

Simple Groups with Prescribed Local Structure

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1. Introduction.

In recent years the construction of simple groups with local structure of prescribed type has been investigated in some works of Dixon, Evans and Smith ([DES1], [DES3]), while the problem of embedding a given group into a simple group whose local structure is, to some extent, known, was considered in [DES2]. These papers provide methods to construct simple groups which are locally-(\mathcal{P} -by-finite), when \mathcal{P} is some relevant group-theoretical property (e.g. solubility, nilpotence, residual finiteness etc..) and embedding theorems for some classes of groups are proved. In this short note we describe an alternative construction which yields the same kind of results, using a more direct approach. Our approach works for groups of any infinite order, so that the restrictions on cardinalities needed in [DES1] and [DES3] can be dropped. This note is based upon a generalization of an idea developed in [D].

2. Simple locally-(\mathcal{P} -by-finite) groups.

When \mathcal{P} is a group-theoretical property, we say that the group G is \mathcal{P} -by-finite if G has a *proper* and *non-trivial* normal \mathcal{P} -subgroup N of finite index.

We say that a set \mathcal{F} of subgroups of the infinite group G is a *directed residual system* if its members are normal subgroups of finite index, $\bigcap\{H \mid H \in \mathcal{F}\} = 1$ and, for every $H, K \in \mathcal{F}$ there exists $L \in \mathcal{F}$ such that $L \leq H \cap K$.

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Let G be any infinite group. We shall consider G embedded into $S = \text{Sym}(G)$ via the right regular representation. Let \mathcal{F} be a set of subgroups of G and assume that, given $H, K \in \mathcal{F}$, there exists $L \in \mathcal{F}$ such that $L \leq H \cap K$. It is clear that, under this assumption, the set $M(\mathcal{F}) = \bigcup_{H \in \mathcal{F}} C_S(H)$ is a subgroup of S . The group M has been studied in [D], when $G = \mathbb{Z}$ and $\mathcal{F} = \{H \leq \mathbb{Z} \mid H \neq 0\}$. We call $M(\mathcal{F})$ the group of \mathcal{F} -modular permutations on G .

The set $\mathcal{L} = \{C_S(H) \mid H \in \mathcal{F}\}$ is a directed local system of $M(\mathcal{F})$, and the next lemma describes the structure of each $C_S(H)$.

LEMMA 2.1. *Let G be embedded into $S = \text{Sym}(G)$ via the right regular representation ρ . Choose H any normal subgroup of G of finite index n . Then $C_S(H) \simeq \text{Hwr}S_n$ and G embeds into $C_S(H)$.*

PROOF. Choose a left transversal $T = \{t_1, \dots, t_n\}$ for H in G and, for every $\sigma \in S_n$ define $\bar{\sigma} \in S$ by setting $(t_i h)\bar{\sigma} = t_{i\sigma} h$ for all $h \in H$. The set $\bar{S}_n = \{\bar{\sigma} \mid \sigma \in S_n\}$ is a subgroup of $C_S(H)$ isomorphic to S_n . Choose $x \in H$ and $t_i \in T$. The map $\tau_{i,x}$ defined by $(t_j h)\tau_{i,x} = t_j h$ when $j \neq i$ and $(t_i h)\tau_{i,x} = t_i x h$, is easily seen to be in $C_S(H)$. The function $\theta_i : H \rightarrow C_S(H)$ sending x to $\tau_{i,x}^{-1}$ is an injective group homomorphism. Moreover $B = \langle H^{\theta_i} \mid i = 1, \dots, n \rangle$ is isomorphic to the direct power of n copies of H , since $[H^{\theta_i}, H^{\theta_j}] = 1$ whenever $i \neq j$. The group B is contained in the setwise stabilizer of the H -orbits $\{t_i H \mid i = 1, \dots, n\}$. Conversely choose $\eta \in C_S(H)$ stabilizing the cosets of H and, for each i , define $x_i \in H$ by $(t_i)\eta = t_i x_i^{-1}$. A generic element of G can be written uniquely as $t_i h$ for suitable $t_i \in T$ and $h \in H$. Thus

$$(t_i h)\eta = (t_i)h^\rho \eta = (t_i)\eta h^\rho = (t_i x_i^{-1})h^\rho = t_i x_i^{-1} h$$

Hence $\eta = \prod_{i=1}^n (x_i)^{\theta_i}$, proving that B is the setwise stabilizer of the H -orbits.

The H -orbits are preserved by $C_S(H)$; hence, for every $\alpha \in C_S(H)$, $(t_i H)\alpha$ is still a coset of H . Let $\sigma \in S_n$ be such that $(t_i H)\alpha = t_{i\sigma} H$. The permutation $\bar{\sigma}^{-1}\alpha$ stabilizes all the H -cosets, hence $\bar{\sigma}^{-1}\alpha = b$ for a suitable $b \in B$. Whence $C_S(H) = \bar{S}_n B$ and the group $\bar{S}_n B$ is isomorphic to the permutational wreath product $\text{Hwr}S_n$ with S_n acting naturally. Finally we recall that the image of the left regular representation λ of G , is the centralizer in S of G^ρ , whence $G^\lambda \leq C_S(H)$. \square

The following lemma contains some easy facts which can be found, in a slightly more general setting, in [DES2]. We give a proof for the reader's convenience.

LEMMA 2.2. *Let H be any group, $G = \text{Hwr}S_n$ a permutational wreath product where $n \geq 7$ and B the base group of G . Then*

1. $W = G' = [B, A_n]A_n = (\text{Hwr}A_n)'$;
2. if $x \in W \setminus [B, A_n]$ then $x^W = W$.

PROOF. The base group B is the direct product $H_1 \times \dots \times H_n$ of n copies of H . If we define $K_i = \{(h_1, \dots, h_n) \mid h_j = 1 \forall j \neq i\}$, the subgroup B can be written as $B = \prod_{i=1}^n K_i$. Choose $x, y \in H$, $\sigma = (123)$, $\tau = (132) \in A_n$ and set $u = (x^{-1}, 1, \dots, 1)$, $v = (y^{-1}, 1, \dots, 1)$. Then $[[u, \sigma], [v, \tau]] = ([x, y], 1, \dots, 1)$, proving that $K'_1 \leq [B, A_n] \leq [B, S_n]$. In particular $B' \leq [B, A_n]$. Since $G' = B'[B, S_n]A_n$ we get $G' = [K'_1, S_n]A_n$. For any given $\sigma \in S_n$, there exists $\tau \in A_n$ such that $1\sigma = 1\tau$. Hence, for all $k \in K_1$, one has $[k, \sigma] = k^{-1}k^\sigma = k^{-1}k^\tau = [k, \tau]$, showing that $[K_1, S_n] = [K_1, A_n]$. Therefore $W = [K_1, A_n]A_n = [B, A_n]A_n = (\text{Hwr}A_n)'$, proving (1).

In order to prove (2) choose $r, s \in H$, set $u = (r^{-1}, r, 1, \dots, 1)$, $v = (s^{-1}, 1, s, 1, \dots, 1)$ and consider the even permutations $\sigma = (14)(25)$, $\tau = (16)(37)$. The quotient $W/[B, A_n]$ is simple, hence $W = x^W[B, A_n]$ and we can find $a, b \in [B, A_n]$ such that $\sigma a, \tau b \in x^W$. Since u, v are in W , the commutator $[[u, \sigma a], [v, \tau b]]$ belongs to x^W . On the other hand the elements $u^{\sigma a}$ and $v^{\tau b}$ have support $\{4, 5\}$ and $\{6, 7\}$ resp., so that they commute and centralize both u and v . Thus $[[u, \sigma a], [v, \tau b]] = [u^{-1}, v^{-1}] = ([r, s], 1, \dots, 1)$, showing that $K'_1 \leq x^W$. Since K'_1 is normalized by B , we have

$$(K'_1)^{x^W} = (K'_1)^{[B, A_n]x^W} = (K'_1)^{[B, A_n]A_n} = (K'_1)^{A_n} = B'.$$

Thus $B' \leq x^W$ and, to complete the proof, we may assume H to be abelian. The group A_n stabilizes the series $[B, A_n, A_n] \leq [B, A_n] \leq B$. On the other hand A_n is simple because $n \geq 7$, hence $[B, A_n, A_n] = [B, A_n]$. The group $W/x^W = x^W[B, A_n]/x^W$ is isomorphic to a quotient of $[B, A_n]$ so that is abelian. But $W' = [B, A_n, A_n]A_n = [B, A_n]A_n = W$, hence W is perfect and has no non-trivial abelian images. Thus $x^W[B, A_n] = x^W$, proving (2). \square

We can now extend Theorem 2 of [DES2].

THEOREM 2.3. *Let G be any residually finite group. There exists a simple group S such that*

1. G' embeds in S ;
2. each finitely generated subgroup of S is isomorphic to a finite extension of a subgroup of a finite direct power of G ;
3. $|S| = |G|$.

PROOF. Let \mathcal{F} be any directed residual system. Without loss of generality we may assume that G/H has order at least 7 when H belongs to \mathcal{F} . Since \mathcal{F} is directed, $M(\mathcal{F}) = \bigcup_{H \in \mathcal{F}} C_{\text{Sym}(G)}(H)$ is a group. Moreover, by lemma 2.1, each $C_{\text{Sym}(G)}(H)$ is isomorphic to $HwrS_n$ where $n = |G : H|$. We prove that $S(\mathcal{F}) = M(\mathcal{F})'$ is simple. For $H \in \mathcal{F}$ write $C_{\text{Sym}(G)}(H) = B_H S_H$ where S_H is the symmetric group of degree $|G : H|$ and B_H indicates the base group of $HwrS_H$. It is useful to notice that, if A_H is the alternating subgroup of S_H , we have

$$S(\mathcal{F}) = \bigcup_{H \in \mathcal{F}} (C_{\text{Sym}(G)}(H))' = \bigcup_{H \in \mathcal{F}} [B_H, A_H]A_H.$$

Since G embeds in $M(\mathcal{F})$ via the left regular representation, G' embeds into $S(\mathcal{F})$. Let N be a normal subgroup of $S(\mathcal{F})$. Two cases can occur.

1. For each $K \in \mathcal{F}$, there exists $H \in \mathcal{F}$ with $H \leq K$ and $N \cap [B_H, A_H]A_H$ not contained in $[B_H, A_H]$.

Let $r \in S(\mathcal{F})$ be any element, and choose K such that $r \in [B_K, A_K]A_K$. Select $H \in \mathcal{F}$ in such a way that $H \leq K$ and $N \cap [B_H, A_H]A_H$ is not contained in $[B_H, A_H]$. It is then possible to find $x \in N \cap [B_H, A_H]A_H \setminus [B_H, A_H]$. By lemma 2.2, $x^{[B_H, A_H]A_H} = [B_H, A_H]A_H$ hence $r \in x^{[B_H, A_H]A_H} \leq x^{S(\mathcal{F})} \leq N$. Since this holds for every r , we have that $N = S(\mathcal{F})$.

2. For each $K \in \mathcal{F}$, $N \cap [B_K, A_K]A_K \leq [B_K, A_K]$.

Let H be in \mathcal{F} and choose $x \in N \cap [B_H, A_H] \leq B_H$. If x is not the identity, there exists $r \in G$ such that $(r)x \neq x$. The set \mathcal{F} is a directed residual system, so that we can find $L \in \mathcal{F}$ such that $L \leq H$ and $rL \neq (r)xL$. Since $B_H A_H \leq B_L A_L$, the element x belongs to $[B_L, A_L]$ so that x normalizes the L -cosets. On the other hand, for any given $a \in L$, $(ra)x = (ra^p)x = (r)a^p x = (r)xa^p = (r)xa$. Thus $(rL)x = (r)xL$ and this contradicts the fact that x normalizes the left L -cosets. Hence $N \cap [B_H, A_H]A_H = 1$ for every $H \in \mathcal{F}$, showing that $N = 1$.

Since \mathcal{F} has cardinality at least $\kappa = |G|$, the group $S(\mathcal{F})$ has cardinality at least κ . By proposition 3.1 of [DES3], G' is contained in a simple subgroup S of $S(\mathcal{F})$, of cardinality κ . It is clear that S has the claimed properties. \square

COROLLARY 2.4. Let \mathcal{P} be a group-theoretical property closed under taking direct products and subgroups. If there exists an infinite residually-finite \mathcal{P} -group then, for every infinite cardinal κ , there exists a simple group $S(\kappa)$ of cardinality κ , with a local system of \mathcal{P} -by-finite subgroups.

PROOF. Suppose there exists an infinite residually finite \mathcal{P} -group R . There is no loss of generality in assuming that R is countable. Given an infinite cardinal κ let G be the direct product of κ copies of R . The group G is a residually finite \mathcal{P} -group of cardinality κ . Apply theorem 2.3 to G to construct a simple group $S(\kappa)$ of cardinality κ , with a local system of \mathcal{P} -by-finite subgroups. \square

Also theorem 1 of [DES2] can be generalized to groups of any infinite cardinality.

THEOREM 2.5. *Let \mathcal{B} be a variety in which the free groups of finite rank are residually finite. For every cardinal κ there exists a simple group $S(\kappa)$ such that*

1. $S(\kappa)$ is locally- $(\mathcal{B}$ -by-finite);
2. $S(\kappa)$ has cardinality $\lambda = \max\{\kappa, \aleph_0\}$;
3. $S(\kappa)$ contains a copy of $V(\kappa)$, the \mathcal{B} -free group of rank κ .

PROOF. If κ is an infinite cardinal, the group $V(\kappa)$ is residually- $(\mathcal{B}$ -free of finite rank), and then $V(\kappa)$ is residually-finite. Thus $V(\kappa)$ is residually-finite for every cardinal κ , finite or infinite. Let C be a cyclic group of order 2 and set $G = V(\kappa) \text{ wr } C$. The group G is residually finite and has cardinality $\lambda = \max\{\kappa, \aleph_0\}$. It is readily seen that $[B, C]$ has an image isomorphic to $V(\kappa)$ so that $V(\kappa)$ embeds into $[B, C] \leq G'$. Theorem 2.3 can now be invoked to get the claim. \square

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