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On the Homogeneity of Global Minimizers for the Mumford-Shah Functional when K is a Smooth Cone

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ABSTRACT - We show that if (u, K) is a global minimizer for the Mumford-Shah functional in $\mathbb{R}^N$, and if K is a smooth enough cone, then (modulo constants) u is a homogenous function of degree $\frac{1}{2}$. We deduce some applications in $\mathbb{R}^3$ as for instance that an angular sector cannot be the singular set of a global minimizer, that if K is a half-plane then u is the corresponding cracktip function of two variables, or that if K is a cone that meets S^2 with an union of C^∞ curvilinear convex polygones, then it is a $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$.

Introduction.

The functional of D. Mumford and J. Shah [18] was introduced to solve an image segmentation problem. If Ω is an open subset of $\mathbb{R}^2$, for example a rectangle, and $g \in L^\infty(\Omega)$ is an image, one can get a segmentation by minimizing

$$J(K,u) := \int_{\Omega\setminus K} |\nabla u|^2 dx + \int_{\Omega\setminus K} (u-g)^2 dx + H^1(K)$$

over all the admissible pairs $(u, K) \in \mathcal{A}$ defined by

$$\mathcal{A} := \{(u,K);\ K \subset \Omega \text{ is closed },\ u \in W^{1,2}_{loc}(\Omega\setminus K)\}.$$

Any solution (u, K) that minimizes J represents a "smoother" version of the image and the set K represents the edges of the image.

Existence of minimizers is a well known result (see for instance [11]) using SBV theory.

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The question of regularity for the singular set K of a minimizer is more difficult. The following conjecture is currently still open.

CONJECTURE 1 (Mumford-Shah). [18] *Let (u, K) be a reduced minimizer for the functional J. Then K is the finite union of C^1 arcs.*

The term "reduced" just means that we cannot find another pair $(\tilde{u}, \tilde{K})$ such that $K \subset \tilde{K}$ and $\tilde{u}$ is an extension of u in $\Omega \backslash \tilde{K}$.

Some partial results are true for the conjecture. For instance it is known that K is C^1 almost everywhere (see [7], [4] and [2]). The closest to the conjecture is probably the result of A. Bonnet [4]. He proves that if (u, K) is a minimizer, then every isolated connected component of K is a finite union of C^1-arcs. The approach of A. Bonnet is to use blow up limits. If (u, K) is a minimizer in Ω and y is a fixed point, consider the sequences (u_k, K_k) defined by

$$u_k(x) = \frac{1}{\sqrt{t_k}}\, u(y + t_k x), \quad K_k = \frac{1}{t_k}(K - y), \quad \Omega_k = \frac{1}{t_k}(\Omega - y).$$

When $\{t_k\}$ tends to infinity, the sequence (u_k, K_k) may tend to a pair (u_∞, K_∞), and then (u_∞, K_∞) is called a Global Minimizer. Moreover, A. Bonnet proves that if K_∞ is connected, then (u_∞, K_∞) is one of the list below:

- 1ST CASE: $K_\infty = \emptyset$ and u_∞ is a constant.
- 2ND CASE: K_∞ is a line and u_∞ is locally constant.
- 3RD CASE: "*Propeller*": K_∞ is the union of 3 half-lines meeting with 120 degrees and u_∞ is locally constant.
- 4TH CASE: "*Cracktip*": $K_\infty = \{(x, 0); x \leq 0\}$ and $u_\infty(r\cos(\theta), r\sin(\theta)) = \pm\sqrt{\frac{2}{\pi}}\, r^{1/2} \sin\frac{\theta}{2} + C$, for $r > 0$ and $|\theta| < \pi$ (C is a constant), or a similar pair obtained by translation and rotation.

We don't know whether the list is complete without the hypothesis that K_∞ is connected. This would give a positive answer to the Mumford-Shah conjecture.

The Mumford-Shah functional was initially given in dimension 2 but there is no restriction to define Minimizers for the analogous functional in $\mathbb{R}^N$. Then we can also do some blow-up limits and try to think about what should be a global minimizer in $\mathbb{R}^N$. Almost nothing is known in this direction and this paper can be seen as a very preliminary step. Let state some definitions.

Definition 2. *Let $\Omega \subset \mathbb{R}^N$, $(u, K) \in \mathcal{A}$ and B be a ball such that $\bar{B} \subset \Omega$. A competitor for the pair (u, K) in the ball B is a pair $(v, L) \in \mathcal{A}$ such that*

$$\left.\begin{array}{r} u = v \\ K = L \end{array}\right\} \text{ in } \Omega \backslash B$$

and in addition if x and y are two points in $\Omega\backslash(B \cup K)$ that are separated by K then they are also separated by L.

The expression "be separated by K" means that x and y lie in different connected components of $\Omega\backslash K$.

Definition 3. *A global minimizer in $\mathbb{R}^N$ is a pair $(u, K) \in \mathcal{A}$ (with $\Omega = \mathbb{R}^N$) such that for every ball B in $\mathbb{R}^N$ and every competitor (v, L) in B we have*

$$\int_{B\backslash K} |\nabla u|^2 dx + H^{N-1}(K \cap B) \leq \int_{B\backslash L} |\nabla v|^2 dx + H^{N-1}(L \cap B)$$

where H^{N-1} denotes the Hausdorff measure of dimension $N - 1$.

Proposition 9 on page 267 of [8] ensures that any blow up limit of a minimizer for the Mumford-Shah functional in $\mathbb{R}^N$, is a global minimizer in the sense of Definition 3. As a beginning for the description of global minimizers in $\mathbb{R}^N$, we can firstly think about what should be a global minimizer in $\mathbb{R}^3$. If u is locally constant, then K is a minimal cone, that is, a set that locally minimizes the Hausdorff measure of dimension 2 in $\mathbb{R}^3$. Then by [9] we know that K is a cone of type $\mathbb{P}$ (hyperplane), $\mathbb{Y}$ (three half-planes meeting with 120 degrees angles) or of type $\mathbb{T}$ (cone over the edges of a regular tetraedron centered at the origin). Those cones became famous by the theorem of J. Taylor [20] which says that any minimal surface in $\mathbb{R}^3$ is locally C^1 equivalent to a cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$.

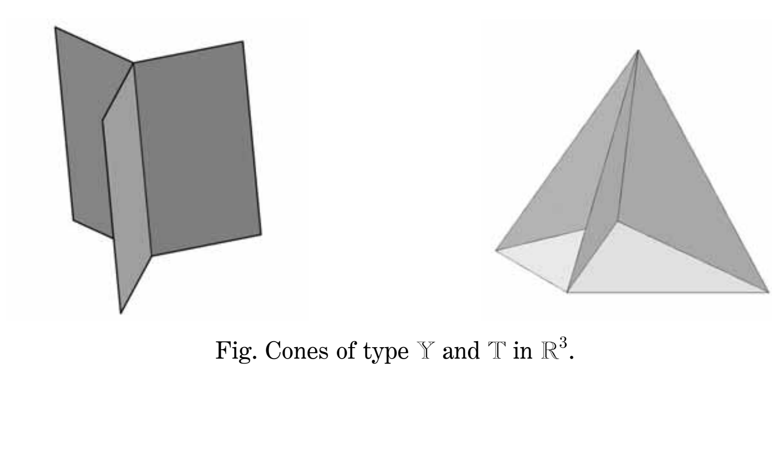

Fig. Cones of type $\mathbb{Y}$ and $\mathbb{T}$ in $\mathbb{R}^3$.

To be clearer, this is a more precise definition of $\mathbb{Y}$ and $\mathbb{T}$, as in [10].

DEFINITION 4. *Define Prop* $\subset \mathbb{R}^2$ *by*

$$\begin{aligned} Prop &= \{(x_1,x_2); x_1 \geq 0, x_2 = 0\} \\ &\cup \{(x_1,x_2); x_1 \leq 0, x_2 = -\sqrt{3}x_1\} \\ &\cup \{(x_1,x_2); x_1 \leq 0, x_2 = \sqrt{3}x_1\}. \end{aligned}$$

Then let $Y_0 = Prop \times \mathbb{R} \subset \mathbb{R}^3$. *The spine of* Y_0 *is the line* $L_0 = \{x_1 = x_2 = 0\}$. *A cone of type* $\mathbb{Y}$ *is a set* $Y = R(Y_0)$ *where* R *is the composition of a translation and a rotation. The spine of* Y *is then the line* $R(L_0)$.

DEFINITION 5. *Let* $A_1 = (1,0,0)$, $A_2 = \left(-\frac{1}{3}, \frac{2\sqrt{2}}{3}, 0\right)$, $A_3 = \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3}\right)$, *and* $A_4 = \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3}\right)$ *the four vertices of a regular tetrahedron centered at* 0. *Let* T_0 *be the cone over the union of the* 6 *edges* $[A_i, A_j]$ $i \neq j$. *The spine of* T_0 *is the union of the four half lines* $[0, A_j[$. *A cone of type* $\mathbb{T}$ *is a set* $T = R(T_0)$ *where* R *is the composition of a translation and a rotation. The spine of* T *is the image by* R *of the spine of* T_0.

So the pairs (u, Z) where u is locally constant and Z is a minimal cone, are examples of global minimizers in $\mathbb{R}^3$. Another global minimizer can be obtained with K_∞ a half-plane, by setting $u := Craktip \times \mathbb{R}$ (see [8] section 76). These examples are the only global minimizers in $\mathbb{R}^3$ that we know.

Note that if (u, K) is a global minimizer in $\mathbb{R}^N$, then u locally minimizes the Dirichlet integral in $\mathbb{R}^N \backslash K$. As a consequence, u is harmonic in $\mathbb{R}^N \backslash K$. Moreover, if B is a ball such that $K \cap B$ is regular enough, then the normal derivative of u vanishes on $K \cap B$.

In this paper we wish to study global minimizers (u, K) for which K is a cone. It seems natural to think that any singular set of a global minimizer is a cone. But even if all known examples are cones, there is no proof of this fact. In addition, we will add some regularity on K. We denote by S^{N-1} the unit sphere in $\mathbb{R}^N$ and, if Ω is a open set, $W^{1,2}(\Omega)$ is the Sobolev space. We will say that a domain Ω on S^{N-1} has a piecewise C^2 boundary, if the topological boundary of Ω, defined by $\partial\Omega = \bar{\Omega}\backslash\Omega$, consists of an union of $N-2$ dimensional hypersurfaces of class C^2. This allows some cracks, i.e. when Ω lies in each sides of its boundary. We will denote by $\tilde{\Sigma}$ the set of all the singular points of the boundary, that is

$$\tilde{\Sigma} := \{x \in \partial\Omega; \forall r > 0, B(x,r) \cap \partial\Omega \text{ is not a } C^2 \text{ hypersurface}\}.$$

Definition 6. *A smooth cone is a set K of dimension $N-1$ in $\mathbb{R}^N$ such that K is conical, centered at the origin, and such that $S^{N-1}\backslash K$ is a domain with piecewise C^2 boundary. Moreover we assume that the embedding $W^{1,2}(S^{N-1}\backslash K) \to L^2(S^{N-1}\backslash K)$ is compact. Finally we suppose that we can strongly integrate by parts in $B(0,1)\backslash K$. More precisely, denoting by Σ the set of singularities*

$$\Sigma := \{tx; (t,x) \in \mathbb{R}^+ \times \tilde{\Sigma}\},$$

we want that

$$\int_{B(0,1)\backslash K} \langle \nabla u, \nabla \varphi \rangle = 0$$

for every harmonic function u in $B(0,1)\backslash K$ with $\frac{\partial}{\partial n} u = 0$ on $K\backslash\Sigma$, and for all $\varphi \in W^{1,2}(B(0,1)\backslash K)$ with vanishing trace on $S^{N-1}\backslash K$.

Remark 7. For instance if K is the cone over a finite union of C^2-arcs on S^2, then we can strongly integrate by parts in $B(0,1)\backslash K$. Another example in $\mathbb{R}^N$ is given by the union of admissible set of faces (as in Definition (22.2) of [5]).

Now this is the main result.

Theorem 15. *Let (u,K) be a global minimizer in $\mathbb{R}^N$ and assume that K is a smooth cone. Then there is a $\frac{1}{2}$-homogenous function u_1 such that $u-u_1$ is locally constant.*

As we shall see, this result implies that if (u,K) is a global minimizer in $\mathbb{R}^N$, and if K is a smooth cone other than a minimal cone, then $\frac{3-2N}{4}$ is an eigenvalue for the spherical Laplacian in $S^{N-1}\backslash K$ with Neumann boundary conditions. In section 2 we will give some applications about global minimizers in $\mathbb{R}^3$, using the estimates on the first eigenvalue that can be found in [6], [5] and [14]. More precisely, we have:

Proposition 17. *Let (u,K) be a global Mumford-Shah minimizer in $\mathbb{R}^3$ such that K is a smooth cone. Moreover, assume that $S^2 \cap K$ is a union of convex curvilinear polygons with C^∞ sides. Then u is locally constant and K is a cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$.*

Another consequence of the main result is the following.

PROPOSITION 19. *Let (u,K) be a global Mumford-Shah minimizer in $\mathbb{R}^3$ such that K is a half plane. Then u is equal to a function of type cracktip $\times\ \mathbb{R}$, that is, in cylindrical coordinates,*

$$u(r,\theta,z) = \pm\sqrt{\frac{2}{\pi}}r^{\frac{1}{2}}\sin\frac{\theta}{2} + C$$

for $0<r<+\infty$, $-\pi<\theta<\pi$ where C is a constant.

Finally, we deduce two other consequences from Theorem 15. Let $(r,\theta,z)\in\mathbb{R}^+\times[-\pi,\pi]\times\mathbb{R}$ be the cylindrical coordinates in $\mathbb{R}^3$. For all $\omega\in[0,\pi]$ set

$$\partial\Gamma_\omega := \{(r,\theta,z)\in\mathbb{R}^3; \theta = -\omega \text{ or } \theta=\omega\}.$$

and

$$S_\omega := \{(r,\theta,z)\in\mathbb{R}^3; z=0,\ r>0,\ \theta\in[-\omega,\omega]\,\} \tag{1}$$

Observe that S_0 is a half line, $S_{\frac{\pi}{2}}$, $\partial\Gamma_0$ and $\partial\Gamma_\pi$ are half-planes, and that S_π and $\partial\Gamma_{\frac{\pi}{2}}$ are planes.

PROPOSITION 18. *There is no global Mumford-Shah minimizer in $\mathbb{R}^3$ such that K is wing of type $\partial\Gamma_\omega$ with $\omega\notin\left\{0,\frac{\pi}{2},\pi\right\}$.*

PROPOSITION 23. *There is no global Mumford-Shah minimizer in $\mathbb{R}^3$ such that K is an angular sector of type (u,S_ω) for $\omega\notin\left\{\frac{\pi}{2},\pi\right\}$.*

1. If *K* is a cone then *u* is homogenous.

In this section we want to prove Theorem 15. Notice that this result is only useful if the dimension $N\geq 3$. Indeed, in dimension 2, if K is a cone then it is connected thus it is in the list described in the introduction.

1.1 – *Preliminary.*

Let us recall a standard uniqueness result about energy minimizers.

PROPOSITION 8. *Let Ω be an open and connected set of $\mathbb{R}^N$ and let $I\subset\partial\Omega$ be a hypersurface of class C^∞. Suppose that u and v are two* *functions in* $W^{1,2}(\Omega)$ *such that* $u = v$ *a.e. on* I *(in terms of trace), solving the minimizing problem*

$$\min E(w) := \int_{\Omega} |\nabla w(x)|^2 dx$$

over all the functions $w \in W^{1,2}(\Omega)$ *that are equal to* u *and* v *on* I*. Then*

$$u = v.$$

PROOF. This comes from a simple convexity argument which can be found for instance in [8], but let us write the proof since it is very short. By the parallelogram identity we have

$$E\Big(\frac{u+v}{2}\Big) = \frac{1}{2}E(u) + \frac{1}{2}E(v) - \frac{1}{4}E(u-v). \tag{2}$$

On the other hand, since $\frac{u+v}{2}$ is equal to u and v on I, and by minimality of u and v we have

$$E\Big(\frac{u+v}{2}\Big) \geq E(u) = E(v).$$

Now by (2) we deduce that $E(u-v) = 0$ and since Ω is connexe, this implies that $u - v$ is a constant. But $u - v$ is equal to 0 on I thus $u = v$. □

REMARK 9. The existence of a minimizer can also be proved using the convexity of $E(v)$.

1.2 – *Spectral decomposition.*

The key ingredient to obtain the main result will be the spectral theory of the Laplacian on the unit sphere. Since u is harmonic, we will decompose u as a sum of homogeneous harmonic functions just like we usually use the classical spherical harmonics. The difficulty here comes from the lack of regularity of $\mathbb{R}^N \backslash K$.

It will be convenient to work with connected sets. So let Ω be a connected component of $S^{N-1}\backslash K$, and let $A(r)$ be

$$A(r) := \{tx; (x,t) \in \Omega \times [0, r[\ \}.$$

We also set

$$A(\infty) := \{tx; (x,t) \in \Omega \times \mathbb{R}^+\ \}.$$

All the following results are using that the embedding $W^{1,2}(\Omega)$ in $L^2(\Omega)$ is compact. Recall that this is the case by definition, since K is a smooth cone. Notice that for instance the cone property insures that the embedding is compact (see Theorem 6.2. p. 144 of [1]).

Consider the quadratic form

$$Q(u) = \int_\Omega |\nabla u(x)|^2 dx$$

of domain $W^{1,2}(\Omega)$ dense into the Hilbert space $L^2(\Omega)$. Since Q is a positive and closed quadratic form (see for instance Proposition 10.61 p. 129 of [16]) there exists a unique selfadjoint operator denoted by $-\Delta_n$ of domain $D(-\Delta_n) \subset W^{1,2}(\Omega)$ such that

$$\forall u \in D(-\Delta_n),\ \forall v \in W^{1,2}(\Omega), \qquad \int_\Omega \langle \nabla u, \nabla v\rangle = \int_\Omega \langle -\Delta_n u, v\rangle.$$

PROPOSITION 10. *The operator $-\Delta_n$ has a countably infinite discrete set of eigenvalues, whose eigenfunctions span $L^2(\Omega)$.*

PROOF. The proof is the same as if Ω was a regular domain. Consider the new quadratic form

$$\tilde{Q}(u) := Q(u) + \|u\|_2^2$$

with the same domain $W^{1,2}(\Omega)$. The form $\tilde{Q}$ has the same properties than Q and the associated operator is $\mathrm{Id} - \Delta_n$. Moreover $\tilde{Q}$ is coercive. As a result, the operator $\mathrm{Id} - \Delta_n$ is bijective and its inverse goes from $L^2(\Omega)$ to $D(-\Delta_n) \subset W^{1,2}(\Omega)$. By hypothesis the embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact. Thus the resolvant $(\mathrm{Id} - \Delta_n)^{-1}$ is a compact operator, and we conclude using the spectral theory of operators with a compact resolvant (see [19] Theorem XIII. 64 p. 245). □

REMARK 11. The domain of $-\Delta_n$ is not known in general. If Ω was smooth, then we could show that the domain is exactly $D(-\Delta_n) = \left\{u \in W^{2,2}(\Omega); \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}$. Here, the boundary of Ω has some singularities so this result doesn't apply directly. But knowing exactly the domain of $-\Delta_n$ will not be necessary for us.

Now we want to study the link between the abstract operator Δ_n and the classical spherical Laplacian Δ_S on the unit sphere. Recall that if we compute the Laplacian in spherical coordinates, we obtain the following equality

$$\Delta = \frac{\partial^2}{\partial r} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S. \tag{3}$$

PROPOSITION 12. *For every function* $f \in D(-\Delta_n)$ *such that* $-\Delta_n f = \lambda f$ *we have*

i) $f \in C^\infty(\Omega)$

ii) $-\Delta_S f = -\Delta_n f = \lambda f$ *in* Ω

iii) $\dfrac{\partial f}{\partial n}$ *exists and is equal to* 0 *on* $K \cap \overline{\Omega} \backslash \Sigma$

PROOF. Let φ be a C^∞ function with compact support in Ω and $f \in D(-\Delta_n)$. Then the Green formula in the distributional sense gives

$$\int_\Omega \nabla f . \nabla \varphi = \langle -\Delta_S f, \varphi \rangle$$

where the left and right brackets mean the duality in the distributional sense. On the other hand, by definition of $-\Delta_n$ and since f is in the domain $D(-\Delta_n)$, we also have

$$\int_\Omega \nabla f . \nabla \varphi = \langle -\Delta_n f, \varphi \rangle$$

where this time the brackets mean the scalar product in L^2. Therefore

$$\Delta_n f = \Delta_S f \quad \text{in } \mathcal{D}'(\Omega).$$

In other words, $-\Delta_S f = \lambda f$ in $\mathcal{D}'(\Omega)$. But now since $f \in W^{1,2}(\Omega)$, by hypoellipticity of the Laplacian we know that f is C^∞ and that $-\Delta_S f = \lambda f$ in the classical sense. That proves *i)* and *ii)*. We even know by the elliptic theory that, since $K \backslash \Sigma$ is regular, f is regular at the boundary on $K \backslash \Sigma$.

Now consider a ball B such that the intersection with $K \cap \overline{\Omega}$ does not meet Σ. Assume that B is cut in two parts B^+ and B^- by K, and that B^+ is one part in Ω. Possibly by modifying B in a neighborhood of the intersection with K, we can assume that the boundary of B^+ and B^- is C^2. The definition of Δ_n implies that for all function $\varphi \in C^2(\bar{\Omega})$ that vanishes out of B^+ we have

$$\int_{B^+} \langle \nabla f, \nabla \varphi \rangle dx = \int_{B^+} \langle -\Delta_n f, \varphi \rangle dx = \lambda \int_{B^+} \langle f, \varphi \rangle dx.$$

On the other hand, integrating by parts,

$$\begin{aligned}\int_{B^+} \langle \nabla f, \nabla \varphi \rangle dx =& \int_{B^+} \langle -\Delta_S f, \varphi \rangle + \int_{\partial B^+} \frac{\partial u}{\partial n}\varphi \\ =&\lambda \int_{\partial B^+} \langle f, \varphi \rangle + \int_{\partial B^+} \frac{\partial f}{\partial n}\varphi\end{aligned}$$

thus

$$\int_{\partial B^+} \frac{\partial f}{\partial n}\varphi = 0.$$

In other words the function f is a weak solution of the mixed boundary value problem

$$\begin{aligned}-\Delta_S u =& \lambda f \text{ in } B^+ \\ u =& f \text{ on } \partial B^+ \backslash K \\ \frac{\partial u}{\partial n} =& 0 \text{ on } K \cap \partial B^+\end{aligned}$$

Therefore, some results from the elliptic theory imply that f is smooth in B and is a strong solution (see [21]). □

Let us recapitulate what we have obtained. For all function $f \in L^2(\Omega)$, there is a sequence of numbers a_i such that

$$f = \sum_{i=0}^{+\infty} a_i f_i \tag{4}$$

where the sum converges in L^2. The functions f_i are in $C^\infty(\Omega) \cap W^{1,2}(\Omega)$, verify $-\Delta_S f_i = \lambda_i f_i$ and $\frac{\partial f_i}{\partial n} = 0$ on $K \cap \overline{\Omega} \backslash \Sigma$. Moreover, we can normalize the f_i in order to obtain an orthonormal basis on $L^2(\Omega)$, in particular we have the following Parseval formula

$$\|f\|_2^2 = \sum_{i=0}^{+\infty} |a_i|^2.$$

Note that if f belongs to the kernel of $-\Delta_n$ (i.e. is an eigenfunction with eigenvalue 0), then

$$\langle \nabla f, \nabla f \rangle = \langle -\Delta_n f, f \rangle = 0$$

and since Ω is connected that means that f is a constant. Thus 0 is the first eigenvalue and the associated eigenspace has dimension 1. Then we can suppose that $\lambda_0 = 0$ and that all the λ_i for $i > 0$ are positive.

We define the scalar product in $W^{1,2}(\Omega)$ by

$$\langle u, v\rangle_{W^{1,2}} := \langle u, v\rangle_{L^2} + \langle \nabla u, \nabla v\rangle_{L^2}.$$

PROPOSITION 13. *The family $\{f_i\}$ is orthogonal in $W^{1,2}(\Omega)$. Moreover if $f \in W^{1,2}(\Omega)$ and if its decomposition in $L^2(\Omega)$ is $f = \sum_{i=0}^{+\infty} a_i f_i$, then the sum $\sum_{i=0}^{+\infty} |a_i|^2 \|\nabla f_i\|_2^2$ converges and*

$$\sum_{i=0}^{+\infty} |a_i|^2 \|\nabla f_i\|_2^2 = \|\nabla f\|_2^2. \tag{5}$$

PROOF. We know that $\{f_i\}$ is an orthogonal family in $L^2(\Omega)$. In addition if $i \neq j$ then

$$\begin{aligned}\int_\Omega \nabla f_i \nabla f_j &= \int_\Omega -\Delta_n f_i f_j \\ &= \lambda_i \int_\Omega f_i f_j \\ &= 0\end{aligned}$$

thus $\{f_i\}$ is also orthogonal in $W^{1,2}(\Omega)$ and

$$\|f_i\|_{W^{1,2}}^2 := \|f_i\|_2^2 + \|\nabla f_i\|_2^2 = 1 + \lambda_i.$$

Consider now the orthogonal projection (for the scalar product of L^2)

$$P_k : f \mapsto \sum_{i=0}^{k} a_i f_i.$$

The operator P_k is the orthogonal projection on the closed subspace A_k generated by $\{f_0, \dots, f_k\}$. More precisely, we are interested in the restriction of P_k to the subspace $W^{1,2}(\Omega) \subset L^2(\Omega)$. Also denote by $\tilde{P}_k : W^{1,2} \to A_k$ the orthogonal projection on the same subspace but for the scalar product of $W^{1,2}$. We want to show that $P_k = \tilde{P}_k$. To prove this, it suffice to show that for all sets of coefficients $\{a_i\}_{i=1\dots k}$ and $\{b_i\}_{i=1\dots k}$,

$$\left\langle f - \sum_{i=0}^{k} a_i f_i, \sum_{i=0}^{k} b_i f_i \right\rangle_{W^{1,2}} = 0.$$

Since we already have

$$\left\langle f - \sum_{i=0}^{k} a_i f_i, \sum_{i=0}^{k} b_i f_i \right\rangle_{L^2} = 0,$$

all we have to show is that

$$\int_{\Omega} \left\langle \nabla f - \sum_{i=0}^{k} a_i \nabla f_i, \sum_{i=0}^{k} b_i \nabla f_i \right\rangle dx = 0.$$

Now

$$\begin{aligned}
\int_{\Omega} \left\langle \nabla f - \sum_{i=0}^{k} a_i \nabla f_i, \sum_{i=0}^{k} b_i \nabla f_i \right\rangle &= \int_{\Omega} \left\langle \nabla f, \sum_{i=0}^{k} b_i \nabla f_i \right\rangle - \sum_{i=0}^{k} a_i b_i \|\nabla f_i\|_2^2 \\
&= \sum_{i=0}^{k} b_i \langle -\Delta_n f_i, f \rangle_{L^2} - \sum_{i=0}^{k} a_i b_i \lambda_i \\
&= \sum_{i=0}^{k} a_i b_i \lambda_i - \sum_{i=0}^{k} a_i b_i \lambda_i \\
&= 0
\end{aligned}$$

thus $P_k = \tilde{P}_k$ and therefore, by Pythagoras

$$\|P_k(f)\|_{W^{1,2}}^2 \leq \|f\|_{W^{1,2}}^2.$$

By letting k tend to infinity we obtain

$$\sum_{i=0}^{+\infty} a_i^2 \|\nabla f_i\|_2^2 \leq \|\nabla f\|_2^2. \tag{6}$$

From this inequality we deduce that the sum is absolutely converging in $W^{1,2}(\Omega)$. Therefore, the sequence of partial sum $\sum_{i=0}^{K} a_i f_i$ is a Cauchy sequence for the norm $W^{1,2}(\Omega)$. Thus, since the sum $\sum a_i f_i$ already converges to f in $L^2(\Omega)$, by uniqueness of the limit the sum converges to f in $W^{1,2}(\Omega)$, so we deduce that (6) is an equality and the proof is over. □

Once we have a basis $\{f_i\}$ on $\Omega \subset S^{N-1}$, we consider for a certain $r_0 > 0$, the functions

$$h_i(x) = r_0^{\alpha_i} f_i\left(\frac{x}{r_0}\right)$$

defined on $r_0\Omega$. The exponent α_i is defined by

$$\alpha_i = \frac{-(N-2) + \sqrt{(N-2)^2 + 4\lambda_i}}{2}\,. \tag{7}$$

The functions h_i form a basis of $W^{1,2}(r_0\Omega)$. Indeed, if $f \in W^{1,2}(r_0\Omega)$, then $f(r_0x) \in W^{1,2}(\Omega)$ thus applying the decomposition on Ω we obtain

$$f(r_0x) = \sum_{i=0}^{+\infty} b_i f_i(x)$$

thus

$$f(x) = \sum_{i=0}^{+\infty} a_i h_i(x)$$

with

$$a_i = b_i r_0^{-\alpha_i}. \tag{8}$$

Notice that since $\|h_i\|_2^2 = r_0^{2\alpha_i+N-1}$ we also have

$$\sum_{i=0}^{\infty} a_i^2 \|h_i\|_2^2 = \sum_{i=0}^{\infty} a_i^2 r_0^{2\alpha_i+N-1} = \|f\|^2_{L^2(r_0\Omega)} < +\infty. \tag{9}$$

Moreover, applying Proposition 13 we have that

$$\sum_{i=0}^{\infty} b_i^2 \|\nabla f_i\|_2^2 = \|\nabla f(r_0x)\|_2^2 < +\infty. \tag{10}$$

We are now able to state our decomposition in $A(r_0)$.

PROPOSITION 14. *Let K be a smooth cone in $\mathbb{R}^N$, centered at the origin and let Ω be a connected component of $S^{N-1}\backslash K$. Then there exist some harmonic homogeneous functions g_i, orthogonal in $W^{1,2}(A(1))$, such that for every function $u \in W^{1,2}(A(1))$ harmonic in $A(1)$ with $\frac{\partial u}{\partial n} = 0$ on $K \cap A(1)\backslash\Sigma$, and for every $r_0 \in]0,1[$, we have that*

$$u = \sum_{i=0}^{+\infty} a_i\, g_i \quad \text{in } A(r_0)$$

where the a_i do not depend on radius r_0 and are unique. The sum converges in $W^{1,2}(A(r_0))$ and uniformly on all compact sets of $A(1)$. Moreover

$$\|u\|^2_{W^{1,2}(A(r_0))} = \sum_{i=0}^{+\infty} a_i^2 \|g_i\|^2_{W^{1,2}(A(r_0))}. \tag{11}$$

PROOF. Since $u \in W^{1,2}(A(1))$ then for almost every r_0 in $]0,1]$ we have that

$$u|_{r_0\Omega} \in W^{1,2}(r_0\Omega).$$

Thus we can apply the decomposition on $r_0\Omega$ and say that

$$u = \sum_{i=0}^{+\infty} a_i h_i \quad \text{on } r_0\Omega.$$

Define g_i by

$$g_i(x) := \|x\|^{\alpha_i} f_i\left(\frac{x}{\|x\|}\right)$$

where α_i is defined by (7). Since the f_i are eigenfunctions for $-\Delta_S$, we deduce from (3) that

$$\begin{aligned}\Delta g_i &= \frac{\partial^2}{\partial r} g_i + \frac{N-1}{r}\frac{\partial}{\partial r} g_i + \frac{1}{r^2}\Delta_S g_i\\ &= \alpha_i(\alpha_i - 1)r^{\alpha_i-2} f_i + \frac{N-1}{r}\alpha_i r^{\alpha_i-1} f_i - r^{\alpha_i-2}\lambda_i f_i\\ &= (\alpha_i^2 + (N-2)\alpha_i - \lambda_i)r^{\alpha_i-2} f_i\\ &= 0\end{aligned}$$

by definition of α_i, thus the g_i are harmonic in $A(+\infty)$. Notice that the g_i are orthogonal in $L^2(A(1))$ because they are homogeneous and orthogonal in $L^2(\Omega)$. Note also that h_i is equal to g_i on $r_0\Omega$. Moreover for all $0 < r \leq 1$ we have

$$\begin{aligned}(12)\quad \|g_i\|^2_{L^2(A(r))} &= \int_{A(r)} |g_i|^2 = \int_0^r \int_{\partial B(t)\cap A(1)} |g_i(w)|^2 dw dt\\ &= \int_0^r \int_\Omega t^{N-1}|g_i(ty)|^2 dy dt = \int_0^r t^{2\alpha_i+N-1} \int_\Omega |g_i(y)|^2 dy dt\\ &= \frac{r^{2\alpha_i+N}}{2\alpha_i+N}\|f_i\|^2_{L^2(\Omega)} = \frac{r^{2\alpha_i+N}}{2\alpha_i+N} \leq 1.\end{aligned}$$

On the other hand, since the f_i and their tangential gradients are orthogonal in $L^2(\Omega)$, we deduce that the gradients of g_i are orthogonal in $A(1)$.

Then, by a computation similar to (12) we obtain for all $0<r\leq 1$

$$
\begin{aligned}
(13)\ \|\nabla g_i\|^2_{L^2(A(r))} &= \int_0^r \int_{\partial B(t)\cap A(1)} \left|\frac{\partial g_i}{\partial r}\right|^2 + |\nabla_\tau g_i|^2 dw dt \\
&= \int_0^r \int_{\partial B(t)\cap A(1)} \left|\alpha_i t^{\alpha_i-1} f_i\left(\frac{w}{t}\right)\right|^2 + \left|t^{\alpha_i}\nabla_\tau f_i\left(\frac{w}{t}\right)\frac{1}{t}\right|^2 dw dt \\
&= \alpha_i^2 \int_0^r t^{2(\alpha_i-1)} \int_{\partial B(t)\cap A(1)} \left|f_i\left(\frac{w}{t}\right)\right|^2 dw dt \\
&\qquad + \int_0^r t^{2(\alpha_i-1)} \int_{\partial B(t)\cap A(1)} \left|\nabla_\tau f_i\left(\frac{w}{t}\right)\right|^2 dw dt \\
&= \alpha_i^2 \int_0^r t^{2(\alpha_i-1)} \int_\Omega |f_i(w)|^2 t^{N-1} dw dt \\
&\qquad + \int_0^r t^{2(\alpha_i-1)} \int_\Omega |\nabla_\tau f_i(w)|^2 t^{N-1} dw dt \\
&= \alpha_i^2 \frac{r^{2(\alpha_i-1)+N}}{2(\alpha_i-1)+N}\|f_i\|^2_{L^2(\Omega)} + \frac{r^{2(\alpha_i-1)+N}}{2(\alpha_i-1)+N}\|\nabla_\tau f_i\|^2_{L^2(\Omega)} \\
&= \frac{r^{2(\alpha_i-1)+N}}{2(\alpha_i-1)+N}(\alpha_i^2+\lambda_i)\|f_i\|^2_{L^2(\Omega)} \\
&\leq C r^{2\alpha_i}(\alpha_i^2+\lambda_i)
\end{aligned}
$$

because $\|\nabla_\tau f_i\|_2^2 = \lambda_i\|f_i\|_2^2$, $r\leq 1$ and $\alpha_i \geq 0$. Moreover the constant C depends on the dimension N but does not depend on i.

We denote by g the function defined in $A(\infty)$ by

$$g := \sum_{i=0}^{+\infty} a_i g_i.$$

Then g lies in $L^2(A(r_0))$ because using (12) and (9)

$$\|g\|^2_{L^2(A(r_0))} = \sum_{i=0}^{+\infty} |a_i|^2 \|g_i\|^2_{L^2(A(r_0))} \leq \sum_{i=0}^{+\infty} |a_i|^2 r_0^{2\alpha_i+N} < +\infty.$$

We want now to show that $g = u$.

• *First step:* We claim that g is harmonic in $A(r_0)$. Indeed, since the g_i are all harmonic in $A(r_0)$, the sequence of partial sums $s_k := \sum_{i=0}^{k} a_i g_i$ is a sequence of harmonic functions, uniformly bounded for the L^2 norm in each compact set of $A(r_0)$. By the Harnack inequality we deduce that the sequence of partial sums is uniformly bounded for the uniform norm in each compact set. Thus there is a subsequence that converges uniformly to a harmonic function, which in fact is equal to g by uniqueness of the limit. Therefore, g is harmonic in $A(r_0)$.

• *Second step:* We claim that g belongs to $W^{1,2}(A(r_0))$. Firstly, since $u \in W^{1,2}(r_0\Omega)$, by (8) and (10) we have that

$$\sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} \|\nabla_\tau f_i\|^2_{L^2(\partial B(0,1)\backslash K)} < +\infty. \tag{14}$$

In addition, since $\|\nabla_\tau f_i\|_2^2 = \lambda_i \|f_i\|_2^2$ and $\|f_i\|_2 = 1$, we deduce

$$\sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} \lambda_i < +\infty \tag{15}$$

and since α_i and λ_i are linked by the formula (7) we also have that

$$\sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} \alpha_i^2 < +\infty. \tag{16}$$

Now, since $\sum a_i g_i$ converges absolutely on every compact set, we can say that

$$\nabla g = \sum_{i=0}^{+\infty} a_i \nabla g_i$$

thus using (13), (15), (16), and orthogonality,

$$\begin{aligned}\|\nabla g\|^2_{L^2(A(r_0))} &= \sum_{i=0}^{+\infty} a_i^2 \|\nabla g_i\|^2_{L^2} \\ &\leq C \sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i}(\alpha_i^2 + \lambda_i) < +\infty\,.\end{aligned}$$

Therefore, $g \in W^{1,2}(A(r_0))$.

• *Third step :* We claim that $\frac{\partial g}{\partial n} = 0$ on $K \cap \overline{A(r_0)} \backslash \Sigma$. We already know that $\frac{\partial g_i}{\partial n} = 0$ on $K \backslash \Sigma$ (because the f_i have this property). We want to show that g is so regular that we can exchange the order of $\frac{\partial}{\partial n}$ and $\sum$. So let x_0 be a point of $K \cap \overline{A(r_0)} \backslash \Sigma$ and let B be a neighborhood of x_0 in $\mathbb{R}^N$ that doesn't meet Σ and such that K separates B in two parts B^+ and B^-. Assume that B^+ is a part in $A(r_0)$. The sequence of partial sums $s_k := \sum_{i=0}^{k} a_i\, g_i$ is a sequence of harmonic functions in B^+. Since $\partial B^+ \cap K$ is C^2 we can do a reflection to extend s_k in B^-. For all k, this new function s_k is the solution of a certain elliptic equation whose operator become from the composition of the Laplacian with the application that makes $\partial B^+ \cap K$ flat. Thus since $\sum a_i\, g_i$ converges absolutely for the L^2 norm, by the Harnack inequality $\sum a_i\, g_i$ converges absolutely for the uniform norm in a smaller neighborhood $B' \subset B$ that still contains x_0. Thus s_k converges to a C^1 function denoted by s, which is equal to g on B^+. And since $\frac{\partial s_k}{\partial n}(x_0) = 0$, by the absolute convergence of the sum we can exchange the order of the derivative and the symbol $\sum$ so we deduce that $\frac{\partial s}{\partial n}(x_0) = 0$. Finally, since s is equal to g on B^+ we deduce that g is C^1 at the boundary and $\frac{\partial g}{\partial n} = 0$ at x_0.

• *Fourth step:* we claim that g is equal to u on $r_0\Omega$. Let r be a radius such that $r < r_0$. Then the function $x \mapsto g_r(x) := g\left(r\frac{x}{r_0}\right)$ is well defined for $x \in r_0\Omega$, and since the g_i are homogeneous we have

$$g\left(r\frac{x}{r_0}\right) = \sum_{i=0}^{+\infty} a_i\, g_i\left(r\frac{x}{r_0}\right) = \sum_{i=0}^{+\infty}\left(\frac{r}{r_0}\right)^{\alpha_i} a_i\, g_i(x) = \sum_{i=0}^{+\infty}\left(\frac{r}{r_0}\right)^{\alpha_i} a_i h_i(x).$$

We deduce that the function $x \mapsto g\left(\frac{r}{r_0}x\right)$ is in $L^2(r_0\Omega)$ and its coefficients in the basis $\{h_i\}$ are $\left\{\left(\frac{r}{r_0}\right)^{\alpha_i} a_i\right\}$. We want to show that $\|g_r - u\|_{L^2(r_0\Omega)}$ tend to 0. Indeed, writing u in the basis $\{h_i\}$

$$u = \sum_{i=0}^{+\infty} a_i h_i,$$

we obtain

$$\|g_r - u\|_2^2 = \sum_{i=0}^{+\infty}\left(\left(\frac{r}{r_0}\right)^{\alpha_i} - 1\right)^2 a_i^2 \|h_i\|_2^2$$

which tends to zero when r tends to r_0 by the dominated convergence theorem because $\left(\left(\frac{r}{r_0}\right)^{\alpha_i} - 1\right)^2 \leq 1$. Therefore, there is a subsequence for which g_r tends to u almost everywhere. On the other hand, since g is harmonic, the limit of g_r exists and is equal to g. That means that g tends to u radially at almost every point of $r_0\Omega$.

• *Fifth step:* The functions u and g are harmonic functions in $A(r_0)$, with finite energy, with a normal derivative equal to zero on $K \cap \overline{A(r_0)} \backslash \Sigma$ and that coïncide on $\partial A(r_0)\backslash K$. To show that $u = g$ in $A(r_0)$ we shall prove that g is an energy minimizer. Proposition 8 will then give the uniqueness.

Let $\varphi \in W^{1,2}(A(r_0))\backslash K)$ have a vanishing trace on $\partial B(0, r_0)$. Then, setting $J(v) := \int_{A(r_0)} |\nabla v|^2$ for $v \in W^{1,2}(A(r_0))$ we have

$$J(g + \varphi) = J(g) + \int_{A(r_0)} \nabla g \nabla \varphi + J(\varphi).$$

Now since g is harmonic with Neumann condition on $K\backslash\Sigma$ and since φ vanishes on $r_0\Omega$, integrating by parts we obtain

$$J(g + \varphi) = J(g) + J(\varphi).$$

Since J is non negative and $g + \varphi$ describes all the functions in $W^{1,2}(A(r_0))$ with trace equal to u on $r_0\Omega$, we deduce that g minimizes J. We can do the same with u thus u and g are two energy minimizers with same boundary conditions. Therefore, by Proposition 8 we know that $g = u$.

• *Sixth step:* The decomposition do not depends on r_0. Indeed, let r_1 be a second choice of radius. Then we can do the same work as before to obtain a decomposition

$$u(x) := \sum_{i=0}^{+\infty} b_i g_i(x) \quad \text{in } B(0, r_1)\backslash K.$$

Now by uniqueness of the decomposition in $B(0, min(r_0, r_1))$ we deduce that $b_i = a_i$ for all i.

In addition, r_0 was initially chosen almost everywhere in $]0, 1[$. But since the decomposition does not depend on the choice of radius, r_0 can be chosen anywhere in $]0, 1[$, by choosing a radius almost everywhere in $]r_0, 1[$. □

THEOREM 15. *Let (u, K) be a global minimizer in $\mathbb{R}^N$ such that K is a smooth cone. Then for each connected component of $\mathbb{R}^N\backslash K$ there is a constant u_k such that $u - u_k$ is $\frac{1}{2}$-homogenous.*

PROOF. Let Ω be a connected component of $\mathbb{R}^N \backslash K$. We apply the preceding proposition to u. Thus

$$u(x) = \sum_{i=0}^{+\infty} a_i\, g_i(x) \quad \text{in } A(r_0).$$

for a certain radius r_0 chosen in $]0,1[$. Let us prove that the same decomposition is true in $A(\infty)$. Applying Proposition 14 to the function $u_R(x) = u(Rx)$ we know that there are some coefficients $a_i(R)$ such that

$$u_R(x) = \sum_{i=0}^{+\infty} a_i(R) g_i(x) \text{ in } A(r_0).$$

Now since $u_R\left(\frac{x}{R}\right) = u(x)$ we can use the homogeneity of the g_i to identify the terms in $B(0, r_0)$ thus $a_i(R) = a_i R^{\alpha_i}$. Now we fix $y = Rx$ and we obtain that

$$u(y) = \sum_{i=0}^{+\infty} a_i\, g_i(y) \text{ in } A(Rr_0).$$

Since R is arbitrary the decomposition is true in $A(\infty)$.

In addition for every radius R we know that

$$\|\nabla u\|^2_{L^2(A(R))} = \sum_{i=0}^{+\infty} a_i^2 \|\nabla g_i\|^2_{L^2(A(R))} \tag{17}$$

and since g_i is α_i-homogenous,

$$\|\nabla g_i\|^2_{L^2(A(R))} = R^{2(\alpha_i - 1)+N} \|\nabla g_i\|^2_{L^2(A(1))}.$$

Now, since u is a global minimizer, a classical estimate on the gradient obtained by comparing (u, K) with (v, L) where $v = \mathbf{1}_{B(0,R)^c} u$ and $L = \partial B(0, R) \cup (K \backslash B(0, R))$ gives that there is a constant C such that for all radius R

$$\|\nabla u\|^2_{L^2(B(0,R)\backslash K)} \leq CR^{N-1}.$$

We deduce

$$\sum_{i=0}^{+\infty} a_i^2 R^{2(\alpha_i-1)+N} \|\nabla g_i\|^2_{L^2(A(1))} \leq CR^{N-1}.$$

Thus

$$\sum_{i=0}^{+\infty} a_i^2 R^{2\alpha_i - 1} \|\nabla g_i\|^2_{L^2(A(1))} \leq C.$$

This last quantity is bounded when R goes to infinity if and only if $a_i = 0$ whenever $\alpha_i > 1/2$. On the other hand, this quantity is bounded when R goes to 0, if and only if $a_i = 0$ whenever $0<\alpha_i<1/2$. Therefore, $u - a_0$ is a finite sum of terms of degree $\frac{1}{2}$. □

REMARK 16. In Chapter 65 of [8], we can find a variational argument that leads to a formula in dimension 2 that links the radial and tangential derivatives of u. For all $\xi \in K \cap \partial B(0,r)$, we call $\theta_\xi \in \left[0, \frac{\pi}{2}\right]$ the non oriented angle between the tangent to K at point ξ and the radius $[0,\xi]$. Then we have the following formula

$$\int_{\partial B(0,r)\setminus K} \left(\frac{\partial u}{\partial r}\right)^2 dH^1 = \int_{\partial B(0,r)\setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 dH^1 + \sum_{\xi \in K\cap\partial B(0,r)} \cos\theta_\xi - \frac{1}{r} H^1(K \cap B(0,r)).$$

Notice that for a global minimizer in $\mathbb{R}^2$ with K a centered cone we find

$$\int_{\partial B(0,r)\setminus K} \left(\frac{\partial u}{\partial r}\right)^2 dH^1 = \int_{\partial B(0,r)\setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 dH^1. \tag{18}$$

Now suppose that (u,K) is a global minimizer in $\mathbb{R}^N$ with K a smooth cone centered at 0. Then by Theorem 15 we know that u is harmonic and $\frac{1}{2}$-homogenous. Its restriction to the unit sphere is an eigenfunction for the spherical Laplacian with Neumann boundary condition and associated to the eigenvalue $\frac{2N-3}{4}$. We deduce that

$$\|\nabla_\tau u\|^2_{L^2(\partial B(0,1))} = \frac{2N-3}{4} \|u\|^2_{L^2(\partial B(0,1))}.$$

On the other hand

$$\frac{\partial u}{\partial r}(x) = \frac{1}{2} \|x\|^{-\frac{1}{2}} u\left(\frac{x}{\|x\|}\right)$$

thus

$$\| \frac{\partial u}{\partial r} \|^2_{L^2(\partial B(0,1))} = \frac{1}{4} \|u\|^2_{L^2(\partial B(0,1))}.$$

So

$$\|\nabla_\tau u\|^2_{L^2(\partial B(0,1))} = (2N-3) \| \frac{\partial u}{\partial r} \|^2_{L^2(\partial B(0,1))}.$$

In particular, for $N = 2$ we have the same formula as (18).

2. Some applications.

As it was claimed in the introduction, here is some few applications of Theorem 15.

PROPOSITION 17. *Let (u, K) be a global minimizer in $\mathbb{R}^3$ such that K is a smooth cone. Moreover, assume that $S^2 \cap K$ is a union of convex curvilinear polygons with C^∞ sides. Then u is locally constant and K is a cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$.*

PROOF. In each polygon we know by Proposition 4.5. of [6] that the smallest positive eigenvalue for the operator minus Laplacian with Neumann boundary conditions is greater than or equal to 1. Thus it cannot be $\frac{3}{4}$ and u is locally constant. Then K is a minimal cone in $\mathbb{R}^3$ and we know from [9] that it is a cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$. □

Let $(r, \theta, z) \in \mathbb{R}^+ \times [-\pi, \pi] \times \mathbb{R}$ be the cylindrical coordinates in $\mathbb{R}^3$. For every $\omega \in [0, \pi]$ set

$$\Gamma_\omega := \{(r, \theta, z) \in \mathbb{R}^3; -\omega < \theta < \omega\}$$

of boundary

$$\partial\Gamma_\omega := \{(r, \theta, z) \in \mathbb{R}^3; \theta = -\omega \text{ or } \theta = \omega\}.$$

Consider $\Omega_\omega = \Gamma_\omega \cap S^2$ and let λ_1 be the smallest positive eigenvalue of $-\Delta_S$ in Ω_ω with Neumann conditions on $\partial\Omega_\omega$. Then by Lemma 4.1. of [6] we have that

$$\lambda_1 = \min(2, \lambda_\omega)$$

where

$$\lambda_\omega = \left(\frac{\pi}{2\omega} + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

In particular for the cone of type $\mathbb{Y}$, $\omega = \frac{\pi}{3}$ thus $\lambda_1 = 2$.

Observe that for $\omega \neq \pi$, $\lambda_\omega \neq \frac{3}{4}$. So we get this following proposition.

PROPOSITION 18. *There is no global Mumford-Shah minimizer in $\mathbb{R}^3$ such that K is wing of type $\partial\Gamma_\omega$ with $\omega \notin \left\{0, \frac{\pi}{2}, \pi\right\}$.*

Another consequence of Theorem 15 is the following. Let P be the half plane

$$P := \{(r,\theta,z) \in \mathbb{R}^3; \theta = \pi\}.$$

PROPOSITION 19. *Let (u,K) be a global Mumford-Shah minimizer in $\mathbb{R}^3$ such that $K = P$. Then u is equal to cracktip $\times\, \mathbb{R}$, that is in cylindrical coordinates*

$$u(r,\theta,z) = \pm\sqrt{\frac{2}{\pi}}\, r^{\frac{1}{2}} \sin\frac{\theta}{2} + C$$

for $0 < r < +\infty$ *and* $-\pi < \theta < \pi$.

REMARK 20. In Section 3 we will give a second proof of Proposition 19.

REMARK 21. We already know that $u = cracktip \times \mathbb{R}$ is a global minimizer in $\mathbb{R}^3$ (see [8]).

To prove Proposition 19 we will use the following well known result.

PROPOSITION 22 ([5], [13]) . *The smallest positive eigenvalue for $-\Delta_n$ in $S^2 \backslash P$ is $\frac{3}{4}$, the corresponding eigenspace is of dimension 1 generated by the restriction on S^2 of the following function in cylindrical coordinates*

$$u(r,\theta,z) = r^{\frac{1}{2}} \sin\frac{\theta}{2}$$

for $0 < r < +\infty$ *and* $-\pi < \theta < \pi$.

Now the proof of Proposition 19 can be easily deduce from Proposition 22 and Theorem 15.

PROOF OF PROPOSITION 19. If (u,P) is a global minimizer, we know that after removing a constant the restriction of u to the unit sphere is an eigenfunction for $-\Delta_n$ in $S^2 \backslash P$ associated to the eigenvalue $\frac{3}{4}$. Therefore, from Proposition 22 we know that

$$u(r,\theta,z) = C r^{\frac{1}{2}} \sin\frac{\theta}{2}$$

so we just have to determinate the constant C. But by a well known argument about Mumford-Shah minimizers we prove that C must be equal to $\pm\sqrt{\frac{2}{\pi}}$ (see [8] Section 61 for more details). □

Now set

$$S_\omega := \{(r,\theta,0); r>0, \theta\in[-\omega,\omega]\}$$

PROPOSITION 23. *There is no global Mumford-Shah minimizer in* $\mathbb{R}^3$ *such that* K *is an angular sector of type* (u, S_ω) *for* $0<\omega<\frac{\pi}{2}$ *or* $\frac{\pi}{2}<\omega<\pi$.

PROOF. According to Theorem 15, if (u, S_ω) is a global minimizer, then $u-u_0$ is a homogenous harmonic function of degree $\frac{1}{2}$, thus its restriction to $S^2\backslash S_\omega$ is an eigenfunction for $-\Delta_n$ associated to the eigenvalue $\frac{3}{4}$. Now if $\lambda(\omega)$ denotes the smallest eigenvalue on $\partial B(0,1)\backslash S_\omega$, we know by Theorem 2.3.2. p. 47 of [14] that $\lambda(\omega)$ is non decreasing with respect to ω. Since $\lambda\left(\frac{\pi}{2}\right)=\frac{3}{4}$, we deduce that for $\omega<\frac{\pi}{2}$, we have

$$\lambda(\omega)\geq\frac{3}{4}. \tag{19}$$

In [14] page 53 we can find the following asymptotic formula near $\omega=\frac{\pi}{2}$

$$\lambda(\omega)=\frac{3}{4}+\frac{2}{\pi}\cos\omega+O(\cos^2\omega). \tag{20}$$

this proves that the case when (19) is a equality only arises when $\omega=\frac{\pi}{2}$. Thus such eigenfunction u doesn't exist.

Consider now the case $\omega>\frac{\pi}{2}$. For $\omega=\pi$ there are tow connected components. Thus 0 is an eigenvalue of multiplicity 2. The second eigenvalue is equal to 2. Therefore, for $\omega=\pi$ the spectrum is

$$0\leq 0\leq 2\leq\lambda_3\leq\ldots \qquad \omega=\pi$$

By monotonicity, when ω decreases, the eigenvalues increase. Since the domain becomes connexe, 0 become of multiplicity 1 thus the second eigenvalue become positive. The spectrum is now

$$0\leq\lambda_1\leq\lambda_2\leq\ldots \qquad \omega<\pi$$

with $\lambda_i\geq 2$ for $i\geq 2$. Thus the only eigenvalue that could be equal to $\frac{3}{4}$ is λ_2 which is increasing from from 0 to $\frac{3}{4}$, reached for $\omega=\frac{\pi}{2}$. Now (20) says that the increasing is strict near $\omega=\frac{\pi}{2}$. Therefore there is no eigenvalue equal to $3/4$ for $\omega>\frac{\pi}{2}$ and there is no possible global minimizer. $\square$

3. Second proof of Propositions 19 and 22.

Here we want to give a second proof of Proposition 19, without using Theorem 15, and which do not use Proposition 22. In a remark at the end of this section, we will briefly explain how to use this proof of Proposition 19 in order to obtain a new proof of Proposition 22 as well.

Let assume that K is a half plane in $\mathbb{R}^3$. We can suppose for instance that

$$K = P := \{x_2 = 0\} \cap \{x_1 \leq 0\} \tag{21}$$

We begin by studying the harmonic measure in $\mathbb{R}^3 \backslash P$.

Let $B := B(0, R)$ be a ball of radius R and let γ be the trace operator on $\partial B(0, R) \backslash P$. We denote by T the image of $W^{1,2}(B \backslash K)$ by γ. We also denote by $C^0_b(\partial B \backslash K)$ the set of continuous and bounded functions on $\partial B(0, 1) \backslash P$. Finally set $A := T \cap C^0_b$. Obviously A is not empty. To every function $f \in A$, Proposition 15.6. of [8] associates a unique energy minimizing function $u \in W^{1,2}(B \backslash K)$ such that $\gamma(u) = f$ on $\partial B \backslash P$. Since u is harmonic we know that it is C^∞ in $B \backslash K$. Let $y \in B \backslash K$ be a fixed point and consider the linear form μ_y defined by

$$\begin{aligned} \mu_y : A &\to \mathbb{R} \\ f &\mapsto u(y). \end{aligned} \tag{22}$$

By the maximum principle for energy minimizers, we know that for all $f \in A$ we have

$$|\mu_y(f)| \leq \|f\|_\infty$$

thus μ_y is a continuous linear form on A for the norm $\| \ \|_\infty$. We identify μ_y with its representant in the dual space of A and we call it *harmonic measure*.

Moreover, the harmonic measure is positive. That is, if $f \in A$ is a non negative function, then (by the maximum principle) $\mu_y(f)$ is non negative. By positivity of μ_y, if $f \in A$ is a non negative function and $g \in A$ is such that $fg \in A$, then since $(\|g\|_\infty + g)f$ and $(\|g\|_\infty - g)f$ are two non negative functions of A we deduce that

$$|\langle fg, \mu_y \rangle| \leq \|g\|_\infty \langle f, \mu_y \rangle. \tag{23}$$

Now here is an estimate on the measure μ^R_y.

LEMMA 24. *There is a dimensional constant C_N such that the following holds. Let R be a positive radius. For $0<\lambda<\dfrac{R}{2}$ consider the spherical domain*

$$\mathcal{C}_\lambda := \{x \in \mathbb{R}^3 \; ; \; |x| = R \text{ and } d(x,P) \leq \lambda\}.$$

Let $\varphi_\lambda \in C^\infty(\partial B(0,R))$ be a function between 0 and 1, that is equal to 1 on $\mathcal{C}_\lambda$ and 0 on $\partial B(0,R)\backslash\mathcal{C}_{2\lambda}$ and that is symmetrical with respect to P. Then for every $y \in B\left(0,\dfrac{R}{2}\right)\backslash P$ we have

$$\mu_y^R(\varphi_\lambda) \leq C\frac{\lambda}{R}.$$

PROOF. Since φ_λ is continuous and symmetrical with respect to P, by the reflection principle, its harmonic extension φ in $B(0,R)$ has a normal derivative equal to zero on P in the interior of $B(0,R)$. Moreover φ_λ is clearly in the space A. Thus by definition of μ_y,

$$\varphi(y) = \langle \varphi_\lambda, \mu_y^R \rangle.$$

On the other hand, since φ_λ is continuous on the entire sphere, we also have the formula with the classical Poisson kernel

$$\varphi(y) = \frac{R^2 - |y|^2}{N\omega_N R} \int_{\partial B_R} \frac{\varphi_\lambda(x)}{|x-y|^3} ds(x)$$

with ω_N equal to the measure of the unit sphere. In other words

$$\mu_y^R(\varphi_\lambda) = \frac{R^2 - |y|^2}{N\omega_N R} \int_{\partial B_R} \frac{\varphi_\lambda(x)}{|x-y|^3} ds(x).$$

For $x \in \partial B_R$ we have

$$\frac{1}{2}R \leq |x| - |y| \leq |x-y| \leq |x| + |y| \leq \frac{3}{2}R.$$

We deduce that

$$\mu_y^R(\varphi_\lambda) \leq C_N \frac{1}{R^2} \int_{\mathcal{C}_{2\lambda}} ds.$$

Now integrating by parts,

$$\begin{aligned}\int_{\mathcal{C}_\lambda} ds =& 2\int_0^\lambda 2\pi\sqrt{R^2-w^2}dw\\ =& 4\pi\,\frac{\lambda}{2}\sqrt{R^2-\lambda^2}+R^2\arcsin\left(\frac{\lambda}{R}\right)\\ \leq& CR\lambda\end{aligned}$$

because $\arcsin(x)\leq\frac{\pi}{2}x$. The proposition follows. □

Now we can prove the uniqueness of *cracktip* $\times\,\mathbb{R}$.

Second Proof of Proposition 19. Let us show that u is vertically constant. Let t be a positive real. For $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ set $x_t:=(x_1,x_2,x_3+t)$. We also set

$$u_t(x):=u(x)-u(x_t).$$

Since u is a function associated to a global minimizer, and since K is regular, we know that for all $R>0$, the restriction of u to the sphere $\partial B(0,R)\backslash K$ is continuous and bounded on $\partial B(0,R)\backslash K$ with finite limits on each sides of K. It is the same for u_t. Thus for all $x\in\mathbb{R}^3\backslash P$ and for all $R>2\|x\|$ we can write

$$u_t(x):=\langle u_t|_{\partial B(0,R)\backslash P},\mu_x^R\rangle$$

where μ_x is the harmonic measure defined in (22). We want to prove that for $x\in\mathbb{R}^3\backslash P$, $\langle u_t|_{\partial B(0,R)\backslash P},\mu_x^R\rangle$ tends to 0 when R goes to infinity. This will prove that $u_t=0$.

So let $x\in\mathbb{R}^3\backslash P$ be fixed. We can suppose that $R>100(\|x\|+t)$. Let $\mathcal{C}_\lambda$ and φ_λ be as in Lemma 24. Then write

$$u_t(x)=\langle u_t|_{\partial B(0,R)\backslash P}\varphi_\lambda,\mu_x^R\rangle+\langle u_t|_{\partial B(0,R)\backslash P}(1-\varphi_\lambda),\mu_x^R\rangle.$$

Now by a standard estimate on Mumford-Shah minimizers (that comes from Campanato's Theorem, see [3] p. 371) we have for all $x\in\mathbb{R}^N\backslash P$,

$$|u_t(x)|\leq C\sqrt{t}.$$

Then, using Lemma 24 we obtain

$$|\langle u_t|_{\partial B(0,R)\backslash P}\;\varphi_\lambda,\;\mu_x^R\rangle|\leq C\sqrt{t}\,\frac{\lambda}{R}.$$

On the other hand, for the points y such that $d(y,P) \geq \lambda$, since $\tilde{u} : u(.) - u(y)$ is harmonic in $B(y, d(y,P))$ we have, by a classical estimation on harmonic functions (see the introduction of [12])

$$|\nabla \tilde{u}(y)| \leq C \frac{1}{d(y,P)} \|\tilde{u}\|_{L^\infty(\partial B(y, \frac{1}{2} d(y,P)))}.$$

Now using Campanato's Theorem again we know that

$$\|\tilde{u}\|_{L^\infty(\partial B(y, \frac{1}{2} d(y,P)))} \leq C d(y,P)^{\frac{1}{2}}$$

thus

$$|\nabla u(y)| \leq C \frac{1}{d(y,P)^{\frac{1}{2}}}$$

and finally by the mean value theorem we deduce that for all the points y such that $d(y,P) \geq \lambda$,

$$|u_t(y)| \leq C \sup_{z \in [y, y_t]} |\nabla u(z)|.|y - y_t| \leq t \frac{1}{\lambda^{\frac{1}{2}}}.$$

Therefore,

$$|\langle u_t|_{\partial B(0,R) \backslash P}(1 - \varphi_\lambda), \mu_x^R \rangle| \leq Ct \frac{1}{\lambda^{\frac{1}{2}}}.$$

So

$$|u_t(x)| \leq C\sqrt{t} \frac{\lambda}{R} + Ct \frac{1}{\lambda^{\frac{1}{2}}}$$

thus by setting $\lambda = R^{\frac{1}{2}}$ and by letting R go to $+\infty$ we deduce that $u_t(x) = 0$ thus $z \mapsto u(x,y,z)$ is constant.

Now we fix $z_0 = 0$ and we introduce $P_0 := P \cap \{z = 0\}$. We want to show that $(u(x,y,0), P_0)$ is a global minimizer in $\mathbb{R}^2$. Let $(v(x,y), \Gamma)$ be a competitor for $u(x,y,0)$ in the 2-dimensional ball B of radius ρ. Let $\mathcal{C}$ be the cylinder $\mathcal{C} := B \times [-R, R]$. Define $\tilde{v}$ and $\tilde{\Gamma}$ in $\mathbb{R}^3$ by

$$\tilde{v}(x,y,z) = \begin{cases} v(x,y) & \text{if } (x,y,z) \in \mathcal{C} \\ u(x,y,z) & \text{if } (x,y,z) \notin \mathcal{C} \end{cases}$$

$$\tilde{\Gamma} := (\mathcal{C} \cap [\Gamma \times [-R,R]]) \cup (P \backslash \mathcal{C}) \cup (B \times \{\pm R\}).$$

It is a topological competitor because $\mathbb{R}^3 \backslash P$ is connected (thus P doesn't separate any points). Now finally let $\tilde{B}$ be a ball that contains $\mathcal{C}$. Then $(\tilde{v}, \tilde{\Gamma})$

is a competitor for (u, P) in $\tilde{B}$. By minimality we have:

$$\int_{\tilde{B}} |\nabla u|^2 + H^2(P \cap \tilde{B}) \leq \int_{\tilde{B}} |\nabla \tilde{v}|^2 + H^2(\tilde{\Gamma} \cap \tilde{B}).$$

In the other hand u is equal to $\tilde{v}$ in $\tilde{B} \backslash \mathcal{C}$ and it is the same for Γ and $\tilde{\Gamma}$. We deduce

$$\int_{\mathcal{C}} |\nabla u|^2 dxdydz + H^2(P \cap \mathcal{C}) \leq \int_{\mathcal{C}} |\nabla \tilde{v}|^2 dxdydz + H^2(\tilde{\Gamma} \cap \mathcal{C}).$$

Now, since u and $\tilde{v}$ are vertically constant, $\nabla_z u = \nabla_z \tilde{v} = 0$, and $\nabla_x u$, $\nabla_y u$ are also constant with respect to the variable z (as for $\tilde{v}$). Thus

$$2R \int_B |\nabla u(x, y, 0)|^2 dxdy + H^2(P \cap \mathcal{C}) \leq 2R \int_B |\nabla v(x, y)|^2 dxdy + H^2(\tilde{\Gamma} \cap \mathcal{C}).$$

To conclude we will use the following lemma.

LEMMA 25. *If Γ is rectifiable and contained in a plane Q then*

$$H^2(\Gamma \times [-R, R]) = 2RH^1(\Gamma).$$

PROOF. We will use the coarea formula (see Theorem 2.93 of [3]). We take $f : \mathbb{R}^3 \to \mathbb{R}$ the orthogonal projection on the coordinate orthogonal to Q. By this way, if $E := \Gamma \times [-R, R]$, we have $E \cap f^{-1}(t) = \Gamma$ for all $t \in [-R, R]$. E is rectifiable (because Γ is by hypothesis). So we can apply the coarea formula. To do this we have to calculate the jacobian $c_k d^E f_x$. By construction, the approximate tangente plane in each point of E is orthogonal to Q. We deduce that if T_x is a tangent plane, then there is a basis of T_x $(\overrightarrow{b_1}, \overrightarrow{b_2})$ such that $\overrightarrow{b_1}$ is orthogonal to Q. Since the function f is the projection on $\overrightarrow{b_1}$, and its derivative as well (because f is linear) we obtain that the matrix of $d^E f_x : T_x \to \mathbb{R}$ in the basis $(\overrightarrow{b_1}, \overrightarrow{b_2})$ is

$$d^E f_x = (1, 0)$$

thus

$$c_k d^E f_x = \sqrt{det[(1, 0).^t(1, 0)]} = 1.$$

Therefore

$$H^2(E) = \int_{-R}^{R} H^1(\Gamma) = 2RH^1(\Gamma). \qquad \square$$

Here we can suppose that Γ is rectifiable. Indeed, the definition of Mumford-Shah minimizers is equivalent if we only allow rectifiables competitors. This is because the jump set of a SBV function is rectifiable and in [11] it is proved that the relaxed functional on the SBV space has same minimizers.

So we have

$$2R\int_B |\nabla u(x,y,0)|^2 dxdy + 2RH^1(P\cap B) \leq 2R\int_B |\nabla v(x,y)|^2 dxdy + 2RH^1(\Gamma\cap B) + H^2(B\times\{\pm R\}).$$

Then, dividing by $2R$,

$$\int_B |\nabla u(x,y,0)|^2 dxdy + H^1(P\cap B) \leq \int_B |\nabla v(x,y)|^2 dxdy + H^1(\Gamma\cap B) + \pi\frac{\rho^2}{R}$$

thus, letting R go to infinity,

$$\int_B |\nabla u(x,y,0)|^2 dxdy + H^1(P\cap B) \leq \int_B |\nabla v(x,y)|^2 dxdy + H^1(\Gamma\cap B).$$

This last inequality proves that $(u(x,y,0),P_0)$ is a global minimizer in $\mathbb{R}^2$, and since P_0 is a half-line, u is a *cracktip*. $\square$

REMARK 26. Using a similar argument as the preceding proof, we can show that the first eigenvalue for $-\Delta$ in $S^2\backslash P$ with Neumann boundary conditions (where P is still a half-plane), is equal to $\frac{3}{4}$. Moreover we can prove that the eigenspace is of dimension 1, generated by a function of type *cracktip* $\times\,\mathbb{R}$, thus we have a new proof of Proposition 22. The argument is to take an eigenfunction f in $S^2\backslash P$, then to consider $u(x) := \|x\|^\alpha f\left(\frac{x}{\|x\|}\right)$ with a good coefficient $\alpha\in\left];0,\frac{1}{2}\right]$ that makes u harmonic. Finally we use the same sort of estimates on the harmonic measure to prove that u is vertically constant. Thus we have reduced the problem in dimension 2 and we conclude using that we know the eigenfunctions on the circle. A detailed proof is done in [15].

4. Open questions.

As it is said in the introduction, this paper is a very short step in the discovering of all the global minimizers in $\mathbb{R}^N$. This final goal seems rather far but nevertheless some open questions might be accessible in a more reasonable time. All the following questions were pointed out by Guy David in [8], and unfortunately they are still open after this paper.

• Suppose that (u, K) is a global minimizer in $\mathbb{R}^N$. Is it true that K is conical?

• Suppose that (u, K) is a global minimizer in $\mathbb{R}^N$, and K is a cone. Is it true that $\frac{3-2N}{4}$ is the smallest eigenvalue of the Laplacian on $S^{N-1}\backslash K$?

• Suppose that (u, K) is a global minimizer in $\mathbb{R}^3$, and suppose that K is contained in a plan (and not empty). Is it true that K is a plane or a half-plane?

• Could one found an extra global minimizer in $\mathbb{R}^3$ by blowing up the minimizer described in section 76.c. of [8] (see also [17])?

One can find other open questions on global minimizers in the last page of [8].

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