# Notes On Generalized $(\sigma, \tau)-$Derivation

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ABSTRACT - Let $R$ be a prime ring with $charR \neq 2$ and let $\sigma, \tau$ be automorphisms of $R$. An additive mapping $f : R \to R$ is called a generalized $(\sigma, \tau)-$derivation if there exists a $(\sigma, \tau)-$derivation $d : R \to R$ such that $f(xy) = f(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$. In this paper, some well known results concerning generalized derivations of prime rings are extended to generalized $(\sigma, \tau)-$derivations.

## 1. Introduction.

Let $R$ will be an associative ring with center $Z$. For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$ and for a non-empty subset $S$ of $R$, we put $C_R(S) = \{x \in R \mid [x, s] = 0, \text{ for all } s \in S\}$. The set of all commutators of elements of $S$ will be written as $[S, S]$. Recall that a ring $R$ is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. An additive subgroup $U$ of $R$ is said to be a Lie ideal of $R$ if $[u, r] \in U$, for all $u \in U, r \in R$. Let $\sigma$ and $\tau$ be any two automorphisms of $R$. For any $x, y \in R$ we set $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. An additive mapping $d : R \to R$ is called a $(\sigma, \tau)-$derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$. Of course a $(1, 1)-$derivation where 1 is the identity map on $R$ is a derivation.

An additive function $f : R \to R$ is called a generalized inner derivation if $f(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping $f$, it is easy to see that

$$f(xy) = f(x)y + x[y, b] = f(x)y + xI_b(y) \text{ for all } x, y \in R.$$

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This observation leads to the following definition, given in [8]: An additive mapping $f : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$f(xy) = f(x)y + xd(y), \quad \text{for all } x, y \in R.$$

Familiar examples of generalized derivations are derivations and generalized inner derivations, and the later include left multipliers and right multipliers. Hence it should be interesting to extend some results concerning these notions to generalized derivations.

Inspired by the definition $(\sigma, \tau)-$derivation, the notion of generalized derivation was extended as follows: Let $\sigma, \tau$ two automorphisms of $R$. An additive mapping $f : R \to R$ is called a generalized $(\sigma, \tau)-$derivation on $R$ if there exists a $(\sigma, \tau)-$derivation $d : R \to R$ such that

$$f(xy) = f(x)\sigma(y) + \tau(x)d(y), \quad \text{for all } x, y \in R.$$

Of course a generalized $(1, 1)-$derivation is a generalized derivation on $R$, where 1 is the identity mapping on R.

Let $S$ be a nonempty subset of $R$. A mapping $F$ from $R$ to $R$ is said to be centralizing on $S$ if $[F(x), x] \in Z$ for all $x \in S$, in the special case when $[F(x), x] = 0$, the mapping $F$ is said to be commuting on $S$. There are several results in the existing literature dealing with centralizing and commuting mappings in rings. The study of centralizing mappings was initiated by E. C. Posner [12] which states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (Posner's second theorem). During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of $R$ (see [5] for a partial bibliography). In [10], N. Rehman showed that if $R$ is a prime ring with characteristic different from two and $(f, d)$ generalized derivation of $R$ such that $[f(u), u] = 0$ for all $u \in U$ where $U$ is a Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$ and $d \neq 0$, then $U \subset Z$. Also, he proved this theorem for prime ring $R$ even without the characteristic assumption on ring. Meanwhile in [1], N. Argaç and E. Albaş showed the same theorem for $(f, d)$ generalized derivation of prime ring $R$ with char$R \neq 2$, and they obtained that $f(x) = \lambda x$, for all $x \in R$, $\lambda \in C$ (extended centroid of $R$). Our primary purpose is to determine possible analogues of this result for generalized $(\sigma, \tau)-$derivation of $R$, and we also study for a nonzero Lie ideal $U$ of $R$.

On the other hand, in [6], H. E. Bell and L. C. Kappe have proved that $d$ is a derivation of $R$ which is either an homomorphism or anti-homo-

morphism in semi-prime ring $R$ or a nonzero right ideal of $R$ then $d = 0$. Latter on, several authors proved this result for a $(\sigma, \tau)-$derivation of $R$ or a Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$ (see [3], [11], [7], [2], [4]). Our second aim to prove this result for generalized $(\sigma, \tau)-$derivation of $R$ and also for a Lie ideal of $R$ without the condition $u^2 \in U$ for all $u \in U$.

Throughout the paper, $R$ will be a prime ring with right Martindale ring of quotients $Q_r(R)$, extended centroid $C$ and central closure $R_C = RC$. $U$ be a nonzero Lie ideal of $R$. We denote a generalized $(\sigma, \tau)-$derivation $f : R \to R$ determined by a $(\sigma, \tau)-$derivation $d$ of $R$ with $(f, d)$ and make some extensive use of the basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z$$
$$[xy, z] = [x, z]y + x[y, z]$$
$$[xy, z]_{\sigma, \tau} = x[y, z]_{\sigma, \tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma, \tau}y$$
$$[x, yz]_{\sigma, \tau} = \tau(y)[x, z]_{\sigma, \tau} + [x, y]_{\sigma, \tau}\sigma(z)$$

## 2. Preliminaries.

LEMMA 1 [9, Lemma 1.1].   *Let $R$ be a prime ring of characteristic not two, $U$ a nonzero Lie ideal of $R$ and $d$ a $(\sigma, \tau)-$derivation of $R$. If $d(U) = 0$ then $U \subset Z$.*

LEMMA 2 [9, Lemma 1.2].   *Let $R$ be a prime ring of characteristic not two, $U$ a nonzero Lie ideal of $R$ and $d$ a nonzero $(\sigma, \tau)-$derivation of $R$. If $t \in R$ such that $td(U) = 0$ (or $d(U)t = 0$) then $t = 0$ or $U \subset Z$.*

LEMMA 3 [8, Lemma 2].   *Let $f : R \to R_C$ be an additive map satisfying $f(xy) = f(x)y$, for all $x, y \in R$. Then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$ for all $x \in R$.*

LEMMA 4 [7, Theorem 1].   *Let $R$ be a prime ring. If $d$ is a $(\sigma, \tau)-$derivation of $R$ which is endomorphism on $R$ then $d = 0$.*

## 3. Results.

The following theorem gives a partial generalization of Posner's well known result [12, Lemma 3].

THEOREM 1.   *Let $(f, d)$ be a generalized $(\sigma, \tau)-derivation$ of prime ring $R$. If $f(x)y = xf(y)$, for all $x, y \in R$ and $d \neq 0$, then there exists $q \in Q_r(R_C)$ such that $f(x) = qx$ for all $x \in R$.*

PROOF.   Assume that

(3.1)                     $$f(x)y = xf(y), \text{ for all } x, y \in R.$$

Replacing $x$ by $zx$ in (3.1) and applying (3.1), we have

$$f(z)\sigma(x)y + \tau(z)d(x)y = zf(x)y$$

and so

$$(zf(\sigma(x)) - zf(x) + \tau(z)d(x))y = 0, \text{ for all } x, y, z \in R.$$

Since $R$ is prime, we get

(3.2)         $$z(f(\sigma(x)) - f(x)) = -\tau(z)d(x), \text{ for all } x, z \in R.$$

Substituting $yz$ for $z$ in (3.2) and using (3.2), we get

(3.3)             $$(y - \tau(y))\tau(z)d(x) = 0, \text{ for all } x, y, z \in R.$$

By the primeness of $R$, we have $d = 0$ or $y = \tau(y)$ for all $y \in R$.

Since $d \neq 0$, we get $y = \tau(y)$ for all $y \in R$. Now replacing $y$ by $yz$ in (3.1) and using $\tau(z) = z$, we have

$$f(x)yz = xf(yz).$$

Using the hyphothesis, we arrive at

$$xf(y)z = xf(y)\sigma(z) + xyd(z)$$
$$xyf(z) = xf(y)\sigma(z) + xyd(z)$$

and so

$$x(yf(z) - f(y)\sigma(z) - yd(z)) = 0, \text{ for all } x, y, z \in R.$$

Again the primenessly of $R$, we have

$$yf(z) = f(y)\sigma(z) + yd(z) = f(yz), \text{ for all } y, z \in R.$$

By the hypothesis we can write the above

(3.4)             $$f(yz) = yf(z) = f(y)z, \text{ for all } y, z \in R.$$

Hence we obtain that there exists $q \in Q_r(R_C)$ such that $f(x) = qx$ for all $x \in R$, by Lemma 3 for any cases.                                    □

THEOREM 2.  *Let $(f, d)$ be a generalized $(\sigma, \tau)-$derivation of prime ring. If $(f, d)$ acts as a homomorphism on $R$, then $d = 0$.*

PROOF.    Assume that $f$ acts as a homomorphism on $R$. Then one obtains

(3.5)       $f(xy) = f(x)\sigma(y) + \tau(x)d(y) = f(x)f(y)$, for all $x, y \in R$.

Replacing $y$ by $yr$ in (3.5), we have $f(x)f(yr) = f(x)\sigma(yr) + \tau(x)d(yr)$ for all $x, y, r \in R$. Since $(f, d)$ is a generalized $(\sigma, \tau)-$derivation and $d$ is a $(\sigma, \tau)-$derivation, we obtain that, for all $x, y, r \in R$

(3.6)   $f(x)f(y)\sigma(r) + f(x)\tau(y)d(r) = f(x)\sigma(y)\sigma(r) + \tau(x)d(y)\sigma(r) + \tau(x)\tau(y)d(r).$

Using (3.5) in the last equation gives us

$$(\tau(x) - f(x))\tau(y)d(r) = 0, \text{ for all } x, y, r \in R.$$

Thus either $d = 0$ or $f = \tau$ by the primeness of $R$. Now we assume that $f = \tau$. Using this in (3.6), we have

$$\tau(x)\tau(y)\sigma(r) + \tau(x)\tau(y)d(r) = \tau(x)\sigma(y)\sigma(r) + \tau(x)d(y)\sigma(r) + \tau(x)\tau(y)d(r)$$

and so

$$\tau(x)(\tau(y) - \sigma(y) - d(y))\sigma(r) = 0, \text{ for all } x, y, r \in R.$$

Since $R$ is prime ring, we arrive at $\tau = \sigma + d$, for all $y \in R$. Thus $d = \tau - \sigma$, therefore $\tau - \sigma$ is an automorphism of $R$, we get $d$ acts as a homomorphism on $R$. Hence we obtain that $d = 0$ by Lemma 4.                    □

THEOREM 3.  *Let $(f, d)$ be a generalized $(\sigma, \tau)-$derivation of prime ring $R$. If $(f, d)$ acts as an anti-homomorphism on $R$, then $d = 0$.*

PROOF.    Assume that $f$ acts as an anti-homomorphism on $R$. If $f(R) \subset Z$, then there is nothing to prove by Theorem 2. Hence we may assume that $f(R) \nsubseteq Z$.
Now we have

(3.7)       $f(xy) = f(x)\sigma(y) + \tau(x)d(y) = f(y)f(x)$, for all $x, y \in R$.

Replacing $x$ by $xy$ in (3.7), we obtain $f(y)f(xy) = f(xy)\sigma(y) + \tau(xy)d(y)$ for all $x, y \in R$. Using the hyphothesis and $(f, d)$ is a generalized $(\sigma, \tau)-$derivation of $R$ in the last relation gives

(3.8)               $\tau(xy)d(y) = f(y)\tau(x)d(y)$, for all $x, y \in R$.

Taking $rx$ instead of $x$ in (3.8) and using (3.8), we arrive at

$$[\tau(r), f(y)]\tau(x)d(y) = 0, \text{ for all } x, y, r \in R.$$

By the primeness of $R$, we obtain that $f(y) \in Z$ or $d(y) = 0$, for each $y \in R$. We set $K = \{y \in R \mid f(y) \in Z\}$ and $L = \{y \in R \mid d(y) = 0\}$. Clearly each of $K$ and $L$ is additive subgroup of $R$. Morever, $R$ is the set-theoretic union of $K$ and $L$. But a group can not be the set-theoretic union of two proper subgroups, hence $K = R$ or $L = R$. If $K = R$, it contradicts $f(R) \not\subseteq Z$. So we have $L = R$. Hence $d = 0$. The proof is completed. $\qquad\square$

## 4. Some Other Results.

We now propose to extend some of these results of Section 3 on $U$ a nonzero Lie ideal of prime ring $R$. Parallel results are be obtained using the same tecniques. However, we couldn't showed that $(f, d)$ (or $d$) acts as an anti-homomorphism on $U$ without the condition $u^2 \in U$, for all $u \in U$.

THEOREM 4. *Let $d$ be a $(\sigma, \tau)$–derivation of prime ring $R$ with $charR \neq 2$ and $U$ be a nonzero Lie ideal of $R$. If $d$ acts as a homomorphism on $U$, then $d = 0$ or $U \subset Z$.*

PROOF. By the hyphothesis, we have

(4.1) $\qquad d(uv) = d(u)d(v) = d(u)\sigma(v) + \tau(u)d(v), \text{ for all } u, v \in U.$

Substituing $u[x, u], x \in R$ for $u$ in (4.1), we get

$$d(u[x, u])d(v) = d(u[x, u])\sigma(v) + \tau(u[x, u])d(v)$$
$$d(u)d([x, u])d(v) = d(u)d([x, u])\sigma(v) + \tau(u[x, u])d(v)$$
$$d(u)d([x, u]v) = d(u)d([x, u])\sigma(v) + \tau(u[x, u])d(v)$$
$$d(u)d([x, u])\sigma(v) + d(u)\tau([x, u])d(v) = d(u)d([x, u])\sigma(v) + \tau(u[x, u])d(v)$$

and so

(4.2) $\qquad (d(u) - \tau(u))\tau([x, u])d(v) = 0, \text{ for all } u, v \in U, x \in R.$

By Lemma 2, we obtain that

$\qquad d = 0$ or $(d(u) - \tau(u))\tau([x, u]) = 0, \text{ for all } u \in U, x \in R$ or $U \subset Z.$

Now assume that $(d(u) - \tau(u))\tau([x, u]) = 0$, for all $u \in U, x \in R$. Replacing $x$ by $xy$ in the last relation and using this gives

$\qquad (d(u) - \tau(u))\tau(x)\tau([y, u]) = 0, \text{ for all } u \in U, x, y \in R.$

By the primeness of $R$, we obtain that $d(u) = \tau(u)$ or $u \in Z$, for each $u \in U$. We set $K = \{u \in U \mid d(u) = \tau(u)\}$ and $L = \{u \in U \mid u \in Z\}$. Clearly each

of $K$ and $L$ is additive subgroup of $U$. By a standart arguments one of these must hold for $u \in U$. In the first case, if $U = K$, then we have $d(u) = \tau(u)$, for all $u \in U$. Hence $d([v, u]) = \tau([v, u])$, for all $u, v \in U$. That is

$$d(vu) - d(uv) = \tau(vu) - \tau(uv)$$
$$d(v)d(u) - d(uv) = \tau(v)\tau(u) - \tau(u)\tau(v),$$

Using the $d(u) = \tau(u)$, for all $u \in U$ in the last equation, we have

$$\tau(v)\tau(u) - d(uv) = \tau(v)\tau(u) - \tau(u)d(v),$$

and so,

$$d(uv) = \tau(u)d(v)$$
$$d(u)\sigma(v) + \tau(u)d(v) = \tau(u)d(v)$$

and so,

$$d(u)\sigma(v) = 0, \text{ for all } u, v \in U.$$

Again appliying Lemma 2, we obtain that $d = 0$ or $U \subset Z$. This completes the proof. $\qquad\square$

LEMMA 5. *Let $R$ be a prime ring and $U$ a nonzero Lie ideal of $R$. If $a \in R$ such that $Ua = (0)$ (or $aU = (0)$), then $a = 0$.*

PROOF. Assume that $Ua = (0)$. Then we get $[u, x]a = 0$, for all $u \in U, x \in R$. That is $uxa - xua = 0$, and so

$$URa = 0$$

Since $R$ is prime ring and $U \neq 0$, we have $a = 0$. $\qquad\square$

THEOREM 5. *Let $(f, d)$ be a generalized $(\sigma, \tau)-$derivation of prime ring $R$ with char$R \neq 2$ and $U$ be a nonzero Lie ideal of $R$. If $(f, d)$ acts as a homomorphism on $U$, then $d = 0$ or $U \subset Z$.*

PROOF. Using the same methods in the begining of Theorem 4, we arrive at

(4.3) $\qquad (f(u) - \tau(u))\tau([x, u])d(v) = 0, \text{ for all } u, v \in U, x \in R.$

By Lemma 2, we obtain that

$$d = 0 \text{ or } (f(u) - \tau(u))\tau([x, u]) = 0, \text{ for all } u \in U, x \in R \text{ or } U \subset Z.$$

Assume that $(f(u) - \tau(u))\tau([x, u]) = 0$, for all $u \in U, x \in R$. Again using the same tecniques the above theorem, we obtain that

$$f(u) = \tau(u) \text{ or } u \in Z, \text{ for each } u \in U.$$

By a standart arguments one of these must hold for $u \in U$. In the first case, we have $f(u) = \tau(u)$, for all $u \in U$, and so

$$f(uv) = f(u)f(v) = f(u)\sigma(v) + \tau(u)d(v)$$

$$\tau(u)\tau(v) = \tau(u)\sigma(v) + \tau(u)d(v),$$

and so,

$$u\tau^{-1}(\tau(v) - \sigma(v) - d(v)) = 0, \text{ for all } u, v \in U.$$

Hence we obtain that $d(v) = (\tau - \sigma)(v)$, for all $v \in U$ or $U \subset Z$ by Lemma 5. If $d(v) = (\tau - \sigma)(v)$, for all $v \in U$, then we get $d$ acts as a homomorphism on $U$ because of $d = \tau - \sigma$ is an automorphism of $R$. Hence we obtain that $d = 0$ by Theorem 4.      □

THEOREM 6.    *Let $(f, d)$ and $(g, h)$ be two generalized $(\sigma, \tau)-$derivations of prime ring $R$ with charR $\neq 2$ and $U$ be a nonzero Lie ideal of $R$. If $f(u)\sigma(v) = \tau(u)g(v)$, for all $u, v \in U$, then $U \subset Z$.*

PROOF.    Assume that

(4.4)                  $$f(u)\sigma(v) = \tau(u)g(v), \text{ for all } u, v \in U.$$

Replacing $v$ by $[x, v]v$ in (4.4) and applying (4.4), we have

$$U[x, v]\tau^{-1}(h(v)) = 0, \text{ for all } v \in U, x \in R.$$

By Lemma 5, we obtain that

$$[x, v]\tau^{-1}(h(v)) = 0, \text{ for all } v \in U, x \in R.$$

Taking $xy$ instead of $x$ in this relation, we arrive at

$$[x, v]R\tau^{-1}(h(v)) = 0, \text{ for all } v \in U, x \in R.$$

By the primeness of $R$, we obtain that $v \in Z$ or $h(v) = 0$, for each $v \in U$. We set $K = \{v \in U \mid v \in Z\}$ and $L = \{v \in U \mid h(v) = 0\}$. Clearly each of $K$ and $L$ is additive subgroup of $U$. Using the standart argument, we have $K = U$ or $L = U$. If $L = U$, then $U \subset Z$ by Lemma 1. Hence we find that $U \subset Z$ for any cases.      □

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