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Periodic-by-Nilpotent Linear Groups

B. A. F. WEHRFRITZ

ABSTRACT - Let G be a linear group of (finite) degree n and characteristic $p \geq 0$. Suppose that for every infinite subset X of G there exist distinct elements x and y of X with $\langle x, x^y \rangle$ periodic-by-nilpotent. Then G has a periodic normal subgroup T such that if $p > 0$ then G/T is torsion-free abelian and if $p = 0$ then G/T is torsion-free nilpotent of class at most max $\{1, n-1\}$ and is isomorphic to a linear group of degree n and characteristic zero. We also discuss the structure of periodic-by-nilpotent linear groups.

In [4] Rouabhi and Trabelsi prove that if G is a finitely generated soluble-by-finite group such that for every infinite subset X of G there exist distinct elements x and y of X with $\langle x, x^y \rangle$ periodic-by-nilpotent, then G is periodic-by-nilpotent, work that ultimately was prompted by very much earlier work of B.H.Neumann, see [4]. Throughout for any positive integer n we set $n' = \max\{1, n-1\}$. Here we prove the following.

THEOREM. *Let G be a linear group of (finite) degree n and characteristic $p \geq 0$. Suppose that for every infinite subset X of G there exist distinct elements x and y of X with $\langle x, x^y \rangle$ periodic-by-nilpotent. Then G is periodic-by-nilpotent. Further G has a periodic normal subgroup T such that if $p > 0$ then G/T is torsion-free abelian and if $p = 0$ then G/T is torsion-free nilpotent of class at most $n' = \max\{1, n-1\}$ and is isomorphic to a linear group of degree n and characteristic zero.*

If T is a periodic normal subgroup of some linear group G of degree n and characteristic zero, then G/T is always isomorphic to a linear group of

Indirizzo dell'A.: School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, England.
E-mail: b.a.f.wehrfritz@qmul.ac.uk
MSC: 20H20, 20F45.

characteristic zero (see [9]), but not necessarily of degree n, so the above situation is unusual. As a simple example, SL$(2,5)$ has a faithful representation of degree 2 over the complex numbers (indeed over $\boldsymbol{Q}(\sqrt{5}, \sqrt[5]{1})$), but the least degree of a faithful representation in characteristic zero of its image PSL(2, 5) $\cong$ Alt(5) is 3.

Not every torsion-free nilpotent group is isomorphic to a linear group; for example, a direct product D of infinitely many copies of the full unitriangular group $\mathrm{Tr}_1(3, \boldsymbol{Z})$ over the integers $\boldsymbol{Z}$ is torsion-free nilpotent of class 2, but is not isomorphic to any linear group of any characteristic. Every torsion-free abelian group is isomorphic to a linear group of arbitrary characteristic ([6] 2.2) and any finitely generated torsion-free nilpotent group is isomorphic to a unipotent linear group over the integers (see [6] Page 23), so the counter example D above is about as small as one can get.

For any group G denote its hypercentre by $\zeta(G)$ and the i-th terms of its upper and lower central series by $\zeta_i(G)$ and $\gamma^i G$ respectively (where $\zeta_0(G) = \langle 1\rangle$ and $\gamma^1 G = G$). Let G be a group. If $\gamma^{m+1}G$ is finite, then $G/\zeta_{2m}(G)$ is finite (if G is finitely generated even $G/\zeta_m(G)$ is finite) and if $G/\zeta_m(G)$ is finite, then $\gamma^{m+1}G$ is finite, see [3] 4.25, 4.24 and 4.21, Corollary 2. Something similar happens for periodic-by-nilpotent linear groups.

PROPOSITION. *Let G be a linear group of degree n and characteristic $p \geq 0$.*

a) Suppose that $\gamma^{m+1}G$ is periodic for some integer $m \geq 0$ and that $O_p(G) = \langle 1\rangle$ if $p > 0$. Then $G/\zeta_m(G)$ (and $\gamma^{m+1}G$) are locally finite.

b) If $G/\zeta(G)$ is periodic, then $\gamma^{n'+1}G$ and $G/\zeta_{n'}(G)$ are locally finite.

c) If $G/\zeta_m(G)$ is periodic for some integer $m \geq 0$, then $\gamma^{m+1}G$ and $G/\zeta_m(G)$ are locally finite.

Of course Part *c*) only adds to Part *b*) in the Proposition for $m < n'$ and Part *a*) for $m > n'$ adds nothing to the case $m = n'$ by the Theorem. Perhaps Part *b*) is a slight surprise, since if $G/\zeta(G)$ is finite there is no need for any $\gamma^i G$ to be finite, even if G is also linear (consider the infinite locally dihedral 2-group). If G is any group with $G/\zeta_m(G)$locally finite, then $\gamma^{m+1}G$ is easily seen to be locally finite.

If G is the wreath product of a cyclic group of prime order p by an infinite cyclic group, then G is isomorphic to a triangular linear group of degree 2 and characteristic p with $\gamma^2 G$ periodic (even elementary abelian) and yet $\zeta(G) = \langle 1\rangle$. Thus the extra hypothesis if $p > 0$ in Part a) cannot be removed. There is no obvious analogue to Part b) in the context of Part a); if G is a non-cyclic free group, then G is isomorphic to a linear group of degree 2 in any characteristic and yet $\gamma^{\omega}G = \langle 1 \rangle = \zeta(G)$. Note that there exist hypercentral linear groups of infinite central height, even periodic ones, see [6] 8.3, so for example in Part *b*) there is no need for $\zeta(G)$ and $\zeta_{n'}(G)$ to be equal.

Let G be any group. Denote its unique maximal periodic normal subgroup by $\tau(G)$ and its unique maximal normal π-subgroup for π some set of primes by $O_\pi(G)$. If G is linear, G^0 denotes its connected component containing the identity (relative to the Zariski topology).

Proof of the Theorem. To begin with, note that if G is a torsion-free, locally nilpotent group with a normal subgroup H such that G/H is periodic and such that H is nilpotent of class c, then G is nilpotent of class c. This can be derived either from isolator theory, see [2] 2.3.9, or from the Zariski topology using [5]. 5.11 and Point 3 on Page 23.

Suppose G is a subgroup of $GL(n, F)$, where F is an algebraically closed field of characteristic $p \geq 0$. If G is not soluble-by-(locally finite), then G contains a free subgroup on an infinite set X by Tits' Theorem, see [6] 10.17. Then $\langle x, x^y \rangle$ is free of rank 2 for every pair of distinct elements x and y of X. Consequently G is soluble-by-(locally finite). By Rouabhi & Trabelsi's theorem, see [4], the group G is locally (periodic-by-nilpotent) and hence is locally (periodic-by-(torsion-free nilpotent)). Therefore G is periodic-by-(torsion-free, locally nilpotent).

Set $T = \tau(G)$ and suppose p is positive. Clearly $O_p(G) \leq T$, so G/T is isomorphic to a torsion-free, locally nilpotent, linear group of characteristic p by Corollary 1 of [9]. Then G/T is also abelian-by-finite by [6] 3.6 and consequently G/T is abelian by the remark at the beginning of this proof. This settles the positive characteristic case.

From now on assume that $p = 0$. Set $C = C_G(T)$. Then G/CT is finite by [5] 5.1.6. Also C is locally nilpotent, so C has a Jordan decomposition

$$C \leq C_u \times C_d = C_u C = C C_d = GL(n, F),$$

see [6] Chapter 7, especially 7.14 and 7.13 (recall F here is algebraically closed). Here C_u is unipotent, torsion-free and nilpotent of class less than n. Set $P = \tau(C_d)$. Then C_d/P is torsion-free, locally nilpotent and abelian-by-finite ([6] 7.7 & 3.5). Therefore C_d/P is abelian. Consequently $P = \tau(C_u C_d)$, $C \cap T = C \cap P$ and $CT/T \cong C/(C \cap P)$ is nilpotent of class at most $\max\{n-1, 1\} = n'$. But then G/T is torsion-free, locally nilpotent and has a nilpotent subgroup CT/T of finite index and class at most n'. Therefore G

is torsion-free and nilpotent of class at most n', again by our remark at the beginning.

Finally G/T is isomorphic to a linear group over F of n-bounded degree by the theorem of [9], but we need to ensure it is actually isomorphic to a linear group of degree n and characteristic zero. If $n = 1$, then G is abelian and clearly G/T embeds into $F^* = GL(1, F)$, since F^* is divisible and splits over $\tau(F^*)$. Suppose $n > 1$. If K is an extension field of F, then the centre of the unitriangular group $\mathrm{Tr}_1(n, K)$ is isomorphic to the additive group of K and hence is equal to $Z \times R$ for Z the centre of $\mathrm{Tr}_1(n, F)$ and R a direct sum of copies of the additive group of the rationals. Further C_u is isomorphic to a subgroup of $\mathrm{Tr}_1(n, F)$ and C_d/P is embeddable in R for a suitably large K. In which case C_uC_d/P is isomorphic to a subgroup of $\mathrm{Tr}_1(n, K)$ and hence so too is $CT/T \cong CP/P \leq C_uC_d/P$. Now $\mathrm{Tr}_1(n, K)$ is torsion-free, nilpotent and divisible. Thus $\mathrm{Tr}_1(n, K)$ contains a divisible completion D of CT/T. Since G/T is torsion-free, nilpotent and of finite index over CT/T, so D contains a copy of G/T. (See [2] Chap. 2, especially 2.1.1, for divisible completions of nilpotent groups.). Therefore G/T is isomorphic to a linear group of degree n and characteristic zero. The proof is complete.

As an example of an application of this theorem, we have the following.

Corollary. *Let G be a soluble-by-finite group with finite Hirsch number. Suppose that for every infinite subset X of G there exist distinct elements x and y of X with $\langle x, x^y \rangle$ periodic-by-nilpotent. Then G is periodic-by-nilpotent.*

Proof. For if $T = \tau(G)$, then G/T has a torsion-free soluble normal subgroup of finite rank and index. Then G/T is isomorphic to a linear group over the rationals (e.g. [7] 1.2) and applying the theorem to G/T yields the corollary. Alternatively it follows from [4] as follows. By [4] the group G is locally periodic-by-nilpotent, so G is periodic-by-(torsion-free and locally nilpotent of finite rank). Consequently G is periodic-by-nilpotent by a theorem of Mal'cev ([3] 6.36).

The Proposition follows at once from the following three lemmas.

Lemma 1. *Let G be a linear group of degree n and characteristic $p \geq 0$ such that $O_p(G) = \langle 1 \rangle$ if $p > 0$. If there exists an integer $m \geq 0$ such that $\gamma^{m+1}G$ is periodic, then $G/\zeta_m(G)$ is locally finite.*

Note that here $\gamma^{m+1}G$ is locally finite by [6] 4.9.

Proof. Let X be a finitely generated subgroup of G. Suppose first that $p = 0$. Then $\gamma^{m+1}X$ is finite by [6] 4.8 and therefore $X/\zeta_m(X)$ is finite by [3] 4.24. Also $\zeta_m(X)$ is closed in X by [6] 5.10, so $X^0 \leq \zeta_m(X)$. If Y is a finitely generated subgroup of G containing X, then

$$X^0 \leq X \cap Y^0 \leq \zeta_m(Y).$$

Thus $X^0 \leq \zeta_m(G)$. Set $G^* = \bigcup_X X^0$. Then G^* is a normal subgroup of G with G/G^* locally finite and $G^* \leq \zeta_m(G)$. The case $p = 0$ follows.

Now assume that $p > 0$. Here [6] 4.8 only yields that $\gamma^{m+1}X$ is a finite extension of a p-group. Define $Z_i(X)$ by

$$Z_i(X)/O_p(X) = \zeta_i(X/O_p(X)).$$

Then $X/Z_m(X)$ is finite by [3] 4.24 and $O_p(X)$ and $Z_m(X)$ are closed in X, so $X^0 = Z_m(X)$. Thus $X^0 \leq \bigcap_{Y \geq X} X \cap Z_m(Y)$ and a simple localizing argument (cf. the previous paragraph) yields that $[G^*, {}_mG]$ is a p-group. Clearly it is normal in G and $O_p(G) = \langle 1 \rangle$. Therefore $G^* \leq \zeta_m(G)$ and the lemma follows.

Lemma 2. *Let G be a linear group of degree n with $G/\zeta_m(G)$ periodic for some integer $m \geq 0$. Then $\gamma^{m+1}G$ and $G/\zeta_m(G)$ are locally finite.*

Proof. Now $\zeta_m(G)$ is closed in G, so $G/\zeta_m(G)$ is isomorphic to a periodic linear group ([6] 6.4) and so is locally finite ([6] 4.9). Then [3] 4.21, Corollary 2, yields that $\gamma^{m+1}X$ is finite for every finitely generated subgroup X of G. Consequently $\gamma^{m+1}G$, which equals $\bigcup_X \gamma^{m+1}X$, is locally finite.

Lemma 3. *Let G be a linear group of degree n and characteristic $p \geq 0$ such that $G/\zeta(G)$ is periodic. Then $\gamma^{n'+1}G$ and $G/\zeta_{n'}(G)$ are locally finite.*

Proof. We may assume that the ground field F of G is algebraically closed. If $g \in GL(n, F)$, let $g = g_u g_d = g_d g_u$ be its Jordan decomposition (see [6] Chapter 7). Set

$$K = \langle g_u, g_d : g \in \zeta(G) \rangle \leq GL(n, F)$$

Then $K = \zeta(GK)$, $G \cap K = \zeta(G)$ and $K = K_u \times K_d$, where K_u is unipotent and K_d is a d-subgroup, see [6] 7.17, 7.14 and 7.13. Also $GK/K \cong G/\zeta(G)$, which is periodic.

By the theorem of [8] we have $K_u \leq \zeta_{n'}(GK)$. Set $D = (K_d)^0$. Then D is a diagonalizable normal subgroup of GK by [6] 7.7 and 5.8. Let π denote the finite set of all primes not exceeding n. Then $O_\pi(D)$ has finite rank (at most n), so D splits over $O_\pi(D)$ by [1] 21.2 and 27.5, say $D = O_\pi(D) \times E$. Also

$GK/C_{GK}(D)$ is a finite π-group by [6] 1.12. Then $H = \bigcap_{g \in GK} E^g$ is a normal subgroup of GK with $O_\pi(H) = \langle 1 \rangle$ and D/H a periodic π-group. Also $[H, GK]$ is a π-group by [6] 4.14. Therefore $[H, GK] = \langle 1 \rangle$. We have now shown that $K_u \times H = \cong \zeta_{n'}(GK)$. Since GK/K and D/H are periodic and K_d/D is finite, so $GK/\zeta_{n'}(GK)$ is periodic. It follows that $G/\zeta_{n'}(G)$ is periodic. Finally $\gamma^{n'+1}G$ and $G/\zeta_{n'}(G)$ are locally finite by Lemma 2.

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