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An Integral Formula Related to Inner Isoptics

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ABSTRACT - An isoptic C_α of a strictly convex C^2-curve in the plane is the locus of all points from which C is seen under the same fixed angle. The two supporting lines of C through such a point determine a secant of C, and the envelope of all these secants is the inner isoptic of C and C_α. We describe an integral formula for inner isoptics in terms of quantities that naturally occur in this geometric configuration.

Let C be an oval, i.e., a simple closed C^2-curve in the plane with positive curvature. We take a coordinate system with the origin O in the interior of C. Let $p(t), t \in \mathbb{R}$, be the distance from O to the supporting line $l(t)$ of C perpendicular to the vector $e^{it} = \cos t + i \sin t$. It is well-known that $p(t)$ is of class C^2 and that the parametrization of C is then given by $z(t) = p(t)e^{it} + p'(t)ie^{it}$, where $ie^{it} = -\sin t + i \cos t$. Note that $p(t)$, called the *support function* of C, is a periodic function on $\mathbb{R}$ with the period 2π. We introduce the notations

$$\begin{aligned} q(t) &:= z(t) - z(t+\alpha)\,,\\ b(t) &:= [q(t), e^{it}]\,,\\ B(t) &:= [q(t), ie^{it}]\,, \end{aligned}$$

where $[\cdot,\cdot]$ means determinant. Let C_α be the locus of apices of a fixed angle $\pi - \alpha$, where $\alpha \in (0, \pi)$, formed by two supporting lines of C. Then C_α is called *α-isoptic of C*; see Fig. 1. In this paper we want to describe geometric properties of so called inner isoptics which are derived from isoptics of the given curve C. To avoid possible confusion, we note that the notion of

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isoptic is sometimes also used with different meanings, e.g. in classical illumination geometry (see [6]) or in the theory of the light field (cf. [2]), for example with respect to so called isophotic families of area elements.

Regarding notations our paper is based on [4], where also the notion of inner isoptic was introduced; see also [5]. In particular, we use the symbols $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to denote the Euclidean norm and the Euclidean dot product of the arguments, respectively.

We assume that the oval C is parametrized as above, that the origin coincides with the Steiner point of C, and that the coordinate system is placed in such a way that the tangent line at $z(0)$ is perpendicular to the x-axis. Thus, in particular we have that $z(0)=(a,0)$ for some $a>0$ and $p'(0)=0$. Then the distance from O to the line $d(t)$ determined by the points $z(t)$ and $z(t+\alpha)$, $\alpha\in(0,\pi)$, is given by

$$P(t)=-\frac{[z(t),q(t)]}{|q(t)|}\,.$$

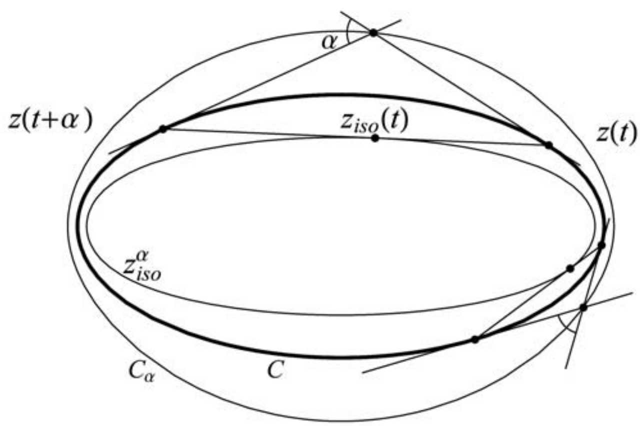

Fig. 1. Deriving an inner isoptic from C_α.

Then we can determine the envelope of the lines $d(t)$ (see Fig. 1), obtaining

$$z^{\alpha}_{\mathrm{iso}}(t)=z_{\mathrm{iso}}(t)=P(t)E^{it}+P_{\mathrm{prim}}(t)iE^{it}\,,$$

where

$$E^{it}=\left\{-\frac{[z(0),q]}{|z(0)|\cdot|q|}\,,\ \frac{\langle z(0),q\rangle}{|z(0)|\cdot|q|}\right\}(t)=J\left(\frac{q}{|q|}(t)\right),$$

with J denoting the positive rotation about $\frac{\pi}{2}$,

$$iE^{it} = \left\{ -\frac{\langle z(0), q\rangle}{|z(0)| \cdot |q|}\,,\;\; -\frac{\langle z(0), q\rangle}{|z(0)| \cdot |q|} \right\}(t) = -\frac{q}{|q|}(t)\,,$$

and

$$P_{\text{prim}}(t) = -\frac{[z', q] \cdot |q|^2 + \langle q, z\rangle \cdot [q, q']}{|q| \cdot [q, q']}(t)\,. \tag{1}$$

Note that the function P for the inner isoptic z^{α}_{iso} is a counterpart of a support function of an oval. Therefore the term P_{prim} is used to have also a similar formula for its parametrization. Moreover, it is in fact a derivative of P with respect to a some naturally arising parameter as explained in formula (3.10) in [4]. We call this envelope the *inner isoptic associated to C_α and C*; see also Fig. 2.

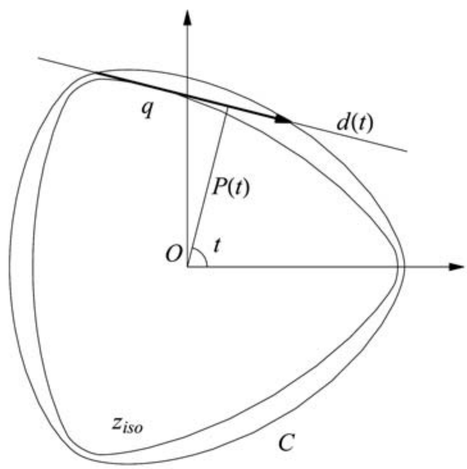

Fig. 2. An inner isoptic as envelope of the vectors q.

Dropping the point t, we get

$$z'_{\text{iso}} = \frac{\partial z_{\text{iso}}}{\partial t} = P'E^{it} + PiE^{it}\frac{[q, q']}{|q|^2} + P_{\text{prim}|t}iE^{it} - P_{\text{prim}}E^{it}\frac{[q, q']}{|q|^2}\,.$$

LEMMA 1. *With the above notations we have*

$$P'\frac{|q|^2}{[q, q']} = P_{\text{prim}}\,.$$

Hence we get

$$z'_{\rm iso} = \left(P\frac{[q,q']}{|q|^2} + P_{{\rm prim}|t}\right) iE^{it}\,.$$

We define

$$P\frac{[q,q']}{|q|^2} + P_{{\rm prim}|t} =: K\,.$$

Thus $\|z'_{\rm iso}\| = |K|$, but for small values of α (cf. [4]) the number K is positive.

LEMMA 2. *The curvature of* $z_{\rm iso}$ *is given by*

$$\kappa_{\rm iso} = \frac{1}{|K|}\cdot\frac{[q,q']}{|q|^2}\,.$$

PROOF. From the above considerations we have

$$z'_{\rm iso} = KiE^{it}.$$

Then we obtain

$$z''_{\rm iso} = K'iE^{it} - KE^{it}\frac{[q,q']}{|q|^2}\,,$$

and next

$$\kappa_{\rm iso} = \frac{[z'_{\rm iso}, z''_{\rm iso}]}{|z'_{\rm iso}|^3} = -\frac{[iE^{it}, E^{it}]\cdot[q,q']}{|K|\cdot|q|^2} = \frac{[q,q']}{|K|\cdot|q|^2}\,,$$

since $[E^{it}, iE^{it}] = \left[J\left(\frac{q}{|q|}\right), \frac{q}{|q|}\right] = \left\langle \frac{q}{|q|}, \frac{q}{|q|}\right\rangle = 1.$ □

Let us write

$$\alpha_0 = \sup\left\{\alpha > 0 \colon z^{\beta}_{\rm iso} \text{ is convex for every } \beta \in (0,\alpha)\right\},$$

and let the angle α_0 be called the *limit angle* for inner isoptics.

EXAMPLES:

a) Since one can show that $|z'_{\rm iso}| = 0$ iff the curvature of the corresponding usual α-isoptic vanishes, in case of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we get

$$\cos\alpha_0 = \frac{b^2}{b^2 - a^2}\,,$$

and for $1 < \frac{a}{b} < \sqrt{2}$ all inner isoptics are convex with $K > 0$ everywhere. Note that usual isoptics are convex under the same condition.

b) This is no longer true for curves with $p(t) = a + b\cos 3t$, where $a > 8b$. And inner isoptics can also have cusps (i.e., points with $K = 0$) when their generating isoptics are still convex.

Recall that

$$\begin{aligned} z_{\text{iso}} &= PE^{it} + P_{\text{prim}} iE^{it} \\ &= PJ\left(\frac{q}{|q|}\right) + P_{\text{prim}}\left(-\frac{q}{|q|}\right) \\ &= \left[\frac{q}{|q|}, z\right] J\left(\frac{q}{|q|}\right) + \left(\frac{[z', q]\cdot|q|}{[q, q']} + \frac{\langle q, z\rangle}{|q|}\right)\frac{1}{|q|} \\ &= z\left\langle\frac{q}{|q|}, \frac{q}{|q|}\right\rangle - \frac{q}{|q|}\langle z, q\rangle + \left(\frac{[z', q]\cdot|q|}{[q, q']} + \frac{\langle q, z\rangle}{|q|}\right)\frac{q}{|q|} \\ &= z + \frac{[z', q]}{[q, q']} q\,, \end{aligned}$$

where we used the formula

$$v\langle w, w\rangle - w\langle v, w\rangle = [w, v]\,J(w)\,. \tag{2}$$

Note that

$$z_{\text{iso}} - z = \frac{[z', q]}{[q, q']}\cdot q\,.$$

Thus $\overrightarrow{zz_{\text{iso}}} = -\frac{[z', q]}{[q, q']}\cdot q$. Moreover, we have $\frac{[z', q]}{[q, q']} < 0$.

Thus, if z_{iso} is convex, then $t_1 = -\frac{[z', q]}{[q, q']}\cdot|q|$ and $t_2 = -\frac{[q, z'(t+\alpha)]}{[q, q']}\cdot|q|$ (see Fig. 3 for the quantities t_1, t_2).

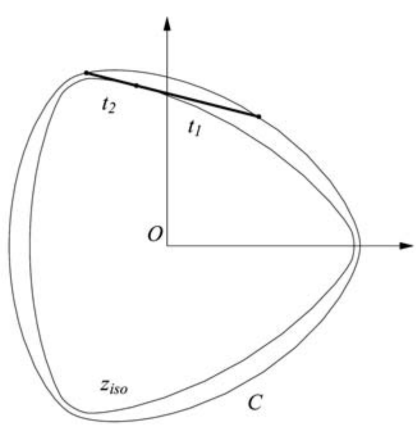

Fig. 3. Geometric illustration of the numbers t_1 and t_2.

For a given oval C the interval $(0,\alpha_0)$ yields only inner isoptics that are convex. Having this, the inner isoptics fill an "annulus" (see Fig. 4), and we can parametrize this "annulus" CC_α minus some set of measure zero (i.e., minus a graph of the curve $\alpha \mapsto z^\alpha_{\text{iso}}(0)$) by means of $z^\alpha_{\text{iso}}(t)$.

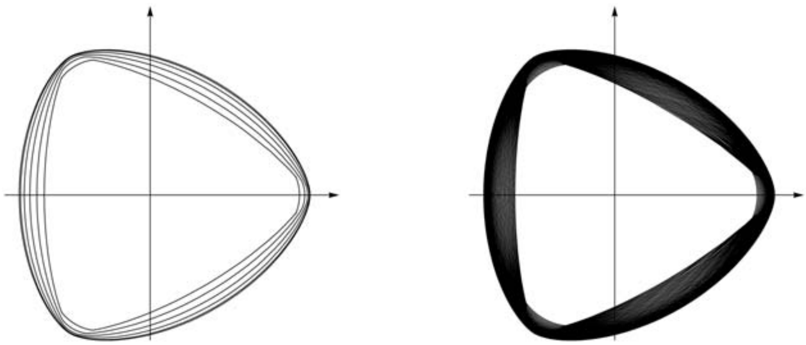

Fig. 4. Several inner isoptics and the whole annulus CC_α.

Let

$$F(t,\alpha) = z^\alpha_{\text{iso}}(t)\,, \quad \alpha \in (0,\alpha_0)\,,\ t \in (0,2\pi),$$

and calculate the Jacobian matrix

$$\frac{\partial(F)}{\partial(t,\alpha)} = \left[\frac{\partial z_{\text{iso}}}{\partial t}, \frac{\partial z_{\text{iso}}}{\partial \alpha}\right].$$

We have

$$\frac{\partial z_{\text{iso}}}{\partial t} = -K\frac{q}{|q|}\,.$$

Let us calculate

$$\frac{\partial z_{\text{iso}}}{\partial \alpha} = \frac{\partial P}{\partial \alpha}\cdot J\left(\frac{q}{|q|}\right) + P\cdot J\left(\left(\frac{q}{|q|}\right)_{|\alpha}\right) + \frac{\partial P_{\text{prim}}}{\partial \alpha}\cdot\left(\frac{-q}{|q|}\right) + P_{\text{prim}}\left(\frac{-q}{|q|}\right)_{|\alpha}.$$

To this aim we have by (2)

$$\left(\frac{q}{|q|}\right)_{|\alpha} = \frac{q_{|\alpha}\langle q,q\rangle - q\langle q, q_{|\alpha}\rangle}{|q|^3} = \frac{[q,q_{|\alpha}]}{|q|^2}\left(\frac{q}{|q|}\right),$$

but $q(t) = z(t) - z(t+\alpha)$, and so

$$q_{|\alpha} = -z'(t+\alpha)\,.$$

Hence

$$\left(\frac{q}{|q|}\right)_{|\alpha} = -\frac{[q, z'(t+\alpha)]}{|q|^2} J\left(\frac{q}{|q|}\right) = -\frac{[q, z'(t+\alpha)]}{|q|^2} E^{it}$$

and

$$\left(iE^{it}\right)_{|\alpha} = \left(-\frac{q}{|q|}\right)_{|\alpha} = +\frac{[q, z'(t+\alpha)]}{|q|^2} E^{it},$$

yielding

$$\left(E^{it}\right)_{\alpha} = J\left(\left(\frac{q}{|q|}\right)_{|\alpha}\right) = \frac{[q, z'(t+\alpha)]}{|q|^2}\frac{q}{|q|} = -\frac{[q, z'(t+\alpha)]}{|q|^2} iE^{it}.$$

Also we have

$$\frac{\partial P}{\partial \alpha} = -\left[z, \left(\frac{q}{|q|}\right)_{|\alpha}\right] = \frac{[q, z'(t+\alpha)]}{|q|^2} \left\langle z, \frac{q}{|q|}\right\rangle,$$

where we used the formula

$$\langle w, v\rangle = [w, J(v)].$$

Hence

$$\begin{aligned}\left[\frac{\partial z_{\text{iso}}}{\partial t}, \frac{\partial z_{\text{iso}}}{\partial \alpha}\right] &= \left[KiE^{it}, \frac{[q, z'(t+\alpha)]}{|q|^2}\left(\left\langle z, \frac{q}{|q|}\right\rangle + P_{\text{prim}}\right)E^{it}\right], \\ &= -K\frac{[q, z'(t+\alpha)]}{|q|^2}\left(\left\langle z, \frac{q}{|q|}\right\rangle + P_{\text{prim}}\right),\end{aligned}$$

but by formula (1) we get

$$\begin{aligned}\left[\frac{\partial z_{\text{iso}}}{\partial t}, \frac{\partial z_{\text{iso}}}{\partial \alpha}\right] &= K\frac{[q, z'(t+\alpha)]}{|q|^2} \cdot \frac{[z', q]}{[q, q']} \cdot |q| \\ &= K\frac{[q, z'(t+\alpha)][z', q]}{|q|[q, q']} = \frac{K}{|q|} \cdot \frac{[q, q']}{|q|^2} \cdot t_1 \cdot t_2 > 0.\end{aligned}$$

Moreover, we have

$$\frac{\partial(F)}{\partial(t, \alpha)} = \frac{1}{\kappa_{\text{iso}}} \cdot \frac{[q, z'(t+\alpha)][z', q]}{|q|^3},$$

where κ_{iso} is the curvature of $z_{\text{iso}}^{\alpha}(t)$.

Since

$$z'(t+\alpha) = R(t+\alpha)ie^{i(t+\alpha)}\,,\ z'(t) = R(t)ie^{it}\,,$$

then

$$\begin{aligned}\frac{\partial(F)}{\partial(t,\alpha)} &= \frac{R(t)\cdot R(t+\alpha)}{\kappa_{\text{iso}}}\cdot\frac{[q, ie^{it}(\cos\alpha + i\sin\alpha)][ie^{it}, q]}{|q|^3}\\ &= \frac{R\cdot R\alpha}{\kappa_{\text{iso}}}\cdot\frac{(-\mu)\cdot\lambda\cdot\sin^2\alpha}{|q|^3}\,,\end{aligned}$$

where $R = R(t)$ is the curvature radius of C at $z(t), R_\alpha = R(t+\alpha)$ is the curvature radius of C at $z(t+\alpha)$, λ is the length of the segment $\overline{z_\alpha(t)z(t)}$ on the tangent line of C at $z(t)$ (forming an angle $\pi - \alpha$ with the tangent line at $z(t+\alpha)$ and apex $z_\alpha(t)$), and $-\mu$ is then the length of the second tangential segment $\overline{z_\alpha(t)z(t+\alpha)}$ bounding this angle; see Fig. 5.

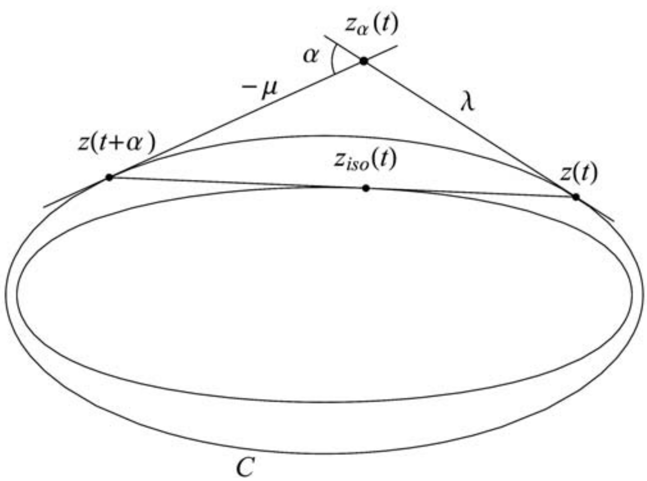

Fig. 5. Notations for λ and μ.

Recall the sine theorem for isoptics, namely

$$\frac{|q|}{\sin\alpha} = \frac{\lambda}{\sin\alpha_1} = \frac{-\mu}{\sin\alpha_2}\ ;$$

see [1] and Fig. 6.

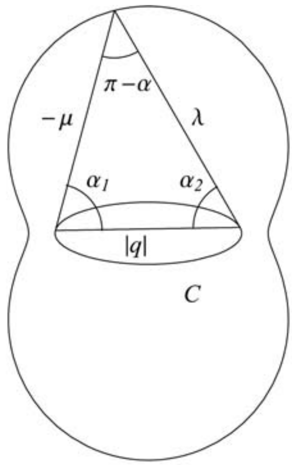

Fig. 6. Notations used for the sine theorem.

Hence, we have

$$
\begin{aligned}
\frac{\partial F}{\partial(t,\alpha)} &= \frac{R\cdot R_\alpha}{\kappa_{\mathrm{iso}}\cdot|q|}\cdot(-\mu)\cdot\lambda\cdot\frac{\sin\alpha}{|q|}\cdot\frac{\sin\alpha}{|q|}\\
&= \frac{R\cdot R_\alpha}{\kappa_{\mathrm{iso}}\cdot|q|}\cdot\sin\alpha_1\cdot\sin\alpha_2 \qquad (3)\\
&= \frac{1}{|q|}\cdot R\cdot R_\alpha\cdot R_{\mathrm{iso}}\cdot\sin\alpha_1\cdot\sin\alpha_2\,.
\end{aligned}
$$

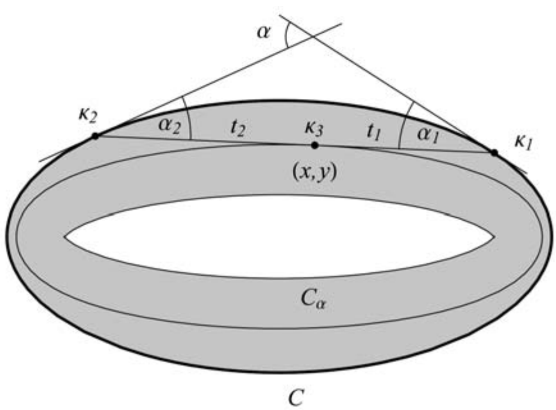

Fig. 7. Notations for $\alpha_1,\alpha_2,\kappa_1,\kappa_2,\kappa_3$.

Note that for a given point (x, y) inside the curve C it would be interesting to know the angle α generating an inner isoptic containing this point. We will consider this problem in a forthcoming paper, but we know already the following: in case of an ellipse one has to consider all chords passing trough this point, looking for α giving the maximum of the angle $\pi - \alpha$ between the tangents at the endpoints of the corresponding chord. We conjecture that this is true for all ovals.

Our derivations above yield the following theorem.

THEOREM. *We have*

$$\iint\limits_{CC_{\alpha_0}} \frac{\kappa_1 \cdot \kappa_2 \cdot \kappa_3}{\sin\alpha_1 \cdot \sin\alpha_2} \cdot (t_1 + t_2) \cdot dxdy = 2\pi \cdot \alpha_0 \,,$$

where κ_i are curvatures at suitable points, the α_i present the adequate angles, and t_i are determined as shown in Fig. 7.

PROOF. Using the formula (3), we have

$$\iint\limits_{CC_{\alpha_0}} \frac{\kappa_1 \cdot \kappa_2 \cdot \kappa_3}{\sin\alpha_1 \cdot \sin\alpha_2} dxdy$$

$$= \int_0^{2\pi} \int_0^{\alpha_0} \frac{\kappa(t) \cdot \kappa(t+\alpha)\kappa_{\text{iso}}(t)(t_1 + t_2)}{\sin\alpha_1 \cdot \sin\alpha_2} \cdot \frac{\sin\alpha_1 \cdot \sin\alpha_2}{\kappa(t) \cdot \kappa(t+\alpha) \cdot \kappa_{\text{iso}}(t) \cdot |q|} d\alpha dt$$

$$= \int_0^{2\pi} \int_0^{\alpha_0} d\alpha dt = 2\pi\alpha_0 \,,$$

since $|q| = t_1 + t_2$. □

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