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Rad-supplemented Modules

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ABSTRACT - Let τ be a radical for the category of left R-modules for a ring R. If M is a τ-coatomic module, that is, if M has no nonzero τ-torsion factor module, then $\tau(M)$ is small in M. If V is a τ-supplement in M, then the intersection of V and $\tau(M)$ is $\tau(V)$. In particular, if V is a Rad-supplement in M, then the intersection of V and $\mathrm{Rad}(M)$ is $\mathrm{Rad}(V)$. A module M is τ-supplemented if and only if the factor module of M by $P_\tau(M)$ is τ-supplemented where $P_\tau(M)$ is the sum of all τ-torsion submodules of M. Every left R-module is Rad-supplemented if and only if the direct sum of countably many copies of R is a Rad-supplemented left R-module if and only if every reduced left R-module is supplemented if and only if $R/P(R)$ is left perfect where $P(R)$ is the sum of all left ideals I of R such that $\mathrm{Rad}\, I = I$. For a left duo ring R, R is a Rad-supplemented left R-module if and only if $R/P(R)$ is semiperfect. For a Dedekind domain R, an R-module M is Rad-supplemented if and only if M/D is supplemented where D is the divisible part of M.

1. Introduction.

All rings considered in this paper will be associative with an identity element. Unless otherwise stated R denotes an arbitary ring and all modules will be *left* unitary R-modules. By R-$\mathcal{M}od$, we denote the category of left R-modules. Unless otherwise stated, τ is a radical on

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R-$\mathcal{M}od$. For fundamentals on module theory, see for example [17], [4] and [30]. Let R be a ring and M be an R-module. Denote by $X \leq M$ that X is a submodule of M. As usual, Rad$M$ denotes the radical of M and $J(R)$ denotes the Jacobson radical of the ring R. A submodule K of M is called *small* in M (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule T of M. For an index set I, $M^{(I)}$ denotes as usual the direct sum $\bigoplus_{i\in I} M$. The set of natural numbers is denoted by $\mathbb{N}$. See [30, § 41] and the recent monograph [10] for results (and the definitions) related to (weak) supplements and (weakly) supplemented modules. Given submodules $K \leq L \leq M$, the inclusion $K \leq L$ is called *cosmall in M* if $L/K \ll M/K$ (see [10, 3.1]). A submodule $L \leq M$ is called *coclosed in M* if L has no proper submodule K for which the inclusion $K \leq L$ is cosmall in M (see [10, 3.6]).

We shall investigate some properties of Rad-supplemented modules and in general τ-supplemented modules where τ is a radical for R-$\mathcal{M}od$. The motivation for considering Rad-supple-ments (coneat submodules) and τ-supplements in general is given in the next section. One of the main questions we shall answer is when are all left R-modules Rad-supplemented. In the investigation of this problem, the notion of radical modules, reduced modules and coatomic modules turn out to be useful; see [32, pp. 47]. In the definitions and properties for reduced and coatomic modules, instead of Rad, we can use any (pre)radical τ on R-$\mathcal{M}od$ (see Section 3), and these will be useful in the investigation of the properties of τ-supplemented modules. For a module M, the sum of all radical submodules of M is denoted by $P(M)$, that is, $P(M)$ is the sum of all submodules U of M such that Rad$U = U$. For submodules U and V of a module M, the submodule V is said to be a *Rad-supplement* of U in M or U is said to *have a Rad-supplement* V in M if $U + V = M$ and $U \cap V \leq$ Rad$V$. A module M is called a *Rad-supplemented module* if every submodule of M has a Rad-supplement in M. See also [29]; Rad-supplemented modules are called generalized supplemented modules there. In Section 6, we shall prove that every left R-module is Rad-supplemented if and only if $R/P(R)$ is left perfect. In [9], it is proved that the class of Rad-supplemented rings lies properly between those of the semiperfect and the semilocal rings. We show that a left duo ring R is Rad-supplemented as a left R-module if and only if $R/P(R)$ is semiperfect. Whenever possible the related results are given in general for a radical τ for R-$\mathcal{M}od$. See [1] and [10, § 10] for some properties of τ-supplements and τ-supplemented modules. We shall investigate some further properties of τ-supplemented modules in Section 4. For some rings R, we shall also determine when all left R-modules are τ-supplemented in Section 5. We are also going to study the property $\mathrm{Rad}V = V \cap \mathrm{Rad}M$ for a submodule V of M. It is known that this holds if V is a supplement in M (see [30, 41.1]) and moreover if V is coclosed in M (see [10, 3.7]). We show that this property also holds when V is a Rad-supplement in M (Corollary 4.2); in general for a radical τ for R-$\mathcal{M}od$, we show that if V is a τ-supplement in M, then $\tau(V) = V \cap \tau(M)$. It is clear that every supplemented module is Rad-supplemented. But the converse implication fails to be true. For example, the $\mathbb{Z}$-module $\mathbb{Q}$ is Rad-supplemented but not supplemented. Since $\mathrm{Rad}\mathbb{Q} = \mathbb{Q}$ (see for example [17, 2.3.7]), $\mathbb{Q}$ is Rad-supplemented (by Proposition 4.5-(i)). But $\mathbb{Q}$ is not supplemented by example [10, 20.12]. In Section 7, we understand this example clearly and describe Rad-supplemented modules over Dedekind domains using the structure of supplemented modules over Dedekind domains which was completely determined in [32].

For definitions and elementary properties of preradicals, see [26, Ch. VI], [6] or [10, § 6]. A *preradical* τ for R-$\mathcal{M}od$ is defined to be a subfunctor of the identity functor on R-$\mathcal{M}od$. Let τ be a preradical for R-$\mathcal{M}od$. The following module classes are defined: the *preradical* or *(pre)torsion class* of τ is

$$\mathbb{T}_\tau = \{N \in R\text{-}\mathcal{M}od \mid \tau(N) = N\}$$

and the *preradical free* or *(pre)torsion free class* of τ is

$$\mathbb{F}_\tau = \{N \in R\text{-}\mathcal{M}od \mid \tau(N) = 0\}.$$

τ is said to be *idempotent* if $\tau(\tau(N)) = \tau(N)$ for every R-module N. τ is said to be a *radical* if $\tau(N/\tau(N)) = 0$ for every R-module N. For the main elementary properties that we shall use frequently for a (pre)radical, see for example [10, pp. 55]. For R-modules $K \leq M$, we always have $(\tau(M) + K)/K \leq \tau(M/K)$. If moreover τ is a *radical* and $K \leq \tau(M)$, then $\tau(M/K) = \tau(M)/K$ [26, Ch. VI, Lemma 1.1]. When we consider a ring R as a left R-module, we already have that $A = \tau({}_RR)$ is a left ideal of R; indeed it is a two-sided ideal of R [26, Ch. VI, § 1, Examples (3), pp. 139] so that we can consider the quotient ring R/A which we shall use in the results for τ-supplemented modules. For a free R-module F, the property $\tau(F) = \tau(R)F$ is easily obtained. This also holds for projective modules. See also [13] and [7] for some related concepts in torsion theories (mostly for a hereditary preradical).

2. Coneat submodules and Rad-supplements.

Neat subgroups of abelian groups (introduced in [15, pp. 43-44]) have been generalized to modules in [28, 9.6] (and [27, § 3]). The class of coneat submodules has been introduced in [21] and [3]: A monomorphism $f : K \to L$ is called *coneat* if each module M with $\mathrm{Rad}M = 0$ is *injective* with respect to it, that is, the Hom sequence

$$\mathrm{Hom}(L, M) \to \mathrm{Hom}(K, M) \to 0$$

is exact. See [21, Proposition 3.4.2] or [10, 10.14] or [1, 1.14] for a characterization of coneat submodules. This characterization will be the particular case $\tau = \mathrm{Rad}$ in Proposition 2.1 and this is the reason for considering Rad-supplements and in general τ-supplements given below. For more results on coneat submodules see [21], [3], [10, § 10 and 20.7-8], [1] and [24].

Proper classes of monomorphisms and short exact sequences were introduced in [8] to do *relative* homological algebra. In [27, Remark after Proposition 6], it is pointed out that supplement submodules induce a proper class of short exact sequences (the term 'low' is used for supplements dualizing the term 'high' used in abelian groups). [12] uses the terminology 'cohigh' for supplements and gives more general definitions for proper classes of supplements related to another given proper class (motivated by the considerations as pure-high extensions and neat-high extensions in [14]). For the definition and properties of *proper classes*, see [25], [20, Ch. 12, § 4], [28] and [22]. We shall follow the terminology and notation as in [10, § 10] and [1] since we will mainly refer to these for τ-supplemented modules and Rad-supplemented modules.

Denote by $\mathbb{E}_{Suppl}$ the class of all short exact sequences induced by supplement submodules; that is $\mathbb{E}_{Suppl}$ is the class of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms such that $\mathrm{Im}(f)$ is a supplement in B. Then as mentioned above, the class $\mathbb{E}_{Suppl}$ forms a *proper* class, see for example [10, 20.7]. Every module M with $\mathrm{Rad}M = 0$ is $\mathbb{E}_{Suppl}$-injective that is M is injective with respect to every short exact sequence in $\mathbb{E}_{Suppl}$. Thus supplement submodules are coneat submodules by the definition of coneat submodules. In the definition of coneat submodules, using any radical τ instead of Rad, the following result is obtained. It gives us the definition of a *τ-supplement* in a module because the last condition is like the usual supplement condition except that, instead of $U \cap V \ll V$, the condition $U \cap V \leq \tau(V)$ is required.

PROPOSITION 2.1 (see [10, 10.11] or [1, 1.11]). *Let τ be a radical for R-$\mathcal{M}od$. For a submodule $V \leq M$, the following statements are equivalent.*

(i) *Every module N with $\tau(N) = 0$ is injective with respect to the inclusion $V \hookrightarrow M$;*

(ii) *there exists a submodule $U \leq M$ such that*

$$U + V = M \text{ and } U \cap V = \tau(V);$$

(iii) *there exists a submodule $U \leq M$ such that*

$$U + V = M \text{ and } U \cap V \leq \tau(V).$$

If these conditions are satisfied, then V is called a τ-supplement in M.

The usual definitions are then given as follows. For submodules U and V of a module M, the submodule V is said to be a *τ-supplement* of U in M or U is said to *have a τ-supplement* V in M if $U + V = M$ and $U \cap V \leq \tau(V)$. A module M is called a *τ-supplemented module* if every submodule of M has a τ-supplement in M. We call M *totally τ-supplemented* if every submodule of M is τ-supplemented. A submodule N of M is said to have *ample τ-supplements in* M if for every $L \leq M$ with $N + L = M$, there is a τ-supplement L' of N with $L' \leq L$. A module M is said to be *amply τ-supplemented* if every submodule of M has ample τ-supplements in M.

For $\tau = \text{Rad}$, the above definitions give *Rad-supplement submodules* of a module, *Rad-supplemented modules*, etc. By these definitions, a submodule V of a module M is a *coneat* submodule of M if and only if V is a *Rad-supplement* of a submodule U of M in M.

3. τ-reduced and τ-coatomic modules, and the largest τ-torsion submodule $P_\tau(M)$.

Let τ be a preradical for R-$\mathcal{M}od$ and let M be an R-module. By taking τ instead of Rad in the definitions of reduced and coatomic module definitions in [32, pp. 47], we define the following:

(i) M is said to be a *τ-torsion* module if $\tau(M) = M$, that is M is in the pretorsion class $\mathbb{T}_\tau$.

(ii) By $P_\tau(M)$ we denote the sum of *all* τ-torsion submodules of M, that is,

$$P_\tau(M) = \sum\{U \leq M \mid \tau(U) = U\}.$$

(iii) M is said to be a *τ-reduced* module if it has *no* nonzero τ-torsion submodule, that is, for every submodule U of M, $\tau(U) = U$ implies $U = 0$; equivalently, $\tau(U) \neq U$ for every nonzero submodule U of M. Clearly, M is τ-reduced if and only if M is P_τ-torsion free, that is, $P_\tau(M) = 0$.

(iv) M is said to be a *τ-coatomic* module if it has *no* nonzero τ-torsion factor module, that is, for every submodule U of M, $\tau(M/U) = M/U$ implies $U = M$; equivalently, $\tau(M/U) \neq M/U$ for every proper submodule U of M.

For $\tau = \text{Rad}$, $P_\tau(M)$ will be denoted by just $P(M)$, a Rad-torsion module is called a *radical module*, a Rad-reduced module will be called a *reduced module* and a Rad-coatomic module will be called a *coatomic module* following the terminology in [32]. Coatomic modules appear in the theory of supplemented, semiperfect, and perfect modules. See [32, Lemma 1.5] for some properties of reduced and coatomic modules. For the structure of coatomic modules over commutative Noetherian rings see [33]; the Noetherian assumption is needed to have that every submodule of a coatomic module over a commutative Noetherian ring is coatomic [33, Lemma 1.1].

For completeness note the following elementary properties of $P_\tau(M)$:

THEOREM 3.1. *Let τ be a preradical for R-$\mathcal{M}od$ and let M be an R-module.*

(i) P_τ is an idempotent preradical.

(ii) *If $M \leq N$ for a module N, then $P_\tau(M) \leq \tau(N)$. In particular, $P_\tau(M) \leq \tau(M)$.*

(iii) *$\tau(P_\tau(M)) = P_\tau(M)$, that is, $P_\tau(M)$ is τ-torsion, and so by its definition $P_\tau(M)$ is the largest τ-torsion submodule of M.*

(iv) *If $P_\tau(M) \leq V$ for a submodule V of M, then $P_\tau(M) \leq \tau(V)$.*

(v) *$P_\tau(\tau(M)) = P_\tau(M)$*

(vi) *The pretorsion class of P_τ equals the pretorsion class of τ and the pretorsion free class of P_τ contains the pretorsion free class of τ:*

$$\mathbb{T}_{P_\tau} = \mathbb{T}_\tau \qquad \text{and} \qquad \mathbb{F}_{P_\tau} \supseteq \mathbb{F}_\tau.$$

(vii) *Moreover, if τ is a radical, then the factor module $M/P_\tau(M)$ is τ-reduced, that is, $P_\tau\big(M/P_\tau(M)\big) = 0$ and so P_τ is an* idempotent radical.

REMARK 3.2. In general, given any class $\mathbb{A}$ of modules, a *preradical* $\tau^{\mathbb{A}}$ is defined by setting for each module N,

$$\tau^{\mathbb{A}}(N) = \sum\{\operatorname{Im} f \mid f : A \to N \text{ in } R\text{-}\mathcal{M}od,\ A \in \mathbb{A}\}.$$

and if $\mathbb{A}$ is a pretorsion class, then $\tau^{\mathbb{A}}$ is an idempotent preradical (see for example [10, 6.5-6]). In our case, the preradical P_τ is equal to $\tau^{\mathbb{A}}$ when the pretorsion class $\mathbb{A} = \mathbb{T}_\tau$, the torsion class of τ. See also [26, Ch. VI, § 1]; P_τ is the largest idempotent preradical that is smaller than τ and see [26, Ch. VI, Exercise 4, p. 157] for the properties Theorem 3.1-(iii,v). Since P_τ is an idempotent radical when τ is a radical, it gives a torsion theory for $R\text{-}\mathcal{M}od$ with torsion class $\mathbb{T}_{P_\tau} = \mathbb{T}_\tau$ and torsion free class $\mathbb{F}_{P_\tau}$. By the results in [26, Ch. VI, § 2], the properties for τ-torsion and τ-reduced modules in the following Proposition 3.4 are obtained because τ-torsion modules equate with P_τ-torsion modules and τ-reduced modules form the torsion free class $\mathbb{F}_{P_\tau}$.

REMARK 3.3. See [13, pp. 29, 63] for the definitions and properties of τ-dense submodules of a module and τ-cotorsionfree modules for a *hereditary* idempotent preradical τ on $R\text{-}\mathcal{M}od$: A submodule N of a module M is said to be τ-*dense* in M if M/N is τ-torsion, that is, $\tau(M/N) = M/N$, and a module M is said to be τ-*cotorsionfree* if it has *no* proper τ-dense submodules. Our definition of τ-coatomic module coincides with τ-cotorsionfree module but in our case, τ need not be idempotent or hereditary. Observe that since being τ-torsion is the same with being P_τ-torsion and P_τ is an idempotent preradical, the idempotent assumption is not a problem. But in our case τ is not assumed to be hereditary; in particular, Rad is not hereditary. The properties for τ-cotorsionfree modules given in [13] hold under this *hereditary* assumption. For example, arbitary direct sum of τ-cotorsionfree modules is τ-cotorsionfree when τ is a *hereditary* idempotent preradical but in our case, for just an (idempotent) preradical τ, arbitrary direct sum of τ-coatomic modules need not be τ-coatomic.

Note also the following properties of τ-reduced and τ-coatomic modules which are easily proved:

PROPOSITION 3.4. *Let τ be a preradical for $R\text{-}\mathcal{M}od$.*

(i) *The class of τ-torsion modules is closed under quotients and direct sums. Moreover, if τ is a radical, then the class of τ-torsion modules is closed under extensions.*

(ii) *The class of τ-reduced modules is closed under submodules, direct products and direct sums.*

(iii) *Every factor module of a τ-coatomic module is τ-coatomic.*

(iv) *The class of τ-reduced, respectively τ-coatomic, modules is closed under extensions, that is, if*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of modules such that A and C are τ-reduced, respectively τ-coatomic, then B is also τ-reduced, respectively τ-coatomic.

PROPOSITION 3.5. *Let τ be a radical for R-$\mathcal{M}od$. If a module M is τ-coatomic, then $\tau(M) \ll M$.*

PROOF. Suppose $\tau(M) + L = M$ for some submodule $L \leq M$. Since $M/L = (\tau(M) + L)/L \leq \tau(M/L)$, we obtain $M/L = \tau(M/L)$. This gives $L = M$ since M is τ-coatomic. Hence $\tau(M) \ll M$. □

4. τ-supplemented modules.

Throughout the rest of the paper, τ denotes a radical on R-$\mathcal{M}od$ (where R is an arbitrary ring). See [1] and [10, § 10] for properties of τ-supplements and τ-supplemented modules. In this section, we shall see some other properties of τ-supplemented modules. We shall frequently use the fact that any factor module of a τ-supplemented module is τ-supplemented [1, 2.2(2)].

THEOREM 4.1. *If V is a τ-supplement in a module M, then $\tau(V) = V \cap \tau(M)$.*

PROOF. $\tau(V) \leq V \cap \tau(M)$ always holds. To show the converse we only require to show that $(V \cap \tau(M))/\tau(V) = 0$. Since V is a τ-supplement in M, there exists a submodule $U \leq M$ such that $U + V = M$ and $U \cap V = \tau(V)$ by Proposition 2.1-(ii)). Then

$$M/(U \cap V) = (U/(U \cap V)) \oplus ((V/U \cap V)) = (U/\tau(V) \oplus (V/\tau(V)).$$

Since τ is a radical, we obtain:

$$\tau(M/\tau(V)) = \tau(U/\tau(V)) \oplus \tau(V/\tau(V)) = \tau(U/\tau(V)) \oplus 0 = \tau(U/\tau(V)).$$

By properties of a radical, since $\tau(V) \leq \tau(M)$, we have:

$$\tau(M)/\tau(V) = \tau(M/\tau(V)) = \tau(U/\tau(V)), \quad \textit{and}$$

$$\begin{aligned}(V \cap \tau(M))/\tau(V) &= (V/\tau(V)) \cap (\tau(M)/\tau(V)) = (V/\tau(V)) \cap \tau(U/\tau(V)) \\ &\leq (V/\tau(V)) \cap (U/\tau(V)) \\ &= (U \cap V)/\tau(V) = \tau(V)/\tau(V) = 0.\end{aligned}$$

□

Corollary 4.2. *If V is a Rad-supplement in a module M, then*

$$\mathrm{Rad}V = V \cap \mathrm{Rad}M.$$

Proposition 4.3. *Let K, L, M be modules such that $K \leq L \leq M$.*

(i) *If K is a τ-supplement in M, then it is a τ-supplement in L.*

(ii) *If $K \leq \tau(L)$ and L/K is a τ-supplement in M/K, then L is a τ-supplement in M.*

(iii) *If K is a τ-supplement in L and L is a τ-supplement in M, then K is a τ-supplement in M.*

Proof. (i) Since K is a τ-supplement in M, there exists a submodule $U \leq M$ such that $U + K = M$ and $U \cap K \leq \tau(K)$. So $L = L \cap M = L \cap (U + K) = L \cap U + K$ and $(L \cap U) \cap K = U \cap K \leq \tau(K)$.

(ii) Since L/K is a τ-supplement in M/K, there exists a submodule $U \leq M$ with $K \leq U$ such that $U/K + L/K = M/K$ and $(U/K) \cap (L/K) \leq \tau(L/K)$. So we obtain $U + L = M$ and

$$(U \cap L)/K = (U/K) \cap (L/K) \leq \tau(L/K) = \tau(L)/K$$

by properties of a radical since $K \leq \tau(L)$. Hence $U \cap L \leq \tau(L)$ and so L is a τ-supplement (of U) in M.

(iii) Temporarily denote by $\mathbb{E}$ the class induced by τ-supplement submodules; that is $\mathbb{E}$ is the class of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms such that $\mathrm{Im}(f)$ is a τ-supplement in B. For such a short exact sequence in the class $\mathbb{E}, f$ is said to be an $\mathbb{E}$-monomorphism. By Proposition 2.1, the class $\mathbb{E}$ is the proper class injectively generated by all modules M such that $\tau(M) = 0$. By the definition of proper classes, the composition of two $\mathbb{E}$-monomorphisms is an $\mathbb{E}$-monomorphism (see [10, 10.1]). If K is a τ-supplement in L and L is a τ-supplement in M, then the inclusions $K \hookrightarrow L$ and $L \hookrightarrow M$ are $\mathbb{E}$-monomorphisms and so their composition $K \hookrightarrow M$ is also an $\mathbb{E}$-monomorphism, that is, K is a τ-supplement in M. □

PROPOSITION 4.4. *Let M be a module and let N, K be submodules of M such that $M = N + K$. If K is τ-supplemented, then K contains a τ-supplement of N in M.*

PROOF. Since K is τ-supplemented, the submodule $N \cap K$ of K has a τ-supplement in K, that is, there exists a submodule $L \leq K$ such that $(N \cap K) + L = K$ and $(N \cap K) \cap L \leq \tau(L)$. Then $M = N + K = N + (N \cap K) + L = N + L$ and $N \cap L = (N \cap K) \cap L \leq \tau(L)$. Hence L is a τ-supplement of N in M. □

It is trivial to show that:

PROPOSITION 4.5.

(i) *Every τ-torsion module is τ-supplemented.*
(ii) *The module $P_\tau(M)$ is τ-supplemented for every module M.*

THEOREM 4.6. *If a module M is τ-reduced and τ-supplemented, then M is τ-coatomic,* $\mathrm{Rad}M = \tau(M)$ *and M is weakly supplemented.*

PROOF. Let U be a proper submodule of M. Since M is τ-supplemented, there exists a submodule $V \leq M$ such that $U + V = M$ and $U \cap V \leq \tau(V)$. So we have $\tau(V/(U \cap V)) = \tau(V)/(U \cap V)$ by properties of a radical. We also have $\tau(V) \neq V$ since M is τ-reduced, and so $\tau(V)/(U \cap V) \neq V/(U \cap V)$. Therefore, using the fact that $M/U = (U + V)/U \cong V/(U \cap V)$ we obtain

$$\tau(M/U) \cong \tau(V/(U \cap V)) = \tau(V)/(U \cap V) \neq V/(U \cap V),$$

or equivalently, $\tau(M/U) \neq M/U$, that is, M is τ-coatomic. By Proposition 3.5, $\tau(M) \ll M$ and hence $\tau(M) \leq \mathrm{Rad}M$. By [1, 2.2(3)], $M/\tau(M)$ is semisimple since M is τ-supplemented. Then $\mathrm{Rad}(M/\tau(M)) = 0$ and so $\mathrm{Rad}M \leq \tau(M)$. Thus $\mathrm{Rad}M = \tau(M)$. Since $\mathrm{Rad}M = \tau(M) \ll M$ and M is a semilocal module (that is $M/\mathrm{Rad}M = M/\tau(M)$ is semisimple), we obtain that M is weakly supplemented by [19, Theorem 2.7]. □

THEOREM 4.7. *If M is a τ-supplemented module, then* $\mathrm{Rad}M \leq \tau(M)$, *and*

$$\mathrm{Rad}(M/P_\tau(M)) = \tau(M/P_\tau(M)) = \tau(M)/P_\tau(M).$$

PROOF. By [1, 2.2(3)], $M/\tau(M)$ is semisimple and so $\mathrm{Rad}(M/\tau(M)) = 0$ which gives $\mathrm{Rad}M \leq \tau(M)$. The module $M/P_\tau(M)$ is τ-supplemented as a factor module of the τ-supplemented module M. Since $M/P_\tau(M)$ is τ-reduced, $\mathrm{Rad}(M/P_\tau(M)) = \tau(M/P_\tau(M))$ by Theorem 4.6. By properties of a radical, $\tau(M/P_\tau(M)) = \tau(M)/P_\tau(M)$. □

PROPOSITION 4.8. *The following are equivalent for a module M and a submodule $K \leq P_\tau(M)$:*

(i) *M is τ-supplemented;*
(ii) *M/K is τ-supplemented;*
(iii) *$M/P_\tau(M)$ is τ-supplemented.*

PROOF. Since every factor module of a τ-supplemented module is τ-supplemented, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear. To prove $(iii) \Rightarrow (i)$, take $U \leq M$. By hypothesis, there is a submodule $V \leq M$ such that $P_\tau(M) \leq V$,

$$[(U + P_\tau(M))/P_\tau(M)] + [V/P_\tau(M)] = M/P_\tau(M)$$

and

$$\begin{aligned}(U \cap V + P_\tau(M))/P_\tau(M) &= [(U + P_\tau(M))/P_\tau(M)] \cap [V/P_\tau(M)] \\ &\leq \tau(V/P_\tau(M)) = \tau(V)/P_\tau(M).\end{aligned}$$

Note that the last equality holds by Theorem 3.1-(iv). So we have $U + V = M$ and $U \cap V \leq \tau(V)$. That is V is a τ-supplement of U in M. □

COROLLARY 4.9. *The following are equivalent for a ring R:*

(i) *every R-module is τ-supplemented;*
(ii) *every free R-module is τ-supplemented;*
(iii) *every τ-reduced R-module is τ-supplemented.*

PROOF. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are clear. (ii) $\Rightarrow$ (i) follows since every module is an epimorphic image of a free R-module and being τ-supplemented is preserved under passage factor modules. To prove (iii) $\Rightarrow$ (i) take an R-module M. Since $M/P_\tau(M)$ is τ-reduced, we obtain that $M/P_\tau(M)$ is τ-supplemented by the hypothesis. So M is τ-supplemented by Proposition 4.8. □

PROPOSITION 4.10. *If V is a τ-supplement in a module M and V is τ-coatomic, then V is a supplement in M.*

PROOF. Since V is a τ-supplement in M, there exists $U \leq M$ such that $U + V = M$ and $U \cap V \leq \tau(V)$. Since V is τ-coatomic, we have by Proposition 3.5 that $\tau(V) \ll V$. Then $U \cap V \leq \tau(V) \ll V$ and so V is a supplement in M. □

PROPOSITION 4.11. *If M is a τ-reduced module that is totally τ-supplemented, then M is totally supplemented.*

PROOF. Since being τ-reduced is inherited by submodules, it is enough to prove that M is supplemented. Let $U \leq M$ and V be a τ-supplement of U in M. Then $U + V = M$ and $U \cap V \leq \tau(V)$. By hypothesis, V is τ-supplemented and τ-reduced. So by Theorem 4.6, V is τ-coatomic. Then $\tau(V) \ll V$ by Proposition 3.5. Therefore $U \cap V \ll V$ and so V is a supplement of U in M. Hence M is supplemented. □

Clearly supplemented modules are Rad-supplemented and so we obtain the following:

COROLLARY 4.12. *If M is a reduced module, then M is totally Rad-supplemented if and only if M is totally supplemented.*

5. When are all left R-modules τ-supplemented?

In this section, we shall characterize the rings all of whose (left) modules are τ-supplemented for some particular radicals τ including Rad.

An epimorphism $f : P \to M$ is said to be a *projective cover* if P is projective and $\text{Ker} f \ll P$. A property that we shall use is that if P is projective and P/U has a projective cover, then U has a supplement V in P such that V is a direct summand of P and hence projective (see [30, 42.1]). A ring R is called *left perfect* if every left R-module has a projective cover. Recall that, a subset I of a ring R is said to be *left T-nilpotent* in case for every sequence $\{a_k\}_{k=1}^{\infty}$ in I there is a positive integer n such that $a_1 \cdots a_n = 0$. A ring R is said to be a *left max ring* if every left R-module has a maximal submodule, equivalently $\text{Rad}(M) \ll M$ for every left R-module M. A ring R is said to be a *semilocal* ring if $R/J(R)$ is a semisimple ring (that is a left (and right) semisimple R-module), see [18, § 20]. Semilocal rings are also referred to as rings semisimple modulo their radical (see [4, § 15, pp. 170-172]). For a semilocal ring R, $\text{Rad}M = JM$ for every left R-module M where $J = J(R)$ (see for example [4, Corollary 15.18]). By a characterization of left perfect rings by Bass, as in for example [4, Theorem 28.4], a ring R is left perfect if and only if R is a semilocal ring and $J(R)$ is left T-nilpotent if and only if R is a semilocal left max ring. A ring R is called *left semiperfect* if every finitely generated left R-module has a projective cover. A ring R is (left or right) semiperfect if and only if the left (or right) R-module R is supplemented (see [30, 42.6]).

An epimorphism $f : N \to M$ is said to be a *τ-cover* if $\operatorname{Ker} f \leq \tau(N)$. If moreover N is projective, then f is called a *projective τ-cover*. A ring R is called left *τ-perfect* if every left R-module has a projective τ-cover. These rings are studied in [5] and [31] for the radical $\tau = \mathrm{Rad}$, and in [23] for a larger class of preradicals. A ring R is called left *τ-semiperfect* if every finitely generated left R-module has a projective τ-cover. The relation between τ-cover and τ-supplements is the following:

PROPOSITION 5.1 [1, 2.14]. *For an R-module L and $U \leq L$, the following are equivalent:*

(i) *L/U has a projective τ-cover;*
(ii) *U has a τ-supplement V which has a projective τ-cover.*

It is clear from the definitions and Proposition 5.1 that, if R is a left τ-(semi)perfect ring then every (finitely generated) left R-module is τ-supplemented. But the converse need not be true, for example when $\tau = \mathrm{Rad}$; see Example 6.2.

LEMMA 5.2. *If R is a ring that is a τ-reduced left R-module and if the free left R-module $F = R^{(\mathbb{N})}$ is τ-supplemented, then $\tau(R)$ is left T-nilpotent.*

PROOF. Since $P_\tau(R) = 0$ and $P_\tau(F) = (P_\tau(R))^{(\mathbb{N})} = 0$, F is τ-reduced. Then F is τ-coatomic by Theorem 4.6, and so by Proposition 3.5

$$\tau(R)F = (\tau(R))^{(\mathbb{N})} = \tau(F) \ll F.$$

Therefore $\tau(R)$ is left T-nilpotent by [4, Lemma 28.3]. □

THEOREM 5.3. *If R is a ring that is a τ-reduced left R-module, then the free left R-module $F = R^{(\mathbb{N})}$ is τ-supplemented if and only if R is left perfect and $\tau(R) = J(R)$.*

PROOF. Suppose $F = R^{(\mathbb{N})}$ is τ-supplemented. Then R is τ-supplemented as a direct summand of F. Since R is also τ-reduced by hypothesis, we obtain $\tau(R) = J(R)$ by Theorem 4.6. By Lemma 5.2, $J(R) = \tau(R)$ is left

T-nilpotent. Since R is τ-supplemented, $R/J(R) = R/\tau(R)$ is semisimple by [1, 2.2(3)]. Hence R is left perfect by [4, Theorem 28.4]. Conversely suppose R is left perfect and $\tau(R) = J(R)$. Let $U \leq F = R^{(\mathbb{N})}$. Since R is left perfect, every left R-module, and in particular, F/U has a projective cover. Then by [30, 42.1]), U has a supplement V in the free module F such that V is a direct summand of F. Since F is free, its direct summand V is projective. So $\tau(V) = \tau(R)V$ by properties of radicals. Since V is a supplement of U in M, $U + V = M$ and $U \cap V \ll V$. So $U \cap V \leq \text{Rad}(V)$. Since R is a left perfect ring, it is a semilocal ring and so $\text{Rad}(V) = J(R)V$. Thus $U \cap V \leq \text{Rad}(V)$ $= J(R)V = \tau(R)V = \tau(V)$. Hence V is a τ-supplement of U in M. □

Note that the above proof for the converse implication works for every free left R-module F, not necessarily countably generated. Moreover, since every factor module of a τ-supplemented module is τ-supplemented and every module is isomorphic to a factor module of a free module, we have:

COROLLARY 5.4. *If R is a ring that is a τ-reduced left R-module, then every (free) left R-module is τ-supplemented if and only if R is left perfect and $\tau(R) = J(R)$.*

It is easy to see that a radical τ on R-modules is also a radical on $R/P_\tau(R)$-modules since every $R/P_\tau(R)$-module can be considered as an R-module (with annihilator containing $P_\tau(R)$). We shall use this fact in the proof of the following theorem:

THEOREM 5.5. *For a ring R with $P_\tau(R) \leq J(R)$, the following are equivalent.*

- (i) *every left R-module is τ-supplemented;*
- (ii) *every free left R-module is τ-supplemented;*
- (iii) *the free left R-module $F = R^{(\mathbb{N})}$ is τ-supplemented;*
- (iv) *the quotient ring $R/P_\tau(R)$ is left perfect and $\tau(R) = J(R)$.*

PROOF. (i) ⇔ (ii) follows by Corollary 4.9. (ii) ⇒ (iii) is clear.

(iii) ⇒ (iv): Since F is τ-supplemented, so is its factor module $\overline{F} = F/P_\tau(F)$ $\cong (R/P_\tau(R))^{(\mathbb{N})}$. The R-module $\overline{F}$ can be considered as an $R/P_\tau(R)$-module and τ can be considered also as a radical on $R/P_\tau(R)$-modules. By Theorem 5.3, since $R/P_\tau(R)$ is τ-reduced, we obtain that the quotient ring $R/P_\tau(R)$ is left perfect and

$$\tau(R/P_\tau(R)) = J(R/P_\tau(R)).$$

Then by properties of radicals, $\tau(R/P_\tau(R)) = \tau(R)/P_\tau(R)$ and $J(R/P_\tau(R)) = J(R)/P_\tau(R)$ since $P_\tau(R) \le J(R)$ by hypothesis. Hence $\tau(R) = J(R)$.

(iv) $\Rightarrow$ (ii): By properties of radicals, since $P_\tau(R) \le \tau(R) = J(R)$ by hypothesis, we obtain for the left perfect quotient ring $S = R/P_\tau(R)$ that:

$$\tau(S) = \tau(R/P_\tau(R)) = \tau(R)/P_\tau(R) = J(R)/P_\tau(R) = J(R/P_\tau(R)) = J(S).$$

By Corollary 5.4, every free S-module is τ-supplemented, where we consider τ also as a radical on S-modules. Let F be a free R-module. Then $F \cong R^{(I)}$ for some index set I. By Proposition 4.8, it is enough to prove that $\overline{F} = F/P_\tau(F) \cong S^{(I)}$ is τ-supplemented. But this holds since $\overline{F}$ can be considered as a free S-module. □

6. When are all left R-modules Rad-supplemented?

Using the results of the previous sections for $\tau = \text{Rad}$, we obtain the following characterization of the rings R over which every R-module is Rad-supplemented. Of course, more work still remains to understand $P(R)$ and the condition that $R/P(R)$ is left perfect.

THEOREM 6.1. *For a ring R, the following are equivalent.*

(i) *every left R-module is Rad-supplemented;*
(ii) *every reduced left R-module is Rad-supplemented;*
(iii) *every reduced left R-module is supplemented;*
(iv) *the free left R-module $R^{(\mathbb{N})}$ is Rad-supplemented;*
(v) *$R/P(R)$ is left perfect.*

PROOF. (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) is obtained by Theorem 5.5 since $P(R) \le \text{Rad}(R) = J(R)$. (i) $\Leftrightarrow$ (ii) follows by Corollary 4.9. (iii) $\Rightarrow$ (ii) holds since supplemented modules are Rad-supplemented. To prove (ii) $\Rightarrow$ (iii) , take any reduced left R-module M. Then every submodule of M is also reduced and Rad-supplemented by hypothesis (ii). So M is a reduced module that is totally Rad-supplemented. By Corollary 4.12, M is totally supplemented and hence supplemented. □

The following is an example of a ring R that is not left perfect (and so not left Rad-perfect by [23, Theorem 1.5]) but where all R-modules are Rad-supplemented.

EXAMPLE 6.2. Let k be a field. In the polynomial ring $k[x_1, x_2, \dots]$ with countably many indeterminates x_n, $n \in \mathbb{Z}^+$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \dots)$ generated by x_1^2 and $x_{n+1}^2 - x_n$ for each $n \in \mathbb{Z}^+$. In the quotient ring $R = k[x_1, x_2, \dots]/I$, the maximal ideal $M = (x_1, x_2, \dots)/I$ of R generated by all $\overline{x_n} = x_n + I$, $n \in \mathbb{Z}^+$, is the *unique* maximal ideal of R. This is because, if K is any maximal ideal of R, then $\overline{x}_1^2 = 0 \in K$ and so $\overline{x}_1 \in K$ since K is a prime ideal. Now $\overline{x}_2^2 = \overline{x}_1 \in K$ and so $\overline{x}_2 \in K$. By induction, we obtain $\overline{x}_n^2 = \overline{x}_{n-1} \in K$ and so $\overline{x}_n \in K$ for all $n \in \mathbb{Z}^+$. Therefore $K = M$, as desired. Since $\overline{x}_n = \overline{x}_{n+1}^2$ for every $n \in \mathbb{Z}^+$, we obtain $M = M^2$. So $\text{Rad}M = M$ and hence $P(R) = M$. Since the ring $R/P(R) = R/M$ is a field (and so perfect), every R-module is Rad-supplemented (by Theorem 6.1). By [4, Lemma 28.3], $M = J(R)$ is not (left) T-nilpotent, and so R is not a (left) perfect ring.

In [9], it is proved that the class of rings that are Rad-supplemented lies properly between the classes of semilocal rings and semiperfect rings. Recall that a ring R is said to be a *left duo ring* if every left ideal of R is a two-sided ideal. We shall characterize the left duo rings R that are Rad-supplemented left R-modules. Firstly, we need the following lemma:

LEMMA 6.3. *If R is a left duo ring and J, A, B are left ideals of R such that $A + B = R$ and $A \cap B = JA \cap JB$, then $A \cap B = J(A \cap B)$.*

PROOF. Clearly $J(A \cap B) \leq A \cap B$. Conversely let $x \in A \cap B = JA \cap JB$. Since $A + B = R$, we have $a + b = 1$ for some $a \in A$ and $b \in B$. Then $x = xa + xb$ and $x = \sum_{i \in I} s_i a_i = \sum_{i \in I'} t_i b_i$ where I, I' are finite index sets, $a_i \in A$, $b_i \in B$ and $s_i, t_i \in J$. Now we have,

$$xb = \sum_{i \in I} s_i a_i b \in J(AB) \text{ and } xa = \sum_{i \in I'} t_i b_i a \in J(BA).$$

Since R is a left duo ring we have $AB \leq A \cap B$ and $BA \leq A \cap B$. So $x = xa + xb \in J(BA) + J(AB) \leq J(A \cap B)$. Thus $A \cap B \leq J(A \cap B)$. □

THEOREM 6.4. *If R is a left duo ring such that $P(R) = 0$, then R is a Rad-supplemented left R-module if and only if R is semiperfect.*

PROOF. If R is semiperfect, then R is a supplemented, and so a Rad-supplemented, left R-module. Conversely, suppose R is a Rad-supplemented left R-module. Then R is semilocal and R is an amply

Rad-supplemented left R-module by [1, 2.2(3) and 2.6(2)]. Let A' be a left ideal of R. Since R is an amply Rad-supplemented left R-module, A' has a Rad-supplement B in R, and B has a Rad-supplement $A \leq A'$ in R. So $R = A' + B = A + B$, $A \cap B \leq A' \cap B \leq \text{Rad}B$ and $A \cap B \leq \text{Rad}A$. Thus $A \cap B = (\text{Rad}A) \cap (\text{Rad}B)$. Let $J = J(R)$. Then $A \cap B = JA \cap JB = J(A \cap B)$ by Lemma 6.3. Since R is a semilocal ring, $\text{Rad}(A \cap B) = J(A \cap B)$. Then $A \cap B$ is a Rad-torsion submodule of R and so $A \cap B \leq P(R) = 0$. This gives that $R = A \oplus B$. Therefore $JB \leq J \ll R$ implies that $\text{Rad}(B) = JB \ll B$ since B is a direct summand of R. Hence B is a supplement of A' in R. This shows that R is a supplemented left R-module and so R is semiperfect (see [30, 42.6]). □

THEOREM 6.5. *For a left duo ring R, the following are equivalent:*

(i) *$R/P(R)$ is semiperfect;*

(ii) *the left R-module R is Rad-supplemented;*

(iii) *every finitely generated free left R-module is Rad-supplemented;*

(iv) *every finitely generated left R-module is Rad-supplemented.*

PROOF. (ii) ⇒ (iii) follows by [1, 2.3(2)]. (iii) ⇒ (iv) holds since every finitely generated module is an epimorphic image of a finitely generated free module and Rad-supplemented modules are closed under epimorphic images. (iv) ⇒ (ii) is clear.

(i) ⇒ (ii): Since the quotient ring $S = R/P(R)$ is semiperfect, $R/P(R)$ is a Rad-supplemented left S-module and so a Rad-supplemented left R-module. Then the left R-module R is Rad-supplemented by Proposition 4.8.

(ii) ⇒ (i): The factor module $R/P(R)$ is also a Rad-supplemented left R-module. So the ring $S = R/P(R)$ is a Rad-supplemented left S-module with $P(S) = 0$ and so $S = R/P(R)$ is semiperfect by Theorem 6.4. □

Note that all implications except $(ii) \Rightarrow (i)$ of Theorem 6.5 hold for any ring R, while the implication $(ii) \Rightarrow (i)$ raises the question whether a Rad-supplemented ring R with $P(R) = 0$ is necessarily semiperfect.

7. Rad-supplemented Modules over Dedekind Domains.

Over Dedekind domains, divisible modules coincide with injective modules as in abelian groups. Note that for a module M over a Dedekind domain R, M is divisible if and only if $\text{Rad}M = M$, and this holds if and only if M is injective; see for example [2, Lemma 4.4]. This is the motivation for the definition of reduced modules in general. A module over a Dedekind domain is *reduced* if it has no nonzero divisible submodules. As in abelian groups (see for example [11, Theorem 21.3]), any module M over a Dedekind domain possesses a unique largest divisible submodule D and $M = D \oplus C$ for a reduced submodule C of M (see [16, Theorem 8]); this D is called the *divisible part* of M. Following the terminology in abelian groups, an R-module M over a Dedekind domain is said to be *bounded* if $rM = 0$ for some nonzero $r \in R$.

The structure of supplemented modules over Dedekind domains is completely determined in [32]:

THEOREM 7.1 [32, Theorem 2.4. and Theorem 3.1]. *Let R be a Dedekind domain with quotient field $K \neq R$. Let M be an R-module.*

(i) *Suppose R is a local Dedekind domain, that is, a discrete valuation ring (DVR) with the unique prime element p. Then M is supplemented if and only if $M \cong R^a \oplus K^b \oplus (K/R)^c \oplus B$ for some R-module B, where a, b, c are nonnegative integers and $p^nB = 0$ for some integer $n > 0$.*

(ii) *Suppose R is non local. Then M is supplemented if and only if M is torsion and every primary component of M is a direct sum of an artinian submodule and a bounded submodule.*

Part (i) of the above theorem for Rad-supplemented modules is obtained as follows:

THEOREM 7.2. *Let R be a DVR with quotient field $K \neq R$, and p be the unique prime element. Then M is Rad-supplemented if and only if $M \cong R^a \oplus K^{(I)} \oplus (K/R)^{(J)} \oplus B$ for some R-module B, where a is a nonnegative integer, I, J are arbitrary index sets and $p^nB = 0$ for some integer $n0$.*

PROOF. ($\Rightarrow$): If M_1 is the divisible part of M, then there exists a reduced submodule M_2 of M such that $M = M_1 \oplus M_2$. Since M_2 is also Rad-supplemented, it is coatomic by Theorem 4.6. Then by [32, Lemma 2.1], $M_2 = R^a \oplus B$, for some nonnegative integer a and a bounded module B. Since M_1 is divisible, $M_1 \cong K^{(I)} \oplus (K/R)^{(J)}$ for some index sets I and J (see [16, Theorem 7]).

($\Leftarrow$): The module $N = K^{(I)} \oplus (K/R)^{(J)}$ is divisible, and so Rad$N = N$. Then N is Rad-supplemented by Proposition 4.5. By Theorem 7.1, the module $R^a \oplus B$ is supplemented, and hence Rad-supplemented. Therefore the direct sum $R^a \oplus K^{(I)} \oplus (K/R)^{(J)} \oplus B$ is Rad-supplemented. □

Over commutative Noetherian rings we have:

PROPOSITION 7.3. *Let R be a commutative noetherian ring and M be a reduced R-module. Then M is Rad-supplemented if and only if M is supplemented.*

PROOF. Suppose M is Rad-supplemented. Then M is coatomic by Theorem 4.6, and so every submodule of M is coatomic by [33, Lemma 1.1] since R is a commutative noetherian ring. Let U be a submodule of M and V be a Rad-supplement of U in M. Then V is coatomic, and so $U \cap V \leq \mathrm{Rad} V \ll V$. Thus V is a supplement of U in M. The converse is clear. □

Since the structure of supplemented modules is known by Theorem 7.1, it is enough to characterize Rad-supplemented modules in terms of supplemented modules. Note that for an R-module M where R is a Dedekind domain, $P(M)$ equals the *divisible part* of M.

THEOREM 7.4. *Let R be a Dedekind domain and M be an R-module. Then M is Rad-supplemented if and only if $M/P(M)$ is (Rad-)supplemented.*

PROOF. Since R is a Dedekind domain, M has a decomposition as $M = P(M) \oplus N$ for some reduced submodule N of M. If M is Rad-supplemented, then $N \cong M/P(M)$ is also Rad-supplemented. Since N is reduced, N is supplemented by Proposition 7.3. Conversely, suppose $N \cong M/P(M)$ is Rad-supplemented. By Proposition 4.5-(ii), the submodule $P(M)$ is already Rad-supplemented. Therefore $M = P(M) \oplus N$ is Rad-supplemented as a sum of two Rad-supplemented modules. □

These characterizations can be used to give examples of Rad-supplemented modules which are not supplemented.

EXAMPLE 7.5. Let R be a Dedekind domain with quotient field $K \neq R$. The R-module $M = K^{(I)}$ is Rad-supplemented for every index set I. If R is a local Dedekind domain (i.e. a DVR), then M is supplemented only when I is finite. If R is a non-local Dedekind domain, then M is not supplemented for every index set I, since M is not torsion.

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