covariant constant with respect to the metric. In this case, the maximal possible holonomy of the WSD manifold X is included in the $\boldsymbol{so}(2,\mathbb{R})$ generated by the operator J. We will show now that J commutes with $\mathcal{L}_{\mathbb{C}}$. Our proof will be strictly algebraic, so that the commutativity between $\boldsymbol{so}(2,\mathbb{R})$ and $\mathcal{L}_{\mathbb{C}}$ will hold also on WSD manifolds for which the holonomy is more general.

DEFINITION 3.1. *Given $n\in\mathbb{Z}$, we indicate with V_n the one dimensional complex representation of $\boldsymbol{SO}(2,\mathbb{R})\cong S^1\ \cong\mathbb{R}/\mathbb{Z}$ given by the character:*

$$\theta\to e^{2\pi\imath n\theta}.$$

PROPOSITION 3.2. *Under the $\boldsymbol{SO}(2,\mathbb{R})$ representation induced by the operator J, for any $p\in X$:*

1) *The space* $\bigwedge^1 (T^*_{\mathbb{C}} X_p)$ *splits as*

$$V_{-1}^{\oplus 3} \bigoplus V_1^{\oplus 3}.$$

2) *The whole space* $\bigwedge^* (T^*_{\mathbb{C}} X_p)$ *splits according to the following picture:*

$$\begin{aligned}
\textstyle\bigwedge^0 (T^*_{\mathbb{C}} X_p) &= && && V_0 && && \\
\textstyle\bigwedge^1 (T^*_{\mathbb{C}} X_p) &= && V_{-1}^{\oplus 3} && \oplus && V_1^{\oplus 3} && \\
\textstyle\bigwedge^2 (T^*_{\mathbb{C}} X_p) &= V_{-2}^{\oplus 3} && \oplus && V_0^{\oplus 9} && \oplus && V_2^{\oplus 3} \\
\textstyle\bigwedge^3 (T^*_{\mathbb{C}} X_p) &= V_{-3} && \oplus && V_{-1}^{\oplus 9} && \oplus && V_1^{\oplus 9} && \oplus && V_3 \\
\textstyle\bigwedge^4 (T^*_{\mathbb{C}} X_p) &= V_{-2}^{\oplus 3} && \oplus && V_0^{\oplus 9} && \oplus && V_2^{\oplus 3} \\
\textstyle\bigwedge^5 (T^*_{\mathbb{C}} X_p) &= && V_{-1}^{\oplus 3} && \oplus && V_1^{\oplus 3} && \\
\textstyle\bigwedge^6 (T^*_{\mathbb{C}} X_p) &= && && V_0 && &&
\end{aligned}$$

PROOF. 1) The space $T^*_{\mathbb{C}} X_p$ is a direct sum of the three W_j, and each one of these is the standard two dimensional real representation of $\boldsymbol{so}(2,\mathbb{R})$. We therefore diagonalize the representation introducing a new basis for each $W_j = \langle v_{1j}, v_{2j} \rangle$:

$$w_j = v_{1j} + \imath\, v_{2j}, \qquad \overline{w}_j = v_{1j} - \imath v_{2j}.$$

From the definition of J, one has then for every $j \in \{0,1,2\}$

$$J(w_j) = -\imath w_j, \qquad J(\overline{w}_j) = \imath \overline{w}_j.$$

Therefore one has for every $j \in \{0,1,2\}$

$$\langle w_j \rangle \cong V_{-1}, \qquad \langle \overline{w}_j \rangle \cong V_1.$$

2) To prove the general case, we use the fact that the operator J determines an almost complex structure on the manifold X, compatible with the metric. From this, following standard arguments, the complex differential forms and also the elements of $\bigwedge^* T^*_{\mathbb{C}} X_y$ for any $y \in Y$ can be divided according to their type:

$$\overset{*}{\bigwedge} T^*_{\mathbb{C}} X_y = \bigoplus_{n=0}^{dimX} \bigoplus_{p+q=n} \overset{p,q}{\bigwedge} T^*_{\mathbb{C}} X_y.$$

In the notation adopted in the proof of the first statement, one has

$$\overset{p,q}{\bigwedge} T^*_{\mathbb{C}} X_y = \langle w_{i_1} \wedge \cdots \wedge w_{i_p} \wedge \overline{w}_{j_1} \wedge \cdots \wedge \overline{w}_{j_q} \mid i_1, \ldots, j_q \ \in \{0,1,2\} \rangle.$$

From the definition of the action of J one has therefore that for any p, q

$$\bigwedge^{p,q} T^*_{\mathbb{C}} X_y \cong V_{q-p}^{\oplus k}$$

with $k = \begin{pmatrix} 3 \\ p \end{pmatrix} \begin{pmatrix} 3 \\ q \end{pmatrix}$ from which the second statement of the proposition can be esily deduced. □

THEOREM 3.3. *The operators L_j, V_j for $j \in \{0,1,2\}$ commute with the generator J of $\boldsymbol{so}(2,\mathbb{R})$.*

PROOF. We prove the statements by a direct computation using the basis v_{ij}; moreover, using the action of $\mathcal{S}_3$ (which permutes the L_j, V_j and fixes J), it is enough to prove the commutativity for L_0 and V_0. It useful to rewrite ω_0 (and hence L_0 which is wedge with ω_0) in terms of the basis generated by the w_j:

$$\omega_0 = v_{11} \wedge v_{12} + v_{21} \wedge v_{22} = \frac{1}{2}(w_1 \wedge \overline{w}_2 - w_2 \wedge \overline{w}_1)$$

and then:

$$[J, L_0](w_{i_1} \wedge \cdots \wedge w_{i_p} \wedge \overline{w}_{j_1} \wedge \cdots \wedge \overline{w}_{j_q}) =$$

$$J\left(\frac{1}{2}(w_1 \wedge \overline{w}_2 - w_2 \wedge \overline{w}_1)\right) \wedge \left(w_{i_1} \wedge \cdots \wedge w_{i_p} \wedge \overline{w}_{j_1} \wedge \cdots \wedge \overline{w}_{j_q}\right)$$

$$+\left(\frac{1}{2}(w_1 \wedge \overline{w}_2 - w_2 \wedge \overline{w}_1)\right) \wedge J\left(w_{i_1} \wedge \cdots \wedge w_{i_p} \wedge \overline{w}_{j_1} \wedge \cdots \wedge \overline{w}_{j_q}\right)$$

$$-\frac{1}{2}(w_1 \wedge \overline{w}_2 - w_2 \wedge \overline{w}_1) \wedge J(w_{i_1} \wedge \cdots \wedge w_{i_p} \wedge \overline{w}_{j_1} \wedge \cdots \wedge \overline{w}_{j_q}).$$

Therefore the result follows from the fact that

$$J(\frac{1}{2}(w_1 \wedge \overline{w}_2 - w_2 \wedge \overline{w}_1) = 0$$

as w_j and $\overline{w}_k$ have opposite weight with respect to J for any j, k.

Similarly, $[J, V_0] = 0$ follows from the fact that for any α

$$V_0(\alpha) = v_{10} \wedge v_{20} \wedge \alpha = \frac{\imath}{2}\ w_0 \wedge \overline{w}_0 \wedge \alpha$$

□

From the previous theorem one obtains the following corollary, which holds on any WSD manifold (not necessarily 2-Kähler):

COROLLARY 3.4. *The algebra $\mathcal{L}_{\mathbb{C}}$ commutes with the action of $\boldsymbol{so}(2,\mathbb{R})$ induced by J.*

PROOF. We already know that $[J,L_j]=[J,V_j]=0$ for $j\in\{0,1,2\}$. The corresponding commutation relations for the adjoint generators Λ_j, A_j of $\mathcal{L}_{\mathbb{C}}$ follow from the fact that $J^*=-J$, as noticed in Proposition 2.5. □

REMARK 3.5. *From Schur's lemma it follows that the columns of the diagram of Proposition* 3.2 *are preserved by the action of $\mathcal{L}_{\mathbb{C}}$.*

4. An irreducible representation of $\mathcal{L}_{\mathbb{C}}$.

Looking at the table in Proposition 3.2 we notice that the second column from the left is a representation of $\mathcal{L}_{\mathbb{C}}$ (by Remark 3.5) of dimension 6:

$$V\cong V_{-2}^{\oplus 6}=\langle w_0\wedge w_1,\ w_0\wedge w_2,\ w_1\wedge w_2,\ w_0\wedge w_1\wedge w_2\wedge\overline{w}_0,$$
$$w_0\wedge w_1\wedge w_2\wedge\overline{w}_1,\ w_0\wedge w_1\wedge w_2\wedge\overline{w}_2\rangle.$$

In this section we will compute explicitely this representation.

Using the above described basis, it is not difficult to compute the matrices by hand:

PROPOSITION 4.1. *Indicating with β the ordered basis for V indicated above, the matrices for the (restrictions to V of) the generators of $\mathcal{L}_{\mathbb{C}}$ are the following:*

$$M_\beta(L_0)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\-\frac12&0&0&0&0&0\\0&-\frac12&0&0&0&0\end{pmatrix},\quad M_\beta(\Lambda_0)=\begin{pmatrix}0&0&0&0&-2&0\\0&0&0&0&0&-2\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$M_\beta(L_1)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\\frac12&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-\frac12&0&0&0\end{pmatrix},\quad M_\beta(\Lambda_1)=\begin{pmatrix}0&0&0&2&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-2\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$M_\beta(L_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_\beta(\Lambda_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_\beta(V_0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\imath}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_\beta(A_0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\imath & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_\beta(V_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\imath}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_\beta(A_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\imath & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_\beta(V_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\imath}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_\beta(A_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2\imath \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

PROOF. Direct computation using the basis generated by the w_j. □

COROLLARY 4.2. *The algebra generated by the restriction of $\mathcal{L}_{\mathbb{C}}$ to V is isomorphic to $\boldsymbol{sl}(6,\mathbb{C})$, with V its natural representation.*

One can sum up the computations above in the following theorem:

THEOREM 4.3. *There is an exact sequence of Lie algebras*

$$0 \to K \to \mathcal{L}_{\mathbb{C}} \to \boldsymbol{sl}(6,\mathbb{C}) \to 0$$

given by the restriction to V.

In the next section we will prove that $K = \{0\}$, and therefore the representation V is faithful and $\mathcal{L}_{\mathbb{C}} \cong \boldsymbol{sl}(6,\mathbb{C})$.

5. Quadratic invariants.

We begin by showing that the action of Lie algebra $\mathcal{L}_{\mathbb{C}}$ is induced by a (non-canonical) Clifford algebra representation. We use for simplicity the canonical identification $T^{**}X_p \cong TX_p$ without further comment, so that if $\{v_{ij}\}$ is a basis for T^*_pX, then $\left\{\dfrac{\partial}{\partial v_{ij}}\right\}$ is the corresponding dual basis for T_pX.

DEFINITION 5.1. *For $p \in X$, the Clifford algebra $\mathcal{C}_p$ is*

$$\mathcal{C}_p = Cl(T_pX \oplus T^*_pX, q)$$

with the quadratic form q induced by the metric

$$\begin{aligned}
&\forall i,j,h,k && \langle v_{ij}, v_{hk}\rangle = 0,\\
&\forall i,j,h,k && \left\langle \frac{\partial}{\partial v_{ij}}, \frac{\partial}{\partial v_{hk}} \right\rangle = 0,\\
&\forall (i,j) \neq (h,k) && \left\langle v_{ij}, \frac{\partial}{\partial v_{hk}} \right\rangle = 0,\\
&\forall i,j && \left\langle v_{ij}, \frac{\partial}{\partial v_{ij}} \right\rangle = -\frac{1}{2}.
\end{aligned}$$

REMARK 5.2. *The Clifford algebras $\mathcal{C}_p$ for varying p define a Clifford bundle $\mathcal{C}$ on X, as the definition of $\mathcal{C}_p$ is independent on the choice of a basis. Indeed, the quadratic form used to define it is simply induced by $-\dfrac{1}{2}$ times the natural bilinear pairing $T_pX \otimes T^*_pX \to \mathbb{R}$.*

PROPOSITION 5.3. *The Clifford algebra $\mathcal{C}_p$ has a canonical representation ρ_p on $\bigwedge T^*_pX$, induced by the operators E_{ij} and I_{ij} via the map*

$$\rho_p(v_{ij}) = E_{ij}, \qquad \rho_p\left(\frac{\partial}{\partial v_{ij}}\right) = I_{ij}.$$

PROOF. The Clifford relations

$$\phi\psi + \psi\phi = -2\langle \phi, \psi\rangle$$

are precisely the content of Proposition 2.4. The representation is canonical, even if the operators E_{ij} and I_{ij} are not, because it can be defined in a basis independent way as

$$\rho_p(v)(\alpha) = v \wedge \alpha, \qquad \rho_p\left(\frac{\partial}{\partial v}\right) = \frac{\partial}{\partial v} \rightharpoonup \alpha$$

□

Abusing slightly the notation, we will identify $\mathcal{C}_p$ with its (faithful) image inside $End_{\mathbb{R}}\left(\bigwedge^* T_p^* X\right)$, and we will omit any reference to the map ρ_p. Actually, as the representation above is a real analogue of the Spinor representation, it is easy to check that the map ρ_p is an isomorphism of associative algebras. One then has:

DEFINITION 5.4. *The linear subspace $\mathcal{C}_p^2$ of $\mathcal{C}_p$ is the image of the natural map $\bigwedge^2 (T_p X \oplus T_p^* X) \to \mathcal{C}_p$. The linear subspace $\mathcal{C}_p^0$ of $\mathcal{C}_p$ is the subspace generated by* 1.

Recall that $\mathcal{C}_p^2$ is a Lie subalgebra of $\mathcal{C}_p$ (with the commutator bracket).

PROPOSITION 5.5. *The Lie algebra $\mathcal{L}_p$ and the operator J sit inside $\mathcal{C}_p^2$ for all $p \in X$.*

PROOF. The operators L_j, the Λ_j, the V_j and the A_j lie inside $\mathcal{C}_p^2 \oplus \mathcal{C}_p^0$ by Proposition 2.4 and the fact that $\omega_1, \omega, \omega_D$ lie in $\bigwedge^2 T_p^* X$. The operator J lies inside $\mathcal{C}_p^2 \oplus \mathcal{C}_p^0$ by Proposition 2.5. By definition the elements $\mathcal{C}_p^2$ are commutators, and therefore have trace zero in any representation, and hence also in the ρ_p. Moreover, again by inspection all the generators of $\mathcal{L}_p$ have trace zero once represented via ρ_p (they are nilpotent), and therefore they must lie inside $\mathcal{C}_p^2$. The operator J is in the Lie algebra of the isometry group, and therefore it too has trace zero and hence sits inside $\mathcal{C}_p^2$. As $\mathcal{C}_p^2$ is closed under the commutator bracket of $\mathcal{C}_p$, and this commutator coincides with the composition bracket of operators, we have the conclusion. □

REMARK 5.6. *Giving degree* 1 *to the operators E_{ij} and degree -1 to the operators I_{ij}, we induce a $\mathbb{Z}$-degree on $\mathcal{C}_p$. This degree coincides with the degree of the operators induced from the grading on the forms from $\bigwedge^* T^* X$.*

REMARK 5.7. *For any $p \in X$, the Clifford algebra $\mathcal{C}_p$ is isomorphic to $\boldsymbol{Cl}_{6,6}$, as the metric used to define it has signature* $(6,6)$. *The*

previous proposition therefore shows that $\mathcal{L}_p$ is a Lie subalgebra of $\boldsymbol{Cl}^2_{6,6} \cong \boldsymbol{spin}(6,6) = \boldsymbol{so}(6,6)$, generated by smooth global sections of the Clifford bundle $\mathcal{C}$.

The operator J acts on all of $\mathcal{C}_p$ by adjunction with respect to the commutator bracket, and sends its quadratic part $\mathcal{C}^2_p$ to itself from Proposition 5.5.

We will show that the space of J-invariants inside $\mathcal{C}^2_p$ (the "quadratic" J-invariants) coincides with $\mathcal{L}_{\mathbb{C}}$. To describe it explicitely, let us introduce the following notation:

DEFINITION 5.8.

$$E_{w_j} = E_{1j} + \imath E_{2j}, \qquad E_{\overline{w}_j} = E_{1j} - \imath E_{2j},$$
$$I_{w_j} = I_{1j} - \imath I_{2j}, \qquad I_{\overline{w}_j} = I_{1j} + \imath I_{2j}.$$

LEMMA 5.9. *The adjoint action of the operator J on $E_{w_j}, I_{w_j}, E_{\overline{w}_j}, I_{\overline{w}_j}$ is:*

$$[J, E_{w_j}] = -\imath E_{w_j}, \quad [J, I_{w_j}] = \imath I_{w_j},$$
$$[J, E_{\overline{w}_j}] = \imath E_{\overline{w}_j}, \quad [J, I_{\overline{w}_j}] = -\imath I_{\overline{w}_j}.$$

PROOF. It is enough to consider the corresponding J-weights of the $w_j, \overline{w}_j$. □

PROPOSITION 5.10. *The following* 36 *operators provide a linear basis for the quadratic J-invariants:*

(1) $[E_{w_0}, E_{\overline{w}_1}], [E_{w_0}, E_{\overline{w}_2}], [E_{w_1}, E_{\overline{w}_2}], [E_{w_1}, E_{\overline{w}_0}], [E_{w_2}, E_{\overline{w}_0}], [E_{w_2}, E_{\overline{w}_1}]$;
(2) $[I_{w_0}, I_{\overline{w}_1}], [I_{w_0}, I_{\overline{w}_2}], [I_{w_1}, I_{\overline{w}_2}], [I_{w_1}, I_{\overline{w}_0}], [I_{w_2}, I_{\overline{w}_0}], [I_{w_2}, I_{\overline{w}_1}]$;
(3) $[E_{w_0}, E_{\overline{w}_0}], [E_{w_1}, E_{\overline{w}_1}], [E_{w_2}, E_{\overline{w}_2}]$;
(4) $[I_{w_0}, I_{\overline{w}_0}], [I_{w_1}, I_{\overline{w}_1}], [I_{w_2}, I_{\overline{w}_2}]$;
(5) $[E_{w_0}, I_{w_1}], [E_{w_0}, I_{w_2}], [E_{w_1}, I_{w_0}], [E_{w_1}, I_{w_2}], [E_{w_2}, I_{w_0}], [E_{w_2}, I_{w_1}]$;
(6) $[E_{\overline{w}_0}, I_{\overline{w}_1}], [E_{\overline{w}_0}, I_{\overline{w}_2}], [E_{\overline{w}_1}, I_{\overline{w}_0}], [E_{\overline{w}_1}, I_{\overline{w}_2}], [E_{\overline{w}_2}, I_{\overline{w}_0}], [E_{\overline{w}_2}, I_{\overline{w}_1}]$;
(7) $[E_{w_0}, I_{w_0}], [E_{w_1}, I_{w_1}], [E_{w_2}, I_{w_2}], [E_{\overline{w}_0}, I_{\overline{w}_0}], [E_{\overline{w}_1}, I_{\overline{w}_1}], [E_{\overline{w}_2}, I_{\overline{w}_2}]$.

PROOF. The J-weight of a bracket of J-homogeneous operators is the sum of the respective weights. The quadratic "monomials" (with respect to the bracket) in the $E_{w_j}, I_{w_j}, E_{\overline{w}_j}, I_{\overline{w}_j}$ are all J-homogeneous, and therefore to find a basis of J-invariant quadratic operators it is enough to identify the J-invariant quadratic monomials. To be J-invariant means simply to have weight zero, and the computation of the J-weight of the quadratic mono-

nials follows immediately from those of $E_{w_j}, I_{w_j}, E_{\overline{w}_j}, I_{\overline{w}_j}$, which are respectively $-\imath, \imath, \imath, -\imath$. □

We end this section with the following:

THEOREM 5.11. *In the exact sequence of Theorem* 4.3 *the kernel* K *is equal to* $\{0\}$. *The algebra* $\mathcal{L}_{\mathbb{C}}$ *is therefore isomorphic to* $\boldsymbol{sl}(6, \mathbb{C})$.

PROOF. Since $\mathcal{L}_{\mathbb{C}}$ is included in the Lie algebra of quadratic invariants, it is enough to show that $J \notin \mathcal{L}_{\mathbb{C}}$, as from this and the previous proposition it follows that $dim_{\mathbb{C}}(\mathcal{L}_{\mathbb{C}}) \leq 35$. As $\mathcal{L}_{\mathbb{C}}$ maps surjectively to $\boldsymbol{sl}(6, \mathbb{C})$ which has dimension 35, the kernel must be zero. When restricted to the subrepresentation V, the generators of $\mathcal{L}_{\mathbb{C}}$ have all trace zero by inspection of their matrices. However, by definition of V, J restricted to it is multiplication by $-2\imath$, and has therefore trace equal to $-12\imath$. □

COROLLARY 5.12. *The Lie algebra* $\mathcal{L}_{\mathbb{C}} \oplus \langle J \rangle$ *equals the Lie algebra of quadratic invariants inside* $\mathcal{C}^2_p$.

6. A geometric presentation of Serre generators.

In this section, to gain a better geometric understanding of the representation $\mathcal{L}_{\mathbb{C}}$ of $\boldsymbol{sl}(6, \mathbb{C})$, we explore in greater detail its relation to the geometric structure of a WSD manifold. In particular, we give a presentation of a natural choice of Cartan subalgebra and Serre generators in terms on the geometric generators L_j, Λ_j, V_j, A_j.

The L_j operators are similar in nature to the Lefschetz operators of a Kähler manifold. This analogy is what provided the initial interest in the algebraic structure of $\mathcal{L}_{\mathbb{C}}$. Similarly to the corresponding standard construction of a representation of $\boldsymbol{sl}(2, \mathbb{C})$, we define

DEFINITION 6.1. *For* $j \in \{0, 1, 2\}$

$$H_j = [L_j, \Lambda_j].$$

These operators are self-adjoint, as $L_j^* = \Lambda_j$ by definition. As in the context of Kählerian geometry, for every j the algebra $\langle L_j, \Lambda_j, H_j \rangle$ turns out to be a copy of $\boldsymbol{sl}(2, \mathbb{C})$. Moreover, the following proposition shows that the operators H_j are semisimple on the whole algebra $\mathcal{L}_{\mathbb{C}}$, and therefore generate a toral subalgebra of $\mathcal{L}_{\mathbb{C}}$:

PROPOSITION 6.2. *The geometric operators H_j generate a toral subalgebra of $\mathcal{L}_{\mathbb{C}}$, and the following relations hold: for $j \neq k \in \{0,1,2\}$*

(1) $[H_j, L_j] = 2L_j, \qquad [H_j, \Lambda_j] = -2\Lambda_j,$
(2) $[H_j, L_k] = L_k, \qquad [H_j, \Lambda_k] = -\Lambda_k,$
(3) $[H_j, V_j] = 0, \qquad [H_j, A_j] = 0,$
(4) $[H_j, V_k] = 2V_k, \qquad [H_j, A_k] = -2A_k.$

PROOF. In view of Theorem 5.11, at this point the quickest method of proof of this proposition is to refer to the explicit matrices of the (faithful) restriction of $\mathcal{L}_{\mathbb{C}}$ to V. □

The whole algebra $\mathcal{L}_{\mathbb{C}}$ splits into a direct sum of weight spaces with respect to $\langle H_0, H_1, H_2\rangle$, as this subalgebra is toral. The weight of L_0 with respect to the basis dual to H_0, H_1, H_2 is:

$$\alpha_{L_0} \;=\; (\alpha_{L_0}(H_0), \alpha_{L_0}(H_1), \alpha_{L_0}(H_2)) \;=\; (2,1,1).$$

The full list is:

$$\begin{aligned}
\alpha_{L_0} &= (2,1,1), & \alpha_{\Lambda_0} &= -\alpha_{L_0},\\
\alpha_{L_1} &= (1,2,1), & \alpha_{\Lambda_1} &= -\alpha_{L_1},\\
\alpha_{L_2} &= (1,1,2), & \alpha_{\Lambda_2} &= -\alpha_{L_2},\\
\alpha_{V_0} &= (0,2,2), & \alpha_{A_0} &= -\alpha_{V_0},\\
\alpha_{V_1} &= (2,0,2), & \alpha_{A_1} &= -\alpha_{V_1},\\
\alpha_{V_2} &= (2,2,0), & \alpha_{A_2} &= -\alpha_{V_2}.
\end{aligned}$$

To find a natural geometric expression for two ad-semisimple elements which complete $\langle H_0, H_1, H_2\rangle$ to a Cartan subalgebra we look at the generators V_j and A_j. However, it turns out that the natural candidates $[V_j, A_j]$ already lie in the algebra $\langle H_0, H_1, H_2\rangle$. We instead build the new operators by "subtracting" from the V_j their weight α_{V_j}:

DEFINITION 6.3. *We define:*

$$\begin{aligned}
S_0 &= \imath[[[V_0, \Lambda_1], \Lambda_2], L_0],\\
S_1 &= \imath[[[V_1, \Lambda_2], \Lambda_0], L_1],\\
S_2 &= \imath[[[V_2, \Lambda_0], \Lambda_1], L_2]
\end{aligned}$$

and denote by $\mathcal{H}$ the Lie algebra (over $\mathbb{C}$):

$$\mathcal{H} = \langle H_0, H_1, H_2, S_0, S_1, S_2\rangle.$$

The coefficients $\imath$ which appear in the formulas above are dictated by the fact that with this choice the (diagonal) matrices of the S_j restricted to V have integer entries.

PROPOSITION 6.4. *The algebra $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{L}_{\mathbb{C}}$. More precisely, the following are the diagonals of the operators $H_0, \dots, S_2$ once restricted to V*

$$H_0 : \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \; H_1 : \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \; H_2 : \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \; S_0 : \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \; S_1 : \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \; S_2 : \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

PROOF. The computation of the matrices above shows that, once restricted to V, the algebra $\mathcal{H}$ spans the space of diagonal matrices of trace zero in the given basis. □

REMARK 6.5. *The computation above shows also that operators S_0, S_1, S_2 safisfy the relation*

$$S_0 + S_1 + S_2 = 0.$$

Even if from the previous proposition we know that $\mathcal{H}$ is maximal toral inside $\mathcal{L}_{\mathbb{C}}$, the natural geometric generators L_j, Λ_j are not eigenvectors for the adjoint action of the S_k. At this point however it is possible to single out in natural geometric terms operators of $\mathcal{L}_{\mathbb{C}}$ which have "pure" weight with respect to the algebra $\mathcal{H}$ and which contain in their linear span the L_j, Λ_j:

DEFINITION 6.6. *For $j \in \{0, 1, 2\}$*

$$\begin{aligned} L_{1j} &= -2L_j + [S_j, L_j], \qquad & L_{2j} &= 2L_j + [S_j, L_j], \\ \Lambda_{1j} &= -2\Lambda_j - [S_j, \Lambda_j], \qquad & \Lambda_{2j} &= 2\Lambda_j - [S_j, \Lambda_j]. \end{aligned}$$

PROPOSITION 6.7. *Indicating with e^h_k the 6×6 matrix with a 1 in position k (row) and h (column) and zero otherwise, the matrices of the operators L_{ij} and Λ_{ij} restricted on V are:*

$$\begin{aligned} L_{10} &= 2e^2_6, & L_{11} &= -2e^1_4, & L_{12} &= -2e^3_5, \\ L_{20} &= -2e^1_5, & L_{21} &= -2e^3_6, & L_{22} &= 2e^2_4, \\ \Lambda_{10} &= 8e^6_2, & \Lambda_{11} &= -8e^4_1, & \Lambda_{12} &= -8e^5_3, \\ \Lambda_{20} &= -8e^5_1, & \Lambda_{21} &= -8e^6_3, & \Lambda_{22} &= 8e^4_2. \end{aligned}$$

COROLLARY 6.8. *We have the following relations for the operators of* $\mathcal{L}_{\mathbb{C}}$ *restricted to* V*:*

$$[H_k, L_{ij}] = (1+\delta_{kj})L_{ij}, \quad [H_k, \Lambda_{ij}] = -(1+\delta_{kj})\Lambda_{ij},$$
$$[S_k, L_{ij}] = (-1)^{i+1}(1-3\delta_{kj})L_{ij}, \quad [S_k, \Lambda_{ij}] = (-1)^{i}(1-3\delta_{kj})\Lambda_{ij},$$
$$[S_k, V_j] = 0, \qquad [S_k, A_j] = 0.$$

Guided by all the explicit computations of the action on the isotypical component $V = V_{-2}^{\oplus 6}$ made up to this point, we now define in terms of the natural geometric operators a set of Serre generators for the algebra $\mathcal{L}_{\mathbb{C}}$.

DEFINITION 6.9.

$$\boldsymbol{e}_1 = \frac{1}{4}[L_{20}, A_1], \qquad \boldsymbol{f}_1 = \frac{1}{4}[V_1, \Lambda_{20}],$$
$$\boldsymbol{e}_2 = \frac{1}{4}[L_{22}, A_0], \qquad \boldsymbol{f}_2 = \frac{1}{4}[V_0, \Lambda_{22}],$$
$$\boldsymbol{e}_3 = V_0, \qquad \boldsymbol{f}_3 = A_0,$$
$$\boldsymbol{e}_4 = \frac{1}{4}[L_{12}, A_0], \qquad \boldsymbol{f}_4 = \frac{1}{4}[V_0, \Lambda_{12}],$$
$$\boldsymbol{e}_5 = \frac{1}{4}[L_{10}, A_1], \qquad \boldsymbol{f}_5 = \frac{1}{4}[V_1, \Lambda_{10}].$$

Moreover, for all $i \in \{1, \ldots, 5\}$ *we define* $\boldsymbol{h}_i = [\boldsymbol{e}_i, \boldsymbol{f}_i]$.

As the $\boldsymbol{e}_i$ have by construction associated matrix e^i_{i+1} once restricted to V and the $\boldsymbol{f}_i$ are their respective adjoints, one gets:

PROPOSITION 6.10. *The operators* $\boldsymbol{e}_i, \boldsymbol{f}_j, \boldsymbol{h}_k$ *satisfy the Serre relations for* $\boldsymbol{sl}(6,\mathbb{C})$ *and the* $\boldsymbol{h}_i$ *span the Cartan subalgebra* $\mathcal{H}$*:*

$$\boldsymbol{h}_1 = \frac{1}{2}(H_1 - H_2 - S_1 - S_2),$$
$$\boldsymbol{h}_2 = \frac{1}{2}(H_0 - H_1 + S_2),$$
$$\boldsymbol{h}_3 = \frac{1}{2}(-H_0 + H_1 + H_2),$$
$$\boldsymbol{h}_4 = \frac{1}{2}(H_0 - H_1 - S_2),$$
$$\boldsymbol{h}_5 = \frac{1}{2}(H_1 - H_2 + S_1 + S_2).$$

It would be interesting as a last remark to identify in the list of quadratic invariants the geometric operators $L_{ij}, \Lambda_{ij}, V_j, A_j$, the algebra $\mathcal{H}$ and the $\boldsymbol{so}(2,\mathbb{R})$ generator J. To do this one could of course use the explicit matrices for the quadratic invariants once restricted to V, which are not difficult to compute. One can however get very quickly a qualitative picture by using the notion of multidegree which we now introduce.

The decomposition $T^*X = W_0 \oplus W_1 \oplus W_2$ induces naturally a multidegree on $\bigwedge^* T^*_{\mathbb{C}}X$ with values in $\mathbb{Z}^3$, which we indicate with *mdeg*. This follows from the equation

$$\bigwedge^n T^*_{\mathbb{C}}X \cong \bigoplus_{p+q+r=n} \bigwedge^p (W_0 \otimes \mathbb{C}) \oplus \bigwedge^q (W_1 \otimes \mathbb{C}) \oplus \bigwedge^r (W_2 \otimes \mathbb{C}).$$

We notice furthermore that the (complexified) decomposition above is preserved by the operator J, and therefore *mdeg* commutes with the action of $\boldsymbol{so}(2,\mathbb{R})$.

PROPOSITION 6.11. *The operators $L_j, V_j, \Lambda_j, A_j, H_j, S_j$ are mdeg-homogeneous, with multi-degrees:*

$$\begin{array}{lll}
mdeg(L_0) = (0,1,1), & mdeg(L_1) = (1,0,1), & mdeg(L_2) = (1,1,0),\\
mdeg(\Lambda_0) = (0,-1,-1), & mdeg(\Lambda_1) = (-1,0,-1), & mdeg(\Lambda_2) = (-1,-1,0),\\
mdeg(V_0) = (2,0,0), & mdeg(V_1) = (0,2,0), & mdeg(V_2) = (0,0,2),\\
mdeg(A_0) = (-2,0,0), & mdeg(A_1) = (0,-2,0), & mdeg(A_2) = (0,0,-2),\\
mdeg(H_0) = (0,0,0), & mdeg(H_1) = (0,0,0), & mdeg(H_2) = (0,0,0),\\
mdeg(S_0) = (0,0,0), & mdeg(S_1) = (0,0,0), & mdeg(S_2) = (0,0,0).
\end{array}$$

PROOF. The values for mdeg for the L_j and the V_j follow immediately from mdeg of the corresponding forms and the dual (contraction) operators have opposite value of mdeg. The remaing values can be computed using the additivity of mdeg with respect to the bracket. □

PROPOSITION 6.12. *Let $\{j,k,l\} = \{0,1,2\}$. Then:*

$$\begin{aligned}
Span\big(L_{1j}, L_{2j}\big) &= Span\big([E_{w_k}, E_{\overline{w}_l}], [E_{w_l}, E_{\overline{w}_k}]\big),\\
Span\big(\Lambda_{1j}, \Lambda_{2j}\big) &= Span\big([I_{w_k}, I_{\overline{w}_l}], [I_{w_l}, I_{\overline{w}_k}]\big),\\
Span\big(V_j\big) &= Span\Big([E_{w_j}, E_{\overline{w}_j}]\Big),\\
Span\big(A_j\big) &= Span\Big([I_{w_j}, I_{\overline{w}_j}]\Big),\\
\mathcal{H} \oplus Span(J) &= \bigoplus_{m=0}^{2} Span\big([E_{w_m}, I_{w_m}], [E_{\overline{w}_m}, I_{\overline{w}_m}]\big).
\end{aligned}$$

PROOF. The *mdeg* of the L_{ij} is the same of the corresponding L_j, and similarly for their adjoints. The mdegs of the quadratic monomials are immediately computed as they are the sum of those of their components. For example, $mdeg(E_{w_0}) = mdeg(E_{\overline{w}_0}) = (1,0,0)$, $mdeg(E_{w_1}) = mdeg(E_{\overline{w}_1}) = (0,1,0)$ and therefore $mdeg([E_{w_0}, E_{\overline{w}_1}]) = (1,1,0)$, equal to that of L_{12} and L_{22}. □

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