REND. SEM. MAT. UNIV. PADOVA, Vol. 122 (2009)

Analytic Solutions of Second Order Nonlinear Difference Equations all of whose Eigenvalues are 1

MAMI SUZUKI (*)

ABSTRACT - For nonlinear difference equations, it is difficult to obtain analytic solutions. Especially, when all the absolute values of the equation are equal to 1, it is quite difficult.

We consider a second order nonlinear difference equation which can be transformed into the following simultaneous system of nonlinear difference equations,

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases}$$

where $X(x, y) = \lambda_1 x + \sum_{i+j \geqq 2} c_{ij} x^i y^j$, $Y(x, y) = \lambda_2 y + \sum_{i+j \geqq 2} d_{ij} x^i y^j$ and we assume some conditions. For these equations, when $|\lambda_1| \neq 1$ or $|\lambda_2| \neq 1$, we have obtained analytic solutions in earlier studies. In the present paper, we will prove the existence of an analytic solution and obtain analytic solutions of the difference equations for the case $\lambda_1 = \lambda_2 = 1$.

1. Introduction.

We start by considering the following second order nonlinear difference equation,

$$\begin{cases} u(t+1) = U(u(t), v(t)), \\ v(t+1) = V(u(t), v(t)), \end{cases} \tag{1.1}$$

where $U(u, v)$ and $V(u, v)$ are entire functions for u and v. We suppose that the

(*) Indirizzo dell'A.: Department of Mathematics and Science, College of Liberal Arts, J. F. Oberlin University. 3758 Tokiwa-cho, Machida-City, Tokyo, 194-0294, Japan.
E-mail: m-suzuki@obirin.ac.jp
2000 *Mathematics Subject Classification*. 39A10, 39A11, 39B32.

equation (1.1) admits an equilibrium point $(u^*, v^*) : \begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{pmatrix} U(u^*, v^*) \\ V(u^*, v^*) \end{pmatrix}$. We can assume, without loss of generality, that $(u^*, v^*) = (0, 0)$. Furthermore we suppose that U and V are written in the following form

$$\begin{pmatrix} u(t+1) \\ v(t+1) \end{pmatrix} = M \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} U_1(u(t), v(t)) \\ V_1(u(t), v(t)) \end{pmatrix},$$

where $U_1(u, v)$ and $V_1(u, v)$ are higher order terms of u and v, and M is a constant matrix. Let λ_1, λ_2 be the characteristic values of the matrix M. For some regular matrix P determined by M, put $\begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}$, then we can transform the system (1.1) into the following simultaneous system of first order difference equations (1.2):

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases} \tag{1.2}$$

where $X(x, y)$ and $Y(x, y)$ are supposed to be holomorphic and expanded in a neighborhood of $(0, 0)$ in the form

$$\begin{cases} X(x, y) = \lambda_1 x + \sum_{i+j \geqq 2} c_{ij} x^i y^j = \lambda_1 x + X_1(x, y), \\ Y(x, y) = \lambda_2 y + \sum_{i+j \geqq 2} d_{ij} x^i y^j = \lambda_2 y + Y_1(x, y), \end{cases} \tag{1.3}$$

or

$$\begin{cases} X(x, y) = \lambda x + y + \sum_{i+j \geqq 2} c'_{ij} x^i y^j = \lambda x + X'_1(x, y), \\ Y(x, y) = \lambda y + \sum_{i+j \geqq 2} d'_{ij} x^i y^j = \lambda y + Y'_1(x, y), \end{cases} \tag{1.4}$$

where, in the second case, $\lambda = \lambda_1 = \lambda_2$.

In this paper we consider analytic solutions of difference system (1.2). In [9] and [8], already, we have obtained general analytic solutions of (1.2) in the case $|\lambda_1| \neq 1$ or $|\lambda_2| \neq 1$. However in the case $|\lambda_1| = |\lambda_2| = 1$, it is difficult to prove the existence of analytic solution or obtain an analytic solution of the equation. For a long time we have not been able to treat equation (1.2) under the latter condition.

For analytic solutions of nonlinear first order difference equations, Kimura [2] has studied the cases in which one eigenvalue is equal to 1, furthermore Yanagihara [10] has studied the cases in which the absolute value of the eigenvalue is equal to 1. Here we shall study analytic solutions of nonlinear second order difference equations in which the absolute values of the eigenvalues of the matrix M are both equal to 1.

As an example of the equation (1.3), we consider the following "population model"

$$u(t+2) = \alpha u(t+1) + \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)}, \text{ for } t > -\infty,$$

where $\alpha = 1 + r > 0$, $\beta > 0$ are constants, r is the net (births minus death) endogenous population growth rate. This model was proposed by Prof. D. Dendrios [1], and may be transformed into a system as in (1.2). In [6], we have investigated properties of a solution assuming its existence when $\alpha = 1$, i.e. $r = 0$. However if both eigenvalues of the equation are equal to 1, i.e. under the condition $r = 0$, the existence of a solution of the model is not established. In the next paper, we will show a solution of the population model in the case $r = 0$.

Moreover we have studied some economic models in the form (1.2), but we had to exclude the case $|\lambda_1| = |\lambda_2| = 1$. For example, in [7], we consider the following "duopoly system"

$$\begin{cases} x(t+1) = \alpha y(t)(1 - y(t)) \\ y(t+1) = \beta x(t)(1 - x(t)), \end{cases}$$

where α and β are constants. For this system, we have $\lambda_1 = -\lambda_2$. In [7], we consider the system under the condition such that $0 < \lambda_1^2 = \lambda_2^2 < 1$.

Therefore we will investigate the equation (1.2) in the case $|\lambda_1| = |\lambda_2| = 1$.

In this present paper, making use of theorems in [2] and [5], we will prove the existence of a solution and obtain an analytic solution of (1.2) under the conditions $\lambda_1 = \lambda_2 = 1$, in which X, Y are defined by (1.3), i.e., we suppose that

$$\text{(1.5)} \qquad \begin{cases} X(x,y) = x + \displaystyle\sum_{i+j \geqq 2,\, i \geqq 1} c_{ij} x^i y^j = x + X_1(x,y), \\ Y(x,y) = y + \displaystyle\sum_{i+j \geqq 2,\, j \geqq 1} d_{ij} x^i y^j = y + Y_1(x,y). \end{cases}$$

Here we suppose that $X_1(x,y) \not\equiv 0$ or $Y_1(x,y) \not\equiv 0$.

Next we consider a functional equation

$$\text{(1.6)} \qquad \Psi(X(x, \Psi(x))) = Y(x, \Psi(x)),$$

where $X(x,y)$ and $Y(x,y)$ are holomorphic functions in $|x| < \delta_1$, $|y| < \delta_1$. We assume that $X(x,y)$ and $Y(x,y)$ are expanded there as in (1.5).

Now we will consider the meaning of equation (1.6).

Consider the simultaneous system of difference equations (1.2). Suppose (1.2) admits a solution $(x(t), y(t))$. If $\frac{dx}{dt} \neq 0$ for some t_0, then we can write $t = \psi(x)$ with a function ψ in a neighborhood of $x_0 = x(t_0)$, and we can write

$$y = y(t) = y(\psi(x)) = \Psi(x), \tag{1.7}$$

there. Then the function Ψ satisfies the equation (1.6).

Conversely we assume that a function Ψ is a solution of the functional equation (1.6). If the following first order difference equation

$$x(t+1) = X(x(t), \Psi(x(t))), \tag{1.8}$$

has a solution $x(t)$, we put $y(t) = \Psi(x(t))$. Then the $(x(t), y(t))$ is a solution of (1.2). Hence if there is a solution Ψ of (1.6), then we can reduce the system (1.2) to a single equation (1.8).

This relation is important in order to derive analytic solutions of nonlinear second order difference equations which are written as in (1.2). We have proved the existence of solutions Ψ of (1.6) in [3] (or [4]) and [8] for other conditions on X and Y.

Hereafter we consider t to be a complex variable, and concentrate on the difference system (1.2). We define a domain $D_1(\kappa_0, R_0)$ by

$$D_1(\kappa_0, R_0) = \{t \,:\, |t| > R_0, |\arg[t]| < \kappa_0\}, \tag{1.9}$$

where κ_0 is any constant such that $0 < \kappa_0 \leqq \frac{\pi}{4}$, and R_0 is a sufficiently large number which may depend on X and Y. Further we define a domain $D^*(\kappa, \delta)$ by

$$D^*(\kappa, \delta) = \{x \,:\, |\arg[x]| < \kappa,\ 0 < |x| < \delta\}, \tag{1.10}$$

where δ is a small constant, and κ is a constant such that $\kappa = 2\kappa_0$, i.e., $0 < \kappa \leqq \frac{\pi}{2}$.

Our aim in this paper is to prove the following Theorem 1.

THEOREM 1. *Suppose $X(x, y)$ and $Y(x, y)$ are expanded in the forms* (1.5) *such that $X_1(x, y) \not\equiv 0$ or $Y_1(x, y) \not\equiv 0$, and put $A = c_{20}$.*

(1) *Suppose that $kc_{20} = d_{11}$ for some $k \in \mathbb{N}$, $k \geqq 2$, then we have a formal solution $x(t)$ of* (1.2) *the following form*

$$-\frac{1}{At}\left(1 + \sum_{j+k \geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right)^{-1}, \tag{1.11}$$

where $\hat{q}_{jk}$ are constants defined by X and Y.

(2) *Suppose* $kc_{20} = d_{11} < 0$ *for some* $k \in \mathbb{N}$, $k \geqq 2$, *and* $R_1 = \max(R_0, 2/(|A|\delta))$, *then there is a solution* $x(t)$ *of* (1.2) *such that*

(i) $x(t)$ *is holomorphic and* $x(t) \in D^*(\kappa, \delta)$ *for* $t \in D_1(\kappa_0, R_1)$,

(ii) $x(t)$ *is expressible in the following form*

$$x(t) = -\frac{1}{At}\left(1 + b\left(t, \frac{\log t}{t}\right)\right)^{-1}, \tag{1.12}$$

where $b(t, \log t/t)$ *is asymptotically expanded for* $t \in D_1(\kappa_0, R_1)$ *such that*

$$b\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k \geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k,$$

as $t \to \infty$ *through* $D_1(\kappa_0, R_1)$.

2. Proof of Theorem 1.

In [2], Kimura considered the following first order difference equation

$$w(t + \lambda) = F(w(t)), \tag{D1}$$

where F is represented in a neighborhood of ∞ by a Laurent series

$$F(z) = z\left(1 + \sum_{j=m}^{\infty} b_j z^{-j}\right), \quad b_m = \lambda \neq 0. \tag{2.1}$$

He defined the following domains

$$\begin{aligned} D(\varepsilon, R) = \{t \,:\, |t| > R, |\arg[t] - \theta| < \frac{\pi}{2} - \varepsilon, \text{ or } \mathbf{Im}(e^{i(\theta-\varepsilon)}t) > R,& \\ \text{or } \mathbf{Im}(e^{i(\theta+\varepsilon)}t) < -R\},& \end{aligned} \tag{2.2}$$

$$\begin{aligned} \hat{D}(\varepsilon, R) = \{t \,:\, |t| > R, |\arg[t] - \theta - \pi| < \frac{\pi}{2} - \varepsilon \text{ or } \mathbf{Im}(e^{-i(\theta+\pi-\varepsilon)}t) > R& \\ \text{or } \mathbf{Im}(e^{-i(\theta+\pi+\varepsilon)}t) < -R\},& \end{aligned} \tag{2.3}$$

where ε is an arbitrarily small positive number, R is a sufficiently large number which may depend on ε and F, and θ is defined by $\theta = \arg \lambda$, (in this present paper, we consider the case $\lambda = 1$ in (D1)). He proved the following Theorems A and B.

THEOREM A. – *Equation* ($D1$) *admits a formal solution of the form*

$$t\left(1+\sum_{j+k\geqq 1}\hat{q}_{jk}t^{-j}\left(\frac{\log t}{t}\right)^k\right) \tag{2.4}$$

containing an arbitrary constant, where $\hat{q}_{jk}$ *are constants defined by* F.

THEOREM B. – *Given a formal solution of the form* (2.4) *of* (D1), *there exists a unique solution* $w(t)$ *satisfying the following conditions:*

(i) $w(t)$ *is holomorphic in* $D(\varepsilon, R)$,
(ii) $w(t)$ *is expressible in the form*

$$w(t)=t\left(1+b\left(t,\frac{\log t}{t}\right)\right), \tag{2.5}$$

where the domain $D(\varepsilon, R)$ *is defined by* (2.2), *and* $b(t,\eta)$ *is holomorphic for* $t\in D(\varepsilon,R)$, $|\eta|<1/R$, *and in the expansion*

$$b(t,\eta)\sim\sum_{k=1}^{\infty}b_k(t)\eta^k.$$

Here $b_k(t)$ *is asymptotically developed into*

$$b_k(t)\sim\sum_{j+k\geqq 1}^{\infty}\hat{q}_{jk}t^{-j},$$

as $t\to\infty$ *through* $D(\varepsilon,R)$, *where* $\hat{q}_{jk}$ *are constants which are defined by* F.

Also there exists a unique solution $\hat{w}$ *which is holomorphic in* $\hat{D}(\varepsilon,R)$ *and satisfies a condition analogous to (ii), where the domain* $\hat{D}(\varepsilon,R)$ *is defined by* (2.3).

In Theorems A and B, Kimura defined the function F as in (2.1). In our method, we do not have a Laurent series for the function F. Hence we derive the following Propositions.

In the following, A will be a constant which is defined as in Theorem 1 such that $A=c_{20}$, where c_{20} is the coefficient in (1.5).

PROPOSITION 2. – *Suppose* $\tilde{F}(t)$ *is formally expanded such that*

$$\tilde{F}(t)=t\left(1+\sum_{j=1}^{\infty}b_jt^{-j}\right),\qquad b_1=\lambda\neq 0. \tag{2.6}$$

Then the equation

$$\psi(\tilde{F}(t))=\psi(t)+\lambda \tag{2.7}$$

has a formal solution

$$\psi(t) = t\left(1 + \sum_{j=1}^{\infty} q_j t^{-j} + q\frac{\log t}{t}\right), \tag{2.8}$$

where q_1 *can be arbitrarily prescribed while other coefficients* q_j $(j \geqq 2)$ *and* q *are uniquely determined by* b_j, $(j = 1, 2, \cdots)$, *independently of* q_1.

PROPOSITION 3. – *Suppose* $\tilde{F}(t)$ *is holomorphic and expanded asymptotically in* $\{t\,:\,-1/(At) \in D^*(\kappa, \delta),\ A < 0\}$ *as*

$$\tilde{F}(t) \sim t\left(1 + \sum_{j=1}^{\infty} b_j t^{-j}\right), \qquad b_1 = \lambda \neq 0,$$

where $D^*(\kappa, \delta)$ *is defined in* (1.10). *Then the equation* (2.7) *has a solution* $w = \psi(t)$, *which is holomorphic in* $\{t\,:\,-1/(At) \in D^*(\kappa/2, \delta/2),\ A < 0\}$ *and has an asymptotic expansion*

$$\psi(t) \sim t\left(1 + \sum_{j=1}^{\infty} q_j t^{-j} + q\frac{\log t}{t}\right),$$

there.

These Propositions are proved as in [2] pp. 212–222. Since $A = c_{20} < 0$ and $\kappa_0 = \kappa/2$, we see that $x = -1/(At) \in D^*(\kappa/2, \delta/2)$ equivalent to $t \in D_1(\kappa/2, 2/(|A|\delta)) = D_1(\kappa_0, 2/(|A|\delta))$, where $D_1(\kappa_0, R_0)$ is defined in (1.9).

We define a function ϕ to be the inverse of ψ, so that $w = \psi^{-1}(t) = \phi(t)$. Then we have $\phi \circ \psi(w) = w, \psi \circ \phi(t) = t$, furthermore ϕ is a solution of the following difference equation

$$w(t + \lambda) = \tilde{F}(w(t)), \tag{D}$$

where $\tilde{F}$ is defined as in Proposition 2 (see pp. 236 in [2]). Hereafter, we put $\lambda = 1$. Since $\theta = 0$, we then have the following Propositions 4 and 5, analogous to Theorems A and B.

PROPOSITION 4. – *Suppose* $\tilde{F}(t)$ *is formally expanded as in* (2.6). *Then the equation* (D) *has a formal solution*

$$w = \phi(t) = t\left(1 + \sum_{j+k \geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right), \tag{2.9}$$

where $\hat{q}_{jk}$ *are constants which are defined by* $\tilde{F}$ *as in Theorem A.*

PROPOSITION 5. – *Suppose a function* ϕ *is the inverse of* ψ *such that* $w = \psi^{-1}(t) = \phi(t)$. *Given a formal solution of the form* (2.9) *of* (D), *there exists a unique solution* $w(t) = \phi(t)$ *which is holomorphic and admits an asymptotic expansion for* $t \in D_1(\kappa_0, 2/(|A|\delta))$ *such that*

$$w = \phi(t) = t\left(1 + b\left(t, \frac{\log t}{t}\right)\right), \tag{2.10}$$

where

$$b\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k \geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k.$$

This function $\phi(t)$ *is a solution of difference equation of* (D).

In [5], we proved the following Theorem C.

THEOREM C. – *Suppose* $X(x, y)$ *and* $Y(x, y)$ *are defined in* (1.5). *Then we have following:*

(1) *If* $kc_{20} \neq d_{11}$ *for any* $k \in \mathbb{N}$, $k \geqq 2$, *the formal solution* $\Psi(x)$ *of* (1.6) *of the following form*

$$\Psi(x) = \sum_{m=1}^{\infty} a_m x^m, \tag{2.11}$$

is identical to 0, *i.e.,* $a_1 = a_2 = \cdots = 0$.

(2) *If* $kc_{20} = d_{11}$ *for some* $k \in \mathbb{N}$, $k \geqq 2$, *we have a formal solution* $\Psi(x)$ *of* (1.6) *of the following form*

$$\Psi(x) = \sum_{m=k}^{\infty} a_m x^m, \tag{2.12}$$

i.e., $a_1 = a_2 = \cdots = a_{k-1} = 0$.

(3) *Suppose*

$$kc_{20} = d_{11} < 0 \quad \text{for some} \quad k \in \mathbb{N},\ k \geqq 2. \tag{2.13}$$

For any κ, $0 < \kappa \leqq \dfrac{\pi}{2}$ *and small* δ, *we define the following domain* $D^*(\kappa, \delta)$,

$$D^*(\kappa, \delta) = \{x : |\arg[x]| < \kappa,\ 0 < |x| < \delta\}. \tag{1.10}$$

There is a constant $\delta > 0$ *and a solution* $\Psi(x)$ *of* (1.6), *which is holomorphic* *and can be expanded asymptotically in $D^*(\kappa, \delta)$ such that*

$$\Psi(x) \sim \sum_{j=k}^{\infty} a_j x^j. \tag{2.14}$$

PROOF OF THEOREM 1. – We will first prove (1) of Theorem 1. From Theorem C (2), we have a formal solution $\Psi(x)$ of (1.6) which can be formally expanded so that one has

$$\Psi(x) = \sum_{j=k}^{\infty} a_j x^j. \tag{2.15}$$

On the other hand putting $w(t) = -\dfrac{1}{Ax(t)}$ in (1.8), since $A = c_{20}$, we have

$$\begin{aligned} w(t+1) &= -\frac{1}{Ax(t+1)} = -\frac{1}{AX\big(x(t), \Psi(x(t))\big)} \\ &= -\frac{1}{AX\left(-\dfrac{1}{Aw(t)}, \Psi\left(-\dfrac{1}{Aw(t)}\right)\right)}. \end{aligned} \tag{2.16}$$

From (1.5), we have

$$\begin{aligned} X\big(x(t), \Psi(x(t))\big) &= x(t) + \sum_{i+j \geqq 2,\, i \geqq 1} c_{ij} x(t)^i (\Psi(x(t)))^j \\ &= x(t)\left\{1 + \sum_{i+j \geqq 2,\, i \geqq 1} c_{ij} x(t)^{i-1} (\Psi(x(t)))^j \right\}. \end{aligned}$$

Thus we have

$$\begin{aligned} &\frac{1}{X\big(x(t), \Psi(x(t))\big)} \\ &= \frac{1}{x(t)\left\{1 - \sum_{i+j \geqq 2,\, i \geqq 1} - c_{ij} x(t)^{i-1} (\Psi(x(t)))^j \right\}} \\ &= \frac{1}{x(t)}\left[1 + \left(\sum_{i+j \geqq 2,\, i \geqq 1} - c_{ij} x(t)^{i-1} (\Psi(x(t)))^j\right) + \left(\sum_{i+j \geqq 2,\, i \geqq 1} - c_{ij} x(t)^{i-1} (\Psi(x(t)))^j\right)^2 \right. \\ &\qquad \left. + \left(\sum_{i+j \geqq 2,\, i \geqq 1} - c_{ij} x(t)^{i-1} (\Psi(x(t)))^j\right)^3 + \cdots \right]. \end{aligned}$$

Since $w(t) = -\dfrac{1}{Ax(t)}$, we have

$$\frac{1}{X\big(x(t),\Psi(x(t))\big)} = -Aw(t)\Bigg[1+\Bigg(\sum_{i+j\geqq 2,\ i\geqq 1} -c_{ij}\Big(-\frac{1}{Aw(t)}\Big)^{i-1}\Big(\Psi\Big(-\frac{1}{Aw(t)}\Big)\Big)^{j}\Bigg)$$
$$+\Bigg(\sum_{i+j\geqq 2,\ i\geqq 1} -c_{ij}\Big(-\frac{1}{Aw(t)}\Big)^{i-1}\Big(\Psi\Big(-\frac{1}{Aw(t)}\Big)\Big)^{j}\Bigg)^{2}$$
$$+\Bigg(\sum_{i+j\geqq 2,\ i\geqq 1} -c_{ij}\Big(-\frac{1}{Aw(t)}\Big)^{i-1}\Big(\Psi\Big(-\frac{1}{Aw(t)}\Big)\Big)^{j}\Bigg)^{3}+\cdots\Bigg].$$

Since $\Psi(x)$ is a formal solution of (1.6) such that $\Psi(x) = \Psi\Big(-\dfrac{1}{Aw}\Big) = \sum_{m=k}^{\infty} a_j\Big(-\dfrac{1}{Aw}\Big)^m$, $(k \geqq 2)$, we have

$$-\frac{1}{AX\big(x,\Psi(x)\big)} = w\Bigg[1 + c_{20}\frac{1}{A}w^{-1} + \sum_{k\geqq 2}\tilde{c}_k(w)^{-k}\Bigg], \tag{2.17}$$

where $\tilde{c}_k$ are defined by c_{ij} and a_k $(i+j \geqq 2, i \geqq 1, k \geqq 2)$. From (2.17) and the definition of A, we can write (2.16) in the following form

$$w(t+1) = \tilde{F}(w(t)) = w(t)\Big\{1 + w(t)^{-1} + \sum_{k\geqq 2}\tilde{c}_k(w(t))^{-k}\Big\}. \tag{2.18}$$

On the other hand, putting $\lambda = 1$ and $m = 1$ in (2.1), i.e. $\theta = \arg[\lambda] = \arg[1] = 0$, and making use of Proposition 4, we have the following formal solution $w(t)$ (2.19) of the first order difference equation (2.18),

$$w(t) = t\Bigg(1 + \sum_{j+k\geqq 1}\hat{q}_{jk}t^{-j}\Big(\frac{\log t}{t}\Big)^{k}\Bigg), \tag{2.19}$$

where $\hat{q}_{jk}$ are defined by $\tilde{F}$ in (2.18). From (2.17) and (1.6), $\tilde{F}$ is defined by X and Y. Hence $\hat{q}_{jk}$ are defined by X and Y.

From $x(t) = -\dfrac{1}{Aw(t)}$, we have a formal solution of (1.2) such that

$$x(t) = -\frac{1}{At}\Bigg(1 + \sum_{j+k\geqq 1}\hat{q}_{jk}t^{-j}\Big(\frac{\log t}{t}\Big)^{k}\Bigg)^{-1}. \tag{2.20}$$

From the relation of (1.2) and (1.8) with (1.6), we have proved (1) of Theorem 1.

Next we will show (2). From the assumption that $kc_{20} = q_{11} < 0$ for some

$k \in \mathbb{N}$, $k \geqq 2$, we suppose that $R_0 > R$ and $\kappa_0 < \dfrac{\pi}{4} - \varepsilon$. Since $\theta = 0$, we have

$$D_1(\kappa_0, R_0) \subset D(\varepsilon, R). \tag{2.21}$$

For a $x \in D^*(\kappa, \delta)$, making use of Theorem C (3), we have a solution $\Psi(x)$ of (1.6) which is holomorphic and can be expanded asymptotically in $D^*(\kappa, \delta)$ such that

$$\Psi(x) \sim \sum_{j=k}^{\infty} a_j x^j. \tag{2, 14}$$

Since $R_1 = \max(R_0, 2/(|A|\delta))$, making use of Proposition 5, we have a solution $w(t)$ of (2.18) which has an asymptotic expansion

$$w(t) = t\left(1 + b\left(t, \frac{\log t}{t}\right)\right),$$

in $t \in D_1(\kappa_0, R_1)$. Therefore we have a solution $x(t)$ of (1.2) which has an asymptotic expansion

$$x(t) = -\frac{1}{At}\left(1 + b\left(t, \frac{\log t}{t}\right)\right)^{-1},$$

there. At first we take a small $\delta > 0$. For sufficiently large R, since $R_1 \geqq R_0 > R$, we will have

$$|x(t)| = \left|\frac{1}{At}\right| \left|1 + b\left(t, \frac{\log t}{t}\right)\right|^{-1} < \frac{1}{|A|R}(1+1) < \delta, \tag{2.22}$$

for $t \in D_1(\kappa_0, R_1)$. Since $A = c_{20} < 0$,

$$\arg[x(t)] = \arg\left[-\frac{1}{At}\left(1 + b\left(t, \frac{\log t}{t}\right)\right)^{-1}\right] = -\arg[t] - \arg\left[1 + b\left(t, \frac{\log t}{t}\right)\right].$$

For sufficiently large R_1, we then have

$$\left|\arg\left[1 + b\left(t, \frac{\log t}{t}\right)\right]\right| < \kappa_0, \quad \text{for } t \in D_1(\kappa_0, R_1).$$

Hence we have

$$-\kappa_0 - \kappa_0 \leq \arg[x(t)] \leq \kappa_0 + \kappa_0.$$

From the assumption of $\kappa = 2\kappa_0$, we have

$$|\arg[x(t)]| < \kappa \leqq \frac{\pi}{2} \quad \text{for} \quad t \in D_1(\kappa_0, R_1). \tag{2.23}$$

From (2.22) and (2.23), we have $x(t) \in D^*(\kappa, \delta)$ for a some κ, $\left(0 < \kappa \leqq \frac{\pi}{2}\right)$. Hence we have a $\Psi(x(t))$ which satisfies the equation (1.6).

Therefore from the existence of a solution Ψ of (1.6) and Proposition 5, we have a holomorphic solution $w(t)$ of first order difference equation (2.18) for $t \in D_1(\kappa_0, R_1)$. Thus we obtain a solution $x(t)$ of (1.2) for t there, which satisfies the following conditions:

(i) $x(t)$ is holomorphic in $D_1(\kappa_0, R_1)$,
(ii) $x(t)$ is expressible in the form

$$x(t) = -\frac{1}{At}\left(1 + b\left(t, \frac{\log t}{t}\right)\right)^{-1}. \tag{2.24}$$

Here $b(t, \log t/t)$ is asymptotically expanded for $t \in D_1(\kappa_0, R_1)$ such that

$$b\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k \geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k,$$

as $t \to \infty$ through $D_1(\kappa_0, R_1)$. □

Finally, we obtain a solution $(u(t), v(t))$ of (1.1) by the transformation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = P \begin{pmatrix} x(t) \\ \Psi(x(t)) \end{pmatrix}.$$

REFERENCES

[1] D. Dendrinos, Private communication.
[2] T. Kimura, *On the Iteration of Analytic Functions*, Funkcialaj Ekvacioj, **14** (1971), pp. 197–238.
[3] M. Suzuki, *Holomorphic solutions of some functional equations*, Nihonkai Math. J., **5** (1994), pp. 109–114.
[4] M. Suzuki, *Holomorphic solutions of some system of n functional equations with n variables related to difference systems*, Aequationes Mathematicae, **57** (1999), pp. 21–36.
[5] M. Suzuki, *Holomorphic solutions of some functional equations II*, Southeast Asian Bulletin of Mathematics, **24** (2000), pp. 85–94.
[6] M. Suzuki, *Difference Equation for a Population Model* , Discrete Dynamics in Nature and Society, **5** (2000), pp. 9–18.

[7] A. MATSUMOTO - M. SUZUKI, *Analytic Solutions of Nonlinear Cournot-Duopoly Game*, Discrete Dynamics in Nature and Society, **2005**, issue 2 (2005), pp.119–133.
[8] M. SUZUKI, *Holomorphic solutions of a functional equation and their applications to nonlinear second order difference equations*, Aequationes mathematicae, **74** (2007), pp. 7–25.
[9] M. SUZUKI, *Analytic General Solutions of Nonlinear Difference Equations*, Annali di Matematica Pura ed Applicata, **187** (2008), pp. 353–368.
[10] N. YANAGIHARA, *Meromorphic solutions of some difference equations*, Funkcialaj Ekvacioj, **23** (1980), pp. 309–326.

Manoscritto pervenuto in redazione il 7 dicembre 2007