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Dihedral F-Tilings of the Sphere by Equilateral and Scalene Triangles-I

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ABSTRACT - The study of dihedral f-tilings of the Euclidean sphere S^2 by triangles and r-sided regular polygons was initiated in 2004 where the case $r = 4$ was considered [6]. In a subsequent paper [1], the study of all spherical f-tilings by triangles and r-sided regular polygons, for any $r \geq 5$, was described. Later on, in [3] the classification of all f-tilings of S^2 whose prototiles are an equilateral triangle (a 3-sided regular polygon) and an isosceles triangle are obtained. In this paper we extend these results presenting the study of all spherical f-tilings by equilateral triangles and scalene triangles in a particular way of adjacency. For this particular case of adjacency, we get two continuous families of f-tilings denoted by $\mathcal{F}_1^{\delta}$ and $\mathcal{F}_2^{\delta}$. The combinatorial structure including the transitive properties under the action of their symmetry groups is given in Table 1.

1. Introduction

Spherical folding tilings or *f-tilings* for short, are edge-to-edge decompositions of the sphere by geodesic polygons, such that all vertices are of even valency and the sums of alternate angles around each vertex are π, i.e., each vertex is of even valency $2n$, $n \geq 2$ and the sums of alternate

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angles are equal, that is

$$\sum_{i=1}^{n} \alpha_{2i} = \sum_{i=1}^{n} \alpha_{2i-1} = \pi,$$

where the angles α_i around any vertex are ordered cyclically. A f-tiling τ is said to be *monohedral* if it is composed by congruent cells, and *dihedral* if every tile of τ is congruent to one of two fixed sets X and Y (prototiles of τ). We shall denote by $\Omega(X, Y)$ the set, up to isomorphism, of all dihedral f-tilings of S^2 whose prototiles are X and Y.

Spherical f-tilings are strongly related to the theory of isometric foldings [9]. The study of this special class of tilings was initiated in [2] with a complete classification of all spherical monohedral f-tilings. The classification of all spherical dihedral folding tilings by rhombi and triangles was recently obtained in 2005, [7].

Here, our interest is focused in spherical triangular dihedral f-tilings whose prototiles are an equilateral triangle and a scalene triangle, in a particular way of adjacency (see I in Figure 1). We end up with two continuous families of f-tilings, illustrated in Figures 18 and 19. The other two cases of adjacency were studied in [4] and [5]. With this study the classification of the spherical f-tilings by two triangular tiles one of which equilateral is now completed.

From now on T_1 denotes an equilateral spherical triangle of angle α and side a and T_2 a scalene spherical triangle of angles δ, γ, β, with the order relation $\delta < \gamma < \beta$ and with sides b (opposite to β), c (opposite to γ) and d (opposite to δ).

We begin by pointing out that any element of $\Omega(T_1, T_2)$ has at least two cells congruent, respectively, to T_1 and T_2, such that they are in adjacent positions in one and only one of the situations illustrated in Figure 1.

The type I edge-adjacency condition can be analytically described by the equation (see [8])

$$\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{\cos\beta + \cos\delta\cos\gamma}{\sin\delta\sin\gamma}. \tag{1.1}$$

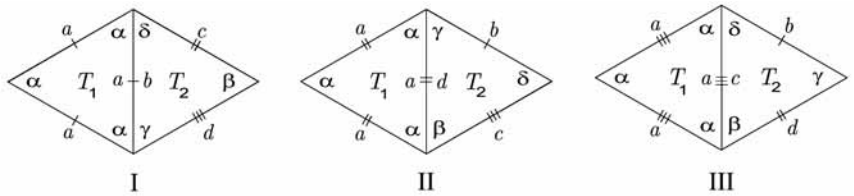

Fig. 1. – Distinct cases of adjacency.

In order to get any dihedral f-tiling $\tau \in \Omega(T_1, T_2)$, we find useful to start by considering one of its *planar representations*, beginning with a common vertex to an equilateral triangle and a scalene triangle in adjacent positions. In the diagrams that follows it is convenient to label the tiles according to the following procedures:

(i) The tiles by which we begin the planar representation of a tiling $\tau \in \Omega(T_1, T_2)$ are labelled by 1 and 2, respectively;

(ii) For $j \geq 2$, the location of tile j can be deduced from the configuration of tiles $(1, 2, \ldots, j-1)$ and from the hypothesis that the configuration is part of a complete planar representation of a f-tiling (except in the cases indicated).

2. Triangular Dihedral F-Tilings with Adjacency of Type I.

Let T_1 and T_2 be tiles with adjacency of type I.

Starting a planar representation of a such tiling $\tau \in \Omega(T_1, T_2)$ we are led to the one illustrated in Figure 2, where $x \in \{\beta, \gamma\}$.

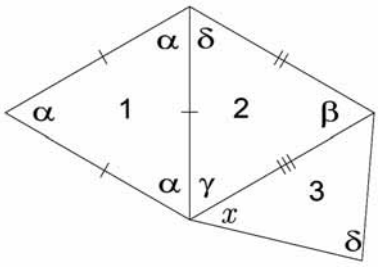

Fig. 2. – Planar representation.

The cases $x = \beta$ and $x = \gamma$ will be studied separately.

Let us begin by assuming that $x = \beta$. Accordingly, $\alpha + \beta \leq \pi$.

PROPOSITION 2.1. *If T_1 and T_2 are tiles with adjacency of type I and $\alpha + x \leq \pi$, with $x = \beta$, then $\Omega(T_1, T_2) = \varnothing$.*

PROOF. Suppose that $\alpha + x = \pi$, with $x = \beta$. The configuration of τ illustrated in Figure 2 expands to the following one (Figure 3).

Since $\alpha + \beta = \gamma + \delta = \pi$ and $\beta > \gamma$, then $\alpha < \delta$. Therefore, $\alpha < \delta < \frac{\pi}{2} < \gamma < \beta$. Accordingly, $\cos\alpha, \cos\delta > 0$ while $\cos\beta, \cos\gamma < 0$ contradicting the adjacency condition 1.1.

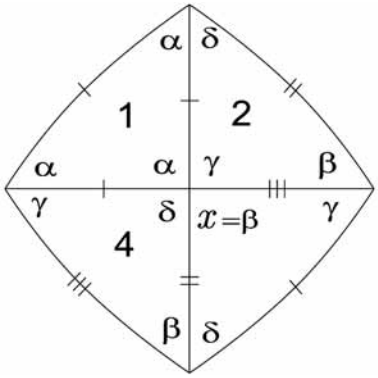

Fig. 3. – Planar representation.

If $\alpha+\beta<\pi$ the edge compatibility around a vertex surrounded by a sequence of angles of type $(\star, \star, \ldots, \alpha, \star, \beta, \star, \ldots)$ implies that all the angles, in this sequence, different from α and β must have amplitude δ. Consequently, $\alpha+\beta+k\delta=\pi$, for some integer $k \geq 1$.

The configuration shown in Figure 2 can be expanded to the one illustrated in Figure 4.

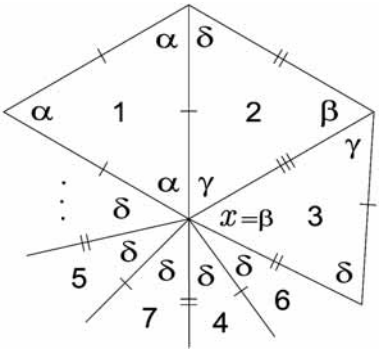

Fig. 4. – Planar representation.

In this situation we end up with the equation $\alpha+\beta+k\delta=\pi=\gamma+(k+1)\delta$ from which we derive the following condition $\alpha+\beta=\gamma+\delta$ and consequently $\frac{\pi}{3}<\alpha<\delta<\gamma<\beta$, contradicting $\alpha+\beta+k\delta=\pi$, for some integer $k \geq 1$. □

Let us assume now that $x=\gamma$, see Figure 2.

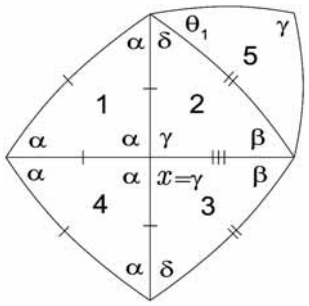

Fig. 5. – Planar representation.

PROPOSITION 2.2. *If T_1 and T_2 are tiles with adjacency of type I and $\alpha+\gamma \leq \pi$, then $\Omega(T_1,T_2) \neq \varnothing$ if and only if $\beta = \frac{\pi}{2}$ and $\alpha+\gamma+\delta = \pi$ ending up with two non isomorphic dihedral f-tilings.*

PROOF. **A**) If $\alpha+\gamma=\pi$, the configuration in Figure 2 expands to the one illustrated in Figure 5.

A decision about the angle $\theta_1 \in \{\delta,\beta\}$ must be taken. Having in account that $\pi = \alpha+\gamma < \alpha+\beta$, then $\theta_1 = \delta$ (Figure 6).

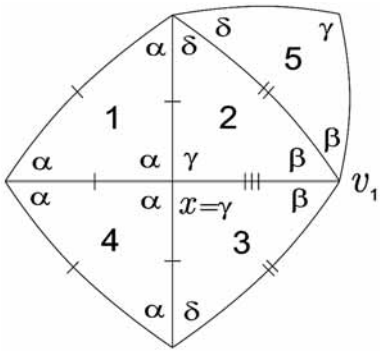

Fig. 6. – Planar representation.

One of the alternate angle sum at vertex v_1 is $2\beta \leq \pi$. If $\beta < \frac{\pi}{2}$ one has $\delta < \gamma < \frac{\pi}{2}$ and $\alpha > \frac{\pi}{2}$, violating the adjacency condition 1.1. Accordingly $\beta = \frac{\pi}{2}$.

B) Consider now that $\alpha+\gamma<\pi$. The angle θ_2 belongs to the set $\{\delta,\beta\}$, see Figure 7.

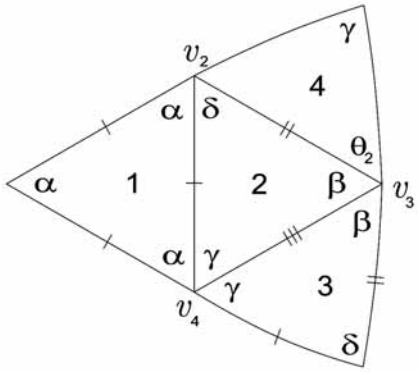

Fig. 7. – Planar representation.

1. *Suppose, firstly, that* $\theta_2 = \delta$.

The alternate angle sum at vertex v_2 containing α and β satisfies $\alpha+\beta<\pi$, otherwise it would become $\alpha+\beta=\pi=\delta+\gamma$. Therefore, we would have $\alpha<\delta<\frac{\pi}{2}<\gamma<\beta$, not satisfying the adjacency condition 1.1. As $\alpha+\beta<\pi$ and having in account that $\alpha,\beta>\frac{\pi}{3}$ and $\alpha+\beta+\gamma>\pi$ then $\alpha+\beta+k\delta=\pi$, for some $k\geq 1$ (vertex v_2). Now the alternate angle sum containing β and δ at vertex v_3 satisfies $\beta+\delta<\pi$. Summarizing, since $\beta+\gamma+\delta>\pi$ and $\alpha+\beta+k\delta=\pi, k\geq 1$, then $\delta<\frac{\pi}{3}<\alpha<\gamma<\beta$.

Consequently, the alternate angle sum, at vertex v_4, containing α and γ is $\alpha+\gamma+t\delta=\pi$, for some $t>k\geq 1$. The configuration extends to the one in Figure 8.

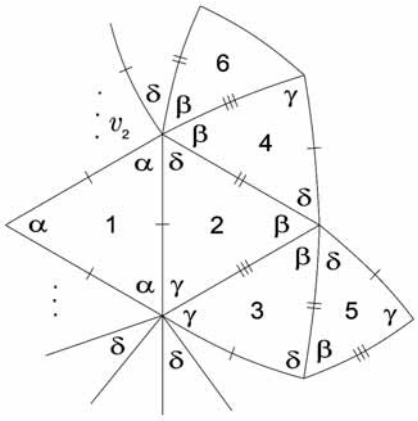

Fig. 8. – Planar representation.

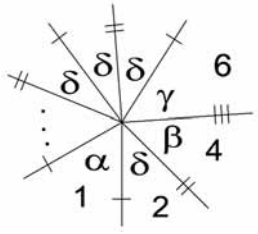

Fig. 9. – Angles arrangement around vertex v_2.

Observe that in tile numbered 6, if the angle β was γ, the vertex v_2 would be surrounded by the sequence of angles illustrated in Figure 9. The alternate angle sums would be $\alpha+\beta+k\delta=\pi=\delta+\gamma+k\delta$. Therefore, $\alpha+\beta=\delta+\gamma$, which is absurd since $\delta<\alpha<\gamma<\beta$.

We may add some new cells to the configuration in Figure 8 and we get the one illustrated in Figure 10.

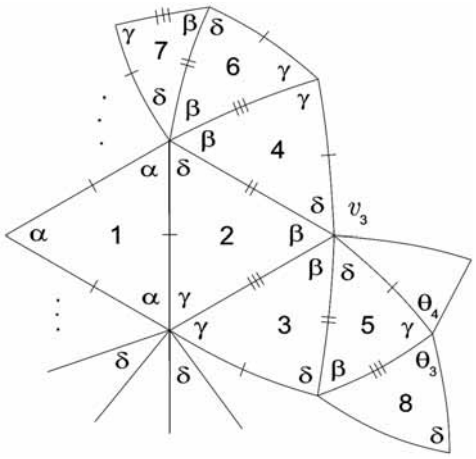

Fig. 10. – Planar representation.

1.1. If $\theta_3=\beta$, then $\theta_4=\gamma$. Observe that $\theta_4=\alpha$ leads to the cyclic sequence $(\alpha,\gamma,\beta,\dots)$, which is absurd (case already studied in the previous proposition). Thus, the alternate angle sums at vertex v_3 are $\beta+r\delta=\pi$, for some $r\geq 1$. Consequently, at vertex v_5 (Figure 11) we have $\beta+\gamma=\pi=2\gamma$ getting again a contradiction, since $\gamma<\beta$.

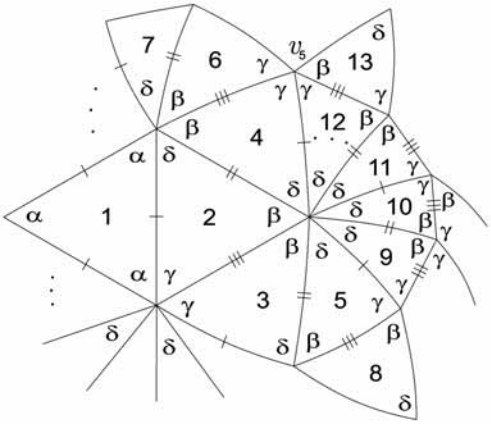

Fig. 11. – Planar representation.

1.2 If $\theta_3 = \gamma$, the configuration in Figure 10 gives rise to the one in Figure 12.

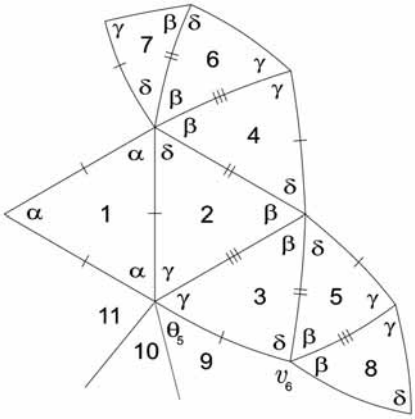

Fig. 12. – Planar representation.

Now, θ_5 is different from δ, otherwise, at vertex v_6 we get the cyclic sequence $(\gamma, \delta, \beta, \beta)$ and the alternate angle sums would be $\beta + \gamma = \pi = \beta + \delta$, which is an absurd. Due to the compatibility of sides in tiles 10 and 11, θ_5 must be different from γ. Therefore, $\theta_5 = \alpha$ and extending the previous configuration, we conclude that $\beta = \dfrac{\pi}{2}$, see Figure 13.

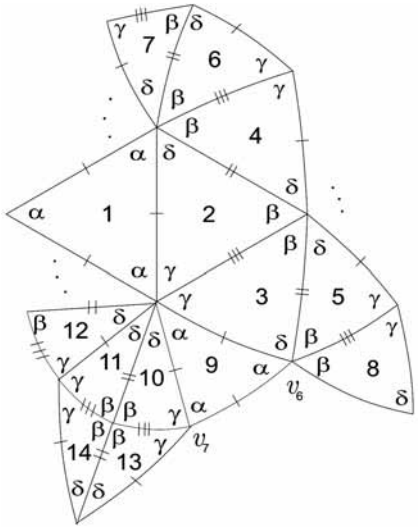

Fig. 13. – Planar representation.

Consequently $\gamma+\delta>\frac{\pi}{2}$ and $\gamma<\frac{\pi}{2}$. Accordingly, $\delta<\frac{\pi}{3}<\alpha<\gamma<\frac{\pi}{2}=\beta$ and all the alternate angle sums containing two angles γ are $2\gamma+\delta=\pi$. Having in account that, $\beta=\frac{\pi}{2}, 2\gamma+\delta=\pi, \alpha+\gamma+t\delta=\pi$ and $\alpha+\beta+k\delta=\pi$, one has $\alpha+k\delta=\frac{\pi}{2}$ and $(1-2t-2k)\delta=0$, which is absurd since $t>k\geq 1$.

2. Suppose now that $\theta_2=\beta$ (see Figure 7). Then, $\beta=\frac{\pi}{2}$ and $\alpha<\frac{\pi}{2}$ $\left(\text{otherwise the adjacency condition 1.1 is violated, since } \delta<\gamma<\frac{\pi}{2}\right)$. Also

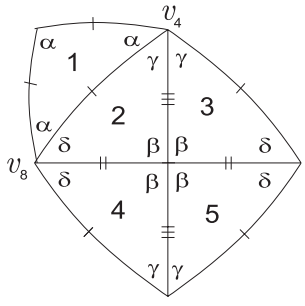

Fig. 14. – Planar representation.

$\delta < \alpha$, otherwise $\delta > \frac{\pi}{3}$ and $\alpha + \delta + \rho > \pi$, for any $\rho \in \{\alpha, \delta, \gamma, \beta\}$. Looking at vertex v_4 in Figure 7, the alternate angle sum containing α and γ is $\alpha + \gamma + \alpha = \pi$ or $\alpha + \gamma + \gamma = \pi$ or $\alpha + \gamma + t\delta = \pi$, for some $t \geq 1$, see Figure 14.

2.1 If $\alpha + \gamma + \alpha = \pi$, then $\frac{\pi}{6} < \delta < \gamma < \frac{\pi}{3} < \alpha$. At vertex v_8, the alternate angle sum containing α and δ must be

$$2\alpha + \delta = \pi \quad \text{or} \quad \alpha + \delta + \gamma = \pi \quad \text{or} \quad \alpha + 2\delta + \gamma = \pi \quad \text{or} \quad \alpha + 3\delta = \pi.$$

Clearly, the first two cases are impossible. Therefore, $\alpha + 2\delta + \gamma = \pi$ or $\alpha + 3\delta = \pi$. As $\delta + \gamma > \frac{\pi}{2}$ and $\alpha > \gamma$, then $\alpha + 2\delta + \gamma > \pi$ and so $\alpha + 3\delta = \pi$, that is $\delta < \frac{2\pi}{9}$. The angular sequence around vertex v_8 of valency 8, is of the form $(\alpha, \alpha, \delta, \ldots, \delta)$ or $(\alpha, \delta, \delta, \alpha, \delta, \ldots)$.

Either way, in the expanded configuration, having in account the edge compatibility we get a vertex surrounded by an angular sequence containing four consecutive γ angles, as illustrated in Figure 15. As $\gamma + \delta > \frac{\pi}{2}$ and $\delta < \gamma < \alpha$ then, $2\gamma + \delta < 2\gamma + \gamma < \pi$, $2\gamma + 2\gamma > 2\gamma + 2\delta > \pi$, $2\gamma + \alpha < \gamma + 2\alpha = \pi$ and $3\gamma + \alpha > 3\gamma + \gamma > \pi$. Consequently it is impossible to continue extending the configuration.

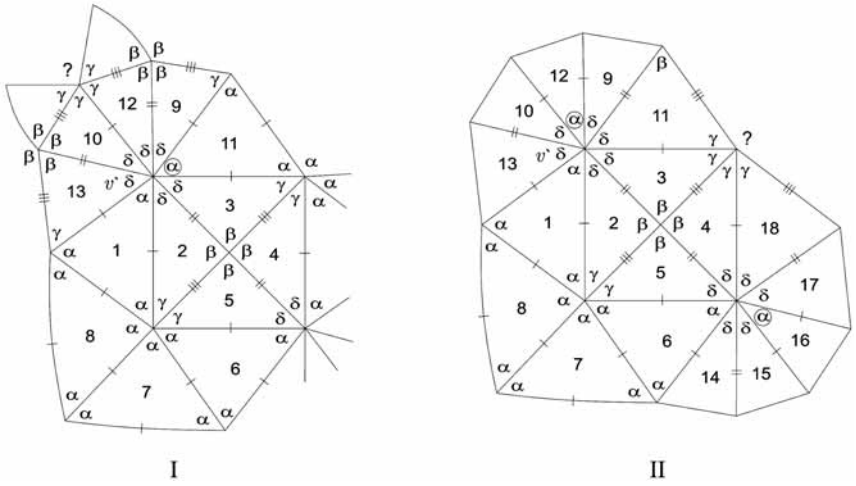

Fig. 15. – Planar representations.

2.2 If $\alpha + 2\gamma = \pi$, the angular order relation is exactly the same,

$$\frac{\pi}{6} < \delta < \gamma < \frac{\pi}{3} < \alpha.$$

The alternate angle sum at vertex v_8 containing α and δ must be

$$\alpha + \delta + \gamma = \pi \quad \text{or} \quad \alpha + 2\delta + \gamma = \pi \quad \text{or} \quad \alpha + 3\delta = \pi, \quad \text{or} \quad 2\alpha + \delta = \pi.$$

The first case is incompatible with the assumption $\alpha + 2\gamma = \pi$.

As $\gamma + \delta > \frac{\pi}{2}$ and $\alpha > \gamma$, then $\alpha + 2\delta + \gamma > \pi$ and the second case must be eliminated. The third case should be discarded, using a similar reasoning to the one made in case 2.1 under the same assumption. Suppose now that $2\alpha + \delta = \pi$. By the adjacency condition we get $\gamma \approx 56.489^\circ$, $\alpha \approx 67.022^\circ$ and $\delta \approx 45.956^\circ$.

Starting from configuration in Figure 14 we are led, according to the edge position of tile 6, to one of the configurations illustrated in Figure 16.

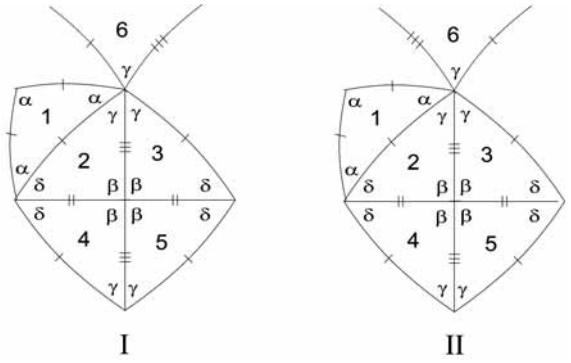

Fig. 16. – Planar representations.

Taking as starting point the configuration I, in Figure 16 we get the one in Figure 17-I, which is impossible to be expanded, since the alternate angle sums at vertex v_9 do not satisfy the angle folding relation (δ is not a submultiple of π and there are no other way of combining δ with anyone of the other angles).

The other possible position for tile 6 gives rise to configuration II. However, the alternate angle sum containing α, γ and δ at vertex v_{10} do not satisfy the angle folding relation and so it is impossible to expand the configuration.

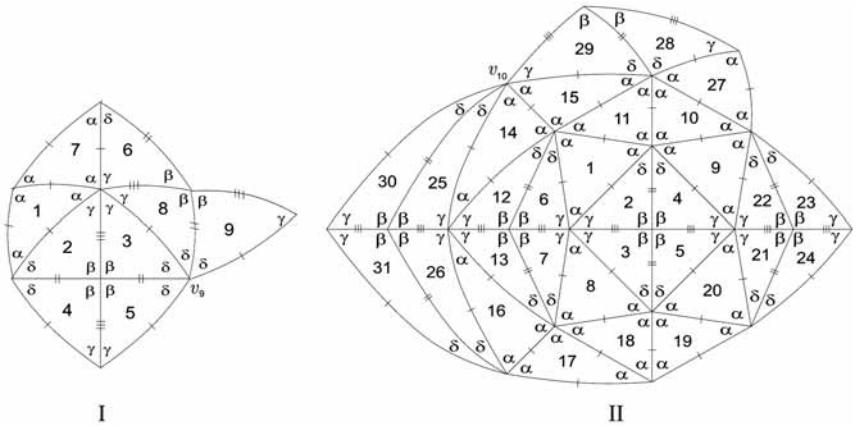

Fig. 17. – Planar representations.

2.3 Assume now that $\alpha + \gamma + t\delta = \pi, t \geq 1$. The discussion of the case $t > 2$ follows a similar reasoning to the one done for $t = 2$ and so we will focus our attention in the cases $t = 1$ and $t = 2$.

2.3.1 If $t = 1$, the configuration illustrated in Figure 18, having as a starting point the one given in Figure 14 corresponds to a particular choice of the position of tiles 6, 11 and 21. It gives rise to a tiling $\tau \in \Omega(T_1, T_2)$ composed by 24 scalene triangles and 8 equilateral triangles and will be denoted by $\mathcal{F}_1^{\delta}$.

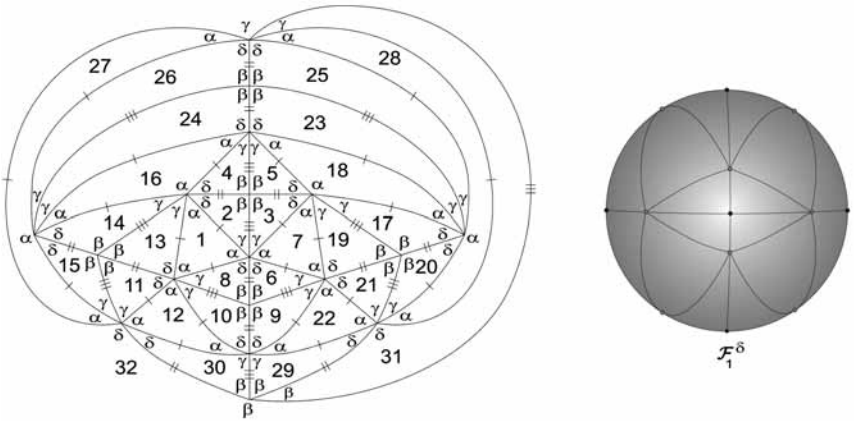

Fig. 18. – 2D and 3D representation of $\mathcal{F}_1^{\delta}$.

The other possible arrangement for the length sides of tile 21 leads us to a global configuration of a tiling $\tau \in \Omega(T_1, T_2)$. This f-tiling is composed by 24 scalene triangles and 8 equilateral triangles and is denoted by $\mathcal{F}_2^{\delta}$, Figure 19.

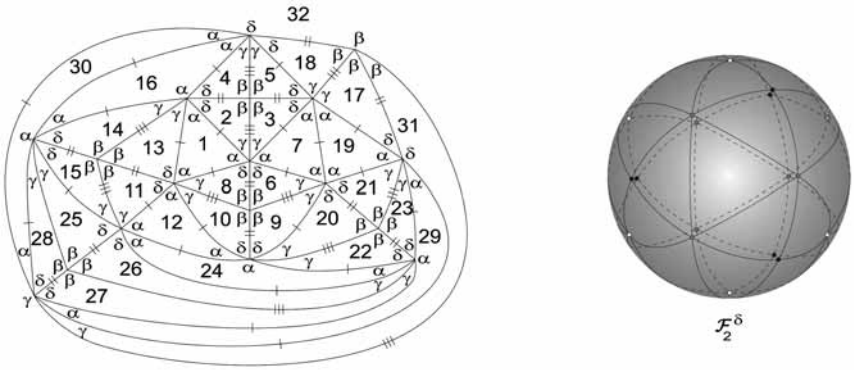

Fig. 19. – 2D and 3D representation of $\mathcal{F}_2^{\delta}$.

The other position of tile 11 leads to the configuration shown in Figure 20.

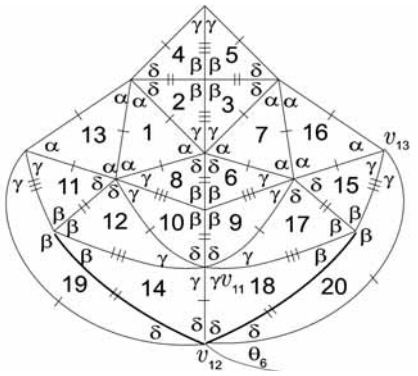

Fig. 20. – Planar representation.

Looking at vertex v_{11}, we conclude that $\gamma + \delta + \gamma = \pi$ (as $2\gamma + \delta + \rho > \pi$, for any $\rho \in \{\alpha, \gamma, \delta, \beta\}$) and so $\alpha = \gamma$. At vertex v_{12}, the angle $\theta_6 \in \{\delta, \gamma, \alpha\}$.

2.3.1.1 If $\theta_6 = \delta$, then at vertex v_{13}, we obtain the alternate angle sum $2\gamma + \alpha \leq \pi$, which is an absurd.

2.3.1.2 If $\theta_6 = \gamma$, v_{12} is a vertex of valency greater than 6, otherwise $\alpha = \gamma = \delta$. Consequently, one of the alternate angle sums is $\gamma + q\delta = \pi$, $q \geq 3$. Since $2\gamma + \delta = \pi$ and $\gamma + q\delta = \pi$ one has $\delta = \dfrac{\pi}{2q-1}$ $\alpha = \gamma = \dfrac{(q-1)\pi}{2q-1}$. By the adjacency condition 1.1, we get $\dfrac{1+\cos((q-1)\delta)}{\sin((q-1)\delta)} = \dfrac{\cos\delta}{\sin\delta}$, that is, $\sin\delta = \sin((q-2)\delta)$. Therefore, $q = 3$ and as a result $\delta = \dfrac{\pi}{5}$, $\alpha = \gamma = \dfrac{2\pi}{5}$.

However, extending configuration in Figure 20, we are led to an absurd, since the vertex surrounded by the angular sequence $(\gamma, \gamma, \gamma, \gamma, \alpha)$ violates the angle folding relation, see below (Figure 21).

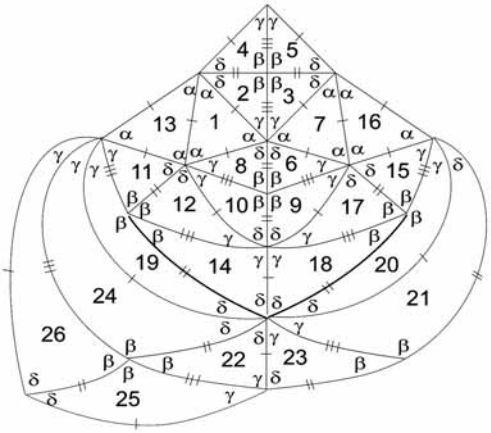

Fig. 21. – Planar representation.

2.3.1.3 If $\theta_6 = \alpha$, since $\alpha = \gamma$, we get exactly the same absurd.

The other possibility for the length sides of tile 6 permits to extend a bit more the configuration in Figure 14 and a decision on $\theta_7 \in \{\delta, \alpha\}$ must be taken (see Figure 22).

If $\theta_7 = \delta$, the configuration is the one illustrated in Figure 23-I. This configuration expands globally and we get a representation of the tiling $\mathcal{F}_2^{\delta}$.

If $\theta_7 = \alpha$, we get the configuration II illustrated in Figure 23. Observe that tile 16 obliges that one of the alternate angle sums at vertex v_{14} is $2\alpha + \delta = \pi$. Adding some new cells to this configuration we get a vertex

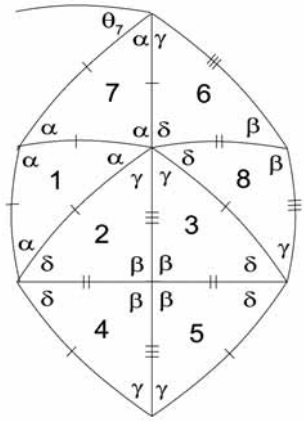

Fig. 22. – Planar representation.

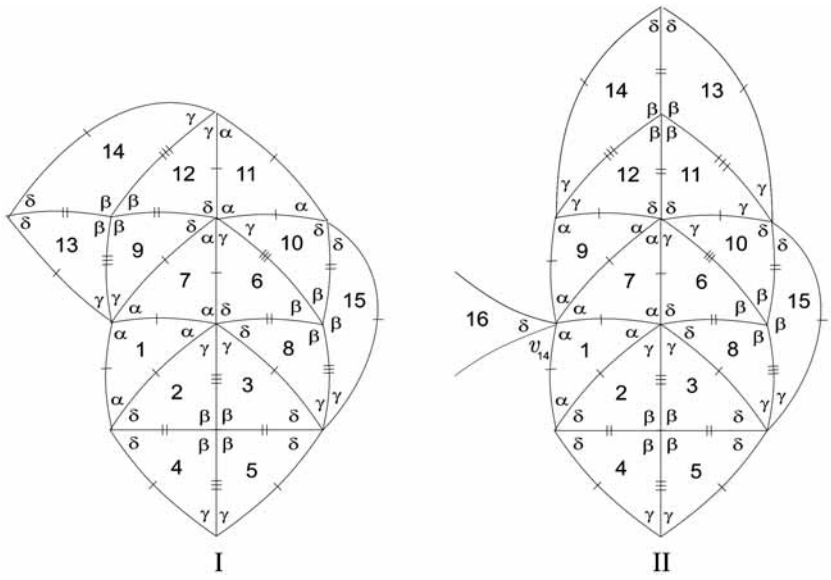

Fig. 23. – Planar representations.

surrounded by alternate angles $(\gamma, \gamma, \alpha, \gamma, \gamma)$, which is absurd (see Figure 24). Suppose now, that $\alpha + \gamma + t\delta = \pi$, for $t > 1$. Having in account that $\gamma + \delta > \frac{\pi}{2}$, then $\gamma > \alpha + (t-2)\delta \geq \alpha$.

Summarizing, $\delta < \frac{\pi}{6} < \frac{\pi}{3} < \alpha < \gamma < \beta = \frac{\pi}{2}$.

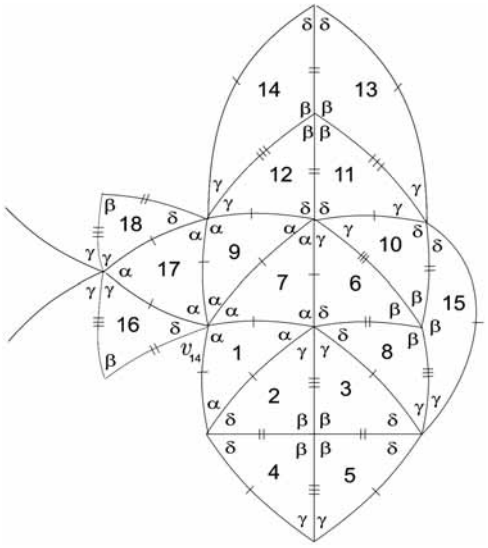

Fig. 24. – Planar representation.

2.3.2 Suppose $t = 2$.

2.3.2.1 The configuration in Figure 25 corresponds to a choice for the positions of tiles 6 and 19, with $\theta_8 \in \{\delta, \gamma, \alpha\}$.

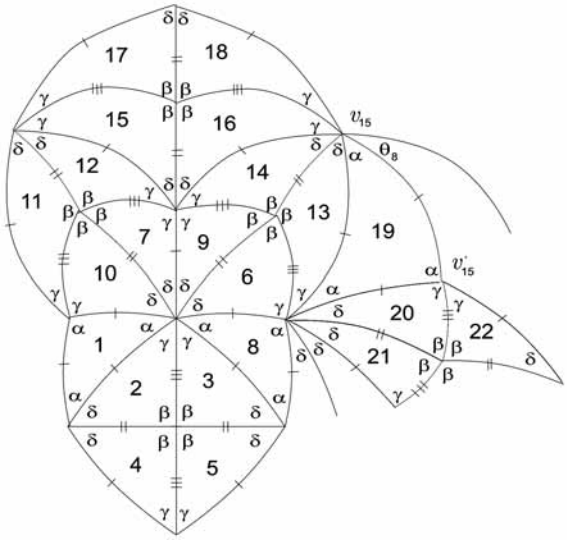

Fig. 25. – Planar representation.

If $\theta_8 = \delta$, then vertex v'_{15} is of valency 6 with alternate angle sums $2\gamma + \delta = \pi$ and $\alpha + \delta + \gamma = \pi$, contradicting our assumption.

If $\theta_8 = \gamma$, one of the alternate angle sum at vertex v_{15} is $2\gamma + \delta + \mu = \pi$, for $\mu \in \{\alpha, \delta, \gamma, \beta\}$, which is an absurd. Therefore, $\theta_8 = \alpha$. Adding some new cells to the configuration we are led to the one in Figure 26, containing a vertex surrounded by the angular sequence $(\gamma, \gamma, \alpha, \gamma, \gamma, \dots)$. Since $\gamma + \alpha + \gamma > \pi$, this configuration does not give rise to a f-tiling.

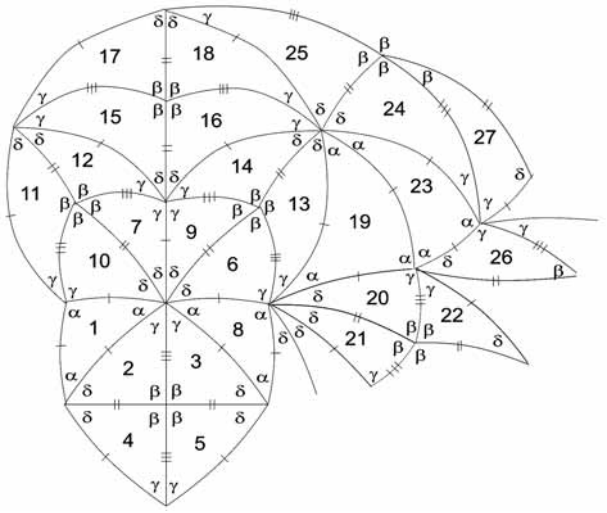

Fig. 26. – Planar representation.

2.3.2.2 If tiles 19 and 21 are, respectively, a scalene triangle and an equilateral triangle, then tile 27 is an equilateral triangle. We are led to a vertex surrounded by the angular sequence $(\gamma, \gamma, \alpha, \gamma, \gamma, \dots)$, which, as seen before, is an absurd (see Figure 27).

2.3.2.3 If tile 21 is a scalene triangle, adding some tiles to the configuration we end up to the one illustrated in Figure 28. Vertex v_{16} is surrounded in circular order by the sequence $(\gamma, \delta, \delta, \delta, \delta, \delta, \delta)$ and $\gamma + 3\delta = \pi$ implies that $\delta = \alpha > \frac{\pi}{3}$, which is an absurd and $\gamma + 4\delta = \pi$ implies that $\delta = \frac{\pi}{7}, \gamma = \frac{3\pi}{7}$ and $\alpha = \frac{2\pi}{7}$, which is also an absurd.

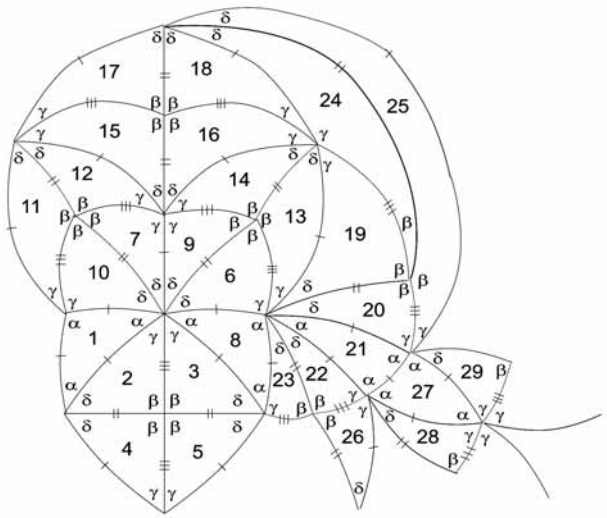

Fig. 27. – Planar representation.

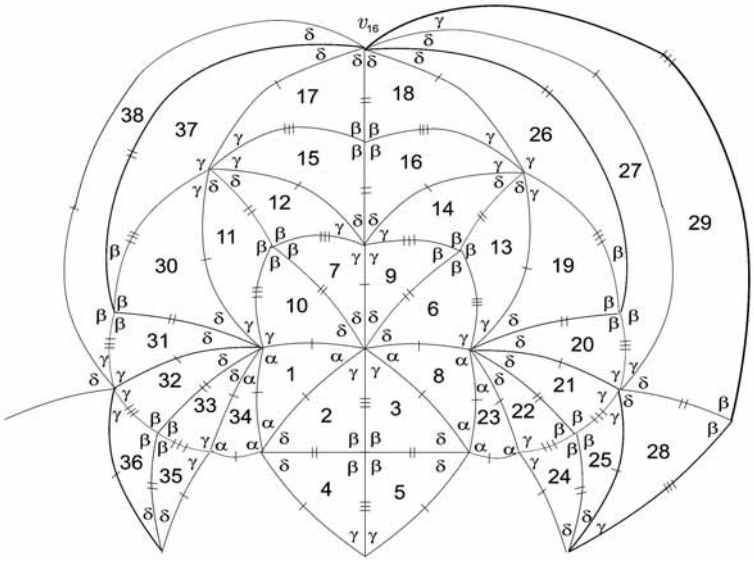

Fig. 28. – Planar representation.

2.3.2.4 Changing the order sides of tile 6 and assuming that tile 9 is a scalene triangle, we end up with configuration Figure 29, where $\theta_9 \in \{\gamma, \alpha, \delta\}$.

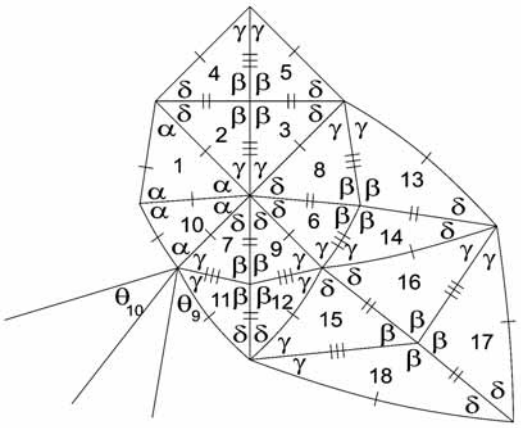

Fig. 29. – Planar representation.

If $\theta_9 = \alpha$, we are led to a configuration with a vertex surrounded by alternate angles $\gamma, \delta, \delta, \delta$, which, as seen before, is an absurd. If $\theta_9 = \gamma$, then one of the alternate angle sum at the vertex does not satisfy $\alpha + \gamma + 2\delta = \pi$. Since $\theta_9 = \delta$, then $\theta_{10} \in \{\alpha, \delta\}$. Suppose $\theta_{10} = \alpha$. Then, extending the configuration we end up in an absurd similar to the one in 2.3.2.2. Therefore, $\theta_{10} = \delta$ and the configuration is the one illustrated below (Figure 30). At vertex v_{17}, one of the alternate angle sum is $\gamma + \alpha + \gamma > \pi$, which is an absurd.

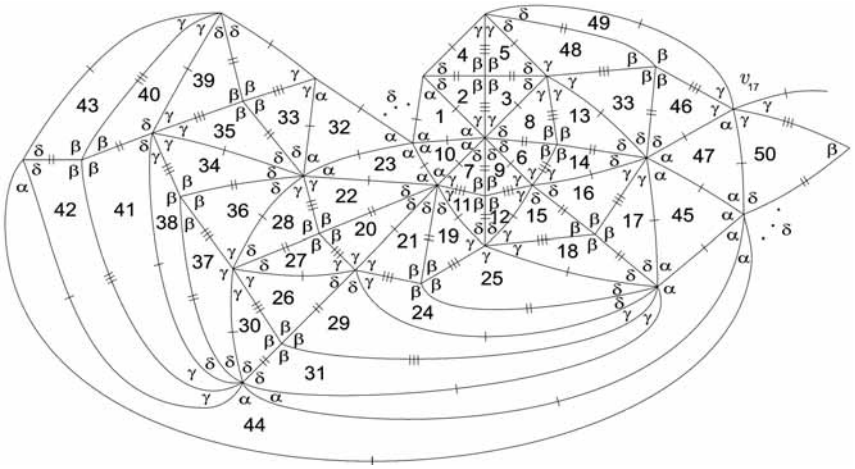

Fig. 30. – Planar representation.

2.3.2.5 Finally, if tile 9 is an equilateral triangle, in order to have, at vertices v_{18} and v_{19}, the alternate angle sum $\alpha+\gamma+2\delta=\pi$, then $\theta_{11}=\alpha$ (see Figure 31).

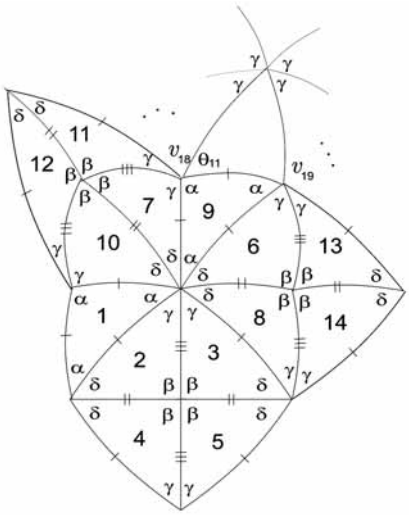

Fig. 31. – Planar representation.

Looking at vertex surrounded by angles $(\gamma,\gamma,\alpha,\gamma,\gamma)$, we conclude that this configuration will not expand, since $2\gamma+\alpha>\pi$. □

REMARK. – The equation (1.1) with $\beta=\dfrac{\pi}{2}$ and $\alpha=\pi-(\gamma+\delta)$ is writing in the form

$$\frac{-\cos(\gamma+\delta)}{1+\cos(\gamma+\delta)}=\frac{\cos\gamma\cos\delta}{\sin\gamma\sin\delta} \tag{2.1}$$

and one the angles γ or δ is completely determined when the remaining is fixed.

In [3] it was proved that (2.1) has the solution $\delta=\gamma\approx 54.735^\circ$ (case when the triangle T_2 is isosceles). The graphic illustration in Figure 32 shows the variations of γ and $\alpha=\pi-(\gamma+\delta)$, when $\delta\in\,]0,54.735^\circ[$. Therefore the f-tilings $\mathcal{F}_1^\delta$ and $\mathcal{F}_2^\delta$ represented in Figure 18 and Figure 19, respectively, are continuous families with $0<\delta<54.735^\circ$.

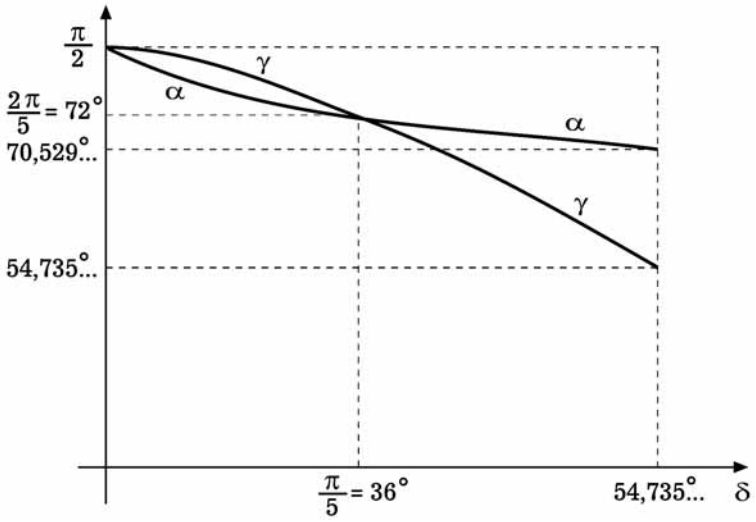

Fig. 32. – α and γ as functions of δ.

3. Symmetry Groups

Here we present the group of symmetries of the spherical f-tilings obtained $\mathcal{F}_1^\delta$ and $\mathcal{F}_2^\delta$. We also determine the transitivity classes of isogonality and isohedrality.

In **Table 1** it is shown a list of the dihedral f-tilings, whose prototiles are an equilateral triangle T_1 of angle α and a scalene triangle T_2 of angles δ, γ, β, $(\delta < \gamma < \beta)$ in a particular way of adjacency. We have used the following notation.

- M and N are, respectively, the number of triangles congruent to T_1 and the number of triangles congruent to T_2 used in such dihedral f-tilings;
- $G(\tau)$ is the symmetry group of the f-tiling τ. We shall say that the tiles T and T' of τ are in the same *transitivity class* if the symmetry group $G(\tau)$ contains a transformation that maps T into T'. If all the tiles of τ form one transitivity class we say that τ is *tile-transitive* or *isohedral*. If there are k transitivity classes of tiles, then τ is *k-isohedral*. On the other hand, if $G(\tau)$ contains transformations that map every vertex of τ into another vertex, then we say that the vertices form one transitivity class or that τ is *isogonal*. If there are k transitivity classes of vertices, then τ is *k-*

isogonal. The numbers of isohedrality-classes and isogonality-classes for the symmetry group are denoted, respectively, by # isoh. and # isog..

With respect to tiling $\mathcal{F}_1^\delta$ the reflections on the coordinate planes generate $G(\mathcal{F}_1^\delta)$. It follows that $G(\mathcal{F}_1^\delta)$ is isomorphic to $C_2 \times C_2 \times C_2$. Now, there are 1 transitivity class of equilateral triangles and 3 transitivity classes of scalene triangles, which means that $\mathcal{F}_1^\delta$ is 4-isohedral. On the other hand, numbering the vertices of the first octant, we conclude that $\mathcal{F}_1^\delta$ is 6-isogonal.

Concerning to tiling $\mathcal{F}_2^\delta$, the rotation of an angle $\dfrac{2\pi}{3}$ around the xx axis generate $G(\mathcal{F}_2^\delta)$, and so it is a cyclic group. It follows that $\mathcal{F}_2^\delta$ is 12-isohedral (2 transitivity classes of equilateral triangles and 10 transitivity classes of scalene triangles) and 6-isogonal.

- The angles δ and γ obey (2.1). Besides, $\alpha = \pi - (\gamma + \delta)$.
- $\delta_0 \approx 54.735^\circ$ and $\gamma_0 \approx 70.529^\circ$.

TABLE 1. –The Combinatorial Structure of the Dihedral F-Tilings of the Sphere by Equilateral and Scalene Triangles with adjacency of type I

f-tilings	δ	γ	α	β	M	N	$G(\tau)$	# isoh.	# isog.
$\mathcal{F}_1^\delta$	$]0, \delta_0[$	$]\delta_0, \frac{\pi}{2}[$	$]\gamma_0, \frac{\pi}{2}[$	$\frac{\pi}{2}$	8	24	$C_2 \times C_2 \times C_2$	4	6
$\mathcal{F}_2^\delta$	$]0, \delta_0[$	$]\delta_0, \frac{\pi}{2}[$	$]\gamma_0, \frac{\pi}{2}[$	$\frac{\pi}{2}$	8	24	C_3	12	6

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