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A Note on Posner's Theorem with Generalized Derivations on Lie Ideals

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ABSTRACT - Let R be a prime ring of characteristic different from 2, $Z(R)$ its center, U its Utumi quotient ring, C its extended centroid, G a non-zero generalized derivation of R, L a non-central Lie ideal of R. We prove that if $\big[[G(u), u], G(u)\big] \in Z(R)$ for all $u \in L$ then one of the following holds:

1. there exists $\alpha \in C$ such that $G(x) = \alpha x$, for all $x \in R$;
2. R satisfies the standard identity $s_4(x_1, \ldots, x_4)$ and there exist $a \in U$, $\alpha \in C$ such that $G(x) = ax + xa + \alpha x$, for all $x \in R$.

1. Introduction.

The motivation for this paper lies in an attempt to extend in some way the well known first Posner's Theorem contained in [15]: there Posner proved that if d is a derivation of a prime ring R such that $[d(x), x]$ falls in the center of R, for all $x \in R$, then either $d = 0$ or R is a commutative ring. Recently in [3] Cheng studied derivations of prime rings that satisfy certain special Engel type conditions: he showed that if R is a prime ring of characteristic different from 2 and d a non-zero derivation of R which satisfies the condition $[[d(x), x], d(x)] = 0$ for all $x \in R$, then R must be commutative.

Our purpose here is to continue this line of investigation by studying the set $S = \{[[G(x), x], G(x)], x \in L\}$, where G is a generalized derivation

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defined on R and L is a non-central Lie ideal of R. More specifically an additive map $G : R \longrightarrow R$ is said to be a generalized derivation if there is a derivation d of R such that, for all $x, y \in R$, $G(xy) = G(x)y + xd(y)$. A significative example is a map of the form $G(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called inner. Our goal is to confirm that there is a relationship between the structure of the prime ring R and the behaviour of suitable additive mappings defined on R that satisfy certain special identities. We will show that if any element of S is central in R, then some informations about the form of the generalized derivation G and the structure of R can be obtained. More precisely we will prove the following:

THEOREM. *Let R be a prime ring of characteristic different from 2, $Z(R)$ its center, U its Utumi quotient ring, C its extended centroid, G a non-zero generalized derivation of R, L a non-central Lie ideal of R. We prove that if $\big[[G(u), u], G(u)\big] \in Z(R)$ for all $u \in L$ then one of the following holds:*

1. *there exists $\alpha \in C$ such that $G(x) = \alpha x$, for all $x \in R$;*
2. *R satisfies the standard identity $s_4(x_1, \ldots, x_4)$ and there exist $a \in U$, $\alpha \in C$ such that $G(x) = ax + xa + \alpha x$, for all $x \in R$.*

For sake of clearness we premit the following:

FACT 1. Denote by $T = U *_C C\{X\}$ the free product over C of the C-algebra U and the free C-algebra $C\{X\}$, with X a coutable set consisting of non-commuting indeterminates $\{x_1, \ldots, x_n, \ldots\}$. The elements of T are called generalized polynomial with coefficients in U. Moreover if I is a non-zero ideal of R, then I, R and U satisfy the same generalized polynomial identities with coefficients in U. For more details about these objects we refer the reader to [1] and [4].

FACT 2. Let $a_1, \ldots, a_k \in U$ be linearly independent over C and $a_1 g_1(x_1, \ldots, x_n) + \ldots + a_k g_k(x_1, \ldots, x_n) = 0 \in T$, for some $g_1, \ldots, g_k \in$ $\in T = U *_C C\{X\}$. As a consequence of the result in [4], if for any i, $g_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j h_j(x_1, \ldots, x_n)$ and $h_j(x_1, \ldots, x_n) \in T$, then $g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)$ are the zero element of T. The same conclusion holds if $g_1(x_1, \ldots, x_n)a_1 + \ldots + g_k(x_1, \ldots, x_n)a_k = 0 \in T$, and $g_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} h_j(x_1, \ldots, x_n)x_j$ for some $h_j(x_1, \ldots, x_n) \in T$.

2. The case of Inner Generalized Derivations.

In this section we study the case when the generalized derivation G is inner defined as follows: $G(x) = ax + xb$ for all $x \in R$, where a, b are fixed elements of U.

In all that follows we denote

$$P(x_1, x_2, x_3) = \Big[\big[a[x_1, x_2] + [x_1, x_2]b, [x_1, x_2]\big], a[x_1, x_2] + [x_1, x_2]b\Big] x_3 - x_3\Big[\big[a[x_1, x_2] + [x_1, x_2]b, [x_1, x_2]\big], a[x_1, x_2] + [x_1, x_2]b\Big]$$

and assume that R satisfies the generalized identity $P(x_1, x_2, x_3)$.

Lemma 1. *If R does not satisfy any non trivial generalized polynomial identity, then $a, b \in C$ and $G(x) = \alpha x$, for all $x \in R$ and for $\alpha = a + b$.*

Proof. Denote by $T = U *_C C\{x_1, x_2, x_3\}$ the free product over C of the C-algebra U and the free C-algebra $C\{x_1, x_2, x_3\}$. Any element of T is a generalized polynomial with coefficients in U.

Suppose that R does not satisfy any non trivial generalized polynomial identity. Thus

$$\begin{aligned} P(x_1, x_2, x_3) = &\Big[a[x_1, x_2]^2 + [x_1, x_2](b-a)[x_1, x_2] - [x_1, x_2]^2 b, a[x_1, x_2] + [x_1, x_2]b\Big] x_3 \\ &- x_3\Big[a[x_1, x_2]^2 + [x_1, x_2](b-a)[x_1, x_2] - [x_1, x_2]^2 b, a[x_1, x_2] + [x_1, x_2]b\Big] \\ &= a\Big([x_1, x_2]^2 a[x_1, x_2] + [x_1, x_2]^3 b - [x_1, x_2]a[x_1, x_2]^2 \\ &\quad - [x_1, x_2]^2(b-a)[x_1, x_2] + [x_1, x_2]^3 b\Big) x_3 \\ &+ [x_1, x_2]\Big((b-a)[x_1, x_2]a[x_1, x_2] + (b-a)[x_1, x_2]^2 b - [x_1, x_2]ba[x_1, x_2] \\ &- [x_1, x_2]b[x_1, x_2]b - ba[x_1, x_2]^2 - b[x_1, x_2](b-a)[x_1, x_2] + b[x_1, x_2]^2 b\Big) x_3 \\ &- x_3\Big[a[x_1, x_2]^2 + [x_1, x_2](b-a)[x_1, x_2] - [x_1, x_2]^2 b, a[x_1, x_2] + [x_1, x_2]b\Big] = 0 \in T. \end{aligned}$$

Suppose that $\{1, a\}$ are linearly C-independent. Since $P(x_1, x_2, x_3)$ is a trivial generalized polynomial identity for R, then

$$\Big([x_1, x_2]^2 a[x_1, x_2] + [x_1, x_2]^3 b - [x_1, x_2]a[x_1, x_2]^2 - [x_1, x_2]^2(b-a)[x_1, x_2] + [x_1, x_2]^3 b\Big) x_3 = 0 \in T$$

that is

$$[x_1,x_2]^2a[x_1,x_2]+[x_1,x_2]^3b-[x_1,x_2]a[x_1,x_2]^2$$
$$-[x_1,x_2]^2(b-a)[x_1,x_2]+[x_1,x_2]^3b=0\in T.$$

This implies that $\{1,b\}$ are linearly C-dependent. In fact, if not it follows that $[x_1,x_2]^3$ is an identity for R, a contradiction. Thus $b=\beta\in C$ and R satisfies

$$[x_1,x_2]^2a[x_1,x_2]+\beta[x_1,x_2]^3-[x_1,x_2]a[x_1,x_2]^2+[x_1,x_2]^2a[x_1,x_2]$$

which is a non-trivial generalized identity, since we suppose that $\{1,a\}$ are linearly C-independent. This contradiction says that $\{1,a\}$ are linearly C-dependent, that is $a=\alpha\in C$.
Therefore R satisfies the generalized identity

$$\Big[[x_1,x_2](b+\alpha)[x_1,x_2]-[x_1,x_2]^2(b+\alpha),[x_1,x_2](b+\alpha)\Big]x_3$$
$$-x_3\Big[[x_1,x_2](b+\alpha)[x_1,x_2]-[x_1,x_2]^2(b+\alpha),[x_1,x_2](b+\alpha)\Big]$$

that is

$$\begin{aligned}0=&\Big([x_1,x_2]b'[x_1,x_2]^2b'-[x_1,x_2]^2b'[x_1,x_2]b'-[x_1,x_2]b'[x_1,x_2]b'[x_1,x_2]\\&+[x_1,x_2]b'[x_1,x_2]^2b'\Big)x_3\\&-x_3\Big([x_1,x_2]b'[x_1,x_2]^2b'-[x_1,x_2]^2b'[x_1,x_2]b'-[x_1,x_2]b'[x_1,x_2]b'[x_1,x_2]\\&+[x_1,x_2]b'[x_1,x_2]^2b'\Big)\\=&\Big([x_1,x_2]b'[x_1,x_2]^2b'-[x_1,x_2]^2b'[x_1,x_2]b'-[x_1,x_2]b'[x_1,x_2]b'[x_1,x_2]\\&+[x_1,x_2]b'[x_1,x_2]^2b'\Big)x_3\\&-x_3[x_1,x_2]b'[x_1,x_2]b'[x_1,x_2]-x_3\Big(2[x_1,x_2]b'[x_1,x_2]^2-[x_1,x_2]^2b'[x_1,x_2]\Big)b'\end{aligned}$$

where $b'=b+\alpha$. If $\{1,b'\}$ are linearly C-independent, then

$$-x_3\Big(2[x_1,x_2]b'[x_1,x_2]^2-[x_1,x_2]^2b'[x_1,x_2]\Big)$$

is a non-trivial generalized identity for R, a contradiction. Then $\{1,b'\}$ are linearly C-dependent, that is $b'\in C$ as well as b, and we are done. □

LEMMA 2. *Let $R=M_m(F)$ be the ring of $m\times m$ matrices over the field F of characteristic different from 2, with $m>1$, a,b elements of R such that*

$$\big[[au+ub,u],au+ub\big]\in Z(R)$$

for all $u \in [R, R]$*. Then one of the following holds:*

1) $a, b \in Z(R)$;
2) $a - b \in Z(R)$ *and* $m = 2$.

PROOF. The first aim is to prove that $a - b$ is a diagonal matrix. Say $a = \sum_{ij} a_{ij} e_{ij}$, $b = \sum_{ij} b_{ij} e_{ij}$, where $a_{ij}, b_{ij} \in F$, and e_{ij} are the usual matrix units. Let $u = [r_1, r_2] = [e_{ii}, e_{ij}] = e_{ij}$, for any $i \neq j$. Thus

$$\big[[ae_{ij} + e_{ij}b, e_{ij}], ae_{ij} + e_{ij}b\big] = (b_{ji} - a_{ji})(a_{ji} - b_{ji})e_{ij} \in Z(R)$$

that is all the off-diagonal entries of the matrix $a - b$ are zeros.

Let now $\chi \in Aut_F(R)$ with $\chi(x) = (1 + e_{ji})x(1 - e_{ji})$. Of course $\big[[\chi(a)u + u\chi(b), u], \chi(a)u + u\chi(b)\big] \in Z(R)$, for all $u \in [R, R]$. By calculation we have that

$$\chi(a) = a + e_{ji}a - ae_{ji} - e_{ji}ae_{ji}$$
$$\chi(b) = b + e_{ji}b - be_{ji} - e_{ji}be_{ji}$$

and by the previous argument we also have that $\chi(a - b)$ is a diagonal matrix. In particular the (j, i)-entry of $\chi(a - b)$ is zero, tha is $a_{ii} - b_{ii} = a_{jj} - b_{jj}$. By the arbitrariness of $i \neq j$, we have that $a - b = \alpha$ is a central matrix in R and $\big[[au + ua + \alpha u, u], au + ua + \alpha u\big] \in Z(R)$, for all $u \in [R, R]$, that is R satisfies

$$\Big[\big[a[x_1, x_2]^2 - [x_1, x_2]^2 a, a[x_1, x_2] + [x_1, x_2]a + \alpha[x_1, x_2]\big], x_3\Big].$$

In case $m = 2$ we are done. Thus assume that $m \geq 3$.
Suppose that a is not a diagonal matrix, for example let $a_{ji} \neq 0$ for $i \neq j$. Let now $v = [e_{ii}, e_{ij} + e_{ji}] = e_{ij} - e_{ji}$. Thus

$$X = \big[av^2 - v^2 a, av + va + \alpha v\big] \in Z(R)$$

hence for any $k \neq i, j$, the (k, i)-entry X_{ki} of the matrix X is zero. By calculations we have that

$$X_{ki} = a_{ki}(a_{ij} - a_{ji}) + a_{kj}(a_{jj} + a_{ii} + \alpha) = 0 \tag{1}$$

On the other hand for $w = [e_{ii}, e_{ij} - e_{ji}] = e_{ij} + e_{ji}$ we have

$$Y = \big[aw^2 - w^2 a, aw + wa + \alpha w\big] \in Z(R) = 0.$$

Again the (k, i)-entry Y_{ki} of the matrix Y is zero, that is

$$Y_{ki} = a_{ki}(a_{ij} + a_{ji}) + a_{kj}(a_{jj} + a_{ii} + \alpha) = 0 \tag{2}$$

By (1) and (2) it follows that

$$-2a_{ki}a_{ji} = 0 \tag{3}$$

Therefore we have that if $a_{ji} \neq 0$, then $a_{ki} = 0$ for all $k \neq i,j$.
Let $\varphi \in Aut_F(R)$ defined as $\varphi(x) = (1 + e_{kj})x(1 - e_{kj})$. Of course for all $u \in [R,R]$, $\left[\varphi(a)u^2 - u^2\varphi(a), \varphi(a)u + u\varphi(a) + \alpha u\right] \in Z(R)$ Since the (k,i)-entry of the matrix $\varphi(a)$ is equal to $a_{ji} \neq 0$, then by the argument in (3) we have that the (j,i)-entry a'_{ji} of the matrix $\varphi(a)$ is zero. By calculations it follows $0 = a'_{ji} = a_{ji}$, a contradiction. Therefore a must be a diagonal matrix in R. As above, for all $r \neq s$, let $\chi \in Aut_F(R)$ with $\chi(x) = = (1 + e_{sr})x(1 - e_{sr})$. Hence also $\chi(a) = a + e_{sr}a - ae_{sr} - e_{sr}ae_{sr}$ must be a diagonal matrix. In particular the (s,r)-entry of $\chi(a)$ is zero, tha is $a_{rr} = a_{ss}$. By the arbitrariness of $i \neq j$, we have that a is a central matrix in R, and we are done again. □

Proposition 1. *Let R be a prime ring of characteristic different from 2. Suppose that a, b are elements of U such that $\left[[au+ub,u], au+bu\right] \in Z(R)$, for all $u \in [R,R]$. Then one of the following holds:*

1) *$a, b \in C$;*
2) *$a - b \in C$ and R satisfies the standard identity $s_4(x_1,\ldots,x_4)$.*

Proof. By Lemma 1 we may assume that R satisfies the non-trivial generalized polynomial identity

$$P(x_1,x_2,x_3) = \left[a[x_1,x_2]^2 + [x_1,x_2](b-a)[x_1,x_2] - [x_1,x_2]^2 b, a[x_1,x_2] + [x_1,x_2]b\right]x_3 \\ - x_3\left[a[x_1,x_2]^2 + [x_1,x_2](b-a)[x_1,x_2] - [x_1,x_2]^2 b, a[x_1,x_2] + [x_1,x_2]b\right].$$

By a theorem due to Beidar (Theorem 2 in [1]) this generalized polynomial identity is also satisfied by U. In case C is infinite, we have $P(r_1,r_2,r_3) \in C$ for all $r_1,r_2,r_3 \in U \bigotimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \bigotimes_C \overline{C}$ are centrally closed ([6], Theorems 2.5 and 3.5), we may replace R by U or $U \bigotimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem [14], R is a primitive ring having a non-zero socle H with C as the associated division ring. In light of Jacobson's theorem ([10], pag 75) R is isomorphic to a dense ring of linear transformations on some vector space V over C.

Assume first that V is finite-dimensional over C. Then the density of R on V implies that $R \cong M_k(C)$, the ring of all $k \times k$ matrices over C. Since R is not commutative we assume $k \geq 2$. In this case the conclusion follows by Lemma 2.

Assume next that V is infinite-dimensional over C. As in lemma 2 in [16], the set $[R,R]$ is dense on R and so from $P(r_1,r_2,r_3) \in Z(R)$, for all $r_1,r_2,r_3 \in R$, we have $\big[[ar+rb,r],ar+rb\big] \in Z(R)$, for all $r \in R$. Due to the infinity-dimensionality, R cannot satisfies any polynomial identity. In particular the non-zero ideal H cannot satisfies $s_4(x_1,\dots,x_4)$. Suppose that either $a \notin C$ or $b \notin C$, then at least one of them doesn't centralize the non zero ideal H of R, and we will prove that this leads to a contradiction.
Hence we are supposing that there exist $h_1,h_2 \in H$ such that either $[a,h_1] \neq 0$ or $[b,h_2] \neq 0$ and there exist $h_3,h_4,h_5,h_6 \in H$ such that $s_4(h_3,\dots,h_6) \neq 0$.

Let $e^2 = e$ any non-trivial idempotent element of H. For $r = exe$, with any $x \in R$, we have that $\big[[aexe+exeb,exe],aexe+bexe\big] \in Z(R)$. By commuting with $(1-e)$ and then right multiplying by $(1-e)$ it follows $2(1-e)a(exe)^3b(1-e)=0$. Since $char(R) \neq 2$, we have that either $(1-e)ae=0$ or $eb(1-e)=0$. If $(1-e)ae=0$ then $ae=eae$ and $bae=beae$. On the other hand, in case $eb(1-e)=0$, we get $eb=ebe$, and so $eba=abea$. In any case we notice that the ring eRe satisfies the generalized identity $\Big[\big[[(eae)X+X(ebe),X],(eae)X+X(ebe)\big],Y\Big]$.

By Litoff's theorem in [7] there exists $e^2 = e \in H$ such that h_1,h_2, $h_3,h_4,h_5,h_6 \in eRe$, moreover eRe is a central simple algebra finite dimensional over its center. Since $s_4(h_3,\dots,h_6) \neq 0$, then $eRe \cong M_t(C)$, for $t \geq 3$. By the finite dimensional case, we have that $eae, ebe \in Z(eRe)$, but this contradicts with the choices of h_1,h_2 in eRe. □

3. The proof of the Theorem.

In this final section we will make use of the result of Kharchenko [11] about the differential identities on a prime ring R. We refer to Chapter 7 in [2] for a complete and detailed description of the theory of generalized polynomial identities involving derivations.
It is well known that any derivation of a prime ring R can be uniquely extended to a derivation of its Utumi quotients ring U, and so any derivation of R can be defined on the whole U ([2], pg. 87).

Now, we denote by Der(Q) the set of all derivations on Q. By a derivation word we mean an additive map Δ of the form $\Delta = d_1d_2\dots d_m$, with each $d_i \in$ Der(Q). Then a differential polynomial is a generalized polynomial, with coefficients in Q, of the form $\Phi(\Delta_j(x_i))$ involving noncommutative indeterminates x_i on which the derivations words Δ_j act as unary operations. The differential polynomial $\Phi(\Delta_j(x_i))$ is said a differential identity on a subset T of Q if it vanishes for any assignment of values from T to its indeterminates x_i.

Let D_{int} be the C-subspace of Der(Q) consisting of all inner derivations on Q and let d and δ be two non-zero derivations on R. As a particular case of Theorem 2 in [11] we have the following result (see also Theorem 1 in [13]):

FACT 3. If d is a non-zero derivation on R and $\Phi(x_1, \dots, x_n, d(x_1), \dots, d(x_n))$ is a differential identity on R, then one of the following holds:

1) either $d \in D_{\text{int}}$;
2) or R satisfies the generalized polynomial identity

$$\Phi(x_1, \dots, x_n, y_1, \dots, y_n,).$$

Now we are ready to prove our main result:

THEOREM. *Let R be a prime ring of characteristic different from 2, $Z(R)$ its center, U its Utumi quotient ring, C its extended centroid, G a non-zero generalized derivation of R, L a non-central Lie ideal of R. We prove that if $\big[[G(u), u], G(u)\big] \in Z(R)$ for all $u \in L$ then one of the following holds:*

1) *there exists $\alpha \in C$ such that $G(x) = \alpha x$, for all $x \in R$;*

2) *R satisfies the standard identity $s_4(x_1, \dots, x_4)$ and there exist $a \in U$, $\alpha \in C$ such that $G(x) = ax + xa + \alpha x$, for all $x \in R$.*

PROOF. By Theorem 3 in [12] every generalized derivation g on a dense right ideal of R can be uniquely extended to the Utumi quotient ring U of R, and thus we can think of any generalized derivation of R to be defined on the whole U and to be of the form $g(x) = ax + d(x)$ for some $a \in U$ and d a derivation on U. Thus we will assume in all that follows that there exist $a \in U$ and d derivation on U such that $G(x) = ax + d(x)$. We note that we may assume that R is not commutative, since L is not central. Moreover, since $char(R) \neq 2$, there exists a non-central two-sided ideal I of R such that $[I, I] \subseteq L$ (see p. 4-5 in [8]; Lemma 2 and Proposition 1 in [5]). Therefore $[[G(u), u], G(u)] \in Z(R)$ for all $u \in [I, I]$. Moreover by [13] R and I satisfy the same differential polynomial identities, that is $[[G(u), u], G(u)] \in Z(R)$ for all $u \in [R, R]$.

By assumption R satisfies the differential identity

$$(4)\quad \Big[[a[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)],[x_1,x_2]],a[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)]\Big]x_3$$
$$-x_3\Big[[a[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)],[x_1,x_2]],a[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)]\Big]$$

First suppose that d is not an inner derivation on U. By Kharchenko's theorem [11] R satisfies the polynomial identity

$$(5)\quad \Big[\big[a[x_1,x_2]+[y_1,x_2]+[x_1,y_2],[x_1,x_2]\big],a[x_1,x_2]+[y_1,x_2]+[x_1,y_2]\Big]x_3$$
$$-x_3\Big[\big[a[x_1,x_2]+[y_1,x_2]+[x_1,y_2],[x_1,x_2]\big],a[x_1,x_2]+[y_1,x_2]+[x_1,y_2]\Big]$$

in particular R satisfies any blended component

$$\Big[\big[a[x_1,x_2],[x_1,x_2]\big],a[x_1,x_2]\Big]x_3-x_3\Big[\big[a[x_1,x_2],[x_1,x_2]\big],a[x_1,x_2]\Big]$$

that is

$$\Big[\big[a[x_1,x_2],[x_1,x_2]\big],a[x_1,x_2]\Big]\in Z(R)$$

and by Proposition 1 we have that $a=\alpha\in C$. Thus from (5), it follows that R satisfies the polynomial identity

$$\Big[\big[[y_1,x_2]+[x_1,y_2],[x_1,x_2]\big],\alpha[x_1,x_2]+[y_1,x_2]+[x_1,y_2]\Big]x_3$$
$$-x_3\Big[\big[[y_1,x_2]+[x_1,y_2],[x_1,x_2]\big],\alpha[x_1,x_2]+[y_1,x_2]+[x_1,y_2]\Big].$$

Since R satisfies a polynomial identity, there exists $M_k(F)$, the ring of all matrices over a suitable field F, such that R and $M_k(F)$ satisfy the same polynomial identities (see [9], Theorem 2 p.54 and Lemma 1 p.89). For $x_1=e_{22}$, $x_2=e_{21}$, $y_1=e_{21}$ and $y_2=e_{12}$ we obtain

$$[y_1,x_2]=0,\quad [x_1,y_2]=-e_{12},\quad [x_1,x_2]=e_{21}$$

and it follows the contradiction

$$\Big[\big[-e_{12},e_{21}\big],\alpha e_{21}-e_{12}\Big]=2e_{12}+2\alpha e_{21}\notin Z(R).$$

Now consider the case when d is an inner derivation induced by the element $b\in U$. Since $G(x)=ax+[b,x]=ax+bx-xb=(a+b)x+x(-b)$ and by Proposition 1, we have that either $a,b\in C$ or $a+2b\in C$ and R satisfies $s_4(x_1,\dots,x_4)$. In the first case we conclude that $G(x)=ax$, for $a\in C$; in the second one $G(x)=a'x+xa'+\alpha x$, where $a'=-b$. □

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