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Right Utumi P.P.-Rings

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ABSTRACT - The class of right Utumi p.p.-rings plays a central role when developing a notion of torsion-freeness over non-commutative rings. This paper shows that it also arises naturally when considering divisible modules. Various notions of divisibility were introduced for modules over integral domains. We determine their relation in the non-commutative case and show that there are significant differences between this and the commutative setting. Finally, we determine large classes of semi-prime Goldie rings for which two or more of these notions coincide.

1. Introduction.

The straightforward attempt to extend the notion of torsion-freeness from integral domains to non-commutative rings encounters immediate difficulties. To overcome these, one can concentrate on either the computational or the homological properties of torsion-free modules. Goodearl and others took the first approach when they introduced the notion of a non-singular module [8]. A right R-module M is *non-singular* if $Z(M) = 0$ where $Z(M) = \{x \in M \mid xI = 0 \text{ for some essential right ideal } I \text{ of } R\}$ denotes the *singular submodule of* M. On the other hand, M is *singular* if $Z(M) = M$. Moreover, a submodule U of an R-module M is *$\mathcal{S}$-closed* if M/U is non-singular. Finally, R is a *right non-singular ring* if R_R is non-singular. The right non-singular rings are precisely the rings which have a regular, right self-injective maximal right ring of quotients, which will be denoted by Q^r (see [8] and [12] for details). Following [12, Chapter XI], Q^r is *a perfect left localization of* R if Q^r is flat as a right R-module and the multiplication map $Q^r \otimes_R Q^r \to Q^r$ is an isomorphism. In particular, Q^r is a perfect left localization of R if and only if every finitely generated non-

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singular right R-module can be embedded into a projective module ([8, Theorem 5.17] and [12, Corollary XII.7.3]). We call such a ring *right strongly non-singular*.

Hattori took the second approach by defining M to be *torsion-free* if $\mathrm{Tor}_1^R(M, R/Rr) = 0$ for all $r \in R$ [9]. The classes of torsion-free and non-singular right R-modules coincide if and only if R is a right Utumi p.p.-ring without an infinite set of orthogonal idempotents [4, Theorem 3.7]. Here, R is a right *p.p.-ring* if all principal right ideals of R are projective. Moreover, a right non-singular ring R is a *right Utumi-ring* if every $\mathcal{S}$-closed right ideal of R is a right annihilator. The right and left Utumi-rings are precisely the right and left non-singular rings whose maximal right and left ring of quotients coincide (see [8, Theorem 2.38] and [12, Proposition XII.4.9]).

Section 2 investigates the structure of right Utumi p.p.-rings without an infinite set of orthogonal idempotents. Theorem 2.3 and Corollary 2.4 show that these rings can be viewed as lower triangular matrix rings. Closely related to the notion of torsion-freeness is that of divisibility where a right R-module D is *divisible* if $\mathrm{Ext}_R^1(R/rR, D) = 0$ for all $r \in R$. From [5, Section 2], one obtains that a right R-module M is torsion-free if and only if its *character module* $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is divisible. Moreover, if R is a left p.p.-ring, then a left module N is divisible if and only if N^* is torsion-free. Therefore, it is not surprising that Utumi p.p.-rings also arise in the discussion of divisible modules (Theorem 3.2).

In Section 4, we look at other notions of divisibility. A right R-module D is *divisible in the classical sense* if $Dc = D$ for all regular elements c of R. Finally, it is *h-divisible* if it is an epimorphic image of a direct sum of copies of Q^r. In case of domains, the notions of divisibility and divisibility in the classical sense coincide. Moreover, they also are equivalent to h-divisibility if R is a countable integral domain ([7, Theorem VII.2.8] and [10]). Although we show in Sections 4 and 5 that this need not be the case if R is non-commutative, the various notions of divisibility will, nevertheless, coincide if R is a semi-prime right and left Goldie p.p.-ring whose classical ring of quotients is countably generated as a right and left R-module (Theorems 5.2 and 5.5).

2. The Structure of Right Utumi P.P.-Rings.

A ring R is a *Baer-ring* if the right annihilator of a subset of R is generated by an idempotent. In contrast to the p.p.-condition, being a

Baer-ring is right-left symmetric. Our first result characterizes the right Utumi p.p.-rings without an infinite family of orthogonal idempotents in terms of their idempotents.

PROPOSITION 2.1. *A right non-singular ring R is a right Utumi p.p.-ring without an infinite set of orthogonal idempotents if and only if the following hold:*

i) *R has finite right Goldie dimension.*
ii) *If $f \in Q^r$ is an idempotent, then there is an idempotent $e \in R$ with $ef = f$ and $fe = e$.*

PROOF. Suppose that R is a right Utumi p.p.-ring without an infinite family of orthogonal idempotents. By [4], R is a Baer-ring, and has finite right Goldie dimension. If $f \in Q^r$ is an idempotent, then $I = fQ^r \cap R$ is an $\mathcal{S}$-closed right ideal of R since $R/I \cong [fQ^r + R]/fQ^r \subseteq Q^r/fQ^r \cong (1-f)Q^r$ is non-singular. Because R is a right Utumi-ring, we obtain that I is the right annihilator of a subset of R. Since R is Baer, there is an idempotent $e \in R$ with $I = eR$. Then, $eQ^r = IQ^r \subseteq fQ^r$. Because R is essential in Q^r, IQ^r is an essential R-submodule of fQ^r. In view of the fact that Q^r is semi-simple Artinian, IQ^r is a direct summand of fQ^r, and hence $fQ^r = eQ^r$. But then $ef = f$ and $fe = e$.

Conversely, assume that the right non-singular R satisfies the two conditions in the proposition. Clearly, R contains no infinite set of orthogonal idempotents; and Q^r is semi-simple Artinian by i) [12, Theorem XII.2.5]. It remains to show that R is a right Utumi p.p.-ring. For this, it suffices to show that every $\mathcal{S}$-closed right ideal I of R is generated by an idempotent. Since Q^r is semi-simple Artinian, there is an idempotent $f \in Q^r$ with $IQ^r = fQ^r$. Choose an idempotent $e \in R$ as in ii), and observe $eQ^r = fQ^r$. To show $I = eR$, observe that $e \in IQ^r$ yields that there exists an essential right ideal J of R with $eJ \subseteq I$. Since I is $\mathcal{S}$-closed in R, one has $eR \subseteq I$, and $I = eR \oplus [I \cap (1-e)R]$. However, $I \cap (1-e)R \subseteq eQ^r \cap (1-e)Q^r = 0$. □

A family $\{M_i \mid i \in I\}$ of right modules over a ring R is *semi-rigid* if $\mathrm{Hom}_R(M_i, M_j) \neq 0$ implies $\mathrm{Hom}_R(M_j, M_i) = 0$ for all $i \neq j$ in I. Modules U and V are *quasi-isomorphic* if there exist essential monomorphisms $U \to V$ and $V \to U$. Finally, if $S \subseteq R$, then $r_R(S) = \{r \in R | Sr = 0\}$ denotes the right annihilator of S in R.

LEMMA 2.2. *Let $R = S \times T$ be a right Utumi p.p.-ring without an infinite set of orthogonal idempotents. Then the same holds for the rings S and T.*

PROOF. Since S is a subring of R, it does not contain an infinite set of orthogonal idempotents either. If $s \in S$, then $r_R((s, 1_T)) = r_S(s) \times \{0\}$ is generated by an idempotent $f = (e, 0)$ of R since R is a right p.p.-ring. Clearly, e is an idempotent of S with $r_S(s) = eS$. Thus, S is a right p.p.-ring.

Finally, if I is an essential right ideal of R, then $I = I_S \times I_T$ where I_S and I_T are essential right ideals of S and T respectively. Therefore, whenever J is a $\mathcal{S}$-closed right ideal of S, then $J \times T$ is a $\mathcal{S}$-closed right ideal of R. Since R is a right Utumi-ring, there is a subset X of R with $J \times T = r_R(X)$. Choose a subset Y of S with $X = Y \times \{0\}$. Clearly, $J = r_S(Y)$, and S is a right Utumi-ring. □

THEOREM 2.3. *Let R be a right Utumi p.p.-ring without an infinite family of orthogonal idempotents. Then,*

a) *$R = R_1 \times \ldots \times R_m$ where $R_1 \ldots, R_m$ are indecomposable right Utumi-p.p.-rings such that, for all $i = 1, \ldots, m$, $R_i = I_{i,1} \oplus \ldots \oplus I_{i,n_i}$ for uniform right ideals $I_{i,1}, \ldots, I_{i,n_i}$ of R_i.*
b) *$N(R) = 0$ if and only if, for each $i = 1, \ldots, m$, the right ideals $I_{i,j}$ and $I_{i,k}$ are quasi-isomorphic whenever $1 \leq j, k \leq n_i$.*

PROOF. Since R is a right Utumi p.p.-ring without an infinite set of orthogonal idempotents, its maximal right ring of quotients, Q^r, is semi-simple Artinian by Proposition 2.1. Moreover, R is a Baer-ring by [4, Theorem 3.7]. Hence, [4, Lemma 3.5] yields that every $\mathcal{S}$-closed right ideal of R is generated by an idempotent. Write $Q^r = Q_1 \times \ldots \times Q_m$ where each Q_i is a simple Artinian ring. Let $f_1, \ldots, f_n$ be the central idempotents of Q^r associated with this decomposition, and consider the two-sided ideals $I_i = R \cap Q_i$ of R. Since $R/I_i \cong [R + Q_i]/Q_i$ is non-singular as a right R-module, I_i is an $\mathcal{S}$-closed right ideal of R. Consequently, it is generated by an idempotent $e_i \in R$. Since I_i is an essential submodule of Q_i, one has $f_iQ = Q_i = I_iQ = e_iQ$. Then, $f_i = e_if_i = f_ie_i = e_i$ for $i = 1, \ldots, n$. Therefore, $R = R_1 \times \ldots \times R_m$ where $R_i = e_iRe_i$ is indecomposable as a ring since e_iQe_i is a simple ring. Moreover, $N(R) = N(R_1) \times \ldots \times N(R_n)$. By Lemma 2.2, each R_i is a right Utumi p.p.-ring without an infinite set of orthogonal idempotents. Hence, we may assume for the remainder of the proof that R is indecomposable as a ring.

Since R has finite right Goldie dimension, there exists a non-zero uniform right ideal U_1 of R. If V_1 is the $\mathcal{S}$-closure of U_1 in R, then V_1 is a direct summand of R because R is a Baer and right Utumi-ring. Write $R = V_1 \oplus W_1$. If $W_1 \neq 0$, then W_1 contains an $\mathcal{S}$-closed non-zero uniform submodule V_2. Since V_2 is a direct summand of R, we obtain

$R = V_1 \oplus V_2 \oplus W_2$ for some right ideal W_2 of R. Since R has finite Goldie-dimension, this process has to stop, and $R = V_1 \oplus \ldots \oplus V_\ell$ for some $\ell < \omega$ and uniform right ideals $V_1, \ldots, V_\ell$. This proves a).

On $\{V_1, \ldots, V_\ell\}$, define $V_i \sim V_j$ if and only if $\mathrm{Hom}_R(V_i, V_j) \neq 0$ and $\mathrm{Hom}_R(V_j, V_i) \neq 0$. Since the V_i's are uniform, each V_i is a non-singular right R-module of Goldie dimension 1. Therefore, every non-zero submodule of V_i is essential in V_i. Moreover, if U is a $\mathcal{S}$-closed submodule of V_1, then $1 = dimV = dimU + dimV/U$ yields $U = 0$ or $U = V$. Hence, every non-zero map $\alpha : V_i \to V_j$ is an essential monomorphism. Therefore, $V_i \sim V_j$ if and only if V_i and V_j are quasi-isomorphic. In particular, $\sim$ is an equivalence relation. Without loss of generality, one may assume that there is $m \leq \ell$ such that $V_1, \ldots, V_m$ are the distinct representatives of the equivalence classes of $\sim$. For $j = 1, \ldots, m$, set $I_j = \oplus\{V_i \mid V_i \sim V_j\}$. Since non-zero maps between the V_k's are essential monomorphism, $R = I_1 \oplus \ldots \oplus I_m$ such that $\{I_1, \ldots, I_m\}$ is a semi-rigid family such that, for each j, $I_j = I_{j,1} \oplus \ldots \oplus I_{j,n_j}$ with each $I_{j,k}$ uniform and $I_{j,k}$ and $I_{j,\ell}$ are quasi-isomorphic for all k and ℓ. Select a family $\{e_1, \ldots, e_n\}$ of orthogonal idempotents of R with $I_i = e_i R$.

Suppose that R is semi-prime. If $\mathrm{Hom}_R(I_j, I_k) \neq 0$ for some $j \neq k$, then $\mathrm{Hom}_R(I_k, I_j) = 0$ by the semi-rigidity condition. Because $\mathrm{Hom}_R(I_j, I_k) \neq 0$, there is $r \in R$ with $0 \neq rI_j \subseteq I_k$. In particular, $e_k r e_j \neq 0$. In view of the fact that $N(R) = 0$, we have $e_k r e_j R e_k r e_j \neq 0$. If $s \in R$ satisfies $e_j s e_k \neq 0$, then left multiplication by $e_j s e_k$ induces a non-zero endomorphism α of R with $\alpha(I_k) \subseteq I_j$, which contradicts $\mathrm{Hom}_R(I_k, I_j) = 0$. Therefore, $\mathrm{Hom}_R(I_j, I_k) = 0$ whenever $j \neq k$.

If there is $r \in R$ with $rI_j \not\subseteq I_j$ for some j, then we can find $k \neq j$ such that $e_k r x \neq 0$ for some $x \in I_j$. Thus, multiplication from the left by $e_k r$ induces a non-zero map $I_j \to I_k$, a contradiction. Therefore, each I_j is a two-sided ideal of R. Since R is indecomposable, we have $m = 1$.

Conversely, suppose that R has a decomposition of the stated form. Again, we may assume $m = 1$. Write $R = J_1 \oplus \ldots \oplus J_\ell$ where each J_i is a uniform right ideal of R, and J_j and J_k are quasi-isomorphic for all j and k. Select orthogonal idempotents $d_1, \ldots, d_\ell$ of R such that $J_j = d_j R$. If $N(R) \neq 0$, then there exists a non-zero $a \in R$ such that $aRa = 0$. Select $i \in \{1, \ldots, \ell\}$ with $d_i a \neq 0$. Moreover, there exists k such that $d_i a d_k \neq 0$. Thus, left multiplication by $d_i a$ is a non-zero map from J_k to J_i. Since $\mathrm{Hom}_R(J_i, J_k) \neq 0$, there is $s \in R$ such that $0 \neq sJ_i \subseteq J_k$. Because non-zero maps between the J_j's are monomorphisms, left multiplication by $d_i a s$ is a one-to-one map from J_i to J_i. Thus, $d_i a s d_i a \neq 0$, and the same holds for $a s d_i a$, a contradiction. $\square$

COROLLARY 2.4. *Let R be a right Utumi p.p.-ring without an infinite family of orthogonal idempotents. Then, $R \cong R_1 \times \ldots \times R_m$ where each R_i is an indecomposable ring which is isomorphic to a lower triangular matrix ring over rings $R_{i,1}, \ldots, R_{i,n_i}$ such that each $R_{i,j}$ is a prime ring.*

PROOF. By Theorem 2.3, we may assume that R is indecomposable. Observe that $Q^r(R) \cong \mathrm{Mat}_\ell(D)$ for some division algebra D. Using the notation of Theorem 2.3, write $R = I_1 \oplus \ldots \oplus I_m$ such that $\{I_1, \ldots, I_m\}$ forms a semi-rigid family. We actually may assume that $I_1, \ldots, I_m$ are arranged in such a way that $\mathrm{Hom}_R(I_j, I_k) = 0$ if $j < k$. Setting $M_{jk} = \mathrm{Hom}_R(I_j, I_k)$, one obtains that R is isomorphic to the ring of matrices with entries from M_{ij}. Clearly, this is a lower triangular matrix ring. Furthermore, $R_i = \mathrm{Hom}_R(I_i, I_i)$ has a full matrix ring over D as its maximal right ring of quotients. □

COROLLARY 2.5. *Let R be a right and left Utumi-p.p.-ring without an infinite family of orthogonal idempotents, and A an essential submodule of Q^r which is also a submodule of ${}_RQ^r$. Then, $A = A_1 \oplus \ldots A_m$ such that*

a) *$\{A_1, \ldots, A_m\}$ is semi-rigid.*
b) *For all $i = 1, \ldots, m$, $A_i = A_{i,1} \oplus \ldots \oplus A_{i,n_i}$ where each $A_{i,j}$ is uniform; and $A_{i,j}$ and $A_{i,k}$ are quasi-isomorphic for all $i = 1, \ldots, m$ and all $1 \leq j, k \leq n_i$*

PROOF. Observe that $S = \mathrm{End}_R(A_R)$ is a subring Q^r which contains R. By [1, Proposition 3.1] and Proposition 2.1, S contains primitive idempotents $e_{i,j}$ such that $e_{i,j}S$ has the desired property. Let $A_{i,j} = e_{i,j}A$. □

3. Divisible Modules.

Although the class of divisible modules is closed with respect to epimorphic images if R is an integral domain, this does not hold in our more general setting:

LEMMA 3.1. *A ring R is a right p.p.-ring if and only if the class of divisible right R-modules is closed with respect to epimorphic images.*

PROOF. Let R be a right p.p.-ring, and consider a submodule U of a divisible module D. For all $r \in R$, we have an exact sequence

$$0 = \mathrm{Ext}^1_R(R/rR, D) \to \mathrm{Ext}^1_R(R/rR, D/U) \to \mathrm{Ext}^2_R(R/rR, U) = 0$$

from which we obtain that D/U is divisible.

Conversely, suppose that epimorphic images of divisible modules are divisible. Then, $\mathrm{Ext}^1_R(R/rR, E/U) = 0$ for all submodules U of an injective module E and all $r \in R$. We therefore have the exact sequence

$$\mathrm{Hom}_R(R, E/U) \to \mathrm{Hom}_R(rR, E/U) \to \mathrm{Ext}^1_R(R/rR, E/U) = 0.$$

By [11, Lemma 4.22], rR is projective. □

For integral domains, the notions of divisibility and divisibility in the classical sense obviously coincide. Although all divisible modules are divisible in the classical sense in the non-commutative setting, the converse may fail in view of the last result.

THEOREM 3.2. *The following are equivalent for a right non-singular ring R:*

a) *R is a right Utumi p.p.-ring without an infinite set of orthogonal idempotents.*
b) *Q^r_R is torsion-free and $D \in_R \mathcal{M}$ is divisible if and only if $M \otimes_R D = 0$ for all singular $M \in \mathcal{M}_R$.*
c) *${}_RQ^r$ is divisible, and $D \in_R \mathcal{M}$ is divisible if and only if $ID = D$ for all essential right ideals I of R.*

PROOF. *a) ⇒ c)*: If D is a divisible left R-module, then $D^* = \mathrm{Hom}_{\mathbb{Z}}(D, \mathbb{Q}/\mathbb{Z})$ is a torsion-free right R-module in the sense of Hattori [5]. Since R is a right Utumi p.p.-ring without an infinite set of orthogonal idempotents, D^* is non-singular by [4]. If there is an essential right ideal I of R with $D/ID \neq 0$, then there exists a non-zero morphism $\phi : D/ID \to \mathbb{Q}/\mathbb{Z}$. We view ϕ as a non-zero map $D \to \mathbb{Q}/\mathbb{Z}$ with $\phi(ID) = 0$. Then, $[\phi r](d) = \phi(rd) = 0$ for all $r \in I$, so that $\phi \in Z(D^*) = 0$, a contradiction.

Conversely, suppose $D = ID$ for all essential right ideals I of R. For $\alpha \in Z(D^*)$, there is an essential right ideal J of R such that $\alpha J = 0$. But then $\alpha(JD) = 0$, and D^* is non-singular. By [4], this means that D^* is torsion-free, from which we get that D is divisible [5]. Finally, R has finite right Goldie-dimension. Therefore, Q^r is semi-simple Artinian by [12, Theorem XII.2.5], and $IQ^r = Q^r$ for all essential right ideals I of R by [8, Proposition 2.32].

c) $\Rightarrow$ *b*): Since the condition $D = ID$ is preserved by epimorphic images, the class of divisible left R-modules is closed with respect to epimorphic images. By Lemma 3.1, R is a left p.p.-ring. As in [5], we obtain that ${}_RD$ is divisible if and only if D^* is torsion-free. Since Q^r is divisible as a left R-module by c), we obtain that $(Q^r)^*$ is torsion-free. Moreover, $(I \cap R)Q^r = Q^r$ for all essential right ideals I of Q^r. By [8, Proposition 2.32], Q^r is semi-simple Artinian.

Let $S_1, \ldots, S_n$ be the pairwise non-isomorphic simple right Q^r-modules. Write $Q^r = R_1 \times \ldots \times R_n$ where $R_i = \mathrm{Mat}_{n_i}(D_i)$ and D_i is the division algebra $\mathrm{End}_R(S_i)$. Since $({}_RQ^r)^*$ is a right Q^r-module, we obtain a decomposition $({}_RQ^r)^* = U_1 \oplus \ldots \oplus U_n$ where each U_i is an R_i-module with $U_jR_i = 0$ for $i \neq j$. Therefore, $U_j \cong \oplus_{m_j} S_j$ for some suitable $m_j < \omega$. Suppose that $m_j = 0$ for some j. Let $\phi \in (Q^r)^*$, and consider $x \in Q^r$ and $q_j \in R_j$. Since $(Q^r)^*R_j = 0$, we have $0 = [\phi q_j](x) = \phi(q_jx)$. Thus, $\phi(R_j) = \phi(R_jQ^r) = 0$. This is only possible if $R_j = 0$, a contradiction. Hence, $m_j > 0$ for all j. Since $({}_RQ^r)^*$ is torsion-free, the same holds for $S_1, \ldots, S_n$, and, therefore, for Q^r_R too.

Furthermore, if D is a left R-module and J is a right ideal of R, then the standard homological arguments show $(R/J) \otimes_R D \cong D/JD$. Therefore, D is divisible if and only if $(R/I) \otimes_R D = 0$ for all essential right ideals I of R because of c). Since every singular right R-module M is the epimorphic image of $\oplus_J R/I_j$ for essential right ideals I_j of R, we have $M \otimes_R D = 0$ for every divisible left module D.

b) $\Rightarrow$ *a*): By [4], it suffices to show that the classes of torsion-free and non-singular right R-modules coincide. Using b), we obtain that epimorphic images of divisible left R-modules are divisible, and R is a left p.p.-ring. By [4], all non-singular right R-modules are torsion-free.

Conversely, consider a torsion-free right R-module M, and an essential right ideal I of R. Since M^* is a divisible left module [5], $IM^* = M^*$ because of $0 = (R/I) \otimes_R M^* \cong M^*/IM^*$ by b). Suppose that there is $\phi \in M^{**}$ with $\phi I = 0$. For $x \in M^*$, there are $r_1, \ldots, r_n \in I$ and $y_1, \ldots, y_n \in M^*$ such that $x = r_1y_1 + \ldots r_ny_n$. Then, $\phi(x) = \phi(r_1y_1) + \ldots + \phi(r_ny_n) = [\phi r_1](y_1) + \ldots + [\phi r_n](y_n) = 0$. Therefore, M^{**} is non-singular, and the same holds for its submodule M. By [4], R is a ring as in a). □

COROLLARY 3.3. *Let R be a right Ore-ring which is a right Utumi, right p.p-ring without an infinite set of orthogonal idempotents. Then, Q_{cl} is a divisible left R-module if and only if R is semi-prime.*

PROOF. If Q_{cl} is divisible, then $IQ_{cl} = Q_{cl}$ for all essential right ideals I of R by Theorem 3.2. Thus, there are a regular element c of R and $x \in I$ with $xc^{-1} = 1$, i.e. $c \in I$. By [8, Theorem 3.35], R is semi-prime. □

A right R-module D is *absolutely pure* if $\mathrm{Ext}^1_R(M, D) = 0$ for all finitely presented modules M.

COROLLARY 3.4. *The following are equivalent for a strongly non-singular right Utumi-ring R without an infinite set of orthogonal idempotents:*

a) *R is right semi-hereditary.*
b) *All divisible right R-modules are absolutely pure.*

PROOF. *a)* $\Rightarrow$ *b)*: Let D be a divisible right R-module, and $r \in R$. As in the proof of Theorem 3.2, we use [5] to obtain that D^* is a torsion-free left R-module in the sense of Hattori. By [4], D^* is flat since R is right semi-hereditary. Then, D^{**} is injective, and D is a pure submodule of an injective module. However, the class of absolutely pure modules is closed with respect to pure submodules.

b) $\Rightarrow$ *a)* is established using arguments similar to the ones used in the proof of Lemma 3.1. □

4. The Singular Submodule of a Divisible Module.

Over an integral domain R, the h-divisible modules are the epimorphic images of injective modules. This does not have to hold in our setting:

THEOREM 4.1. *The following are equivalent for a right non-singular ring R:*

a) *R has finite right Goldie-dimension.*
b) *A module is h-divisible if and only if it an epimorphic image of an injective module.*
c) *Non-singular h-divisible modules are injective.*

Furthermore, $Z(D)$ is a direct summand of D whenever D is an h-divisible module over a finite dimensional right non-singular ring.

PROOF. *a)* $\Rightarrow$ *b)*: Suppose that U is a submodule of an injective module E, and consider an epimorphism $\pi : R^{(I)} \to E$ for some index-set I. It extends to an epimorphism $\overline{\pi} : Q^{r(I)} \to E$. Thus, E/U is h-divisible. Conversely, observe that Q^r is Σ-injective since R has finite right Goldie-dimension.

b) $\Rightarrow$ *c*): Let D be a non-singular h-divisible module. By b), there are an injective module E and an epimorphism $\phi : E \to D$. As before, we can find an index-set I and an epimorphism $\pi : Q^{r(I)} \to E$. We define a Q^r-module structure on D, which is compatible with its R-module structure, as follows: Whenever $q \in Q^r$ and $d \in D$, select $x \in Q^{r(I)}$ with $\phi\pi(x) = d$, and set $d * q = \phi\pi(xq)$. To see that this is well-defined, consider $y \in Q^{r(I)}$ with $\phi\pi(y) = d$. There is an essential right ideal J of R with $qJ \subseteq R$. For all $r \in J$, we have $[\phi\pi(xq) - \phi\pi(yq)]r = [\phi\pi(x) - \phi\pi(y)]qr = 0$. Since D is non-singular, $\phi\pi(xq) = \phi\pi(yq)$. Finally, let $d' \in D$ and $s \in R$. If $x' \in Q^{r(I)}$ with $\phi\pi(x') = d'$, then $d' * s = \phi\pi(x's) = \phi\pi(x')s = d's$. In particular, $\phi\pi : Q^{r(I)} \to D$ becomes a Q^r-epimorphism. Suppose that we have already shown that Q^r is a semi-simple Artinian ring. Then, D is a projective Q^r-module, and the epimorphism $\phi\pi$ splits. Consequently, D is injective since it is isomorphic to a direct summand of the injective module E.

To see that Q^r is semi-simple Artinian, it suffices to show that direct sums of copies of Q^r are injective as right R-modules [12, Proposition XIII.3.3] since Q^r_R is the injective hull of R_R by [8, Chapter 2C]. To see this, we consider an index-set J, and apply the construction of the last paragraph to the h-divisible module $D = \oplus_J Q^r$. Observe that the two Q^r-module structures on D actually coincide: Select $d \in D$ and $x \in Q^{r(I)}$ with $\phi\pi(x) = d$ as before. For $q \in Q^r$ select an essential right ideal J with $qJ \subseteq R$. For all $s \in J$, we have

$$(d * q)s = [\phi\pi(xq)]s = \phi\pi(x(qs)) = [\phi\pi(x)](qs) = d(qs) = (dq)s.$$

Hence, $d * q = dq$ as desired. In particular, $(D, *)$ is a free Q^r-module, and the Q^r-epimorphism $\phi\pi : Q^{r(I)} \to D$ has to split, say $\phi\pi\alpha = 1_D$ for some morphism $\alpha : D \to Q^{r(I)}$. But then, $\phi : E \to D$ splits, and $D \cong \oplus_J Q^r$ is injective for all index-sets J.

c) $\Rightarrow$ *a*) is a direct consequence of the previously mentioned result of [12].

Finally, let $\lambda : D \to D/Z(D)$ be the canonical projection. If ϕ and π are chosen as in the proof of *b*) $\Rightarrow$ *c*), then $\lambda\phi\pi : Q^{r(I)} \to D/Z(D)$ splits, say $\lambda\phi\pi\psi = 1_{D/Z(D)}$ for some $\psi : D/Z(D) \to Q^{r(I)}$. Hence, $Z(D)$ is a direct summand of D. □

Although h-divisible modules are divisible in the classical sense, they need not be divisible:

COROLLARY 4.2. *The following are equivalent for a right non-singular ring R of finite right Goldie-dimension:*

a) *R is a right p.p.-ring.*
b) *Every h-divisible right R-module is divisible.*

PROOF. *a)* $\Rightarrow$ *b)*: By Theorem 4.1, every h-divisible module M is an epimorphic image of an injective module, and hence divisible by Lemma 3.1.

b) $\Rightarrow$ *a)*: Since every image of an injective is h-divisible by Theorem 4.1, we obtain that epimorphic images of injective modules are divisible. As in Lemma 3.1, R is a right p.p.-ring. □

We say that a right R-module M is *weakly cotorsion* if $\mathrm{Ext}^1_R(Q^r, M) = 0$.

THEOREM 4.3. *Let R be a right Utumi p.p.-ring without an infinite set of orthogonal idempotents:*

a) *A right R-module D is divisible if and only if $Z(D)$ is divisible and $D/Z(D)$ is injective.*
b) *$Z(D)$ is direct summand whenever D is divisible if and only if all divisible R-modules are weakly cotorsion.*

PROOF. a) Let $r \in R$, and let J be the $\mathcal{S}$-closure of rR in R. Since R is a right Utumi p.p.-ring, J is a direct summand of R by [4], say $R = P \oplus J$. Thus, $\mathrm{Ext}^1_R(R/rR, -) \cong \mathrm{Ext}^1_R(J/rR, -)$. In particular, the divisibility of D guarantees $\mathrm{Ext}^1_R(J/rR, D) = 0$. On the other hand, we have the exact sequence

$$0 = \mathrm{Hom}_R(J/rR, D/Z(D)) \to \mathrm{Ext}^1_R(J/rR, Z(D)) \to \mathrm{Ext}^1_R(J/rR, D) = 0,$$

from which we obtain $\mathrm{Ext}^1_R(R/rR, Z(D)) = 0$. Thus, $Z(D)$ is divisible.

To show that $D/Z(D)$ is injective, observe that it is divisible since R is a right p.p.-ring. Therefore, we may assume that D is non-singular, and the same holds for its injective hull E. Since D is divisible, it is an RD-submodule of E. By [4], RD-submodules of non-singular modules are $\mathcal{S}$-closed. Thus, E/D is non-singular, which contradicts the fact that D is essential in E.

The converse is obvious since the class of divisible modules is closed with respect to extensions.

b) Assume that $Z(D)$ is a direct summand whenever D is divisible, and consider an exact sequence $0 \to D \to M \to Q^r \to 0$. Then, M is divisible, and $Z(M) = Z(D)$. By i), $M = Z(D) \oplus U$. Then, $D = Z(D) \oplus (U \cap D)$. Because of a), $U \cap D$ is injective. Hence, $U = (U \cap D) \oplus D_1$. Then, $M = Z(D) \oplus U = Z(D) \oplus (U \cap D) \oplus D_1 = D \oplus D_1$ as desired.

Conversely, suppose that divisible modules are weakly cotorsion. If D is divisible, then $Z(D)$ is divisible, and $D/Z(D)$ is an injective module by a). Therefore, $\mathrm{Ext}^1_R(Q^r, Z(D)) = 0$. Since R is a finite dimensional ring, $D/Z(D)$ is a direct summand of $\oplus_I Q^r$ for some index-set I. However, $\mathrm{Ext}^1_R(\oplus_I Q^r, Z(D)) = 0$, from which we obtain $\mathrm{Ext}^1_R(D/Z(D), Z(D)) = 0$. □

We now give an example of a ring such that modules which are divisible in the classical sense need not be weakly cotorsion:

EXAMPLE 4.4. Let F be a field, and I an infinite index set. In $Q = F^I$ consider $R = F^{(I)} + F1_Q$. By [8, Example 3.11], R is a right non-singular ring whose maximal ring of quotients is Q. Since all regular elements of R are units, every R-module is divisible in the classical sense. However, Q is not projective, for otherwise it would be finitely generated by Sandomirski's Theorem [6, Proposition 8.24]. In this case, $dim_F Q \leq (dim_F R)\aleph_0 < 2^{|I|} \leq F^{|I|} = dim_F Q$, a contradiction. Hence, we can find an R-module with $\mathrm{Ext}^1_R(Q, M) \neq 0$.

We conclude this section by looking at the semi-prime case:

COROLLARY 4.5. *Let R be a semi-prime right Goldie-ring. The following are equivalent for a non-singular module D:*

a) *D is injective.*
b) *D is divisible.*
c) *D is divisible in the classical sense.*

PROOF. It remains to show $c) \Rightarrow a)$: Let J be an essential right ideal of R. Choose a regular element $c \in J$. For a map $\phi : J \to D$, consider $d = \phi(c)$ and select $x \in D$ with $xc = d$. If $\psi : R \to D$ is defined by $\psi(1) = x$, then $\psi|_{cR} - \phi|_{cR} = 0$. Since D is non-singular, and J/cR is singular, $\psi|_J = \phi$. □

Furthermore, we obtain

COROLLARY 4.6. *Let R be a semi-prime right and left Goldie-ring.*

a) *R is a p.p.-ring if and only if the classes of divisible modules coincides with the class of modules which are divisible in the classical sense.*
b) *All R-modules which are divisible in the classical sense are weakly cotorsion if and only if $Z(D)$ is a direct summand of D whenever D is divisible in the classical sense.*

PROOF. a) In case that R is a p.p.-ring, consider a right R-module D which is divisible in the classical sense. By [5], it is enough to show that $D^* = \mathrm{Hom}_{\mathbb{Z}}(D, \mathbb{Q}/\mathbb{Z})$ is a non-singular left R-module. Suppose that $\phi : D \to \mathbb{Q}/\mathbb{Z}$ such that $c\phi = 0$ for some regular element c of R. For $d \in D$ select $d' \in D$ such that $d = d'c$. Then, $\phi(d) = \phi(d'c) = c\phi(d') = 0$.

Conversely, observe that the class of modules which are divisible in the classical sense is closed with respect to epimorphic images. By Lemma 3.1, R is a right p.p.-ring.

b) Suppose that D is divisible in the classical sense. Then the same holds for $D/Z(D)$, which is injective by Corollary 4.5. To see that $Z(D)$ is divisible in the classical sense, let $x \in Z(D)$. If $c \in R$ is regular, then there is $y \in D$ with $x = yc$. Since $x \in Z(D)$, there is a regular $d \in R$ with $xd = 0$. Then, $ycd = 0$ yields $y \in Z(D)$. We now can adapt the proof of Theorem 4.3 to establish b). □

Example 5.6 will show that the condition that R is semi-prime cannot be removed from the last result. The ring R in Example 5.6 is divisible in the classical sense when viewed as a right module. However, it is not divisible. Otherwise, it would be absolutely pure by Corollary 3.4 since R is hereditary. Because R is right and left Noetherian, absolutely pure modules are injective, a contradiction.

5. The Projective Dimension of Q^r.

Our first result relates the splitting of $Z(D)$ to the projective dimension of Q^r_R:

PROPOSITION 5.1. *Let R be a right non-singular ring. If $Z(D)$ is a direct summand of every module D which is divisible in the classical sense, then $p.d.(Q^r_R) \leq 1$.*

PROOF. Let M be a right R-module with injective hull E. The singular module E/M is divisible in the classical sense. Consider an exact sequence $0 \to E/M \to X \to Q^r \to 0$ of right R-modules. Since E/M and Q^r are divisible in the classical sense, the same holds for X. By hypothesis, the sequence splits, and $\mathrm{Ext}^1_R(Q^r, E/M) = 0$. We obtain

$$0 = \mathrm{Ext}^1_R(Q^r, E/M) \to \mathrm{Ext}^2_R(Q^r, M) \to \mathrm{Ext}^2_R(Q^r, E) = 0,$$

and Q^r has projective dimension at most 1. □

We now give a non-commutative version of [7, Theorem VII.2.8]:

THEOREM 5.2. *Let R be a semi-prime, right and left Goldie-ring such that $(Q/R)_R$ is a direct sum of countably generated modules.*

a) $p.d.(Q_R) \leq 1$.
b) *A right R-module is h-divisible if and only if it is divisible in the classical sense and weakly cotorsion.*

PROOF. Let U be a countably generated submodule of Q_R containing R, say it is generated by $\{u_n \mid n<\omega\}$. Without loss of generality, we may assume $u_0 = 1$. We construct an ascending chain $\{V_n\}_{n<\omega}$ such that $V_0 = R$, $V_n = d_n^{-1}R$ for some regular element d_n of R subject to the conditions $d_0 = 1$ and $d_{n+1} = c_n d_n$ and $u_0, \ldots, u_n \in V_n$. Since R is a semi-prime right and left Goldie-ring, there is a regular element $g_n \in R$ such that $u_{n+1} \in g_n^{-1}R$. Select $c_n, r_n \in R$ with c_n regular such that $d_n g_n^{-1} = c_n^{-1} r_n$ and set $d_{n+1} = c_n d_n = r_n g_n$. Then, $U \subseteq V = \cup_{n<\omega} V_n$. Since each V_n is free, we have $p.d.(V) \leq 1$, and hence $p.d.(V/R) \leq 1$.

Now write $Q/R = \oplus_I A_i/R$ where each A_i is a countably generated submodule of Q_R containing R. Select $V_i \supseteq A_i$ as was constructed in the last paragraph. Then, A_i/R is a direct summand of V_i/R, and hence has projective dimension at most 1, and the same holds for Q_R.

Let D be h-divisible, and select an epimorphism $\pi : Q^{(I)} \to D$ for some index-set I. Since R has finite right Goldie-dimension, $Q^{(I)}$ is injective. Hence, we obtain a sequence

$$0 = \mathrm{Ext}^1_R(Q, Q^{(I)}) \to \mathrm{Ext}^1_R(Q, D) \to \mathrm{Ext}^2_R(Q, \ker\ \pi) = 0$$

which is exact since $p.d.(Q_R) \leq 1$. Consequently, D is weakly cotorsion.

Conversely, suppose that D is weakly cotorsion and divisible in the classical sense. Consider the sequence

$$\mathrm{Hom}_R(Q, D) \to \mathrm{Hom}_R(R, D) \to \mathrm{Ext}^1_R(Q/R, D) \to \mathrm{Ext}^1_R(Q, D) = 0.$$

Once we have shown that $\mathrm{Ext}^1_R(Q/R, D) = 0$, then D is injective with respect to $0 \to R \to Q$. Consequently, D is an image of a direct sum of copies of Q.

Now $\mathrm{Ext}^1_R(Q/R, D) = \Pi_I \mathrm{Ext}^1_R(A_i/R, D)$. Thus, it suffices to show that $\mathrm{Ext}^1_R(A_i/R, D) = 0$. However, $Q/A_i \cong (Q/R)/(A_i/R) \cong \oplus_{j\neq i}(A_j/R)$ has projective dimension at most 1. The exact sequence

$$0 = \mathrm{Ext}^1_R(Q, D) \to \mathrm{Ext}^1_R(A_i, D) \to \mathrm{Ext}^2_R(Q/A_i, D) = 0$$

yields $\mathrm{Ext}^1_R(A_i, D) = 0$. Thus, $\mathrm{Ext}^1_R(A_i/R, D) = 0$ provided that D is injective with respect to $0 \to R \to A_i$. Since A_i/R is countably generated, there are regular elements $u_0 = 1, u_1, \ldots$ of R such that $W_i = \langle u_n^{-1}R \mid n < \omega \rangle$ contains A_i. If $\phi : R \to D$, then it is enough to show that ϕ can be extended to W_i. As before, there are regular elements $d_0, d_1, \ldots$ of R such that $d_{n+1} = c_n d_n$ and $u_n^{-1} \in V_n = d_n^{-1}R$.

Let $\phi_0(1) = \phi(1)$, and assume that we have already defined maps $\phi_i : V_i \to D$ for $i = 0, \ldots, n$ such that $\phi_{i+1}|_{V_i} = \phi_i$ for $i = 0, \ldots, n-1$. Now, $V_{n+1}/V_n = d_n^{-1}c_n^{-1}R/d_n^{-1}R \cong R/c_nR$. Since D is divisible in the classical sense, $\mathrm{Ext}^1_R(R/c_nR, D) = 0$, and hence ϕ_n extends to ϕ_{n+1} as desired. Therefore, ϕ can be extended to the submodule $\cup_{n<\omega} V_n$ of Q_R which contains W_i. □

However, decompositions of Q^r/R need not be as nice as in the case that R is an integral domain. For instance, it was shown in [7], that $Q/R = A/R \oplus B/R$ yields that A and B are subrings of Q. This need not hold in the non-commutative setting:

PROPOSITION 5.3. *Let R be a right and left Utumi-ring of finite Goldie-dimension whose maximal right and left ring of quotients is Q. If $Q/R = A/R \oplus B/R$ for submodules A and B of Q_R containing R, then the following are equivalent:*

a) *A and B are subrings of Q.*
b) *A and B are submodules of ${}_RQ$.*

PROOF. *b) ⇒ a)*: Let J be an essential right ideal of R. Since R has finite Goldie dimension, Q is a semi-simple Artinian ring, and $JQ = Q$ by [12, Proposition Theorem XII.2.5] and [8, Theorem 3.35]. For every $q \in Q$, we can find $r_1, \ldots, r_n \in J$ and $q_1, \ldots, q_n \in Q$ with $q = \Sigma_{i=1}^n r_i q_i$. For each i, there are $a_i \in A$ and $b_i \in B$ with $q_i = a_i + b_i$. Then, $q = \Sigma_{i=1}^n r_i a_i + \Sigma_{i=1}^n r_i b_i \in JA + JB$. Thus, $Q = JA + JB$. Moreover, if q has been chosen from A, then $q - (\Sigma_{i=1}^n r_i a_i) = \Sigma_{i=1}^n r_i b_i \in A \cap B = R$ by b). Hence, $A = JA + R$.

Select $a \in A$, and choose an essential right ideal I of R with $aI \subseteq R$. Then, $aA = a(IA + R) = (aI) + aR \subseteq A$ by what has been shown in the last paragraph and b).

a) ⇒ b) is obvious. □

EXAMPLE 5.4. Let S be an integral domain with field of quotients F which is not a field. The ring $R = Mat_2(S)$ is a prime right and left Goldie-

ring whose classical right and left ring of quotients Q is a simple Artinian ring. Suppose that e_1 and e_2 are the canonical idempotents of R with a 1 in one positions and zero everywhere else. Then it is easy to see that $A = e_1Q + R$ and $B = e_2Q + R$ are not subrings of Q. Moreover, $A + B = e_1Q + e_2Q + R = Q$ and $A \cap B = R$.

THEOREM 5.5. *Let R be a semi-prime right and left Goldie-ring such that Q_R is a countably generated right R-module.*

a) *An R-module is h-divisible if and only if it is divisible in the classical sense. In particular, every R-module M which is divisible in the classical sense is weakly cotorsion.*
b) *$Z(D)$ is a direct summand of D whenever D is divisible in the classical sense.*
c) *R is a p.p.-ring if and only if the classes of h-divisible and divisible modules coincide.*

PROOF. a) By Theorem 5.2, it remains to show that every right R-module, which is divisible in the classical sense, is weakly cotorsion. Since the class of modules, which are divisible in the classical sense, is closed with respect to extensions, it suffices to consider a module D which is divisible in the classical sense, and to show that every epimorphism $\pi : D \to Q$ splits.

Suppose that $Q = \Sigma_{n<\omega} u_n R$ where $u_0 = 1$. We construct an ascending chain $\{V_n\}_{n<\omega}$ of submodules of Q_R such that $V_0 = R$, $V_n = d_n^{-1}R$ for some regular element d_n of R subject to the conditions $d_0 = 1$, $d_{n+1} = c_n d_n$ for some regular element c_n of R, and $u_n \in V_n$.

Suppose that regular elements $d_0, \ldots, d_n$ and submodules $V_0 \subseteq \ldots \subseteq V_n$ of R have already been constructed with the desired properties. Since Q is the classical right and left ring of quotients of R, there are $c_n, r_n \in R$ with c_n regular such that $d_n u_{n+1} = c_n^{-1} r_n$. Set $d_{n+1} = c_n d_n$ and $V_{n+1} = d_{n+1}^{-1}R$. Since $d_n^{-1} = d_{n+1}^{-1} c_n$, we obtain $V_n \subseteq V_{n+1}$. Moreover, $u_{n+1} = d_n^{-1} c_n^{-1} r_n = = d_{n+1}^{-1} r_n \in V_{n+1}$. Therefore, $Q = \cup_{n<\omega} V_n$.

Choose $s_0 \in D$ with $\pi(s_0) = 1 \in R$, and define $\delta_0(1) = s_0$. Suppose that we have already defined morphisms $\{\delta_i : V_i \to D | i = 0, \ldots, n\}$ such that $\delta_j | V_i = \delta_i$ for all $0 \leq i < j \leq n$ and $\pi\delta_i = 1_{V_i}$ for all $i = 0, \ldots, n$. Let $s_n = \delta_n(d_n^{-1})$. Because D is divisible in the classical sense, there is $s_{n+1} \in D$ such that $s_{n+1} c_n = s_n$. Since $V_{n+1} = d_{n+1}^{-1}R$ is a free R-module, setting $\delta_{n+1}(d_{n+1}^{-1}) = s_{n+1}$ induces an R-module map $\delta_{n+1} : V_{n+1} \to D$. In view of $d_n^{-1} = d_{n+1}^{-1} c_n$, we obtain $\delta_{n+1}(d_n^{-1}) = \delta_{n+1}(d_{n+1}^{-1}) c_n = s_{n+1} c_n = s_n = \delta_n(d_n^{-1})$; and $\delta_{n+1} | V_n = \delta_n$. Moreover, $\pi\delta_{n+1}(d_{n+1}^{-1}) c_n = \pi\delta_{n+1}(d_{n+1}^{-1} c_n) = \pi\delta_{n+1}(d_n^{-1}) =$

$= \pi\delta_n(d_n^{-1}) = d_n^{-1} = d_{n+1}^{-1}c_n$. Since Q is non-singular, $\pi\delta_{n+1}(d_{n+1}^{-1}) = d_{n+1}^{-1}$; and $\pi\delta_{n+1} = 1_{V_{n+1}}$. Then, $\delta : Q \to D$ defined by $\delta(x) = \delta_n(x)$ if $x \in V_n$ satisfies $\pi\delta = 1_Q$.

b) follows from Corollary 4.6 and part a).

c) If R is a p.p.-ring, then h-divisible modules are divisible by Corollary 4.2. Conversely, every divisible module D is divisible in the classical sense. By a), D is h-divisible.

On the other hand, if h-divisible modules and divisible modules coincide, then the class of divisible modules is closed with respect to epimorphic images. By Lemma 3.1, R is a p.p.-ring. □

If $R = Mat_2(\mathbb{Z}[x])$, then the class of h-divisible modules coincides with the class of modules which are divisible in the classical sense. However, there exist h-divisible modules which are not divisible since R is not a p.p.-ring [6, Theorem 8.17].

We now give an example that there may exist weakly cotorsion modules which are divisible in the classical sense but not h-divisible if R is not semi-prime:

EXAMPLE 5.6. Let R be any lower triangular matrix ring over a field F of order at least 2. By [8, Theorem 5.27], R is right and left hereditary. Hence, h-divisible modules are injective. However, since R is Artinian, all regular elements of R are units. Thus, every R-module is divisible in the classical sense. Furthermore, another application of [8, Theorem 5.27] yields that Q^r is flat as a right and left R-module. Since Q^r also is finitely generated, it is projective. Therefore, $\mathrm{Ext}_R^1(Q^r, M) = 0$ for all R-modules M. On the other hand, since R is not semi-prime, there exist an R-module which is not injective.

PROPOSITION 5.7. *Let R and S be a rings, and $\mathcal{F} : \mathcal{M}_R \to \mathcal{M}_S$ an equivalence with inverse $\mathcal{G}$.*

a) *If R is a semi-prime right and left Goldie-ring such that Q/R is a direct sum of countably generated modules, then the same holds for S.*
b) *If R has finite right Goldie-dimension, then $\mathcal{F}(D)$ is h-divisible whenever D is h-divisible.*

PROOF. a) It remains to show that $Q(S)/S$ is a direct sum of countably generated modules. Since $Q(S)$ is the injective hull of S, we obtain that $\mathcal{G}(Q(S))$ is the injective hull of $\mathcal{G}(S)$. Because $\mathcal{G}(S)$ is a finitely generated projective R-module, there is a projective module P such that $\mathcal{G}(S) \oplus P \cong R^n$ for some

$n < \omega$. Then, $\mathcal{G}(Q(S)) \oplus E(P) \cong Q^n$. Thus, $\mathcal{G}(Q(S))/\mathcal{G}(S) \oplus E(P)/P \cong (Q/R)^n$. By Kaplansky's Theorem, $\mathcal{G}(Q(S))/\mathcal{G}(S) = \oplus_I U_i$ where each U_i is countably generated. Since $\mathcal{F}(R)$ is finitely generated, $\mathcal{F}(U_i)$ is countably generated.

b) Since R has finite right Goldie-dimension, D is an epimorphic image of an injective. □

Combining the last two results shows that the property that the classes of divisible and h-divisible modules coincide is not Morita-invariant.

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