

## A Property of Generalized McLain Groups

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ABSTRACT - In this short note we show that if  $S$  is a connected unbounded poset and  $R$  a ring with no zero divisors, then a generalized McLain group  $G(R, S)$  is a product of two proper normal subgroups.

### 1. Introduction.

McLain groups were defined in [3] for the first time. These groups are characteristically simple and locally nilpotent with some further interesting properties.

Let  $S$  be an unbounded partially ordered set (poset, in short) and  $R$  be a ring with  $1 \neq 0$ . Define the generalized McLain group  $G(R, S)$  as in [2]. Now every element of  $G(R, S)$  can be uniquely expressed as

$$1 + \sum_{i=1}^n a_i e_{\alpha_i, \beta_i}$$

where  $a_i \in R$ ,  $\alpha_i, \beta_i \in S$ ,  $\alpha_i < \beta_i$  for  $i = 1, \dots, r$  and  $n \in \mathbb{N}$ .

In [5], some properties of  $G(\mathbb{F}_p, S)$  are considered for some orderings where  $\mathbb{F}_p$  is the field of  $p$  elements. The generalized McLain groups are considered in a general context in [2] and the automorphism groups of these groups are considered in [1], [2] and [4].

[2, Theorem 7.1] gives a necessary and sufficient condition to be  $G(R, S)$  indecomposable. In this short note we ask the following question :

Does  $G(R, S)$  have proper normal subgroups  $K$  and  $N$  such that  $G(R, S) = KN$ ?

Two elements  $\alpha, \beta \in S$  are called *connected*, if there are elements  $\alpha_0, \dots, \alpha_n \in S$  such that  $\alpha_0 = \alpha, \alpha_n = \beta$  and for each  $0 \leq i < n$ , either

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$\alpha_i \leq \alpha_{i+1}$  or  $\alpha_{i+1} \leq \alpha_i$ .  $S$  is called *connected* if every pair of elements in  $S$  is connected.

We shall prove the following:

**THEOREM.** Let  $S$  be a connected unbounded poset and  $R$  a ring with no zero divisors. Then  $M := G(R, S)$  has proper normal subgroups  $K$  and  $N$  such that  $M = KN$ ,  $C_M(K) \neq 1$  and  $C_M(N) = 1$ . Furthermore if  $1 + ce_{\zeta\zeta} \in M$ , then  $1 + ce_{\zeta\zeta} \in K$  or  $1 + ce_{\zeta\zeta} \in N$ .

**2. Proof of the Theorem.**

**LEMMA 2.1.** *Let  $S$  be a connected unbounded poset and  $R$  a ring with no zero divisors. Then every finite family of non-trivial normal subgroups of  $M$  intersects non-trivially.*

**PROOF.** Obviously it is sufficient to prove the lemma for two proper non-trivial normal subgroups of  $M$ . Let  $N$  and  $K$  be such subgroups of  $M$ . Assume  $N \cap K = 1$  and follow the proof of [2, Theorem 7.1] to reach a contradiction. □

**PROOF OF THE THEOREM.** Put  $M := G(R, S)$  and let  $w = 1 + ae_{\alpha\beta}$  with  $0 \neq a \in R$ ,  $K := C_M(\langle w^M \rangle)$  and  $N := \langle (1 + ce_{\gamma\delta})^M : 1 + ce_{\gamma\delta} \notin K, c \in R \rangle$ . Then we will prove that  $M = KN$ . Since  $Z(M) = 1$ , we have  $K \neq M$ . Clearly

$$\begin{aligned} \langle w^M \rangle &= \left\langle (1 + ae_{\alpha\beta})^{1 + \sum_{i=1}^r a_i e_{\lambda_i \alpha} + \sum_{j=1}^s b_j e_{\beta \mu_j}} : a, a_i, b_j \in R, 1 \leq i \leq r, 1 \leq j \leq s \right\rangle \\ &= \left\langle 1 + ae_{\alpha\beta} - \sum_{i=1}^r aa_i e_{\lambda_i \beta} + \sum_{j=1}^s ab_j e_{\alpha \mu_j} + \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} aa_i b_j e_{\lambda_i \mu_j} : a, a_i, b_j \in R, \right. \\ &\quad \left. 1 \leq i \leq r, 1 \leq j \leq s \right\rangle. \end{aligned}$$

Let  $\alpha < \sigma < \tau < \beta$ , then we have  $1 + de_{\sigma\tau} \in K$  with  $0 \neq a \in R$  by [2, Lemma 2.2]. Since a generator  $1 + ce_{\gamma\delta}$  of  $N$  is not contained in  $K$ , it must be of the form  $1 + ce_{\beta\delta}$  or  $1 + ce_{\mu\delta}$  ( $\mu > \beta$ ) or  $1 + ce_{\gamma\alpha}$  or  $1 + ce_{\gamma\lambda}$  ( $\lambda < \alpha$ ) and its conjugates must be of the form:

$$(1 + ce_{\beta\delta})^{1 + \sum_{i=1}^r a_i e_{\lambda_i \beta} + \sum_{j=1}^s b_j e_{\delta \mu_j}} = 1 + ce_{\beta\delta} - \sum_{i=1}^r ca_i e_{\lambda_i \delta} + \sum_{j=1}^s cb_j e_{\beta \mu_j} + \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} ca_i b_j e_{\lambda_i \mu_j}$$

or

$$(1 + ce_{\mu\delta})^{1+\sum_{i=1}^r a_i e_{i\mu} + \sum_{j=1}^s b_j e_{\delta\mu_j}} = 1 + ce_{\mu\delta} - \sum_{i=1}^r a_i e_{i\delta} + \sum_{j=1}^s cb_j e_{\mu\mu_j} + \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} ca_i b_j e_{\lambda_i \mu_j}$$

or

$$(1 + ce_{\gamma\alpha})^{1+\sum_{i=1}^r a_i e_{i\gamma} + \sum_{j=1}^s b_j e_{\alpha\mu_j}} = 1 + ce_{\gamma\alpha} - \sum_{i=1}^r ca_i e_{\lambda_i \alpha} + \sum_{j=1}^s cb_j e_{\gamma\mu_j} + \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} ca_i b_j e_{\lambda_i \mu_j}$$

or

$$(1 + ce_{\gamma\lambda})^{1+\sum_{i=1}^r a_i e_{i\gamma} + \sum_{j=1}^s b_j e_{\lambda\mu_j}} = 1 + ce_{\gamma\lambda} - \sum_{i=1}^r ca_i e_{\lambda_i \lambda} + \sum_{j=1}^s b_j e_{\gamma\mu_j} + \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} ca_i b_j e_{\lambda_i \mu_j}.$$

For all terms  $e_{\theta\varepsilon}$  that appear in each case,  $\theta \notin [\alpha, \beta]$  or  $\varepsilon \notin [\alpha, \beta]$ . Hence we have that there is no product of these elements which equals  $1 + e_{\sigma\tau}$ , i.e.,  $1 + e_{\sigma\tau} \notin N$ . Hence  $N \neq M$  and obviously  $M = KN$ .

Clearly we have  $C_M(K) \neq 1$ . Assume  $C_M(N) \neq 1$ , then  $C_M(K) \cap C_M(N) \neq 1$  by Lemma 2.1. But since  $Z(M) = 1$ , this is a contradiction. The final part of the theorem follows by the construction of  $K$  and  $N$ . Now the proof is complete. □

**COROLLARY 2.2.** *Let  $S$  be a connected unbounded poset and  $R$  a ring with no zero divisors. Put  $M := G(R, S)$  and let  $N$  be the subgroup defined in the theorem. Then*

$$C_M(\langle x^M \rangle) / C_M(\langle x^M \rangle) \cap N$$

*is perfect for every generator  $x$  of  $M$  of the form  $1 + ae_{\alpha\beta}$  ( $a \in R$ ).*

**COROLLARY 2.3.** *Let  $S$  be a connected unbounded poset and  $R$  a ring with no zero divisors. Put  $M := G(R, S)$ . Then  $M$  has a decomposable non-trivial epimorphic image.*

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