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Global Weak Solutions of the Navier-Stokes Equations with Nonhomogeneous Boundary Data and Divergence

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ABSTRACT - Consider a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ with boundary $\partial\Omega$, a time interval $[0,T)$, $0<T\leq\infty$, and the Navier-Stokes system in $[0,T)\times\Omega$, with initial value $u_0 \in L^2_\sigma(\Omega)$ and external force $f = \operatorname{div} F$, $F \in L^2(0,T;L^2(\Omega))$. Our aim is to extend the well-known class of Leray-Hopf weak solutions u satisfying $u|_{\partial\Omega} = 0$, $\operatorname{div} u = 0$ to the more general class of Leray-Hopf type weak solutions u with general data $u|_{\partial\Omega} = g$, $\operatorname{div} u = k$ satisfying a certain energy inequality. Our method rests on a perturbation argument writing u in the form $u = v + E$ with some vector field E in $[0,T)\times\Omega$ satisfying the (linear) Stokes system with $f = 0$ and nonhomogeneous data. This reduces the general system to a perturbed Navier-Stokes system with homogeneous data, containing an additional perturbation term. Using arguments as for the usual Navier-Stokes system we get the existence of global weak solutions for the more general system.

1. Introduction and main results.

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, and let $[0,T)$, $0<T\leq\infty$, be a time interval. We consider in $[0,T)\times\Omega$, together with an associated pressure p, the following general Navier-Stokes system

$$
\begin{aligned}
u_t - \Delta u + u\cdot\nabla u + \nabla p &= f, \quad \operatorname{div} u = k \\
u|_{\partial\Omega} &= g, \quad u|_{t=0} = u_0
\end{aligned}
\tag{1.1}
$$

with given data f, k, g, u_0.

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First we have to give a precise characterization of this general system. To this aim, we shortly discuss our arguments to solve this system in the weak sense (without any smallness assumption on the data). Using a perturbation argument we write u in the form

$$u = v + E, \tag{1.2}$$

and the initial value u_0 at time $t = 0$ in the form

$$u_0 = v_0 + E_0. \tag{1.3}$$

Here E is the solution of the (linear) Stokes system

$$\begin{aligned} E_t - \Delta E + \nabla h = 0, \quad \operatorname{div} E = k \\ E|_{\partial\Omega} = g, \quad E|_{t=0} = E_0 \end{aligned} \tag{1.4}$$

with some associated pressure h, and v has the properties

$$\begin{aligned} v \in L^\infty_{\mathrm{loc}}\big([0,T); L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T); W^{1,2}_0(\Omega)\big), \\ v : [0,T) \mapsto L^2_\sigma(\Omega) \quad \text{is weakly continuous,} \ v|_{t=0} = v_0. \end{aligned} \tag{1.5}$$

Inserting (1.2), (1.3) into the system (1.1) we obtain the modified system

$$\begin{aligned} v_t - \Delta v + (v+E)\cdot\nabla(v+E) + \nabla p^* = f, \quad \operatorname{div} v = 0 \\ v|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0 \end{aligned} \tag{1.6}$$

with associated pressure $p^* = p - h$ and homogeneous conditions for v. Thus (1.6) can be called a *perturbed Navier-Stokes system* in $[0,T) \times \Omega$. This system reduces the general system (1.1) to a certain homogeneous system which contains an additional perturbation term in the form

$$(v+E)\cdot\nabla(v+E) = v\cdot\nabla v + v\cdot\nabla E + E\cdot\nabla(v+E).$$

Therefore, the perturbed system (1.6) can be treated similarly as the usual Navier-Stokes system obtained from (1.6) with $E \equiv 0$.

In order to give a precise definition of the general system (1.1) we need the following steps:

First we develop the theory for the perturbed system (1.6) for data f, v_0 and a given vector field E, as general as possible. In the second step we consider the system (1.4) for general given data k, g, E_0 to obtain a vector field E in such a way that $u = v + E$ with v from (1.6) yields a well-defined solution of the general system (1.1) in the (Leray-Hopf type) weak sense.

Thus we start with the definition of a weak solution v of (1.6) under rather weak assumptions on E needed for the existence of such solutions.

DEFINITION 1.1. (Perturbed system). *Suppose*

$$
\begin{aligned}
& f = \operatorname{div} F \quad \textit{with} \quad F = (F_{i,j})_{i,j=1}^3 \in L^2\big(0,T;L^2(\Omega)\big), \\
(1.7) \qquad & v_0 \in L^2_\sigma(\Omega), \\
& E \in L^s\big(0,T;L^q(\Omega)\big),\ \operatorname{div} E = k \in L^4\big(0,T;L^2(\Omega)\big),
\end{aligned}
$$

with $4 \le s < \infty$, $4 \le q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$.

Then a vector field v *is called a weak solution of the perturbed system* (1.6) *in* $[0,T) \times \Omega$ *with data* f, v_0 *if the following conditions are satisfied:*

a) *For each finite* $T^*, 0 < T^* \le T$,

$$
v \in L^\infty\big(0,T^*;L^2_\sigma(\Omega)\big) \cap L^2\big(0,T^*;W_0^{1,2}(\Omega)\big)\,, \tag{1.8}
$$

b) *for each test function* $w \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$,

$$
\begin{aligned}
& - \langle v, w_t\rangle_{\Omega,T} + \langle \nabla v, \nabla w\rangle_{\Omega,T} - \big\langle (v+E)(v+E), \nabla w\big\rangle_{\Omega,T} \\
& - \big\langle k(v+E), w\big\rangle_{\Omega,T} \;=\; \big\langle v_0, w(0)\big\rangle_{\Omega} - \langle F, \nabla w\rangle_{\Omega,T},
\end{aligned} \tag{1.9}
$$

c) *for* $0 \le t < T$,

$$
\begin{aligned}
& \frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\, d\tau \;\le\; \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F, \nabla v\rangle_\Omega\, d\tau \\
& + \int_0^t \big\langle (v+E)E, \nabla v\big\rangle_\Omega\, d\tau + \frac{1}{2}\int_0^t \big\langle k(v+2E), v\big\rangle_\Omega\, d\tau,
\end{aligned} \tag{1.10}
$$

d) *and*

$$
v : [0,T) \to L^2_\sigma(\Omega) \textit{ is weakly continuous and } v(0) = v_0. \tag{1.11}
$$

In the classical case $E \equiv 0$ we obtain with (1.8)-(1.11) the usual (Leray-Hopf) weak solution v. As in this case the condition (1.11) already follows from the other conditions (1.8)-(1.10), after possibly a modification on a null set of $[0,T)$, see, e.g., [16, V, 1.6]. Here (1.11) is included for simplicity. The relation (1.9) and the energy inequality (1.10) are based on formal calculations as for $E \equiv 0$. The existence of an associated pressure p^* such that

$$
v_t - \Delta v + (v+E)\cdot\nabla(v+E) + \nabla p^* = f \tag{1.12}
$$

in the sense of distributions in $(0,T) \times \Omega$ follows in the same way as for $E \equiv 0$.

In the next step we consider the linear system (1.4). A very general solution class for this system, sufficient for our purpose, has been developed by the theory of so-called very weak solutions, see [1], [3, Sect. 4]. In particular, the boundary values g are given in a general sense of distributions on $\partial\Omega$.

LEMMA 1.2 (Linear system for E, [3]). *Suppose*

$$\begin{aligned} &k \in L^s\big(0,T;L^{q^*}(\Omega)\big), \quad g \in L^s\big(0,T;W^{-\frac{1}{q},q}(\partial\Omega)\big),\ E_0 \in L^q(\Omega),\\ &4 \le s < \infty,\ 4 \le q < \infty,\ \frac{2}{s}+\frac{3}{q}=1,\ \frac{1}{q}=\frac{1}{q^*}-\frac{1}{3}, \end{aligned} \tag{1.13}$$

satisfying the compatibility condition

$$\int_\Omega k(t)\,dx = \int_{\partial\Omega} N\cdot g(t)\,dS \quad \textit{for almost all } t\in[0,T), \tag{1.14}$$

where $N = N(x)$ *means the exterior normal vector at* $x \in \partial\Omega$, *and* $\int_{\partial\Omega} \dots dS$ *the surface integral (in a generalized sense of distributions on* $\partial\Omega$*).*

Then there exists a uniquely determined (very) weak solution

$$E \in L^s\big(0,T;L^q(\Omega)\big) \tag{1.15}$$

of the system (1.4) *in* $[0,T)\times\Omega$ *with data* k, g, E_0 *defined by the conditions:*

a) *For each* $w \in C_0^1([0,T);C^2_{0,\sigma}(\bar\Omega))$,

$$-\langle E,w_t\rangle_{\Omega,T} - \langle E,\Delta w\rangle_{\Omega,T} + \langle g, N\cdot\nabla w\rangle_{\Omega,T} = \big\langle E_0, w(0)\big\rangle_\Omega, \tag{1.16}$$

b) *for almost all* $t \in [0,T)$,

$$\operatorname{div} E = k,\ N\cdot E|_{\partial\Omega} = N\cdot g. \tag{1.17}$$

Moreover, E *satisfies the estimate*

$$\|A_q^{-1}P_qE_t\|_{q,s;\Omega,T} + \|E\|_{q,s;\Omega,T} \le C\big(\|E_0\|_q + \|k\|_{q^*,s;\Omega,T} + \|g\|_{-\frac{1}{q};q,s;\partial\Omega,T}\big) \tag{1.18}$$

with constant $C = C(\Omega,T,q) > 0$.

The trace $E|_{\partial\Omega} = g$ *is well-defined at* $\partial\Omega$ *for almost all* $t\in[0,T)$, *and the initial value condition* $E|_{t=0} = E_0$ *is well-defined (modulo gradients) in the sense that* $P_qE : [0,T) \to L^q_\sigma(\Omega)$ *is weakly continuous satisfying*

$$P_qE|_{t=0} = P_qE_0. \tag{1.19}$$

Finally, there exists an associated pressure h *such that*

$$E_t - \Delta E + \nabla h = 0 \tag{1.20}$$

holds in the sense of distributions in $(0,T)\times\Omega$.

To obtain a precise definition for the general system (1.1) we have to combine Definition 1.1 and Lemma 1.2 as follows:

DEFINITION 1.3. (General system). *Let* $k \in L^s(0,T;L^{q^*}(\Omega)) \cap L^4(0,T;L^2(\Omega))$ *with* s, q^* *as in* (1.13) *and suppose that*

(1.21) *E is a very weak solution of the linear system* (1.4) *in* $[0,T) \times \Omega$ *with data* k, g, E_0 *in the sense of Lemma* 1.2,

and

(1.22) *v is a weak solution of the perturbed system* 1.6 *in* $[0,T) \times \Omega$ *in the sense of Definition* 1.1 *with data* f, v_0 *as in* 1.7.

Then the vector field $u = v + E$ is called a weak solution of the general system (1.1) *in* $[0,T) \times \Omega$ *with data f, k, g and initial value* $u_0 = v_0 + E_0$. *Thus it holds*

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \ \operatorname{div} u = k \tag{1.23}$$

in the sense of distributions in $(0,T) \times \Omega$ *with associated pressure* $p = p^* + h$, p^* *as in* (1.12), h *as in* (1.20). *Further,*

$$u|_{\partial\Omega} = v|_{\partial\Omega} + E|_{\partial\Omega} = g \tag{1.24}$$

is well-defined by $E|_{\partial\Omega} = g$, *and the condition*

$$u|_{t=0} = v|_{t=0} + E|_{t=0} = v_0 + E_0 = u_0 \tag{1.25}$$

is well-defined in the generalized sense modulo gradients by (1.19).

Therefore the general system (1.1) has a well-defined meaning for weak solutions u in a generalized sense.

However, if we suppose in Definition 1.3 additionally the regularity properties

$$\begin{aligned} &k \in L^s(0,T;W^{1,q}(\Omega)), \ k_t \in L^s(0,T;L^2(\Omega)), \\ &g \in L^s(0,T;W^{2-1/q,q}(\partial\Omega)), \ g_t \in L^s(0,T;W^{-\frac{1}{q},q}(\partial\Omega)), \\ &E_0 \in W^{2,q}(\Omega), \end{aligned} \tag{1.26}$$

and the compatibility conditions $u_0|_{\partial\Omega} = g|_{t=0}$, $\operatorname{div} u_0 = k|_{t=0}$, then the solution E in Lemma 1.2 satisfies the regularity properties

$$E \in L^s(0,T;W^{2,q}(\Omega)), \ E_t \in L^s(0,T;L^q(\Omega)), \ E \in C([0,T);L^q(\Omega)),$$

and $E|_{\partial\Omega} = g,\ E|_{t=0} = E_0$ are well-defined in the usual sense, see [3, Corollary 5]. Further it holds $\nabla h \in L^s(0,T;L^q(\Omega))$ for the associated pressure h in (1.20). Therefore, $u = v + E$ satisfies in this case the boundary condition $u|_{\partial\Omega} = g$ and the initial condition $u|_{t=0} = v_0 + E_0$ in the usual (strong) sense.

The most difficult problem is the existence of a weak solution v of the perturbed system (1.6). For this purpose we have to introduce, see (2.12) in Sect. 2, an approximate system of (1.6) for each $m \in \mathbb{N}$ which yields such a weak solution when passing to the limit $m \to \infty$. Then the existence of a weak solution $u = v + E$ of the general system (1.6) is an easy consequence.

This yields the following main result.

THEOREM 1.4 (Existence of general weak solutions).

a) *Suppose*

$$\begin{aligned} &f = \operatorname{div} F,\ F \in L^2\big(0,T;L^2(\Omega)\big),\ v_0 \in L^2_\sigma(\Omega),\\ &E \in L^s\big(0,T;L^q(\Omega)\big),\ \operatorname{div} E = k \in L^4\big(0,T;L^2(\Omega)\big),\\ &4 \le s < \infty,\ 4 \le q < \infty,\ \frac{2}{s} + \frac{3}{q} = 1\,. \end{aligned} \tag{1.27}$$

Then there exists at least one weak solution v of the perturbed system (1.6) *in* $[0,T) \times \Omega$ *with data f, v_0 in the sense of Definition* 1.1. *The solution v satisfies with some constant $C = C(\Omega) > 0$ the energy estimate*

$$\begin{aligned} \|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\,d\tau \le C\bigg(\|v_0\|_2^2 + \int_0^t \|F\|_2^2\,d\tau + \int_0^t \|k\|_2^4\,d\tau\\ + \int_0^t \|E\|_4^4\,d\tau\bigg) \exp\big(C\|k\|_{2,4;t}^4 + C\|E\|_{q,s;t}^s\big) \end{aligned} \tag{1.28}$$

for each $0 \le t < T$.

b) *Suppose additionally*

$$\begin{aligned} &k \in L^s\big(0,T;L^{q^*}(\Omega)\big),\ g \in L^s\big(0,T;W^{-\frac{1}{q},q}(\partial\Omega)\big),\ E_0 \in L^q(\Omega),\\ &\int_\Omega k\,dx = \int_{\partial\Omega} N \cdot g\,dS \textit{ for a.a. } t \in [0,T), \end{aligned} \tag{1.29}$$

and let E be the very weak solution of the linear system (1.4) *in* $[0,T) \times \Omega$ *with data k, g, E_0 as in Lemma* 1.2. *Then $u = v + E$ is a weak solution of* *the general system* (1.1) *with data* f, k, g *and initial value* $u_0 = v_0 + E_0$ *in the sense of Definition* 1.3.

There are some partial results with nonhomogeneous smooth boundary conditions $u|_{\partial\Omega} = g \neq 0$ based on an independent approach by Raymond [15]. For the case of weak solutions with constant in time nonzero boundary conditions g see [4]. Further there are several independent results for smooth boundary values $u|_{\partial\Omega} = g \neq 0$ in the context of strong solutions u if g or (equivalently) the time interval $[0, T)$ satisfy certain smallness conditions, see [1], [3], [6], [10]. Our existence result for weak solutions in Theorem 1.4 does not need any smallness condition, like for usual Leray-Hopf weak solutions. But, on the other hand, there is no uniqueness result as for local strong solutions.

A first result on global weak solutions with time-dependent boundary data (and $k = \operatorname{div} u = 0$) can be found in [5]. In that paper, the authors consider general $s > 2$, $q > 3$ with $\frac{2}{s} + \frac{3}{q} = 1$; however, in that case, E has to satisfy the assumptions

$$E \in L^s\big(0, T; L^q(\Omega)\big) \cap L^4\big(0, T; L^4(\Omega)\big),$$

which is automatically fulfilled in the present article, see Theorem 1.4. Moreover, in simply connected domains or under a further assumption on the boundary data g, the energy estimate (1.28) can be improved considerably.

2. Preliminaries.

First we recall some standard notations. Let $C^\infty_{0,\sigma}(\Omega) = \{w \in C^\infty_0(\Omega);\ \operatorname{div} w = 0\}$ be the space of smooth, solenoidal and compactly supported vector fields. Then let $L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$, $1 < q < \infty$, where in general $\|\cdot\|_q$ denotes the norm of the Lebesgue space $L^q(\Omega)$, $1 \le q \le \infty$. Sobolev spaces are denoted by $W^{m,q}(\Omega)$ with norm $\|\cdot\|_{W^{m,q}} = \|\cdot\|_{m,q}$, $m \in \mathbb{N}$, $1 \le q \le \infty$, and $W^{m,q}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{m,q}}$, $1 \le q < \infty$. The trace space to $W^{1,q}(\Omega)$ is $W^{1-1/q,q}(\partial\Omega)$, $1 < q < \infty$, with norm $\|\cdot\|_{1-1/q,q}$. Then the dual space to $W^{1-1/q',q'}(\partial\Omega)$, where $\frac{1}{q'} + \frac{1}{q} = 1$, is $W^{-1/q,q}(\partial\Omega)$; the corresponding pairing is denoted by $\langle\cdot,\cdot\rangle_{\partial\Omega}$.

As spaces of test functions we need in the context of very weak solutions the space $C^2_{0,\sigma}(\overline{\Omega}) = \{w \in C^2(\overline{\Omega});\ w|_{\partial\Omega} = 0,\ \operatorname{div} w = 0\}$; for weak instationary solutions let the space $C^\infty_0([0,T); C^\infty_{0,\sigma}(\Omega))$ denote vector fields

$w \in C_0^\infty([0,T) \times \Omega)$ such that $\operatorname{div}_x w = 0$ for all $t \in [0,T)$ taking the divergence div_x with respect to $x = (x_1, x_2, x_3) \in \Omega$. The pairing of functions on Ω and $(0,T) \times \Omega$ is denoted by $\langle \cdot, \cdot \rangle_\Omega$ and $\langle \cdot, \cdot \rangle_{\Omega,T}$, respectively.

For $1 \le q,\ s \le \infty$ the usual Bochner space $L^s(0,T; L^q(\Omega))$ is equipped with the norm $\| \cdot \|_{q,s;T} = (\int_0^T \| \cdot \|_q^s \, d\tau)^{1/s}$ when $s < \infty$ and $\| \cdot \|_{q,\infty;T} = \operatorname{ess\,sup}_{(0,T)} \| \cdot \|_q$ when $s = \infty$.

Let $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$, $1 < q < \infty$, be the Helmholtz projection, and let $A_q = -P_q \Delta$ with domain $D(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$ and range $R(A_q) = L^q_\sigma(\Omega)$ denote the Stokes operator. We write $P = P_q$ and $A = A_q$ if there is no misunderstanding. For $-1 \le \alpha \le 1$ the fractional powers $A_q^\alpha : \mathcal{D}(A_q^\alpha) \to L^q_\sigma(\Omega)$ are well-defined closed operators with $(A_q^\alpha)^{-1} = A_q^{-\alpha}$. For $0 \le \alpha \le 1$ we have $D(A_q) \subseteq D(A_q^\alpha) \subseteq L^q_\sigma(\Omega)$ and $R(A_q^\alpha) = L^q_\sigma(\Omega)$. Then there holds the embedding estimate

$$\|v\|_q \le C \|A_q^\alpha v\|_\gamma, \quad 0 \le \alpha \le 1,\ 2\alpha + \frac{3}{q} = \frac{3}{\gamma},\ 1 < \gamma \le q, \tag{2.1}$$

for all $v \in D(A_q^\alpha)$. Further, we need the Stokes semigroup $e^{-tA_q} : L^q_\sigma(\Omega) \to L^q_\sigma(\Omega)$, $t \ge 0$, satisfying the estimate

$$\|A_q^\alpha e^{-tA_q} v\|_q \le C t^{-\alpha} e^{-\beta t} \|v\|_q,\ 0 \le \alpha \le 1,\ t > 0, \tag{2.2}$$

for $v \in L^q_\sigma(\Omega)$ with constants $C = C(\Omega, q, \alpha) > 0$, $\beta = \beta(\Omega, q) > 0$; for details see [2, 7, 8, 9, 11].

In order to solve the perturbed system (1.6) we use an approximation procedure based on Yosida's smoothing operators

$$J_m = \big(I + \frac{1}{m} A^{1/2}\big)^{-1} \quad \text{and} \quad \mathcal{J}_m = \left(I + \frac{1}{m}(-\Delta)^{1/2}\right)^{-1},\ \ m \in \mathbb{N}, \tag{2.3}$$

where I denotes the identity and $-\Delta$ the Dirichlet Laplacian on Ω. In particular, we need the properties

$$\begin{aligned} &\|J_m v\|_q \le C \|v\|_q,\ \|A^{1/2} J_m v\|_q \le mC \|v\|_q,\ m \in \mathbb{N}, \\ &\lim_{m \to \infty} J_m v = v \quad \text{for all } v \in L^q_\sigma(\Omega); \end{aligned} \tag{2.4}$$

and analogous results for $\mathcal{J}_m v$, $v \in L^q(\Omega)$; see [8, 9, 16].

To solve the instationary Stokes system in $[0,T) \times \Omega$, cf. [1, 13, 16, 17, 18], let us recall some properties for the special system

$$
\begin{aligned}
V_t - \Delta V + \nabla H &= f_0 + \operatorname{div} F_0, & \operatorname{div} V &= 0 \\
V &= 0 \text{ on } \partial\Omega, & V(0) &= V_0
\end{aligned} \tag{2.5}
$$

with data

$$
f_0 \in L^1\big(0,T;L^2(\Omega)\big),\ F_0 \in L^2\big(0,T;L^2(\Omega)\big),\ V_0 \in L^2_\sigma(\Omega);
$$

here $F_0 = (F_{0,ij})_{i,j=1}^3$ and $\operatorname{div} F_0 = \Big(\sum_{i=1}^3 \frac{\partial}{\partial x_i} F_{0,ij}\Big)_{j=1}^3$. The linear system (2.5) admits a unique weak solution

$$
V \in L^\infty\big(0,T;L^2_\sigma(\Omega)\big) \cap L^2\big(0,T;W^{1,2}_0(\Omega)\big), \tag{2.6}
$$

satisfying the variational formulation

$$
-\langle V, w_t\rangle_{\Omega,T} + \langle \nabla V, \nabla w\rangle_{\Omega,T} = \langle V_0, w(0)\rangle_\Omega + \langle f_0, w\rangle_{\Omega,T} - \langle F_0, \nabla w\rangle_{\Omega,T} \tag{2.7}
$$

for all $w \in C_0^\infty([0,T); C^\infty_{0,\sigma}(\Omega))$, and the energy equality

$$
\frac{1}{2}\|V(t)\|_2^2 + \int_0^t \|\nabla V\|_2^2\, d\tau = \frac{1}{2}\|V_0\|_2^2 + \int_0^t \langle f_0, V\rangle_\Omega\, d\tau - \int_0^t \langle F_0, \nabla V\rangle_\Omega\, d\tau \tag{2.8}
$$

for $0 \le t < T$. As a consequence of (2.8) we get the energy estimate

$$
\frac{1}{2}\|V\|^2_{2,\infty;T} + \|\nabla V\|^2_{2,2;T} \le 8\big(\|V_0\|_2^2 + \|f_0\|^2_{2,1;T} + \|F_0\|^2_{2,2;T}\big), \tag{2.9}
$$

and see that $V : [0,T) \to L^2_\sigma(\Omega)$ is continuous with $V(0) = V_0$. Moreover, it holds the well-defined representation formula

$$
V(t) = e^{-tA}V_0 + \int_0^t e^{-(t-\tau)A} P f_0\, d\tau + \int_0^t A^{1/2} e^{-(t-\tau)A} A^{-1/2} P \operatorname{div} F_0\, d\tau, \tag{2.10}
$$

$0 \le t < T$; see [16, Theorems IV.2.3.1 and 2.4.1, Lemma IV.2.4.2], and, concerning the operator $A^{-1/2}P \operatorname{div}$, [16, Ch. III.2.6].

Consider the perturbed system (1.6) with $f = \operatorname{div} F$, v_0, k and E as in Definition 1.1, here written in the form

$$
v_t - \Delta v + \operatorname{div}(v+E)(v+E) - k(v+E) + \nabla p^* = f,\ \operatorname{div} v = 0 \tag{2.11}
$$

together with the initial-boundary conditions $v = 0$ on $\partial\Omega$ and $v(0) = v_0$.

In order to obtain the following approximate system, see [16, V, 2.2] for the known case $E \equiv 0$, we insert the Yosida operators (2.3) into (2.11) as follows:

$$
\begin{aligned}
&v_t - \Delta v + \operatorname{div}(J_m v + E)(v + E) - (\mathcal{J}_m k)(v+E) + \nabla p^* = f,\ \operatorname{div} v = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v|_{\partial\Omega} = 0,\ v|_{t=0} = v_0
\end{aligned}
\tag{2.12}
$$

with $v = v_m$, $m \in \mathbb{N}$. Setting

$$
F_m(v) = (J_m v + E)(v + E),\ f_m(v) = (\mathcal{J}_m k)(v + E)
\tag{2.13}
$$

we write the approximate system (2.12) in the form

$$
\begin{aligned}
&v_t - \Delta v + \nabla p^* = f_m(v) + \operatorname{div}\big(F - F_m(v)\big),\ \operatorname{div} v = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad v|_{\partial\Omega} = 0,\ v|_{t=0} = v_0,
\end{aligned}
\tag{2.14}
$$

as a linear system, see (2.5), with right-hand side depending on v. In this form we use the properties (2.6)-(2.10) of the linear system (2.5).

The following definition for (2.12) is obtained similarly as Definition 1.1.

DEFINITION 2.1. (Approximate system). *Suppose*

$$
\begin{aligned}
&f = \operatorname{div} F,\ F \in L^2\big(0, T; L^2(\Omega)\big),\ v_0 \in L^2_\sigma(\Omega), \\
&E \in L^s\big(0, T; L^q(\Omega)\big),\ \operatorname{div} E = k \in L^4\big(0, T; L^2(\Omega)\big), \\
&4 \le s < \infty,\ 4 \le q < \infty,\ \frac{2}{s} + \frac{3}{q} = 1.
\end{aligned}
\tag{2.15}
$$

Then a vector field $v = v_m$, $m \in \mathbb{N}$, is called a weak solution of the approximate system (2.12) *in $[0, T) \times \Omega$ with data f, v_0 if the following conditions are satisfied:*

a)

$$
v \in L^\infty_{\mathrm{loc}}\big([0, T); L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0, T); W^{1,2}_0(\Omega)\big),
\tag{2.16}
$$

b) *for each $w \in C^\infty_0([0, T); C^\infty_{0,\sigma}(\Omega))$,*

$$
\begin{aligned}
&- \langle v, w_t\rangle_{\Omega,T} + \langle \nabla v, \nabla w\rangle_{\Omega,T} - \big\langle (J_m v + E)(v + E), \nabla w\big\rangle_{\Omega,T} \\
&\quad - \big\langle (\mathcal{J}_m k)(v + E), w\big\rangle_{\Omega,T} = \big\langle v_0, w(0)\big\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T}\,,
\end{aligned}
\tag{2.17}
$$

c) *for $0 \le t < T$,*

$$
\begin{aligned}
&\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\, d\tau \le \frac{1}{2}\|v_0\|_2^2 - \int_0^t \big\langle F - (J_m v + E)E, \nabla v\big\rangle_\Omega\, d\tau \\
&\quad + \int_0^t \big\langle (\mathcal{J}_m k - \frac{1}{2}k)v, v\big\rangle_\Omega\, d\tau + \int_0^t \big\langle (\mathcal{J}_m k)E, v\big\rangle_\Omega\, d\tau\,,
\end{aligned}
\tag{2.18}
$$

d) *$v : [0, T) \to L^2_\sigma(\Omega)$ is continuous satisfying $v(0) = v_0$.*

3. The approximate system.

The following existence result yields a weak solution $v = v_m$ of (2.12) first of all only in an interval $[0, T')$ where $T' = T'(m) > 0$ is sufficiently small.

LEMMA 3.1. *Let f, k, E, v_0 be as in Definition* 2.1 *and let $m \in \mathbb{N}$. Then there exists some $T' = T'(f, k, E, v_0, m)$, $0 < T' \le \min(1, T)$, such that the approximate system* (2.12) *has a unique weak solution $v = v_m$ in $[0, T') \times \Omega$ with data f, v_0 in the sense of Definition* 2.1 *with T replaced by T'.*

PROOF. First we consider a given weak solution $v = v_m$ of (2.12) in $[0, T') \times \Omega$ with any $0 < T' \le 1$. Hence it holds

$$v \in X_{T'} := L^\infty\big(0, T'; L^2_\sigma(\Omega)\big) \cap L^2\big(0, T'; W^{1,2}_0(\Omega)\big)$$

with

$$\|v\|_{X_{T'}} := \|v\|_{2,\infty;T'} + \|A^{\frac{1}{2}} v\|_{2,2;T'} < \infty. \tag{3.1}$$

Using Hölder's inequality and several embedding estimates, see [16, Ch. V.1.2], we obtain with some constant $C = C(\Omega) > 0$ the estimates

$$\begin{aligned} \|(J_m v)v\|_{2,2;T'} &\le C\|J_m v\|_{6,4;T'}\ \|v\|_{3,4;T'} \\ &\le C\|A^{1/2} J_m v\|_{2,4;T'}\ \|v\|_{X_{T'}} \\ &\le Cm\|v\|_{2,4;T'} \le Cm(T')^{1/4}\|v\|^2_{X_{T'}}, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \|(J_m v)E\|_{2,2;T'} &\le C\|J_m v\|_{4,4;T'}\|E\|_{4,4;T'} \le C\|J_m v\|_{6,4;T'}\|E\|_{4,4;T'} \\ &\le Cm(T')^{1/4}\|v\|_{X_{T'}}\ \|E\|_{4,4;T'}, \end{aligned} \tag{3.3}$$

$$\|Ev\|_{2,2;T'} \le C\|E\|_{q,s;T'}\|v\|_{(\frac{1}{2}-\frac{1}{q})^{-1},(\frac{1}{2}-\frac{1}{s})^{-1},T'} \le C\|E\|_{q,s;T'}\|v\|_{X_{T'}}; \tag{3.4}$$

of course, $\|EE\|_{2,2;T'} \le C\|E\|^2_{4,4;T'}$. Moreover,

$$\begin{aligned} \|(\mathcal{J}_m k)v\|_{2,1;T'} &\le C\|\mathcal{J}_m k\|_{3,2;T'}\|v\|_{6,2;T'} \le C\|(-\Delta)^{\frac{1}{2}}\mathcal{J}_m k\|_{2,2;T'}\|v\|_{X_{T'}} \\ &\le Cm\|k\|_{2,2;T'}\|v\|_{X_{T'}} \le Cm(T')^{\frac{1}{4}}\|k\|_{2,4;T'}\|v\|_{X_{T'}}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \|(\mathcal{J}_m k)E\|_{2,1;T'} &\le C\|\mathcal{J}_m k\|_{4,2;T'}\|E\|_{4,2;T'} \le C\|(-\Delta)^{\frac{1}{2}}\mathcal{J}_m k\|_{2,2;T'}\|E\|_{4,4;T'} \\ &\le Cm\|k\|_{2,2;T'}\|E\|_{4,4;T'} \le Cm(T')^{\frac{1}{4}}\|k\|_{2,4;T'}\|E\|_{4,4;T'}. \end{aligned} \tag{3.6}$$

Using (2.14) and the energy estimate (2.9) with f_0, F_0 replaced by $f_m(v)$, $F - F_m(v)$ we get from (3.2)-(3.5) the estimate

$$\begin{aligned}\|v\|_{X_{T'}} \le C\big(&\|v_0\|_2 + \|F\|_{2,2;T'} + \|E\|^2_{4,4;T'} + m(T')^{\frac{1}{4}}\|v\|^2_{X_{T'}} + \\ &+ m(T')^{\frac{1}{4}}\|v\|_{X_{T'}}\|E\|_{4,4;T'} + \|v\|_{X_{T'}}\|E\|_{q,s;T'} + \\ &+ m(T')^{\frac{1}{4}}\|k\|_{2,4;T'}(\|E\|_{4,4;T'} + \|v\|_{X_{T'}})\big)\end{aligned} \tag{3.7}$$

with $C = C(\Omega) > 0$.

Applying (2.10) to (2.14) we obtain the equation

$$v = \mathcal{F}_{T'}(v) \tag{3.8}$$

where

$$\begin{aligned}\big(\mathcal{F}_{T'}(v)\big)(t) = \ & e^{-tA}v_0 + \int_0^t e^{-(t-\tau)A}Pf_m(v)\, d\tau \\ & + \int_0^t A^{\frac{1}{2}}e^{-(t-\tau)A}A^{-\frac{1}{2}}P \operatorname{div}\big(F - F_m(v)\big)\, d\tau.\end{aligned}$$

Let

$$\begin{aligned}&a = Cm(T')^{\frac{1}{4}},\ b = C\|E\|_{q,s;T'} + Cm(T')^{\frac{1}{4}}\|E\|_{4,4;T'} + Cm(T')^{\frac{1}{4}}\|k\|_{2,4;T'}, \\ &d = C(\|v_0\|_2 + \|E\|^2_{4,4;T'} + \|F\|_{2,2;T'} + m(T')^{\frac{1}{4}}\|k\|_{2,4;T'}\|E\|_{4,4;T'})\end{aligned} \tag{3.9}$$

with C as in (3.7). Then (3.7) may be rewritten in the form

$$\|\mathcal{F}_{T'}(v)\|_{X_{T'}} \le a\|v\|^2_{X_{T'}} + b\|v\|_{X_{T'}} + d. \tag{3.10}$$

Up to now $v = v_m$ was a given solution as desired in Lemma 3.1. In the next step we treat (3.8) as a fixed point equation in $X_{T'}$ and show with Banach's fixed point principle that (3.8) has a solution $v = v_m$ if $T' > 0$ is sufficiently small.

Thus let $v \in X_{T'}$ and choose $0 < T' \le \min(1, T)$ such that the smallness condition

$$4ad + 2b < 1 \tag{3.11}$$

is satisfied. Then the quadratic equation $y = ay^2 + by + d$ has a minimal positive root given by

$$0 < y_1 = 2d\Big(1 - b + \sqrt{b^2 + 1 - (4ad + 2b)}\Big)^{-1} < 2d$$

and, since $y_1 = ay_1^2 + by_1 + d > d$, we conclude that $\mathcal{F}_{T'}$ maps the closed ball $B_{T'} = \{v \in X_{T'} : \|v\|_{X_{T'}} \le y_1\}$ into itself.

Further let v_1, $v_2 \in B_{T'}$. Then we obtain similarly as in (3.10) the estimate

$$\begin{aligned} \|\mathcal{F}_{T'}(v_1) - \mathcal{F}_{T'}(v_2)\|_{X_{T'}} &\le Cm(T')^{\frac{1}{4}}\|v_1 - v_2\|_{X_{T'}}\big(\|v_1\|_{X_{T'}} + \|v_2\|_{X_{T'}}\big) \\ &\quad + C\|v_1 - v_2\|_{X_{T'}}\big(\|E\|_{q,s;T'} + m(T')^{\frac{1}{4}}\|k\|_{2,4;T'} + m(T')^{\frac{1}{4}}\|E\|_{4,4;T'}\big) \\ &\le \|v_1 - v_2\|_{X_{T'}}\big(a(\|v_1\|_{X_{T'}} + \|v_2\|_{X_{T'}}) + b\big) \end{aligned} \tag{3.12}$$

where

$$a\big(\|v_1\|_{X_{T'}} + \|v_2\|_{X_{T'}}\big) + b \le 2ay_1 + b < 4ad + 2b < 1. \tag{3.13}$$

This means that $\mathcal{F}_{T'}$ is a strict contraction on $B_{T'}$. Now Banach's fixed point principle yields a solution $v = v_m \in B_{T'}$ of (3.8) which is unique in $B_{T'}$.

Using (2.6)-(2.10) with $f_0 + \operatorname{div} F_0$ replaced by $f_m(v) + \operatorname{div}(F - F_m(v))$ we conclude from (3.8) that $v = v_m$ is a solution of the approximate system (2.12) in the sense of Definition 2.1.

Finally we show that v is unique not only in $B_{T'}$, but even in the whole space $X_{T'}$. Indeed, consider any solution $\tilde{v} \in X_{T'}$ of (2.12). Then there exists some $0 < T^* \le \min(1, T')$ such that $\|\tilde{v}\|_{X_{T^*}} \le y_1$, and using (3.12), (3.13) with v_1, v_2 replaced by v, $\tilde{v}$ we conclude that $v = \tilde{v}$ on $[0, T^*]$. When $T^* < T'$ we repeat this step finitely many times and obtain that $v = \tilde{v}$ on $[0, T')$. This completes the proof of Lemma 3.1. □

The next preliminary result yields an energy estimate for the approximate solution $v = v_m$ of (2.12). It is important that the right-hand side of this estimate does not depend on $m \in \mathbb{N}$. This will enable us to treat the limit $m \to \infty$ and to get the desired solution in Theorem 1.4, a).

LEMMA 3.2. *Consider any weak solution $v = v_m$, $m \in \mathbb{N}$, of the approximate system* (2.12) *in the sense of Definition* 2.1. *Then there is a constant $C = C(\Omega) > 0$ such that the energy estimate*

$$\begin{aligned} &\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 \, d\tau \\ &\le C\big(\|v_0\|_2^2 + \|F\|_{2,2;t}^2 + \|k\|_{2,4;t}^4 + \|E\|_{4,4;t}^4\big) \exp\big(C\|k\|_{2,4;t}^4 + C\|E\|_{q,s;t}^s\big) \end{aligned} \tag{3.14}$$

holds for $0 \le t < T$.

Proof. The proof of (3.14) is based on the energy inequality (2.18). Using similar arguments as in (3.2)-(3.6) we obtain the following estimates of the right-hand side terms in (2.18); here $\varepsilon > 0$ means an absolute constant, $C_0 = C_0(\Omega) > 0$ and $C = C(\varepsilon, \Omega) > 0$ do not depend on m, and $\alpha = \frac{2}{s} = 1 - \frac{3}{q}$. First of all

$$\begin{aligned}\Big|\int_0^t \langle (J_m v)E, \nabla v\rangle_\Omega \, d\tau\Big| &\le C_0 \int_0^t \|J_m v\|_{(\frac{1}{2}-\frac{1}{q})^{-1}} \|E\|_q \|\nabla v\|_2 \, d\tau \\ &\le C_0 \int_0^t \|v\|_{(\frac{1}{2}-\frac{1}{q})^{-1}} \|E\|_q \|\nabla v\|_2 \, d\tau \\ &\le C_0 \int_0^t \|v\|_2^\alpha \|E\|_q \|\nabla v\|_2^{2-\alpha} \, d\tau \\ &\le \varepsilon \|\nabla v\|_{2,2;t}^2 + C \int_0^t \|E\|_q^s \|v\|_2^2 \, d\tau,\end{aligned} \tag{3.15}$$

and

$$\Big|\int_0^t \langle EE, \nabla v\rangle_\Omega \, d\tau\Big| \le C_0 \int_0^t \|E\|_4^2 \|\nabla v\|_2 \, d\tau \le \varepsilon \|\nabla v\|_{2,2;t}^2 + C\|E\|_{4,4;t}^4,$$

$$\Big|\int_0^t \langle F, \nabla v\rangle_\Omega \, d\tau\Big| \le \varepsilon \|\nabla v\|_{2,2;t}^2 + C\|F\|_{2,2;t}^2.$$

Moreover, since $\|v\|_4 \le C_0 \|\nabla v\|_2^{1/4} \|\nabla v\|_2^{3/4}$,

$$\Big|\int_0^t \langle \mathcal{J}_m k v, v\rangle_\Omega \, d\tau\Big| \le \varepsilon \|\nabla v\|_{2,2;t}^2 + C \int_0^t \|k\|_2^4 \|v\|_2^2 \, d\tau,$$

$$\begin{aligned}\Big|\int_0^t \langle (\mathcal{J}_m k)E, v\rangle_\Omega \, d\tau\Big| &\le C_0 \int_0^t \|(\mathcal{J}_m k)E\|_{\frac{6}{5}} \|v\|_6 \, d\tau \\ &\le C_0 \int_0^t \|k\|_2 \|E\|_3 \|\nabla v\|_2 \, d\tau \\ &\le \varepsilon \|\nabla v\|_{2,2;t}^2 + C\big(\|k\|_{2,4;t}^4 + \|E\|_{4,4;t}^4\big).\end{aligned}$$

A similar estimate as for $\int_0^t \langle \mathcal{J}_m kv, v\rangle_\Omega \, d\tau$ also holds for $\int_0^t \langle kv, v\rangle_\Omega \, d\tau$.

Choosing $\varepsilon > 0$ sufficiently small we apply these inequalities to (2.18) and obtain that

$$\|v(t)\|_2^2 + \|\nabla v\|_{2,2;t}^2 \le C\big(\|v_0\|_2^2 + \|F\|_{2,2;t}^2 + \|E\|_{4,4;t}^4 + \|k\|_{2,4;t}^4\big)$$
$$+ C \int_0^t \big(\|k\|_2^4 + \|E\|_q^s\big)\|v\|_2^2 \, d\tau$$

for $0 \le t < T$. Then Gronwall's lemma implies that

$$\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 \, d\tau \le C\big(\|v_0\|_2^2 + \|F\|_{2,2;t}^2 + \|E\|_{4,4;t}^4 + \|k\|_{2,4;t}^4\big) \tag{3.16}$$
$$\times \exp\big(C\|k\|_{2,4;t}^4 + C\|E\|_{q,s;t}^s\big)$$

for $0 \le t < T$. This yields the estimate (3.14). □

The next result proves the existence of a unique approximate solution $v = v_m$ for the given interval $[0, T)$.

LEMMA 3.3. *Let f, k, E, v_0 be given as in Definition* 2.1 *and let $m \in \mathbb{N}$. Then there exists a unique weak solution $v = v_m$ of the approximate system* (2.12) *in* $[0, T) \times \Omega$ *with data f, v_0.*

PROOF. Lemma 3.1 yields such a solution if $0 < T \le 1$ is sufficiently small. Let $[0, T^*) \subseteq [0, T)$, $T^* > 0$, be the largest interval of existence of such a solution $v = v_m$ in $[0, T^*) \times \Omega$, and assume that $T^* < T$. Further we choose some finite $T^{**} > T^*$ with $T^{**} \le T$, and some T_0 satisfying $0 < T_0 < T^*$. Then we apply Lemma 3.1 with $[0, T')$ replaced by $[T_0, T_0 + \delta)$ where $\delta > 0$, $T_0 + \delta \le T^{**}$, and find a unique weak solution $v^* = v_m^*$ of the system (2.12) in $[T_0, T_0 + \delta) \times \Omega$ with initial value $v^*|_{t=T_0} = v(T_0)$. The length δ of the existence interval $[T_0, T_0 + \delta)$, see the proof of Lemma 3.1, only depends on $\|v(T_0)\|_2 \le \|v\|_{2,\infty;T^*} < \infty$ and on $\|F\|_{2,2;T^{**}}$, $\|E\|_{q,s;T^{**}}$, $\|k\|_{2,4;T^{**}}$, and can be chosen independently of T_0. Therefore, we can choose T_0 close to T^* in such a way that $T^* < T_0 + \delta \le T^{**}$. Then v^* yields a unique extension of v from $[0, T^*)$ to $[0, T_0 + \delta)$ which is a contradiction. This proves the lemma. □

In the next step, see §4 below, we are able to let $m \to \infty$ similarly as in the classical case $E \equiv 0$. This will yield a solution of the perturbed system (1.6).

4. Proof of Theorem 1.4.

It is sufficient to prove Theorem 1.4, a). For this purpose we start with the sequence (v_m) of solutions of the approximate system (2.12) constructed in Lemma 3.3. Then, using Lemma 3.2, we find for each finite T^*, $0 < T^* \leq T$, some constant $C_{T^*} > 0$ not depending on m such that

$$\|v_m\|^2_{2,\infty;T^*} + \|\nabla v_m\|^2_{2,2;T^*} \leq C_{T^*} \ . \tag{4.1}$$

Hence there exists a vector field

$$v \in L^\infty\big(0, T^*; L^2_\sigma(\Omega)\big) \cap L^2\big(0, T^*; W^{1,2}_0(\Omega)\big), \tag{4.2}$$

and a subsequence of (v_m), for simplicity again denoted by (v_m), with the following properties, see, e.g. [16, Ch. V.3.3]:

$$\begin{aligned} &v_m \rightharpoonup v \text{ in } L^2\big(0, T^*; W^{1,2}_0(\Omega)\big) \quad \text{(weakly)} \\ &v_m \to v \text{ in } L^2\big(0, T^*; L^2(\Omega)\big) \quad \text{(strongly)} \\ &v_m(t) \to v(t) \text{ in } L^2(\Omega) \text{ for a.a. } t \in [0, T^*). \end{aligned} \tag{4.3}$$

Moreover, for all $t \in [0, T^*)$ we obtain that

$$\begin{aligned} \|\nabla v\|^2_{2,2;t} &\leq \liminf_{m\to\infty} \|\nabla v_m\|^2_{2,2;t} \, , \\ \|v(t)\|^2_2 &\leq \liminf_{m\to\infty} \|v_m(t)\|^2_2 \, . \end{aligned} \tag{4.4}$$

Further, using Hölder's inequality and (4.2)-(4.4) we get with some further subsequence, again denoted by (v_m), that

$$\begin{aligned} v_m \rightharpoonup v \quad &\text{in } \ L^{s_1}\big(0, T^*; L^{q_1}(\Omega)\big), \ \frac{2}{s_1} + \frac{3}{q_1} = \frac{3}{2}, \ 2 \leq s_1, \, q_1 < \infty, \\ v_m v_m \rightharpoonup vv \quad &\text{in } \ L^{s_2}\big(0, T^*; L^{q_2}(\Omega)\big), \ \frac{2}{s_2} + \frac{3}{q_2} = 3, \ 1 \leq s_2, \, q_2 < \infty, \\ v_m \cdot \nabla v_m \rightharpoonup v \cdot \nabla v \quad &\text{in } \ L^{s_3}\big(0, T^*; L^{q_3}(\Omega)\big), \ \frac{2}{s_3} + \frac{3}{q_3} = 4, \ 1 \leq s_3, \, q_3 < \infty \, , \end{aligned} \tag{4.5}$$

and that with some constant $C = C_{T^*} > 0$:

$$\|(J_m v_m)v_m\|_{q_2,s_2;T^*} \le C\|v_m\|^2_{q_1,s_1;T^*} \tag{4.6}$$

$$\|(J_m v_m)E\|_{(\frac{1}{q}+\frac{1}{q_1})^{-1},(\frac{1}{s}+\frac{1}{s_1})^{-1};T^*} \le C\|v_m\|_{q_1,s_1;T^*}\|E\|_{q,s;T^*} \tag{4.7}$$

$$\|Ev_m\|_{(\frac{1}{q}+\frac{1}{q_1})^{-1},(\frac{1}{s}+\frac{1}{s_1})^{-1};T^*} \le C\|v_m\|_{q_1,s_1;T^*}\|E\|_{q,s;T^*} \tag{4.8}$$

$$\left|\left\langle (J_m v_m)E, \nabla v_m\right\rangle_{\Omega,T^*}\right| \le C\|v_m\|_{q_1,s_1;T^*}\|E\|_{q,s;T^*}\|\nabla v_m\|_{2,2;T^*} \tag{4.9}$$

as well as

$$\begin{aligned} \left|\left\langle k v_m, v_m\right\rangle_{\Omega,T^*}\right| &\le C\|k\|_{2,4;T^*}\|v_m\|^2_{q_1,s_1;T^*} \\ \left|\left\langle (\mathcal{J}_m k)v_m, v_m\right\rangle_{\Omega,T^*}\right| &\le C\|k\|_{2,4;T^*}\|v_m\|^2_{q_1,s_1;T^*} \\ \left|\left\langle (\mathcal{J}_m k)E, v_m\right\rangle_{\Omega,T^*}\right| &\le C\|k\|_{2,4;T^*}\|E\|_{q,s;T^*}\|v_m\|_{q_1,s_1;T^*}\,. \end{aligned} \tag{4.10}$$

The theorem is proved when we show that (2.16)-(2.18) imply letting $m \to \infty$ the properties (1.8)-(1.10) and the estimate (1.28). This proof rests on the above arguments (4.1)-(4.10).

Obviously, (1.8) follows from (4.1), letting $m \to \infty$. Further, the relation (1.9) follows from (2.17) and (2.4) using that

$$\begin{aligned} \langle v_m, w_t\rangle_{\Omega,T^*} &\to \langle v, w_t\rangle_{\Omega,T^*} \\ \langle \nabla v_m, \nabla w\rangle_{\Omega,T^*} &\to \langle \nabla v, \nabla w\rangle_{\Omega,T^*} \\ \left\langle (J_m v_m + E)(v_m + E), \nabla w\right\rangle_{\Omega,T^*} &\to \left\langle (v+E)(v+E), \nabla w\right\rangle_{\Omega,T^*} \\ \left\langle (\mathcal{J}_m k)(v_m + E), w\right\rangle_{\Omega,T^*} &\to \left\langle k(v+E), w\right\rangle_{\Omega,T^*}. \end{aligned} \tag{4.11}$$

To prove the energy inequality (1.10) we need in (2.18), letting $m \to \infty$, the following arguments.

The left-hand side of (1.10) follows obviously from (4.4). To prove the right-hand side limit $m \to \infty$ in (2.18) we first show that

$$\left\langle (J_m v_m)E, \nabla v_m\right\rangle_{\Omega,T^*} \to \langle vE, \nabla v\rangle_{\Omega,T^*}. \tag{4.12}$$

It is sufficient to prove (4.12) with E replaced by some smooth vector field $\tilde{E}$ such that $\|E - \tilde{E}\|_{q,s;T^*}$ is sufficiently small. This follows using (4.9) with E replaced by $E - \tilde{E}$. Thus we may assume in the following that E in (4.12) is a smooth function $E \in C_0^\infty([0,T^*); C_0^\infty(\Omega))$. Using (4.1)-(4.4) and (2.4), we conclude that

$$\begin{aligned}
&\big|\big\langle (J_m v_m)E - vE, \nabla v_m \big\rangle_{\Omega,T^*}\big| \\
&\quad \le \|(J_m v_m)E - vE\|_{2,2;T^*}\ \|\nabla v_m\|_{2,2;T^*} \\
&\quad \le C(E)\|J_m v_m - v\|_{2,2;T^*} \\
&\quad \le C(E)\big(\|J_m(v_m - v)\|_{2,2;T^*} + \|(J_m - I)v\|_{2,2;T^*}\big) \\
&\quad \le C(E)\big(\|v_m - v\|_{2,2;T^*} + \|(J_m - I)v\|_{2,2;T^*}\big) \to 0
\end{aligned}$$

as $m \to \infty$ where $C(E) > 0$ is a constant. This yields (4.12).

Similarly, approximating k by a smooth function $k \in C_0^\infty([0,T^*); C_0^\infty(\Omega))$, we obtain the convergence properties

$$\begin{aligned}
\langle k v_m, v_m\rangle_{\Omega,T^*} &\to \langle kv, v\rangle_{\Omega,T^*}, \\
\big\langle (\mathcal{J}_m k) v_m, v_m\big\rangle_{\Omega,T^*} &\to \langle kv, v\rangle_{\Omega,T^*}, \\
\big\langle (\mathcal{J}_m k) E, v_m\big\rangle_{\Omega,T^*} &\to \langle kE, v\rangle_{\Omega,T^*}.
\end{aligned}$$

Since $E \in L^4(0,T^*; L^4(\Omega))$, the convergence $\langle EE, \nabla v_m\rangle_{\Omega,T^*} \to \langle EE, \nabla v\rangle_{\Omega,T^*}$ is obvious.

This proves that v is a weak solution in the sense of Definition 1.1.

To prove the energy estimate (1.28) we apply (4.4) to (3.14). This completes the proof. □

5. More general weak solutions.

The existence of a weak solution v for the perturbed system (1.6) under the general assumption on E in Theorem 1.4 a) enables us to extend the solution class of the Navier-Stokes system (1.1) using certain generalized data. For simplicity we only consider the case $k = 0$.

THEOREM 5.1 (More general weak solutions). *Consider*

(5.1) $$f = \operatorname{div} F,\ F \in L^2\big(0,T; L^2(\Omega)\big),\ v_0 \in L^2_\sigma(\Omega),$$

(5.2) $$E \in L^s\big(0,T; L^q(\Omega)\big),\ 4 \le s < \infty,\ 4 \le q < \infty,\ \frac{2}{s} + \frac{3}{q} = 1,$$

satisfying

(5.3) $$E_t - \Delta E + \nabla h = 0,\ \operatorname{div} E = 0$$

in $(0,T) \times \Omega$ *in the sense of distributions with an associated pressure* h.

Let v be a weak solution of the perturbed system (1.6) *in* $[0, T) \times \Omega$ *in the sense of Definition* 1.1 *with* E, f, v_0 *from* (5.1)-(5.3).

Then the vector field $u = v + E$ *is a solution of the Navier-Stokes system*

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \ \operatorname{div} u = 0 \tag{5.4}$$

$$u|_{\partial\Omega} = g, \ u|_{t=0} = u_0 \tag{5.5}$$

in $[0, T) \times \Omega$ *with external force* f *and (formally) given data*

$$g := E|_{\partial\Omega}, \quad u_0 := v_0 + E|_{t=0}, \tag{5.6}$$

in the generalized (well-defined) sense that

$$(u - E)|_{\partial\Omega} = 0, \quad (u - E)|_{t=0} = v_0,$$

and (5.4) *is satisfied in the sense of distributions with an associated pressure p.*

REMARK 5.2. (Regularity properties)

a) *Let E in* (5.2) *be regular in the sense that g and* $E_0 = E|_{t=0}$ *in* (5.6) *have the properties in Lemma* 1.2. *Then the solution* $u = v + E$ *has the properties in Theorem* 1.4, b).

b) *Let E in* (5.2) *be regular in the sense that g and* $E_0 = E|_{t=0}$ *in* (5.6) *have the properties in* (1.26). *Then the solution* $u = v + E$ *is correspondingly regular and* (5.5) *is well-defined in the usual strong sense.*

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