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The Obstacle Thermistor Problem with Periodic Data

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ABSTRACT - The object of this paper is the study of an obstacle thermistor problem with a nonlocal term. Using the classical Lax-Milgram theorem, a result of Lions and a fixed point argument we prove existence of weak periodic solutions. Finally, by means of some a-priori estimates in Campanato's spaces, we obtain the regularity of these solutions.

1. Introduction.

Let Ω be a bounded open set of R^n, $n \geqslant 1$, with smooth boundary $\partial\Omega$. For given $\omega > 0$, we set $Q := \Omega \times P$, $\Sigma := \partial\Omega \times P$ and $P := \dfrac{R}{\omega Z}$ denotes the period interval $[0, \omega]$ so the functions defined in Q and Σ are ω-time periodic. In this paper we consider the following parabolic-elliptic system arising from an obstacle evolution thermistor problem

$$u_t - \triangle u + \int_\Omega G(x,y)u(y,t)dy \geqslant \sigma(u)|\nabla\varphi|^2 \text{ in } Q, \tag{1.1}$$

$$u(x,t) = 0 \text{ on } \Sigma, \tag{1.2}$$

$$div(\sigma(u)\nabla\varphi) = 0 \text{ in } Q, \tag{1.3}$$

$$\varphi(x,t) = \varphi_0(x,t) \text{ on } \Sigma. \tag{1.4}$$

Here Ω is the conductor and the unknowns u and φ are the temperature and the electrical potential in Ω, respectively. The nonlocal term $\int_\Omega G(x,y)u(y,t)dy$, with $G(x,y) \geqslant 0$, describes heat losses to the surrounding

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gas while $\sigma(u)$ denotes the temperature dependent electrical conductivity. Finally, the term $\sigma(u)|\nabla\varphi|^2$represents the Joule heating in Ω. We note that if $\varphi \in L^2(P; W^{1,2}(\Omega)) \cap L^\infty(Q)$, then (1.3) implies

$$\sigma(u)|\nabla\varphi|^2 = div(\sigma(u)\varphi\nabla\varphi),$$

in the sense of distribution.

Thus, we shall study the problem

$$u_t - \triangle u + \int_\Omega G(x,y)u(y,t)dy \geqslant div(\sigma(u)\varphi\nabla\varphi) \text{ in } Q, \tag{1.5}$$

$$u(x,t) = 0 \text{ on } \Sigma \tag{1.6}$$

$$div(\sigma(u)\nabla\varphi) = 0 \text{ in } Q, \tag{1.7}$$

$$\varphi(x,t) = \varphi_0(x,t), \text{ on } \Sigma. \tag{1.8}$$

To solve (1.5)-(1.8) we follow the approach of [1], [2] and introduce the penalized problem

$$u_{nt} - \triangle u_n + \left(\int_\Omega G(x,y)u_n(y,t)dy \right) I_n(u_n) = div(\sigma(u_n)\varphi_n\nabla\varphi_n) \text{ in } Q, \tag{1.9}$$

$$u_n(x,t) = 0 \text{ on } \Sigma, \tag{1.10}$$

$$div(\sigma(u_n)\nabla\varphi_n) = 0 \text{ in } Q, \tag{1.11}$$

$$\varphi_n(x,t) = \varphi_0(x,t) \text{ on } \Sigma, \tag{1.12}$$

where it is assumed that I_n satisfies

H$_n$) $I_n \in C(R)$, $0 \leqslant I_n(s) \leqslant 1$, for all $s \in R$, $I_n(s) = 0$ if $s \leqslant 0$ and $I_n(s) \to H(s)$ in L^2, where $H(s)$ denotes the Heaviside function.

The plan of the paper is as follows: in Section 2 we give some notations and preliminary results. In Section 3, we consider a linearized version of the elliptic problem (1.7)-(1.8) (see (3.5)-(3.6)) and prove existence and uniqueness of the weak periodic solution φ by means of the classical Lax-Milgram's theorem. Next, in Section 4, we use a result due to Lions [5, Theorem 6.1] to obtain existence and uniqueness of periodic solutions u_n for the penalized problem (1.9)-(1.10). In Section 5, we deduce some uniform estimates for (u_n, φ_n) and utilize a fixed point argument to obtain the existence of weak periodic solutions (u, φ) of (1.5)-(1.8) by letting $n \to \infty$. Finally, in Section 6, we establish the Hölder regularity for (u, φ). This will be done by proving a-priori estimates in Campanato's spaces as in [6], [7].

2. Preliminaries and auxiliary results.

Let us introduce an useful space of ω-periodic functions in which we look for solutions to our problem.

We consider the Hilbert space $V_0 := L^2(P; W_0^{1,2}(\Omega))$ endowed with the norm

$$\|v\|_{V0} := \left(\int_Q |\nabla v(x,t)|^2 dxdt \right)^{1/2} \tag{2.1}$$

and its dual $V^* := L^2(P; W^{-1,2}(\Omega))$. The duality pairing between V_0 and V^* shall be denoted by $\langle . \rangle$. The space V_0 is the closure with respect to the norm (2.1) of $C_0^\infty(\overline{Q})$, the space of periodic functions vanishing near Σ. We recall some notations concerning the Campanato spaces.

NOTATIONS. Let $Q_{t_0,t_1} := \Omega \times (t_0, t_1]$, $0 \leqslant t_0 < t_1$. A point $(x,t) \in Q_{t_0,t_1}$ shall be denoted by z. Let $B_r(x_0)$ be the ball centered at x_0 with radius r and let $Q_r(z_0)$ be the cylinder $B_r(x_0) \times (t_0 - r^2,\ t_0]$. Moreover, define $Q_r[z_0,\ r] := Q_r(z_0) \cap Q_{t_0,t_1}$ and $\Omega[x_0,\ r] := B_r(x_0) \cap \Omega$.

For $0 \leqslant \mu < n$, the Campanato space

$$\mathcal{L}^{2,\mu}(\Omega) := \left\{ u \in L^2(\Omega) \ : \ \left(\sup_{x_0 \in \overline{\Omega}, r>0} r^{-\mu} \int_{\Omega[x_0,r]} u^2(x)\, dx \right)^{1/2} < +\infty \right\}$$

is a Banach space with the equivalent norm

$$\|u\|_{2,\mu,\Omega[x_0,r]} = \left(\sup_{x_0 \in \overline{\Omega}, r>0} r^{-\mu} \int_{\Omega[x_0,r]} u^2(x)\, dx \right)^{1/2}.$$

(see [3]).

For $0 \leqslant \mu < n+2$, the Campanato space

$$\mathcal{L}^{2,\mu}(Q_{t_0,t_1}) := \left\{ u \in L^2(Q_{t_0,t_1}) : \sup_{z_0 \in Q_{t_0,t_1}, r>0} \left(r^{-\mu} \int_{Q_r[z_0,r]} u^2(x,t)\, dxdt \right)^{1/2} < +\infty \right\}.$$

is a Banach space with the equivalent norm

$$\|u\|_{2,\mu,Q_{t_0,t_1}} = \left(\sup_{z_0 \in Q_{t_0,t_1}, r>0} r^{-\mu} \int\limits_{Q_r[z_0,r]} u^2(x,t)\, dxdt \right)^{1/2}.$$

Furthermore, we need the following result

PROPOSITION A ([7]). The space $\mathcal{L}^{2,n+2+2\mu}(Q_{t_0,t_1})$ is isomorphic topologically and algebraically to $C^{\mu,\mu/2}(\overline{Q}_{t_0,t_1})$, for $\mu \in (0,1)$.

We shall study problem (1.5)-(1.8) under the following assumptions:

H$_\sigma$) $\sigma \in C(R)$, $0 < \sigma_* \leqslant \sigma(s) \leqslant \sigma^*$ for any $s \in R$;
H$_G$) $G \in C(R^2)$, $G(x,y) \geqslant 0$, we set $\widehat{G} := \sup_{R^2} G(x,y)$;
H$_0$) φ_0 is an ω-periodic bounded function with an extension $\widetilde{\varphi}_0$ to Q which satisfies $\widetilde{\varphi}_0 \in L^\infty(P; W^{1,\infty}(\Omega))$.

Moreover, we define the convex set

$$K := \{v \in L^2(P; W_0^{1,2}(\Omega)),\ v \geqslant 0 \text{ a.e. in } Q\}.$$

The notion of weak periodic solution for our problem is the following

DEFINITION 2.1. A pair of functions (u,φ) are a weak periodic solution to (1.5)-(1.8) if the following conditions hold

$$u \in K\ ,\ \ u_t \in L^2(P; W^{-1,2}(\Omega)) \text{ and } \varphi - \varphi_0 \in V_0,$$

$$\text{(2.2)}\quad \int\limits_Q u_t(v-u)dxdt + \int\limits_Q \nabla u \nabla (v-u)dxdt + \int\limits_Q\int\limits_\Omega G(x,y)u(y,t)(v-u)dydxdt$$

$$\geqslant - \int\limits_Q \sigma(u)\varphi\nabla\varphi\nabla(v-u)dxdt, \text{ for any } v \in K$$

and

$$\int\limits_Q \sigma(u)\nabla\varphi\nabla\xi dxdt = 0, \text{ for any } \xi \in V_0.$$

3. The elliptic problem.

In this section we study problem (1.7)-(1.8). Set $v(x,t) := := \varphi(x,t) - \varphi_0(x,t)$. Fixed $w \in L^2(Q)$ with $w \geqslant 0$, we solve the problem

$$div(\sigma(w)\nabla v) = -div(\sigma(w)\nabla\varphi_0 \,) \text{ in } Q, \tag{3.1}$$

$$v(x,t) = 0 \text{ on } \Sigma, \tag{3.2}$$

We state the weak formulation of solution to (3.1)-(3.2).

DEFINITION 3.1. A function $v \in V_0$ is called a weak periodic solution of (3.1)-(3.2) if

$$\int_Q \sigma(w)\nabla v\nabla\xi dxdt = -\int_Q \sigma(w)\nabla\varphi_0\nabla\xi dxdt, \text{ for all } \xi \in V_0. \tag{3.3}$$

The existence and uniqueness of the weak solution v shall be obtained by the classical Lax-Milgram theorem.

In agreement with this result, we define the bilinear form

$$a : V_0 \times V_0 \to R$$

by setting

$$a(v,\xi) := \int_Q \sigma(w)\nabla v\nabla\xi dxdt, \text{ for any } \xi \in V_0$$

We have

PROPOSITION 3.2. If we assume H$_\sigma$), then

i) a is continuous;
ii) a is coercive.

PROOF. Condition i) is a consequence of Hölder's inequality. In fact

$$|a(v,\xi)| \leqslant \sigma^* \left(\int_Q |\nabla v|^2 dxdt\right)^{1/2} \|\xi\|_{V_0}.$$

$$\leqslant \sigma^* \|v\|_{V_0} \|\xi\|_{V_0}.$$

ii) The coercivity of a follows from

$$a(v,v) = \int_Q \sigma(w)|\nabla v|^2 dxdt \geqslant \sigma_* \|v\|^2_{V0}$$

which implies

$$\frac{a(v,v)}{\|v\|_{V0}} \geqslant \sigma_* \|v\|_{V0} \to \infty, \text{ as } \|v\|_{V0} \to \infty. \qquad \square$$

In this way if H_0) holds, problem (3.3) becomes equivalent to the problem

$$a(v,\xi) = \langle G, \xi \rangle \tag{3.4}$$

where $G \in V^*$ is the linear functional defined as follows

$$\langle G, \xi \rangle := -\int_Q \sigma(w)\nabla\varphi_0 \nabla\xi dxdt, \ \forall \xi \in V_0.$$

Now, we can state the main result of this section

THEOREM 3.3. *Let H_σ) and H_0) be satisfied. There exists a unique weak periodic solution to* (3.4).

PROOF. By the theorem of Lax-Milgram we easily conclude the existence and uniqueness of the weak periodic solution. $\square$

Thus, for $w \in L^2(Q)$, $w \geqslant 0$ we have solved the problem

$$div(\sigma(w)\nabla\varphi) = 0 \text{ in } Q, \tag{3.5}$$

$$\varphi(x,t) = \varphi_0(x,t) \text{ on } \Sigma. \tag{3.6}$$

4. The parabolic penalized problem.

We consider the problem

$$u_{nt} - \triangle u_n + \left(\int_\Omega G(x,y)w(y,t)dy \right) I_n(w) = div(\sigma(w)\varphi\nabla\varphi) \text{ in } Q, \tag{4.1}$$

$$u_n(x,t) = 0 \text{ on } \Sigma. \tag{4.2}$$

DEFINITION 4.1. A weak periodic solution of (4.1)-(4.2) corresponding to w, is a function $u_n \in V_0$ such that

$$\int_Q u_{nt}\zeta dxdt + \int_Q \nabla u_n \nabla\zeta dxdt + \int_Q \left(\int_\Omega G(x,y)w(y,t)dy\right) I_n(w)\zeta dxdt \tag{4.3}$$
$$= -\int_Q \sigma(w)\varphi\nabla\varphi\nabla\zeta dxdt, \text{ for any } \zeta \in V_0.$$

The existence and uniqueness of the periodic solution follows from a Lion's result (see [5, Theorem 6.1]).

For any $u_n \in V_0$ we define the mapping $B : V_0 \to V^*$ as follows

$$\langle Bu_n, \zeta\rangle := \int_Q \nabla u_n \nabla\zeta dxdt, \text{ for any } \zeta \in V_0.$$

The set

$$D := \{v \in L^2(P; W_0^{1,2}(\Omega)) : \ v_t \in L^2(P; W^{-1,2}(\Omega))\}$$

is dense in V_0, because of the density of $C^\infty(\overline{Q}) \subset D$ in V_0.

Let

$$L : D \to V^*$$

be the closed skew-adjoint (i.e. $L = -L^*$) linear operator defined by

$$\langle Lu_n, \zeta\rangle := \int_Q u_{nt}\zeta dxdt, \ \forall\ \zeta \in V_0.$$

It is known that this operator is maximal monotone (see [4, Lemma 1.1, p. 318]).

The properties of B are contained in the following result.

PROPOSITION 4.2. Assume H$_\sigma$), then

j) B is continuous;

jj) B is coercive.

PROOF. j) The continuity of B follows from

$$|\langle Bu_n, \zeta\rangle| \leqslant \|u_n\|_{V_0}\|\zeta\|_{V_0}.$$

jj) By

$$\langle Bu_n, u_n\rangle = \int_Q |\nabla u_n|^2 dxdt = \|u_n\|_{V0}^2$$

one has

$$\frac{\langle Bu_n, u_n\rangle}{\|u_n\|_{V0}} = \|u_n\|_{V0} \to \infty, \text{ as } \|u_n\|_{V0} \to \infty. \qquad \square$$

Finally, let $M_n \in V^*$ be the linear functional defined by

$$\langle M_n, \zeta\rangle := \int_Q \left(\int_\Omega G(x,y)w(y,t)dy \right) I_n(w)\zeta dxdt - \int_Q \sigma(w)\varphi\nabla\varphi\nabla\zeta dxdt,$$

then, problem (4.3) can be considered in the framework of the abstract problems of the form

$$Lu_n + Bu_n = M_n \tag{4.4}$$

to which we apply [5, Theorem 6.1] to establish the existence and uniqueness of the weak periodic solution.

5. A fixed point argument.

The periodicity of solutions to (1.5)-(1.8) shall be proved utilizing the Schauder fixed point theorem for a suitable operator equation. To this aim, let

$$\Theta : \mathcal{S} \to \mathcal{S}$$

be the mapping defined by

$$\Theta(w) = u_n$$

and

$$\mathcal{S} := \{v \in L^2(Q) : \|v\|_{L^2(Q)} \leq M\},$$

where u_n is the unique weak periodic solution of (4.1)-(4.2) corresponding to w. The mapping Θ is well-defined. In order to prove its continuity we will prove some crucial estimates and convergences useful to utilize the Schauder fixed point theorem. Let $w_k \in L^2(Q)$ be a nonnegative sequence such that $w_k \to w$ and $\sigma(w_k) \to \sigma(w)$ strongly in $L^2(Q)$ as $k \to \infty$. We denote by u_{nk} and φ_k, respectively, the weak periodic solutions of

$$(5.1) \quad \int_Q u_{nkt}\zeta dxdt + \int_Q \nabla u_{nk}\nabla\zeta dxdt + \int_Q \left(\int_\Omega G(x,y)w_k(y,t)dy\right) I_n(w_k)\zeta dxdt$$
$$= -\int_Q \sigma(w_k)\varphi_k\nabla\varphi_k\nabla\zeta dxdt, \text{ for any } \zeta \in V_0$$

and

$$(5.2) \qquad \int_Q \sigma(w_k)\nabla\varphi_k\nabla\xi dxdt = 0, \text{ for any } \xi \in V_0.$$

An estimate on $\nabla\varphi_k$ is achieved choosing $\varphi_k - \varphi_0$ as a test function in (5.2). In fact,

$$\sigma_* \int_Q |\nabla\varphi_k|^2 dxdt \leqslant \int_Q \sigma(w_k)|\nabla\varphi_k|^2 dxdt = \int_Q \sigma(w_k)\nabla\varphi_k\nabla\varphi_0 dxdt$$
$$\leqslant \sigma^* \left(\int_Q |\nabla\varphi_k|^2 dxdt\right)^{1/2} \left(\int_Q |\nabla\varphi_0|^2 dxdt\right)^{1/2}$$

hence,

$$(5.3) \qquad \int_Q |\nabla\varphi_k|^2 dxdt \leqslant \left(\frac{\sigma^*}{\sigma_*}\right)^2 \|\nabla\varphi_0\|^2_{L^2(Q)}.$$

Moreover, by the weak maximum principle we derive that

$$(5.4) \qquad \|\varphi_k\|_{L^\infty(Q)} \leqslant \|\widetilde{\varphi}_0\|_{L^\infty(Q)}.$$

Combining (5.3) and (5.4), one has the usual energy estimate

$$(5.5) \qquad \int_Q |\varphi_k(x,t)|^2 dxdt + \int_Q |\nabla\varphi_k(x,t)|^2 dxdt \leqslant C.$$

In what follows, C stands for a generic positive constant independent of k and n.

Thanks to (5.3), we deduce that φ_k is uniformly bounded in the V norm, with respect to k. Therefore

$$\varphi_k \rightharpoonup \varphi \text{ in } V \text{ as } k \text{ goes to infinity.}$$

Furthermore, one has

LEMMA 5.1. The sequence $\nabla\varphi_k$ converges strongly to $\nabla\varphi$ in $L^2(P;(L^2(\Omega))^n)$.

PROOF. Taking $\varphi_k - \varphi$ as a test function in (5.2), we get

$$\int_Q \sigma(w_k)|\nabla(\varphi_k-\varphi)|^2 dxdt = \int_Q \sigma(w_k)\nabla(\varphi-\varphi_k)\nabla\varphi dxdt$$

from which it follows that

$$\sigma_* \int_Q |\nabla(\varphi_k-\varphi)|^2 dxdt \leqslant \int_Q \sigma(w_k)\nabla(\varphi-\varphi_k)\nabla\varphi dxdt.$$

The weak convergence of $\sigma(w_k)\nabla(\varphi-\varphi_k)$ to zero as $k\to\infty$ leads to the conclusion. □

Choosing u_{nk} as a test function in (5.1), we have

$$\int_Q |\nabla u_{nk}|^2 dxdt + \int_Q \left(\int_\Omega G(x,y)w_k(y,t)dy\right) I_n(w_k)u_{nk}(x,t)dxdt$$
$$= -\int_Q \sigma(w_k)\varphi_k\nabla\varphi_k\nabla u_{nk} dxdt.$$

Applying Young's and Poincaré's inequalities, one obtains

$$\frac{1}{2}\int_Q |\nabla u_{nk}|^2 dxdt$$
$$\leqslant \frac{\varepsilon}{2}\int_Q |u_{nk}|^2 dxdt + \frac{\widehat{G}^2}{2\varepsilon}\int_Q \left(\int_\Omega w_k(y,t)dy\right)^2 dxdt$$
$$+\frac{(\sigma^*\|\widetilde{\varphi}_0\|_{L^\infty(Q)})^2}{2}\left(\frac{\sigma^*}{\sigma_*}\right)^2 \|\nabla\varphi_0\|^2_{L^2(Q)}$$
$$\leqslant \frac{\varepsilon C}{2}\int_Q |\nabla u_{nk}|^2 dxdt + \frac{(\widehat{G}|\Omega|)^2}{2\varepsilon}\int_0^\omega\int_\Omega |w_k(y,t)|^2 dydt$$
$$+\frac{(\sigma^*\|\widetilde{\varphi}_0\|_{L^\infty(Q)})^2}{2}\left(\frac{\sigma^*}{\sigma_*}\right)^2 \|\nabla\varphi_0\|^2_{L^2(Q)}$$

$$\leqslant \frac{\varepsilon C}{2}\int\limits_Q |\nabla u_{nk}|^2 dxdt + \frac{(\widehat{G}|\Omega|)^2}{2\varepsilon}[\varepsilon + \int\limits_0^\omega \int\limits_\Omega |w(y,t)|^2 dydt$$

$$+\frac{(\sigma^* \|\widetilde{\varphi}_0\|_{L^\infty(Q)})^2}{2}\left(\frac{\sigma^*}{\sigma_*}\right)^2 \|\nabla\varphi_0\|^2_{L^2(Q)}.$$

Thus,

$$\int\limits_Q |\nabla u_{nk}(x,t)|^2 dxdt \leqslant C$$

and we infer the classical energy estimate

$$\int\limits_Q |u_{nk}(x,t)|^2 dxdt + \int\limits_Q |\nabla u_{nk}(x,t)|^2 dxdt \leqslant C. \tag{5.6}$$

By virtue of (5.5), (5.6) and (5.1), u_{nkt} is uniformly bounded in the V^* norm. Therefore u_{nk} belongs to a bounded set of D i.e.

$$\|u_{nk}\|_D \leqslant C.$$

Thus, we can select a subsequences, still denoted by u_{nk}, such that

$$u_{nk} \rightharpoonup u_n, \text{ in } D \text{ as } k \to \infty.$$

A well known result of [4, Lemma 5.1] guarantees that the sequence u_{nk} is precompact in $L^2(Q)$, then

$$u_{nk} \to u_n \text{ in } L^2(Q) \text{ and a.e. in } Q.$$

LEMMA 5.2. The operator Θ is continuous.

PROOF. From the preceding result, we have

$$u_{nk} \to u_n \text{ in } L^2(Q) \text{ and a.e. in } Q.$$

$$u_{nkt} \rightharpoonup u_{nt} \text{ in } L^2(P; W^{-1,2}(\Omega))$$

$$\nabla u_{nk} \rightharpoonup \nabla u_n \text{ in } L^2(P; (L^2(\Omega))^n)$$

$$\nabla \varphi_k \to \nabla\varphi \text{ in } L^2(P; (L^2(\Omega))^n)$$

$$\varphi_k \to \varphi \text{ in } V \text{ and a.e. in } Q$$

$$w_k \to w \text{ in } L^2(Q)$$

$$\sigma(w_k) \to \sigma(w) \text{ in } L^2(Q)$$

$$I_n(w_k) \to I_n(w) \text{ in } L^2(Q).$$

These convergences enable us to conclude that $\Theta(w_k) = u_{nk}$ converges strongly to $\Theta(w) = u_n$ in $L^2(Q)$. □

LEMMA 5.3. There exists a constant $M > 0$ such that

$$\|\Theta(w)\|_{L^2(Q)} \leqslant M, \text{ for all } w \in L^2(Q).$$

PROOF. The assertion is obtained as above. □

Since $\Theta(L^2(Q)) \subset D$ and the embedding $D \hookrightarrow L^2(Q)$ is compact, Θ is a compact operator from $L^2(Q)$ into itself. □

Our main result, is given in the next statement.

THEOREM 5.4. If H_σ) - H_0) are fulfilled, there exists at least one weak periodic solution to (1.5)-(1.8).

PROOF. As a consequence of Lemmas 5.2 and 5.3 the mapping Θ is both continuous and compact. Hence, by the Schauder fixed point theorem, Θ has a fixed points which corresponds to a weak periodic solutions to (1.9)-(1.12). □

As far as previously proved, we can conclude that

$$u_{nt} \rightharpoonup u_t \text{ in } L^2(P; W^{-1,2}(\Omega))$$
$$u_n \to u \text{ in } L^2(Q) \text{ and a.e. in } Q.$$
$$\nabla u_n \rightharpoonup \nabla u \text{ in } L^2(P; (L^2(\Omega))^n)$$
$$\nabla \varphi_n \to \nabla \varphi \text{ in } L^2(P; (L^2(\Omega))^n)$$
$$\varphi_n \to \varphi \text{ in } V \text{ and a.e. in } Q$$
$$w_n \to w \text{ in } L^2(Q)$$
$$\sigma(w_n) \to \sigma(w) \text{ in } L^2(Q)$$
$$I_n(u_n) \to \rho \text{ weak-* in } L^\infty(Q), \text{ with } 0 \leqslant \rho \leqslant 1.$$

These u_n are nonnegative. In fact, for $\zeta = u_n^-$, we have

$$(5.7) \quad \int_Q u_{nt} u_n^- dxdt + \int_Q \nabla u_n \nabla u_n^- dxdt + \int_Q \int_\Omega G(x,y) u_n(y,t) I_n(u_n) u_n^-(x,t) dydxdt$$
$$= \int_Q \sigma(u_n) |\nabla \varphi_n|^2 u_n^- \, dxdt, \ \forall \, \zeta \in V_0$$

from which it follows that

$$-\int_Q |\nabla u_n|^2 dxdt = \int_Q \sigma(u_n)|\nabla \varphi_n|^2 u_n^- \, dxdt.$$

This implies the nonnegativity of u_n. Since $u_n \geqslant 0$, so is u.

Passing to the limit in

$$(5.8) \quad \int_Q u_{nt}\zeta dxdt + \int_Q \nabla u_n \nabla \zeta dxdt + \int_Q \int_\Omega G(x,y)u_n(y,t)I_n(u_n)\zeta(x,t)dydxdt = -\int_Q \sigma(u_n)\varphi_n \nabla \varphi_n \nabla \zeta dxdt$$

we have

$$(5.9) \quad \int_Q u_t\zeta dxdt + \int_Q \nabla u \nabla \zeta dxdt + \int_Q \int_\Omega G(x,y)u(y,t)\rho(x,t)\zeta(x,t)dydxdt = -\int_Q \sigma(u)\varphi \nabla \varphi \zeta dxdt, \ \forall\ \zeta \in V_0.$$

We observe that $\rho(x_0,t_0) = 1$ if $u(x_0,t_0) > 0$ thus $\rho(x,t)(v-u) \leqslant (v-u)$ for all $v \geqslant 0$. If we replace ζ with $v-u$ in (5.9) with $v \in K$, then (2.2) is satisfied. □

6. Regularity.

This part of paper is devoted to the regularity of weak periodic solutions (u,φ) to problem (1.5)-(1.8). We will prove the Hölder continuity of (u,φ) by means of some a priori estimates in the Campanato spaces.

LEMMA 6.1. Let φ_n be the weak periodic solution of (1.11)-(1.12). If $0 \leqslant \mu_0 < n-2+2\delta_0$, $\delta_0 \in (0,1)$ one has

$$(6.1) \quad \|\nabla \varphi_n(t)\|_{2,\mu_0,\Omega} \leqslant c(\|\varphi_0(t)\|_{2,\mu_0,\Omega} + \|\nabla \varphi_0(t)\|_{2,(\mu_0-2)^+,\Omega}),$$

for a.e. $t \in P$.

PROOF. From a result of [6] we get

$$\|\nabla \varphi_n(t)\|_{2,\mu_0,\Omega} \leqslant c(\|\varphi_0(t)\|_{2,\mu_0,\Omega} + \|\nabla \varphi_0(t)\|_{2,(\mu_0-2)^+,\Omega} + \|\varphi_n(t)\|_{H^1(\Omega)})$$

a.e. t. Since $\varphi_0(t)$, $\nabla\varphi_0(t) \in L^\infty(\Omega)$ for a.e. t and the fact that $L^\infty(\Omega)$ is a multiplier for $\mathcal{L}^{2,\mu_1}(\Omega)$ for $\mu_1 < n$, by the inclusion $\mathcal{L}^{2,(\mu_0-2)^+}(\Omega) \hookrightarrow L^2(\Omega)$ and (5.3) we infer that

$$\|\nabla\varphi_n(t)\|_{L^2(\Omega)} \leqslant \left(\frac{\sigma^*}{\sigma_*}\right) \|\nabla\varphi_0(t)\|_{2,(\mu_0-2)^+,\Omega}$$

for a.e. t, which implies

$$\|\varphi_n(t)\|_{H^1(\Omega)} \leqslant c\|\varphi_0(t)\|_{2,\mu_0,\Omega} + \left(\frac{\sigma^*}{\sigma_*}\right) \|\nabla\varphi_0(t)\|_{2,(\mu_0-2)^+,\Omega}.$$

This completes the proof of Lemma 6.1. □

PROPOSITION 6.2. The weak periodic solutions (u, φ) of (1.5)-(1.8) belong to the space $C^{\alpha,\alpha/2}(\overline{Q})$.

PROOF. From (6.1), one has

$$\|\nabla\varphi_n(t)\|_{2,\mu_0,\Omega} \leqslant C$$

for a.e. t. The definition of Campanato's spaces gives us

$$r^{-(\mu_0+2)} \int_{Q[z_0,r]} |\nabla\varphi_n(x,t)|^2 dxdt \leqslant C \tag{6.2}$$

for all $z_0 \in Q$ and $r > 0$.

Being $\sigma(u_n)\varphi_n \in L^\infty(Q)$ and $L^\infty(Q)$ a multiplier of $\mathcal{L}^{2,\mu}(Q)$ if $0 \leqslant \mu \leqslant n + 2\delta_0$, then $\sigma(u_n)\varphi_n\nabla\varphi_n \in \mathcal{L}^{2,\mu}(Q)$ and the inequality (6.2) implies that

$$\|\nabla\varphi_n\|_{2,\mu}^2 \leqslant C.$$

By the comparison principle, we have that the nonnegative weak periodic solution ψ of problem

$$\psi_t - \triangle\psi = div(\sigma(u_n)\varphi_n\nabla\varphi_n) \text{ in } Q, \tag{6.3}$$

$$\psi = 0 \text{ on } \Sigma,$$

is a supersolution for (5.2), so that

$$0 \leqslant u_n \leqslant \psi. \tag{6.4}$$

According to a result in [7, Theorem 1], we get

$$\|\nabla\psi\|_{2,\mu} \leqslant C$$

for $0 < \mu \leqslant n + 2\alpha$ and ψ belongs to $\mathcal{L}^{2,\mu+2}(Q)$ (see [7, Lemma 2.6]). Because of Proposition A, one has that $\psi \in C^{\alpha,\alpha/2}(\overline{Q})$ and consequently

$$\|\psi\|_{L^\infty(Q)} \leqslant C.$$

The boundedness of ψ implies that $u_n \in L^\infty(Q)$. Theorem 1 in [7] allows to conclude that

$$(6.5)\quad \|\nabla u_n\|_{2,\mu} \leqslant c(\|\sigma(u_n)\varphi_n \nabla\varphi_n\|_{2,\mu} + \|I_n(u_n)\left(\int_\Omega G(x,y)u_n(y,t)dy\right)\|_{2,(\mu-2)^+} + \|u_n\|_V)$$

and the embedding of $L^\infty(Q) \hookrightarrow \mathcal{L}^{2,\mu}(Q)$ yields

$$\left\|\int_\Omega I_n(u_n)(G(x,y)u_n(y,t)dy)\right\|_{2,(\mu-2)^+} \leqslant c\left\|\int_\Omega G(x,y)u_n(y,t)dy\right\|_{L^\infty(Q)} \leqslant C.$$

Thus, if $0 < \mu \leqslant n + 2\alpha$ by (6.5), we obtain

$$\|\nabla u_n\|_{2,\mu} \leqslant C$$

and $u_n \in \mathcal{L}^{2,\mu+2}(Q)$, (see [7, Lemma 2.6]). Since we showed that $u_n \to u$ and $\varphi_n \to \varphi$ a.e. in Q with u_n, φ_n in $L^\infty(Q)$, we conclude that u , $\varphi \in \mathcal{L}^{2,\mu}(Q)$ for $\mu = n + 2 + 2\alpha$. Finally, by Proposition A u, $\varphi \in C^{\alpha,\alpha/2}(\overline{Q})$ for $\alpha = \dfrac{\mu - n}{2}$. □

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