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Screen Transversal Lightlike Submanifolds of Indefinite Cosymplectic Manifolds

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ABSTRACT - In this paper, we introduce screen transversal lightlike submanifolds of indefinite Cosymplectic manifolds. Then, we study the geometry of this new class and its subspaces with example.

Introduction.

In the theory of submanifolds of semi-Riemannian manifolds it is interesting to study the geometry of lightlike submanifolds due to the fact that the intersection of normal vector bundle and the tangent bundle is non-trivial making it more interesting and remarkably different from the study of non-degenerate submanifolds.The geometry of lightlike submanifolds of indefinite Kaehler manifolds was studied by Duggal and Bejancu [2] and a general notion of screen transversal lightlike submanifolds of indefinite Kaehler manifolds was introduced by Sahin [3]. However, a general notion of screen transversal lightlike submanifolds of indefinite Cosymplectic manifolds has not been introduced as yet.

In section 1, we have collected the formulae and information which are useful in subsequent sections. In section 2, we define screen transversal, screen transversal anti-invariant and radical transversal lightlike submanifolds. In section 3, we obtained a characterization of screen transversal anti-invariant lightlike submanifolds as well as a condition for in-

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duced connection to be a metric connection and provided an example of ST-anti-invariant lightlike submanifold of R^9_2. In section 4, we have studied radical screen transversal lightlike submanifolds.

1. Preliminaries.

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\phi, V, \eta, \overline{g}\}$, where ϕ is a (1,1) tensor field, V a vector field, η a 1-form and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$(1.1) \qquad \begin{cases} \phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \qquad \phi V = 0, \quad \eta(V) = \varepsilon \\ \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \varepsilon\eta(X)\eta(Y), \quad \overline{g}(X, V) = \eta(X) \end{cases}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$ and $\varepsilon = \pm 1$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite Cosymplectic manifold if [1, 4],

$$(1.2) \qquad (\overline{\nabla}_X \phi)Y = 0 \qquad \text{and} \qquad \overline{\nabla}_X V = 0$$

for any $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denote the Levi-Civita connection on $\overline{M}$.

A submanifold M^m immersed in a semi-Riemannian manifold $\{\overline{M}^{m+k}, \overline{g}\}$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\overline{g}$ whose radical distribution $Rad(TM)$ is of rank r, where $1 \le r \le m$.

Now, $Rad(TM) = TM \cap TM^{\perp}$, where

$$(1.3) \qquad TM^{\perp} = \bigcup_{x \in M} \{u \in T_x\overline{M} : \overline{g}(u, v) = 0, \forall v \in T_x M\}.$$

Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is, $TM = Rad(TM) \perp S(TM)$.

Also, there exists a *screen transversal vector bundle* $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}$(*cf.* [2], *page* 144). Let $\mathrm{tr}(TM)$ be the complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$(1.4)\qquad \begin{cases} tr(TM) = ltr(TM) \perp S(TM^\perp) \\ T\overline{M}|_M = S(TM) \perp [Rad(TM) \oplus ltr(TM)] \perp S(TM^\perp). \end{cases}$$

A submanifold $(M, g, S(TM), S(TM^\perp))$ of $\overline{M}$ is said to be

(i) r-lightlike if	$r < min\{m, k\}$;	
(ii) Coisotropic if	$r = k < m$,	$S(TM^\perp) = \{0\}$;
(iii) Isotropic if	$r = m < k$,	$S(TM) = \{0\}$;
(iv) Totally lightlike if	$r = m = k$,	$S(TM) = \{0\} = S(TM^\perp)$.

Let $\overline{\nabla}$, ∇ and ∇^t denote the linear connections on $\overline{M}$, M and vector bundle tr(TM), respectively. Then the Gauss and Weingarten formulae are given by

$$(1.5)\qquad \overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \forall X, Y \in \Gamma(TM),$$

$$(1.6)\qquad \overline{\nabla}_X U = -A_U X + \nabla^t_X U, \qquad \forall U \in \Gamma(tr(TM)),$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively and A_U is the shape operator of M with respect to U. Moreover, according to the decomposition (1.4), h^l, h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^\perp))$-valued *lightlike second fundamental form* and *screen second fundamental form* of M, respectively, then

$$(1.7)\qquad \overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

$$(1.8)\qquad \overline{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X, N), \quad N \in \Gamma(ltr(TM)),$$

$$(1.9)\qquad \overline{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X, W), \quad W \in \Gamma(S(TM^\perp)),$$

where $D^l(X, W)$, $D^s(X, N)$ are the projections of ∇^t on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^\perp))$, respectively and ∇^l, ∇^s are linear connections on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^\perp))$, respectively. We call ∇^l, ∇^s the lightlike and screen transversal connections on M, and A_N, A_W are shape operators on M with respect to N and W, respectively. Using (1.5) and (1.7)-(1.9), we obtain

$$(1.10)\qquad \overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y),$$

$$(1.11)\qquad \overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X).$$

Let $\overline{P}$ denote the projection of TM on $S(TM)$ and let ∇^*, ∇^{*t} denote the linear connections on $S(TM)$ and $Rad(TM)$, respectively. Then from the decomposition of tangent bundle of lightlike submanifold, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{1.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{1.13}$$

for X, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where h^*, A^* are the second fundamental form and shape operator of distributions $S(TM)$ and $Rad(TM)$, respectively.

From (1.12) and (1.13), we get

$$\overline{g}(h^l(X, \overline{P}Y), \xi) = g(A^*_\xi X, \overline{P}Y), \tag{1.14}$$

$$\overline{g}(h^*(X, \overline{P}Y), N) = g(A_N X, \overline{P}Y), \tag{1.15}$$

$$\overline{g}(h^l(X, \xi), \xi) = 0, \quad A^*_\xi \xi = 0. \tag{1.16}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, from (1.7), we obtain

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y). \tag{1.17}$$

However, it is important to note that ∇^*, ∇^{*t} are metric connections on $S(TM)$ and $Rad(TM)$, respectively.

2. Screen transversal lightlike submanifolds.

In this section, we introduce screen transversal (ST), radical screen transversal and screen transversal anti-invariant lightlike submanifolds of indefinite Cosymplectic manifolds.

LEMMA 2.1. *Let M be an r-lightlike submanifold of an indefinite Cosymplectic manifold $\overline{M}$. Suppose that $\phi RadTM$ is a vector subbundle of $S(TM^\perp)$. Then, $\phi ltrTM$ is also vector subbundle of the screen transversal bundle $S(TM^\perp)$. Moreover, $\phi ltrTM \cap \phi RadTM = \{0\}$.*

PROOF. Let us assume that $ltrTM$ is invariant with respect to ϕ. Then by the definition of a lightlike submanifold, there exist vector fields $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltr(TM))$ such that $\overline{g}(\xi, N) = 1$. Also from (1.1), we get

$$\overline{g}(\phi\xi, \phi N) = \overline{g}(\xi, N) - \varepsilon\eta(N)\eta(\xi) = \overline{g}(\xi, N) = 1.$$

However, if $\phi N \in \Gamma(ltr(TM))$ then by the hypothesis we get $\overline{g}(\phi\xi, \phi N) = 0$.

Hence, we obtain a contradiction which implies that ϕN does not belong to $ltr(TM)$.

Now, suppose that $\phi N \in \Gamma(S(TM))$. Then, in a similar way, we have

$$\overline{g}(\phi\xi, \phi N) = \overline{g}(\xi, N) - \varepsilon\eta(N)\eta(\xi) = \overline{g}(\xi, N) = 1$$

which is again a contradiction. Thus ϕN does not belong to $S(TM)$.

We can also obtain that ϕN does not belong to $RadTM$. Then, from the decomposition of a lightlike submanifold, we conclude that $\phi N \in S(TM^{\perp})$.

Now, suppose that there exists a vector field $X \in \Gamma(\phi ltrTM \cap \phi RadTM)$. Then, we have $\overline{g}(X, \phi N) = 0$, since $X \in \Gamma(\phi ltrTM)$. However, for an r-lightlike submanifold there exists some vector fields $\phi X \in \Gamma(RadTM)$ for $X \in \Gamma\phi(RadTM)$ such that $\overline{g}(\phi X, N) \neq 0$. Since ϕ is skew symmetric, we get

$$0 \neq \overline{g}(\phi X, N) = -\overline{g}(X, \phi N) = 0$$

which is a contradiction. Thus, proof is complete.

DEFINITION 2.1. *Let M be an r-lightlike submanifold of an indefinite Cosymplectic manifold $\overline{M}$. Then M is called screen transversal lightlike (ST-lightlike) submanifold of $\overline{M}$ if there exist a screen transversal vector bundle $S(TM^{\perp})$ such that $\phi RadTM \subset S(TM^{\perp})$.*

DEFINITION 2.2. *Let M be ST-lightlike submanifold of an indefinite Cosymplectic manifold $\overline{M}$. Then*

(a) *M is a radical ST-lightlike submanifold of $\overline{M}$ if $S(TM)$ is invariant with respect to ϕ.*

(b) *M is a ST-anti-invariant lightlike submanifold of $\overline{M}$ if $S(TM)$ is screen transversal with respect to ϕ i.e. $\phi S(TM) \subset S(TM^{\perp})$.*

From Lemma 2.1 and Definition 2.1, it follows that $\phi ltrTM \subset S(TM^{\perp})$. Also it is obvious that there are no co-isotropic and totally lightlike ST-lightlike submanifolds of indefinite Cosymplectic manifolds. It is important to point out that $\phi RadTM$ and $\phi ltrTM$ are not orthogonal otherwise $S(TM^{\perp})$ would be degenerate.

For ST-anti-invariant lightlike submanifold M of an indefinite Cosymplectic manifold $\overline{M}$ with structure vector field tangent to M, we have

$$S(TM^{\perp}) = \phi(RadTM) \oplus \phi(ltrTM) \perp \phi(D') \perp D_0 \tag{2.1}$$

where $S(TM) = D' \perp \{V\}$ and D_0 is complementary distribution orthogonal to $\phi(RadTM) \oplus \phi(ltrTM) \perp \phi(D')$ in $S(TM^{\perp})$.

PROPOSITION 2.1. *Let M be ST-lightlike submanifold of an indefinite Cosymplectic manifold $\overline{M}$. Then the distribution D_0 is invariant with respect to ϕ.*

PROOF. For $X \in \Gamma(D_0)$, $\xi \in \Gamma(RadTM)$, $N \in \Gamma(ltrTM)$, we have

$$\overline{g}(\phi X, \xi) = -\overline{g}(X, \phi\xi) = 0 \quad \text{and} \quad \overline{g}(\phi X, N) = -\overline{g}(X, \phi N) = 0$$

which implies that $\phi(D_0) \cap RadTM = \{0\}$ and $\phi(D_0) \cap ltr(TM) = \{0\}$. From (1.1), we get

$$\overline{g}(\phi X, \phi\xi) = \overline{g}(X, \xi) - \varepsilon\eta(X)\eta(\xi) = \overline{g}(X, \xi) = 0$$

and

$$\overline{g}(\phi X, \phi N) = \overline{g}(X, N) - \varepsilon\eta(X)\eta(N) = \overline{g}(X, N) = 0$$

which shows that $\phi(D_0) \cap \phi(RadTM) = \{0\}$ and $\phi(D_0) \cap \phi(ltr(TM)) = \{0\}$. Moreover, since $\phi(S(TM))$ and D_0 are orthogonal, we obtain

$$\overline{g}(\phi X, Z) = -\overline{g}(X, \phi Z) = 0$$

and

$$\overline{g}(\phi X, \phi Z) = \overline{g}(X, Z) - \varepsilon\eta(X)\eta(Z) = \overline{g}(X, Z) = 0$$

for $Z \in \Gamma(S(TM))$, $\phi Z \in \Gamma(\phi(S(TM)))$, which shows that

$$\phi(D_0) \cap S(TM) = \{0\} \quad \text{and} \quad \phi(D_0) \cap \phi(S(TM)) = \{0\}.$$

Thus, we find that

$$\phi(D_0) \cap TM = \{0\}, \phi(D_0) \cap ltr(TM) = \{0\}$$

and

$$\phi(D_0) \cap \{\phi(S(TM)) \perp \phi(ltr(TM)) \oplus \phi(RadTM)\} = \{0\}$$

which shows that D_0 is invariant.

3. ST-anti-invariant lightlike submanifolds.

Takahasi [5] shows that it suffices to consider those indefinite almost contact manifolds with spacelike structure vector field V. Therefore from now on, we restrict ourselves to the case of being V spacelike unit vector.

A plane section Π in $T_x\overline{M}$ of a Cosymplectic manifold $\overline{M}$ is called a ϕ-section if it is spanned by a unit vector X orthogonal to V and ϕX, where X is a non-null vector field on $\overline{M}$. The sectional curvature $K(\Pi)$ with respect to Π determined by X is called a ϕ-sectional curvature. If $\overline{M}$ has a ϕ-sectional curvature c which does not depend on the ϕ-section at each point, then c is constant in $\overline{M}$. Then, $\overline{M}$ is called an indefinite Cosymplectic space form and is denoted by $\overline{M}(c)$. The curvature tensor of $\overline{R}$ of $\overline{M}(c)$ for spacelike vector field V is given by [1, 4]

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\{&\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \overline{g}(X,Z)\eta(Y)V - \overline{g}(Y,Z)\eta(X)V + \overline{g}(\phi Y,Z)\phi X + \overline{g}(\phi Z,X)\phi Y \\ &- 2\overline{g}(\phi X,Y)\phi Z)\}\end{aligned} \tag{3.1}$$

for any X, Y, Z vector fields on $\overline{M}$.

The $(R_q^{2m+1}, \phi_0, V, \eta, g)$ will denote the manifold R_q^{2m+1} with its usual Cosymplectic structure given by

$$\begin{cases} \eta = dz, \qquad V = \partial z, \\ \overline{g} = \eta\otimes\eta - \sum_{i=1}^{q/2}(dx^i\otimes dx^i + dy^i\otimes dy^i) + \sum_{i=q+1}^{m}(dx^i\otimes dx^i + dy^i\otimes dy^i), \\ \phi_0\Big(\sum_{i=1}^{m}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z\Big) = \sum_{i=1}^{m}(Y_i\partial x^i - X_i\partial y^i) \end{cases}$$

where (x^i, y^i, z) are the Cartesian coordinates.
we have

EXAMPLE 3.1. Let $\overline{M} = (R_2^9, \overline{g})$ be a semi-Euclidean space, where g is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Consider a submanifold M of R_2^9, defined by

$$\begin{cases} x_1 = \sin u_1 \cosh u_2, \quad x_2 = \cos u_1 \cosh u_2, \\ x_3 = \sin u_1 \sinh u_2, \quad x_4 = \cos u_1 \sinh u_2, \\ x_5 = u_1, \quad x_6 = 0, \quad x_7 = \cos u_3, \quad x_8 = \sin u_3 \\ \qquad z = t. \end{cases}$$

Then a local frame of TM is given by

$$\begin{cases} Z_1 = \cos u_1\cosh u_2\partial x_1 - \sin u_1\cosh u_2\partial x_2 + \cos u_1\sinh u_2\partial x_3 \\ \qquad\qquad - \sin u_1\sinh u_2\partial x_4 + \partial x_5, \\ Z_2 = \sin u_1\sinh u_2\partial x_1 + \cos u_1\sinh u_2\partial x_2 + \sin u_1\cosh u_2\partial x_3 + \cos u_1\cosh u_2\partial x_4, \\ Z_3 = -\sin u_3\partial x_7 + \cos u_3\partial x_8, \qquad Z_4 = V = \partial z \end{cases}$$

Thus, M is a 1-lightlike submanifold with $RadTM = span\{Z_1\}$, and screen distribution $S(TM) = \{Z_3, Z_4\}$. It is easy to see that $S(TM)$ is not invariant with respect to ϕ. Since $\{\phi Z_2, \phi Z_3\}$ is non-degenerate it follows that $\phi(S(TM)) \subset S(TM^\perp)$.

The lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = \frac{1}{2}(-\cos u_1 \cos u_2 \partial x_1 + \sin u_1 \cosh u_2 \partial x_2 - \cos u_1 \sinh u_2 \partial x_3 \\ + \sin u_1 \sinh u_2 \partial x_4 + \partial x_5)$$

and the screen transversal bundle is

$$S(TM^\perp) = span\{W_1 = \phi N, W_2 = \phi Z_1, W_3 = \phi Z_3, W_4 = \phi Z_2\}$$

where

$$\begin{cases} W_1 = \frac{1}{2}(-\sin u_1 \cosh u_2 \partial x_1 - \cos u_1 \cosh u_2 \partial x_2 - \sin u_1 \sinh u_2 \partial x_3 \\ \qquad - \cos u_1 \sinh u_2 \partial x_4 + \partial x_6), \\ W_2 = \sin u_1 \cosh u_2 \partial x_1 + \cos u_1 \cosh u_2 \partial x_2 + \sin u_1 \sinh u_2 \partial x_3 \\ \qquad + \cos u_1 \sinh u_2 \partial x_4 + \partial x_6, \\ W_3 = -\cos u_3 \partial x_7 - \sin u_3 \partial x_8, \\ W_4 = -\cos u_1 \sinh u_2 \partial x_1 + \sin u_1 \sinh u_2 \partial x_2 - \cos u_1 \cosh u_2 \partial x_3 \\ \qquad + \sin u_1 \cosh u_2 \partial x_4. \end{cases}$$

Then it is easy to see that M is a ST-anti-invariant lightlike submanifold.

Now, we give a characterization for ST-anti-invariant lightlike submanifolds of indefinite Cosymplectic space forms.

THEOREM 3.1. *Let M be a lightlike submanifold of an indefinite Cosymplectic space form $\overline{M}(c)$. Suppose that $c \neq 0$ and $\phi(RadTM) \subset S(TM^\perp)$. Then M is ST-anti-invariant lightlike submanifold if and only if*

$$\overline{g}(\overline{R}(X, Y)\xi, \phi N) = 0 \tag{3.2}$$

for $X, Y \in \Gamma(S(TM))$, $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltr(TM))$.

PROOF. Since $\phi(RadTM) \subset S(TM^\perp)$, from Lemma 2.1, we have $\phi(ltr(TM)) \subset S(TM^\perp)$. From (1.1), we have

$$\overline{g}(\phi X, N) = -\overline{g}(X, \phi N) = 0,$$

for $X \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. Hence $\phi(S(TM)) \cap RadTM = \{0\}$. Moreover, we find that

$$\overline{g}(\phi X, \phi \xi) = 0 \quad \text{and} \quad \overline{g}(\phi X, \phi N) = 0$$

for $X \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. Hence, we get

$$\phi(S(TM)) \cap RadTM = \{0\} \quad \text{and} \quad \phi(S(TM)) \cap \phi(ltr(TM)) = \{0\}.$$

Similarly, we can obtain that $\phi(S(TM)) \cap ltr(TM) = \{0\}$.

On the other hand, since $\phi\xi \in \Gamma(S(TM^{\perp}))$, from (3.1), we get

$$\overline{g}(\overline{R}(X,Y)\xi, \phi N) = \frac{c}{2}\,\overline{g}(X,\phi Y)\overline{g}(\xi, N).$$

Since $c \neq 0$ and $\overline{g}(\xi, N) \neq 0$, for $\xi \in \Gamma(RadTM)$, $N \in \Gamma(ltr(TM))$. Thus, $\overline{g}(\overline{R}(X,Y)\xi, \phi N) = 0$ if and only if $\phi(S(TM)) \perp S(TM)$.

Therfore, $\phi(S(TM)) \subset S(TM^{\perp})$ as $\phi(S(TM)) \cap ltr(TM) = \{0\}$. Thus, the proof is complete.

If F_1, F_2, F_3 and F_4 be the projection morphisms on $\phi(RadTM)$, $\phi(S(TM)), \phi(ltr(TM))$ and D_0 respectively. Then, in view of (2.1), for $W \in \Gamma(S(TM^{\perp}))$, we have

$$W = F_1W + F_2W + F_3W + F_4W. \tag{3.3}$$

On the other hand, for $W \in \Gamma(S(TM^{\perp}))$ we write

$$\phi W = BW + CW \tag{3.4}$$

where BW and CW are tangential and transversal parts of ϕW. Then applying ϕ to (3.2), we get

$$\phi W = \phi F_1W + \phi F_2W + \phi F_3W + \phi F_4W. \tag{3.5}$$

Separating tangential and transversal parts in (3.4), we find

$$BW = \phi F_1W + \phi F_2W, \qquad CV = \phi F_3W + \phi F_4W. \tag{3.6}$$

We put $\phi F_1 = B_1, \phi F_2 = B_2, \phi F_3 = C_1$ and $\phi F_4 = C_2$, we can write (3.4) as follows:

$$\phi W = B_1W + B_2W + C_1W + C_2W, \tag{3.7}$$

where $B_1V \in \Gamma(RadTM)$, $B_2V \in \Gamma(S(TM))$, $C_1V \in \Gamma(ltrTM)$ and $C_2V \in \Gamma(D_0)$.

THEOREM 3.2. *Let M be a ST-anti-invariant lightlike submanifold of an indefinite Cosymplectic manifold $\overline{M}$. Then the induced connection is a metric connection if and only if $\nabla^s_X\phi\xi$ has no components in $\phi(S(TM))$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$.*

PROOF. From (1.2), we have

$$\overline{\nabla}_X\phi\xi = (\overline{\nabla}_X\phi)\xi + \phi(\overline{\nabla}_X\xi)$$

$$\Rightarrow \qquad \overline{\nabla}_X\phi\xi = \phi(\overline{\nabla}_X\xi)$$

$$\Rightarrow \qquad \overline{\nabla}_X\xi = -\phi\overline{\nabla}_X\phi\xi$$

for $X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$.

Using (1.7), (1.9) and (3.7), we get

$$\nabla_X\xi + h^l(X,\xi) + h^s(X,\xi) = \phi A_{\phi\xi}X - B_1\nabla^s_X\phi\xi - B_2\nabla^s_X\phi\xi - C_1\nabla^s_X\phi\xi - C_2\nabla^s_X\phi\xi - \phi D^l(X,\phi\xi).$$

Taking the tangential parts of above equation, we get

$$\nabla_X\xi = -B_1\nabla^s_X\phi\xi - B_2\nabla^s_X\phi\xi.$$

Thus assertion follows from (cf. [2], Theorem 2.4, p. 161).

4. Radical ST lightlike submanifolds.

THEOREM 4.1. *Let M be a radical ST-lightlike submanifold of an indefinite Cosymplectic manifold $\overline{M}$. Then*

(a) *screen distribution $S(TM)$ is integrable if and only if*

$$\overline{g}(h^s(X,\phi Y),\phi N) = \overline{g}(h^s(\phi Y,X),\phi N) \tag{4.1}$$

for $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

(b) *radical distribution is integrable if and only if*

$$\overline{g}(h^s(\xi_1,\phi X),\phi\xi_2) = \overline{g}(h^s(\xi_2,\phi X),\phi\xi_1)$$

for $X \in \Gamma(S(TM))$ and $\xi_1, \xi_2 \in \Gamma(RadTM)$.

PROOF. (a) From (1.1) and (1.2), we have $\overline{g}([X,Y],N) = \overline{g}(\overline{\nabla}_X\phi Y,\phi N) - \overline{g}(\overline{\nabla}_Y\phi X,\phi N)$ for $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. Then, using (1.5) and (1.7), we get (4.1).

(b) From (1.1), (1.2), (1.13) and (1.16), we have

$$\overline{g}([\xi_1,\xi_2],X) = \overline{g}(\overline{\nabla}_{\xi_1}\phi\xi_2,\phi X) - \overline{g}(\overline{\nabla}_{\xi_2}\phi\xi_1,\phi X).$$

Then, using (1.5), we get $g([\xi_1,\xi_2],X) = -g(A_{\phi\xi_2}\xi_1,\phi X) + g(A_{\phi\xi_1}\xi_2,\phi X)$.

Thus, from (1.10), we obtain

$$g([\xi_1, \xi_2], \phi X) = \overline{g}(h^s(\xi_1, \phi X), \phi\xi_2) - \overline{g}(h^s(\xi_2, \phi X), \phi\xi_1)$$

which proves the assertion.

THEOREM 4.2. *Let M be a radical ST-lightlike submanifold of an indefinite Cosymplectic manifold $\overline{M}$. Then*

(a) *S(TM) defines a totally geodesic foliation on M if and only if $g(h^s(X, \phi Y))$ has no components in $\phi(RadTM)$ for $X, Y \in \Gamma(S(TM))$.*

(b) *RadTM defines a totally geodesic foliation on M if and only if $h^s(\xi_1, \phi X)$ has no components in $\phi(ltr(TM))$ for $\xi_1 \in \Gamma(RadTM)$ and $X \in \Gamma(S(TM))$.*

PROOF. (a) Using (1.1), (1.2), (1.5), (1.7) and (1.10), we obtain

$$\overline{g}(\nabla_X Y, N) = \overline{g}(h^s(X, \phi Y), \phi N)$$

which proves the assertion.

Similarly, we have (b).

Now, we have:

THEOREM 4.3. *Let M be a radical ST-lightlike submanifold of an indefinite Cosymplectic manifold $\overline{M}$. Then, the induced connection is a metric connection if and only if $h^s(X, \phi Y)$ has no components in $\phi(ltr(TM))$ for $X, Y \in \Gamma(S(TM))$.*

PROOF. From (1.2), we have $\overline{\nabla}_X\xi = -\phi\overline{\nabla}_X\phi\xi$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. Hence, using (1.7) and (1.9), we get

$$\nabla_X\xi + h^l(X, \xi) + h^s(X, \xi) = \phi A_{\phi\xi}X - \phi\nabla^s_X\phi\xi - \phi D^l(X, \phi\xi).$$

Taking inner product in above with $Y \in \Gamma(S(TM))$, we obtain

$$g(\nabla_X\xi, Y) = -g(A_{\phi\xi}X, \phi Y).$$

Hence, using (1.10), we get

$$g(\nabla_X\xi, Y) = -\overline{g}(h^s(X, \phi Y), \phi\xi).$$

Thus, the proof is complete.

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