

Dual Properties of Triebel-Lizorkin-Type Spaces and their Applications

Dachun Yang and Wen Yuan

Abstract. Let $s \in \mathbb{R}$, $p \in (1, \infty)$, $\tau \in [0, \frac{1}{p}]$ and $\mathcal{S}_\infty(\mathbb{R}^n)$ be the set of all Schwartz functions φ whose Fourier transforms $\widehat{\varphi}$ satisfy that $\partial^\gamma \widehat{\varphi}(0) = 0$ for all $\gamma \in (\mathbb{N} \cup \{0\})^n$. Denote by ${}_V\dot{F}_{p,p}^{s,\tau}(\mathbb{R}^n)$ the closure of $\mathcal{S}_\infty(\mathbb{R}^n)$ in the Triebel–Lizorkin-type space $\dot{F}_{p,p}^{s,\tau}(\mathbb{R}^n)$. In this paper, the authors prove that the dual space of ${}_V\dot{F}_{p,p}^{s,\tau}(\mathbb{R}^n)$ is the Triebel–Lizorkin–Hausdorff space $F\dot{H}_{p',p'}^{-s,\tau}(\mathbb{R}^n)$ via their φ -transform characterizations together with the atomic decomposition characterization of the tent space $F\dot{T}_{p',p'}^{-s,\tau}(\mathbb{R}_\mathbb{Z}^{n+1})$, where t' denotes the conjugate index of $t \in [1, \infty]$. This gives a generalization of the well-known duality that $(\text{CMO}(\mathbb{R}^n))^* = H^1(\mathbb{R}^n)$ by taking $s = 0$, $p = 2$ and $\tau = \frac{1}{2}$. As applications, the authors obtain the Sobolev-type embedding property, the smooth atomic and molecular decomposition characterizations, boundednesses of both pseudo-differential operators and the trace operators on $F\dot{H}_{p,p}^{s,\tau}(\mathbb{R}^n)$; all of these results improve the existing conclusions.

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1. Introduction

Recently, the Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and the Triebel–Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ were introduced and investigated in [14, 22, 23]. These spaces unify and generalize Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$, Triebel–Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^n)$, Morrey spaces, Morrey–Triebel–Lizorkin spaces and $Q_\alpha(\mathbb{R}^n)$ spaces. Recall that the

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spaces $Q_\alpha(\mathbb{R}^n)$ were originally introduced by Essén, Janson, Peng and Xiao [9]; see also [7, 9, 19, 20] for the history of Q spaces and their properties.

Let $s \in \mathbb{R}$, $p \in (1, \infty)$, $q \in [1, \infty)$ and $\tau \in [0, \frac{1}{(\max\{p, q\})^\gamma}]$. Here and in what follows, for any $t \in [1, \infty]$, t' denotes its *conjugate index*, namely, $\frac{1}{t} + \frac{1}{t'} = 1$. The Besov–Hausdorff spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and the Triebel–Lizorkin–Hausdorff spaces $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ ($q > 1$) were also introduced in [22, 23]; moreover, it was proved therein that they are respectively the predual spaces of $\dot{B}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$. The spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ unify and generalize Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$, Triebel–Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^n)$ and Hardy–Hausdorff spaces $HH_{-\alpha}^1(\mathbb{R}^n)$ for $\alpha \in (0, 1)$, where $HH_{-\alpha}^1(\mathbb{R}^n)$ was recently introduced by Dafni and Xiao in [7] and was proved to be the predual space of $Q_\alpha(\mathbb{R}^n)$ therein.

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on \mathbb{R}^n and denote by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, namely, the set of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ endowed with the weak *-topology. Let $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. Following Triebel [17], we set

$$\mathcal{S}_\infty(\mathbb{R}^n) \equiv \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi(x) x^\gamma dx = 0 \text{ for all multi-indices } \gamma \in (\mathbb{Z}_+)^n \right\}$$

and consider $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including the topology. Use $\mathcal{S}'_\infty(\mathbb{R}^n)$ to denote the topological dual of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functionals on $\mathcal{S}_\infty(\mathbb{R}^n)$. We also endow $\mathcal{S}'_\infty(\mathbb{R}^n)$ with the weak *-topology. Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all polynomials on \mathbb{R}^n . It is well known that $\mathcal{S}'_\infty(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ as topological spaces.

Let ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ be the *closure of $\mathcal{S}_\infty(\mathbb{R}^n)$ in $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$* . Recall that $\mathcal{S}_\infty(\mathbb{R}^n)$ may not be dense in the space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$; see [22, Remark 3.1(ii)]. Thus, ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ may be a proper subspace of $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and it makes no sense to study the dual space of $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which explains the necessity to introduce the space ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Moreover, the main target of this paper is to show that for all $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in [0, \frac{1}{p}]$, the dual space, denoted by $({}_V\dot{F}_{p,p}^{s,\tau}(\mathbb{R}^n))^*$, of ${}_V\dot{F}_{p,p}^{s,\tau}(\mathbb{R}^n)$ is the space $F\dot{H}_{p',p'}^{-s,\tau}(\mathbb{R}^n)$, which is obtained via their φ -transform characterizations together with the atomic decomposition characterization of the tent space $F\dot{T}_{p',p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})$. This generalizes the well-known result in [6] that $(\text{CMO}(\mathbb{R}^n))^* = H^1(\mathbb{R}^n)$ by taking $s = 0$, $p = 2$ and $\tau = \frac{1}{2}$. Indeed, in order to represent $H^1(\mathbb{R}^n)$ as a dual space, Coifman and Weiss [6] introduced the space $\text{CMO}(\mathbb{R}^n)$, which was originally denoted by $\text{VMO}(\mathbb{R}^n)$ in [6], as the closure of continuous functions with compact supports in the $\text{BMO}(\mathbb{R}^n)$ norm and established this dual relation. We recall that $\text{CMO}(\mathbb{R}^n)$ is also the closure of all smooth functions with compact support in $\text{BMO}(\mathbb{R}^n)$; see, for example, [2, p. 519].

As applications of this new dual theorem, in this paper, we also obtain the Sobolev-type embedding property, the smooth atomic and molecular decomposition characterizations, boundednesses of both pseudo-differential operators and the trace operators on $F\dot{H}_{p,p}^{s,\tau}(\mathbb{R}^n)$; all of these results improve the existing conclusions.

To recall the notions of these spaces, we need some notation. For $k \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$, we denote by Q_{jk} the *dyadic cube* $2^{-j}([0, 1]^n + k)$, $\ell(Q)$ its *side length*, x_Q its *lower left-corner* $2^{-j}k$ and c_Q its *center*. Set $\mathcal{Q}(\mathbb{R}^n) \equiv \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$, $\mathcal{Q}_j(\mathbb{R}^n) \equiv \{Q \in \mathcal{Q}(\mathbb{R}^n) : \ell(Q) = 2^{-j}\}$ for all $j \in \mathbb{Z}$, and $j_Q \equiv -\log_2 \ell(Q)$ for all $Q \in \mathcal{Q}(\mathbb{R}^n)$. When the dyadic cube Q appears as an index, such as $\sum_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ and $\{\cdot\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$, it is understood that Q runs over all *dyadic cubes* in \mathbb{R}^n .

In what follows, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we use $\widehat{\varphi}$ to denote its Fourier transform, namely, for all $\xi \in \mathbb{R}^n$, $\widehat{\varphi}(\xi) \equiv \int_{\mathbb{R}^n} e^{-i\xi x} \varphi(x) dx$. Set $\varphi_j(x) \equiv 2^{jn} \varphi(2^j x)$ for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$.

Assume that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\text{supp } \widehat{\varphi} \subset \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \right\} \quad \text{and} \quad |\widehat{\varphi}(\xi)| \geq C > 0 \quad \text{if} \quad \frac{3}{5} \leq |\xi| \leq \frac{5}{3}. \quad (1)$$

Now we recall the notion of Triebel–Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [23, Definition 1.1].

Definition 1.1. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p \in (0, \infty)$, $q \in (0, \infty]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (1). The *Triebel–Lizorkin-type space* $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'_{\infty}(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^{\tau}} \left\{ \int_P \left[\sum_{j=j_P}^{\infty} (2^{js} |\varphi_j * f(x)|)^q \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}}$$

with suitable modification made when $q = \infty$.

It was proved in [23, Corollary 3.1] that the space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is independent of the choices of φ . Recall that $\dot{F}_{p,q}^{s,0}(\mathbb{R}^n) \equiv \dot{F}_{p,q}^s(\mathbb{R}^n)$, $\dot{F}_{p,q}^{s,1/p}(\mathbb{R}^n) \equiv \dot{F}_{\infty,q}^s(\mathbb{R}^n)$ and $\dot{F}_{2,2}^{\alpha,1/2-\alpha}(\mathbb{R}^n) \equiv Q_{\alpha}(\mathbb{R}^n)$ for all $\alpha \in (0, 1)$; see [23, Proposition 3.1] and [22, Corollary 3.1]. Also, for all $s \in \mathbb{R}$, $q \in (0, \infty]$ and $0 < u \leq p \leq \infty$, $\dot{F}_{u,q}^{s,1/u-1/p}(\mathbb{R}^n) = \dot{\mathcal{E}}_{pqu}^s(\mathbb{R}^n)$, in particular, $\dot{F}_{u,2}^{0,1/u-1/p}(\mathbb{R}^n) = \mathcal{M}_u^p(\mathbb{R}^n)$, where $\dot{\mathcal{E}}_{pqu}^s(\mathbb{R}^n)$ denotes the Triebel–Lizorkin–Morrey space, introduced and investigated in [13, 15], and $\mathcal{M}_u^p(\mathbb{R}^n)$ is the well-known Morrey space; see [14, Theorem 1.1]. Some useful characterizations of $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, including the φ -transform characterization, Sobolev-type embedding property, smooth atomic and molecular decomposition characterizations, were obtained in [23], which generalize the corresponding results on Triebel–Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^n)$; see [3, 4, 10, 11, 16, 17].

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r) \equiv \{y \in \mathbb{R}^n : |x - y| < r\}$. We now recall the notion of Hausdorff capacities; see, for example, [1, 21]. Let $E \subset \mathbb{R}^n$ and $d \in (0, n]$. The d -dimensional Hausdorff capacity of E is defined by

$$H^d(E) \equiv \inf \left\{ \sum_j r_j^d : E \subset \bigcup_j B(x_j, r_j) \right\}, \quad (2)$$

where the infimum is taken over all covers $\{B(x_j, r_j)\}_{j=1}^\infty$ of countable open balls of E . It is well known that H^d is monotone, countably subadditive and vanishes on the empty set. Moreover, H^d in (2) when $d = 0$ also makes sense, and H^0 has the properties that for all sets $E \subset \mathbb{R}^n$, $H^0(E) \geq 1$, and $H^0(E) = 1$ if and only if E is bounded.

For any function $f : \mathbb{R}^n \mapsto [0, \infty]$, the *Choquet integral of f with respect to H^d* is defined by

$$\int_{\mathbb{R}^n} f dH^d \equiv \int_0^\infty H^d(\{x \in \mathbb{R}^n : f(x) > \lambda\}) d\lambda.$$

This functional is not sublinear, so sometimes we need to use an equivalent integral with respect to the d -dimensional dyadic Hausdorff capacity \tilde{H}^d , which is sublinear; see [21] (also [22, 23]) for the definition of dyadic Hausdorff capacities and their properties.

In what follows, for any $p, q \in (0, \infty]$, let $p \vee q \equiv \max\{p, q\}$ and $p \wedge q \equiv \min\{p, q\}$. Set $\mathbb{R}_+^{n+1} \equiv \mathbb{R}^n \times (0, \infty)$. For any measurable function ω on \mathbb{R}_+^{n+1} and $x \in \mathbb{R}^n$, define its *nontangential maximal function* $N\omega$ by setting $N\omega(x) \equiv \sup_{|y-x|<t} |\omega(y, t)|$. We now recall the notion of the spaces $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [22, Definition 5.1].

Definition 1.2. Let $s \in \mathbb{R}$, $p, q \in (1, \infty)$, $\tau \in [0, \frac{1}{(p \vee q)'}]$ and φ be as in Definition 1.1. The *Triebel–Lizorkin–Hausdorff space* $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \inf_\omega \left\| \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left| \varphi_j * f [\omega(\cdot, 2^{-j})]^{-1} \right|^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where ω runs over all nonnegative Borel measurable functions on \mathbb{R}_+^{n+1} satisfying

$$\int_{\mathbb{R}^n} [N\omega(x)]^{(p \vee q)'} dH^{n\tau(p \vee q)'}(x) \leq 1 \quad (3)$$

and with the restriction that for any $j \in \mathbb{Z}$, $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\varphi_j * f$ vanishes.

It was proved in [22, Section 5] that the space $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is independent of the choices of φ . Recall that $F\dot{H}_{p,q}^{s,0}(\mathbb{R}^n) \equiv \dot{F}_{p,q}^s(\mathbb{R}^n)$ and $F\dot{H}_{2,2}^{-\alpha,1/2-\alpha}(\mathbb{R}^n) \equiv HH_{-\alpha}^1(\mathbb{R}^n)$; see also [22, Section 5]. It was proved in [22, Theorem 5.1] that $(F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n))^* = \dot{F}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $p, q \in (1, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)^\tau}]$. Also, the φ -transform characterization, Sobolev-type embedding property, smooth atomic and molecular decomposition characterizations of $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ were obtained in [24].

In what follows, for simplicity, we use $\dot{F}_p^{s,\tau}(\mathbb{R}^n)$ to denote $\dot{F}_{p,p}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ to denote $F\dot{H}_{p,p}^{s,\tau}(\mathbb{R}^n)$, respectively. The main result of this paper is the following dual theorem. Recall that ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the closure of $\mathcal{S}_\infty(\mathbb{R}^n)$ in $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Theorem 1.3. *Let $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in [0, \frac{1}{p}]$. Then the dual space of ${}_V\dot{F}_p^{s,\tau}(\mathbb{R}^n)$ is $F\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)$ in the following sense: if $f \in F\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)$, then the linear map*

$$\nu \mapsto \int_{\mathbb{R}^n} f(x)\nu(x) dx \quad (4)$$

defined initially for all $\nu \in \mathcal{S}_\infty(\mathbb{R}^n)$, has a bounded extension to ${}_V\dot{F}_p^{s,\tau}(\mathbb{R}^n)$ with operator norm no more than a positive constant multiple of $\|f\|_{F\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)}$; conversely, if $L \in ({}_V\dot{F}_p^{s,\tau}(\mathbb{R}^n))^$, then there exists an $f \in F\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)$ with $\|f\|_{F\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)}$ no more than a positive constant multiple of $\|L\|$ such that L has the form (4) for all $\nu \in \mathcal{S}_\infty(\mathbb{R}^n)$.*

Recall that $F\dot{H}_2^{0,1/2}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$; see [22, Remark 5.2]. We also remark that ${}_V\dot{F}_2^{0,1/2}(\mathbb{R}^n) = \text{CMO}(\mathbb{R}^n)$ (see Corollary 2.2 below). Then Theorem 1.3, when taking $s = 0$, $\tau = \frac{1}{2}$ and $p = 2$, generalizes the well-known duality obtained in [6] that $(\text{CMO}(\mathbb{R}^n))^* = H^1(\mathbb{R}^n)$.

Notice that when $\tau = 0$, Theorem 1.3 has a more general version, that is, for all $s \in \mathbb{R}$ and $q \in (0, \infty]$, $({}_V\dot{F}_{p,q}^s(\mathbb{R}^n))^* = \dot{F}_{p',q'}^{-s}(\mathbb{R}^n)$ with $p \in [1, \infty)$ and $({}_V\dot{B}_{p,q}^s(\mathbb{R}^n))^* = \dot{B}_{p',q'}^{-s}(\mathbb{R}^n)$ with $p \in [1, \infty]$, where $q' = \infty$ when $q \in (0, 1]$; see, for example, [16, pp. 121–122] and [17, p. 180, Remark 2]. However, to be surprised, the dual property in Theorem 1.3 are not possible to be correct for all $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $\tau > 0$, $p \in (1, \infty)$, $q \in [1, \infty)$ and $p \neq q$, which is quite different from the above classical cases; see Remark 4.3 below for more details.

Set $\mathbb{R}_{\mathbb{Z}}^{n+1} \equiv \mathbb{R}^n \times \{2^k : k \in \mathbb{Z}\}$. Let $C_c^\infty(\mathbb{R}^n)$ be the set of all smooth functions f on \mathbb{R}^n with compact support. For all $M \in \mathbb{N} \cup \{0\}$, let $C_{c,M}^\infty(\mathbb{R}^n)$ be the set of all $f \in C_c^\infty(\mathbb{R}^n)$ satisfying that $\int_{\mathbb{R}^n} f(x)x^\gamma dx = 0$ for all $|\gamma| \leq M$. We also write $C_{c,-1}^\infty(\mathbb{R}^n) \equiv C_c^\infty(\mathbb{R}^n)$.

In Section 2, we prove that for all admissible indices s , τ , p and q , the space ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ coincides with the closure of $C_{c,M}^\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ for

certain M (see Theorem 2.1 below), which further implies that ${}_V\dot{F}_2^{0,1/2}(\mathbb{R}^n) = \text{CMO}(\mathbb{R}^n)$. In Section 3, we recall the notion and some known results on the tent spaces $F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_Z^{n+1})$ and $F\dot{W}_{p,q}^{s,\tau}(\mathbb{R}_Z^{n+1})$, which are, respectively, corresponding to $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Then for $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in [0, \frac{1}{p}]$, we prove that the dual space of ${}_C F\dot{W}_{p,p}^{s,\tau}(\mathbb{R}_Z^{n+1})$ is just $F\dot{T}_{p',p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})$, where ${}_C F\dot{W}_{p,p}^{s,\tau}(\mathbb{R}_Z^{n+1})$ is the *closure of the set of all functions in $F\dot{W}_{p,p}^{s,\tau}(\mathbb{R}_Z^{n+1})$ with compact support*. Via this, in Section 4, we give the proof of Theorem 1.3. As applications, in Section 5, we establish the Sobolev-type embedding property of $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$. We also obtain its smooth atomic and molecular decomposition characterizations, boundednesses of pseudo-differential operators and the trace operators on $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, which improve the corresponding conclusions in the case that $p = q$ in [24].

Recall that in [6], the atomic decomposition characterization of $H^1(\mathbb{R}^n)$ plays an important role in establishing the duality between $\text{CMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$. However, for the space $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, we have no such analogous atomic decomposition characterization so far. To overcome this difficulty, in this paper, different from [6], by fully using the atomic decomposition characterization of the tent space $F\dot{T}_{p,p}^{s,\tau}(\mathbb{R}_Z^{n+1})$ corresponding to $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, we first obtain the predual space of the tent space $F\dot{T}_{p,p}^{s,\tau}(\mathbb{R}_Z^{n+1})$ (see Theorem 3.4 below), which further induces a dual theorem for the spaces of sequences corresponding to $\dot{F}_p^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ (see Proposition 4.2 below). This combined with the φ -transform characterizations of both $\dot{F}_p^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ then yields the desired conclusion of Theorem 1.3.

Finally we make some conventions on notation. Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line, while $C(\alpha, \beta, \dots)$ denotes a positive constant depending on the parameters α, β, \dots . The symbol $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. If E is a subset of \mathbb{R}^n , we denote by χ_E the characteristic function of E . For a dyadic cube $Q \in \mathcal{Q}(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$, set $\varphi_Q(x) \equiv |Q|^{-\frac{1}{2}}\varphi(2^j(x-x_Q))$ and $\tilde{\chi}_Q(x) \equiv |Q|^{-\frac{1}{2}}\chi_Q(x)$, and for $r > 0$, let rQ be the cube concentric with Q having the side length $r\ell(Q)$. We also set $\mathbb{N} \equiv \{1, 2, \dots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$.

2. An equivalent characterization of ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$

In this section, we establish an equivalent characterization of ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, the *closure of $\mathcal{S}_\infty(\mathbb{R}^n)$ in $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$* ; precisely, we prove that ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ coincides with the *closure of $C_{c,M}^\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$* for certain M , which further implies that $\text{CMO}(\mathbb{R}^n)$ is a special case of ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

For all $M \in \mathbb{Z}_+ \cup \{-1\}$, denote by ${}_M\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ the closure of $C_{c,M}^\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Obviously, ${}_{M_1}\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset {}_{M_2}\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ if $M_1 \geq M_2$. Throughout the whole paper, for all $p, q \in (0, \infty]$ and $s \in \mathbb{R}$, set

$$J \equiv \frac{1}{\min\{1, p, q\}} \quad \text{and} \quad N \equiv \max\{\lfloor J - n - s \rfloor, -1\}, \quad (5)$$

where and in what follows, for any $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the maximal integer no more than a .

The main result of this section is the following theorem.

Theorem 2.1. *Let $s \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0, \infty]$, $M \in \mathbb{Z}_+ \cup \{-1\}$. Let J and N be as in (5).*

- (i) *Let $\tau \in [0, \frac{1}{p} + \frac{1+\lfloor J-s \rfloor - J + s}{n}]$ when $N \geq 0$ or $\tau \in [0, \frac{1}{p} + \frac{s+n-J}{n}]$ when $N < 0$. Then ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset {}_M\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.*
- (ii) *Let $\tau \in [0, \infty)$. Then ${}_M\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset {}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ if $M > \max\{\frac{n}{p \wedge q} - n - 1, -s + \frac{n}{p} - n - 1\}$.*

As an immediately corollary of Theorem 2.1, we have the following conclusion.

Corollary 2.2. *Let $s \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0, \infty]$, and J and N be as in (5). Let $\tau \in [0, \frac{1}{p} + \frac{1+\lfloor J-s \rfloor - J + s}{n}]$ when $N \geq 0$ or $\tau \in [0, \frac{1}{p} + \frac{s+n-J}{n}]$ when $N < 0$ and $M \in \mathbb{Z}_+ \cup \{-1\}$ such that $M > \max\{\frac{n}{p \wedge q} - n - 1, -s + \frac{n}{p} - n - 1\}$. Then ${}_M\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) = {}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.*

Notice that $\dot{F}_2^{0,1/2}(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$ and ${}_{-1}\dot{F}_2^{0,1/2}(\mathbb{R}^n) = \text{CMO}(\mathbb{R}^n)$. Applying Corollary 2.2, we then have ${}_V\dot{F}_2^{0,1/2}(\mathbb{R}^n) = \text{CMO}(\mathbb{R}^n)$.

For all $L \in \mathbb{Z}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, set $\|\varphi\|_{S_L} \equiv \sup_{x \in \mathbb{R}^n} \sup_{|\gamma| \leq L} |\partial^\gamma \varphi(x)| (1 + |x|)^{n+L+|\gamma|}$, where and in what follows, for all $\gamma = (\gamma_1, \dots, \gamma_n) \in (\mathbb{Z}_+)^n$, $\partial^\gamma \equiv \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}}$. To prove Theorem 2.1, we need the following lemma. Its proof is similar to that of [22, Lemma 2.2]. We omit the details.

Lemma 2.3. *Let $M \in \mathbb{Z}_+ \cup \{-1\}$, $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$ and $f \in C_{c,M}^\infty(\mathbb{R}^n)$.*

- (i) *If $j \in \mathbb{Z}_+$, then for all $L \in \mathbb{Z}_+$, there exists a positive constant $C(L, n)$, depending only on L and n , such that for all $x \in \mathbb{R}^n$,*

$$|\varphi_j * f(x)| \leq C(L, n) \|\varphi\|_{S_{L+1}} \|f\|_{S_{L+1}} 2^{-jL} (1 + |x|)^{-n-L-1}.$$

- (ii) *If $j \in \mathbb{Z} \setminus \mathbb{Z}_+$, then there exists a positive constant $C(M, n)$, depending only on M and n , such that for all $x \in \mathbb{R}^n$,*

$$|\varphi_j * f(x)| \leq C(M, n) \|\varphi\|_{S_{M+2}} \|f\|_{S_{M+2}} (2^{-j} + |x|)^{-n-M-1}.$$

We now recall the sequence space corresponding to $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$; see [23, Definition 3.1].

Definition 2.4. Let $s \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0, \infty]$ and $\tau \in [0, \infty)$. The sequence space $\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $t \equiv \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \subset \mathbb{C}$ such that $\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{\infty} \sum_{\ell(Q)=2^{-j}} 2^{j(s+\frac{n}{2})q} |t_Q|^q \chi_Q(x) \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}}.$$

An important tool used in the proof of Theorem 2.1 is the smooth atomic decomposition characterization of $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [23, Theorem 4.3] (see also [10, Theorem 4.1]). Recall that a smooth atom for $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined as follows.

Definition 2.5 ([23, Definition 4.1]). Let $s \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0, \infty]$, and J and N be as in (5). Let $\tau \in [0, \frac{1}{p} + \frac{1+|J-s|-J+s}{n})$ when $N \geq 0$ or $\tau \in [0, \frac{1}{p} + \frac{s+n-J}{n})$ when $N < 0$. A $C^\infty(\mathbb{R}^n)$ function a_Q is called a smooth atom for $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near a dyadic cube Q if there exist integers $K \geq \max\{[s+n\tau+1], 0\}$ and $\tilde{N} \geq N$ such that $\text{supp } a_Q \subset 3Q$, $\int_{\mathbb{R}^n} x^\gamma a_Q(x) dx = 0$ if $|\gamma| \leq \tilde{N}$ and $|\partial^\gamma a_Q(x)| \leq |Q|^{-\frac{1}{2}-\frac{|\gamma|}{n}}$ for all $x \in 3Q$ if $|\gamma| \leq K$.

We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. (i) Since $M_1 \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset M_2 \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ if $M_1 \geq M_2$, we only need to prove (i) when $M \geq N \equiv \max\{[J-n-s], -1\}$; equivalently, it suffices to prove that for any $\varepsilon \in (0, \infty)$ and $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, there exists a function $g \in C_{c,M}^\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $M \geq N$ such that $\|f - g\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \varepsilon$.

Let φ be as in Definition 1.1. By [23, Theorem 4.3] and its proof together with $\mathcal{S}_\infty(\mathbb{R}^n) \subset \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ (see [23, Proposition 3.1(ix)]), we know that each $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ has a representation $f = \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_j(\mathbb{R}^n)} t_Q a_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$, where a_Q is a smooth atom for $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q satisfying that $\int_{\mathbb{R}^n} x^\gamma a_Q(x) dx = 0$ for all $|\gamma| \leq M$,

$$t_Q \equiv C_1 \left[\sum_{\{R \in \mathcal{Q}(\mathbb{R}^n) : \ell(R) = \ell(Q)\}} \frac{|\langle f, \varphi_R \rangle|^{p \wedge q}}{(1 + [\ell(Q)]^{-1} |x_R - x_Q|)^\lambda} \right]^{\frac{1}{p \wedge q}}$$

and $\|\{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C_2 \|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)}$, where $\lambda > n$ can be sufficiently large, which is determined later, and C_1, C_2 are positive constants independent of f .

For $L \in \mathbb{N}$, set

$$f_L \equiv \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} t_Q a_Q \chi_{\{R \in \mathcal{Q}(\mathbb{R}^n) : 2^{-L} \leq \ell(R) \leq 2^L, R \subset [-2^L, 2^L]^n\}}(Q).$$

Obviously, $f_L \in C_{c,M}^\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. By [23, Theorem 4.3] again, we see that

$$\begin{aligned}
 & \|f - f_L\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \\
 & \leq C_2 \left\| \left\{ t_Q \chi_{\{R \in \mathcal{Q}(\mathbb{R}^n): \ell(R) \notin [2^{-L}, 2^L] \text{ or } R \not\subseteq [-2^L, 2^L]^n\}}(Q)\right\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \right\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \\
 & \leq CC_2 \left(\sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{\substack{j=j_P \\ |j|>L}}^{\infty} \sum_{\ell(Q)=2^{-j}} 2^{j(s+\frac{n}{2})q} |t_Q|^q \chi_Q(x) \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \right. \\
 & \quad \left. + \sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{\substack{j=j_P \\ |j|\leq L}}^{\infty} \sum_{\substack{\ell(Q)=2^{-j} \\ Q \not\subseteq [-2^L, 2^L]^n}} 2^{j(s+\frac{n}{2})q} |t_Q|^q \chi_Q(x) \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \right) \\
 & \equiv CC_2 \left(\sup_{P \in \mathcal{Q}(\mathbb{R}^n)} I_P + \sup_{P \in \mathcal{Q}(\mathbb{R}^n)} J_P \right),
 \end{aligned}$$

where C is a positive constant independent of f and L .

Since $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, by [22, Lemma 2.2], we see that for all $j \in \mathbb{Z}$ and $k \in (\mathbb{Z}_+)^n$,

$$t_{Q_{jk}} \lesssim C_1 \left[\sum_{l \in \mathbb{Z}^n} \frac{2^{-[\frac{jn}{2} + |j|K + (j \wedge 0)K](p \wedge q)}}{(1 + |l - k|)^\lambda (2^{-(j \wedge 0)} + |2^{-j}l|)^{(n+K)(p \wedge q)}} \right]^{\frac{1}{p \wedge q}}, \quad (6)$$

where we chose $K \in \mathbb{Z}_+$ such that $K > \max \{n[\frac{1}{p \wedge q} - 1], s + n[\tau + (\frac{1}{q} - \frac{1}{p}) \vee 0], -s + n(\frac{1}{p} - 1)\}$.

If $j_P \geq -L$, we then have

$$\begin{aligned}
 I_P & \lesssim C_1 \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=(j_P \vee L)}^{\infty} \sum_{k \in \mathbb{Z}^n} 2^{j(s-K)q} \chi_{Q_{jk}}(x) \right. \right. \\
 & \quad \left. \left. \times \left(\sum_{l \in \mathbb{Z}^n} \frac{1}{(1 + |l - k|)^\lambda (1 + |2^{-j}l|)^{(n+K)(p \wedge q)}} \right)^{\frac{q}{p \wedge q}} \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}}.
 \end{aligned}$$

Notice that $j \geq (j_P \vee L) \geq 1$. Then $|2^{-j}k| \leq |2^{-j}l| + |l - k|$, which implies that $1 + |2^{-j}l| \geq (1 + |2^{-j}k|)(1 + |k - l|)^{-1}$. In what follows, we always choose

$\lambda > n + (n + K)(p \wedge q)$. By this, we then have

$$\begin{aligned} I_P &\lesssim C_1 \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=(j_P \vee L)}^{\infty} \sum_{k \in \mathbb{Z}^n} 2^{j(s-K)q} \chi_{Q_{jk}}(x) (1 + |2^{-j}k|)^{-(n+K)q} \right. \right. \\ &\quad \left. \left. \times \left(\sum_{l \in \mathbb{Z}^n} \frac{1}{(1 + |l - k|)^{\lambda - (n+K)(p \wedge q)}} \right)^{\frac{q}{p \wedge q}} \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \\ &\lesssim C_1 \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=(j_P \vee L)}^{\infty} \sum_{k \in \mathbb{Z}^n} 2^{j(s-K)q} \chi_{Q_{jk}}(x) (1 + |2^{-j}k|)^{-(n+K)q} \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}}. \end{aligned}$$

When $p \leq q$, by $K > \max \left\{ s + n \left[\tau + \left(\frac{1}{q} - \frac{1}{p} \right) \vee 0 \right], n \left(\frac{1}{p} - 1 \right) \right\}$, $(j_P \vee L) \geq L$ and the inequality that for all $d \in (0, 1]$ and $\{\alpha_j\}_j \subset \mathbb{C}$,

$$\left(\sum_j |\alpha_j| \right)^d \leq \sum_j |\alpha_j|^d, \quad (7)$$

we obtain that $I_P \lesssim C_1 |P|^{-\tau} \left\{ \sum_{j=(j_P \vee L)}^{\infty} 2^{j(s-K)p} \right\}^{\frac{1}{p}} \lesssim C_1 2^{(j_P \vee L)L(s+n\tau-K)} \lesssim C_1 2^{L(s+n\tau-K)}$. When $p > q$, by Minkowski's inequality, we also have

$$I_P \lesssim C_1 \frac{1}{|P|^\tau} \left\{ \sum_{j=(j_P \vee L)}^{\infty} 2^{j(s-K)q} 2^{jn \left(\frac{1}{q} - \frac{1}{p} \right) q} \right\}^{\frac{1}{q}} \lesssim C_1 2^{L[s-K+n(\tau+\frac{1}{q}-\frac{1}{p})]}.$$

If $j_P < -L$, we see that

$$\begin{aligned} I_P &\leq CC_1 \left(\frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{-L-1} \sum_{k \in \mathbb{Z}^n} 2^{j(s+n+K)q} \chi_{Q_{jk}}(x) \right. \right. \right. \\ &\quad \left. \left. \times \left(\sum_{l \in \mathbb{Z}^n} \frac{1}{(1 + |l - k|)^\lambda (1 + |l|)^{(n+K)(p \wedge q)}} \right)^{\frac{q}{p \wedge q}} \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \\ &\quad + \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=L+1}^{\infty} \sum_{k \in \mathbb{Z}^n} 2^{j(s-K)q} \chi_{Q_{jk}}(x) \right. \right. \\ &\quad \left. \left. \times \left(\sum_{l \in \mathbb{Z}^n} \frac{1}{(1 + |l - k|)^\lambda (1 + |2^{-j}l|)^{(n+K)(p \wedge q)}} \right)^{\frac{q}{p \wedge q}} \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \right) \\ &\equiv CC_1 (I_1 + I_2), \end{aligned}$$

where C is a positive constant independent of L . The estimate of I_2 is the same as the estimate for I_P in the case that $j_P \geq -L$, by noticing that in the estimate for I_P , we did not use the fact that $j_P \geq -L$. To estimate I_1 , by $|P|^{-\tau} \leq 1$, $(1 + |l - k|)(1 + |l|) \geq (1 + |k|)$ and $K > -s + n(\frac{1}{p} - 1)$, applying (7) when $p \leq q$ or Minkowski's inequality when $p > q$, we obtain

$$I_1 \lesssim \left\{ \int_P \left[\sum_{j=j_P}^{-L} \sum_{k \in \mathbb{Z}^n} 2^{j(s+n+K)q} (1 + |k|)^{-(n+K)q} \chi_{Q_{jk}}(x) \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \lesssim 2^{-L[s+K+n(1-\frac{1}{p})]}.$$

Therefore, we have $I_P \lesssim C_1 \max \{ 2^{L\{s+n[\tau+(\frac{1}{q}-\frac{1}{p})\vee 0]-K\}}, 2^{-L[s+K+n(1-\frac{1}{p})]} \}$.

Next we estimate J_P . Notice that $(1 + |l - k|)(2^{-(j \wedge 0)} + |2^{-j}l|) \geq 2^{-(j \wedge 0)}(1 + |2^{-(j \vee 0)}k|)$ for all $j \in \mathbb{Z}$ and $l, k \in (\mathbb{Z}_+)^n$. By (6) and $\lambda > n + (n + K)(p \wedge q)$, we have

$$\begin{aligned} J_P &\lesssim C_1 \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{\substack{j=j_P \\ |j| \leq L}}^{\infty} \sum_{\substack{k \in \mathbb{Z}^n \\ Q_{jk} \not\subseteq [-2L, 2L]^n}} 2^{j(s+\frac{n}{2})q} \chi_{Q_{jk}}(x) \right. \right. \\ &\quad \times \left. \left. \left(\sum_{l \in \mathbb{Z}^n} \frac{2^{-[\frac{jn}{2} + |j|K + (j \wedge 0)K](p \wedge q)}}{(1 + |l - k|)^\lambda (2^{-(j \wedge 0)} + |2^{-j}l|)^{(n+K)(p \wedge q)}} \right)^{\frac{q}{p \wedge q}} \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \\ &\lesssim C_1 \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{\substack{j=j_P \\ |j| \leq L}}^{\infty} \sum_{\substack{k \in \mathbb{Z}^n \\ Q_{jk} \not\subseteq [-2L, 2L]^n}} 2^{j sq} \chi_{Q_{jk}}(x) \frac{2^{-[|j|K - (j \wedge 0)n]q}}{(1 + |2^{-(j \vee 0)}k|)^{(n+K)q}} \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}}. \end{aligned}$$

When $p \leq q$, applying (7) yields that

$$\begin{aligned} J_P &\lesssim C_1 \frac{1}{|P|^\tau} \left\{ \sum_{\substack{j=j_P \\ |j| \leq L}}^{\infty} \sum_{\substack{k \in \mathbb{Z}^n \\ Q_{jk} \not\subseteq [-2L, 2L]^n}} 2^{j sp} \frac{2^{-|j|Kp + (j \wedge 0)np - jn}}{(1 + |2^{-(j \vee 0)}k|)^{(n+K)p}} \right\}^{\frac{1}{p}} \\ &\lesssim C_1 \frac{1}{|P|^\tau} \left\{ \sum_{\substack{j=j_P \\ |j| \leq L}}^{\infty} 2^{j sp} 2^{-|j|Kp + (j \wedge 0)np - jn + (j \vee 0)n} 2^{-[L + (j \wedge 0)][(n+K)p - n]} \right\}^{\frac{1}{p}}. \end{aligned}$$

If $j_P > L$, then $J_P = 0$. If $0 \leq j_P \leq L$, then

$$J_P \lesssim C_1 2^{-L[K+n(1-\frac{1}{p})]} 2^{j_P(s+n\tau-K)} \lesssim C_1 2^{-L[K+n(1-\frac{1}{p})]}.$$

If $j_P < 0$, then

$$\begin{aligned} J_P &\lesssim C_1 2^{-L[K+n(1-\frac{1}{p})]} \left[\sum_{j=0}^L 2^{j(s-K)p} + \sum_{j=j_P \vee (-L)}^{-1} 2^{j sp} \right]^{\frac{1}{p}} \\ &\lesssim C_1 \max \left\{ L 2^{-L[K+n(1-\frac{1}{p})]}, 2^{-L[s+K+n(1-\frac{1}{p})]} \right\}. \end{aligned}$$

Thus, we always have $J_P \lesssim C_1 \max\{L2^{-L[K+n(1-\frac{1}{p})]}, 2^{-L[s+K+n(1-1/p)]}\}$.

When $p > q$, applying Minkowski's inequality, we see that

$$\begin{aligned} J_P &\lesssim C_1 \frac{1}{|P|^\tau} \left\{ \sum_{\substack{j=j_P \\ |j|\leq L}}^{\infty} \sum_{\substack{k\in\mathbb{Z}^n \\ Q_{jk}\notin[-2^L, 2^L]^n}} 2^{jsq} \frac{2^{-|j|Kq+(j\wedge 0)nq-\frac{jnq}{p}}}{(1+|2^{-(j\vee 0)k}|)^{(n+K)q}} \right\}^{\frac{1}{q}} \\ &\lesssim C_1 \frac{1}{|P|^\tau} \left\{ \sum_{\substack{j=j_P \\ |j|\leq L}}^{\infty} 2^{jsq} 2^{-|j|Kq+(j\wedge 0)nq-\frac{jnq}{p}+(j\vee 0)n} 2^{-[L+(j\wedge 0)][(n+K)q-n]} \right\}^{\frac{1}{q}}. \end{aligned}$$

Similarly, we have $J_P \lesssim C_1 \max\{L2^{-L[K+n(1-\frac{1}{q})]}, 2^{-L[s+K+n(1-\frac{1}{p})]}\}$.

Combining the estimates of I_P and J_P implies that there exists a positive constant C , independent of L , such that

$$\begin{aligned} &\|f - f_L\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \\ &\leq CC_1C_2 \max\left\{2^{L\{s+n[\tau+(\frac{1}{q}-\frac{1}{p})\vee 0]-K\}}, L2^{-L(K+n[1-\frac{1}{(p\wedge q)}])}, 2^{-L[s+K+n(1-\frac{1}{p})]}\right\}. \end{aligned}$$

For any given $\varepsilon > 0$, choosing L large enough such that

$$CC_1C_2 \max\left\{2^{L\{s+n[\tau+(\frac{1}{q}-\frac{1}{p})\vee 0]-K\}}, L2^{-L(K+n[1-\frac{1}{(p\wedge q)}])}, 2^{-L[s+K+n(1-\frac{1}{p})]}\right\} < \varepsilon,$$

we then have $\|f - f_L\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \varepsilon$, which completes the proof of (i).

(ii) To prove ${}_M\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset {}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, it suffices to show that for any $\varepsilon \in (0, \infty)$ and $f \in C_{c,M}^\infty(\mathbb{R}^n) \cap \dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, there exists a function $g \in \mathcal{S}_\infty(\mathbb{R}^n)$ such that $\|f - g\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \varepsilon$. Since the proof is similar to that of (i), we only give a sketch.

Let φ be as in Definition 1.1. By [11, Lemma (6.9)], there exists a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (1) such that $\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^j\xi) \widehat{\psi}(2^j\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Then by the Calderón reproducing formula in [23, Lemma 2.1], we know that $f = \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_j(\mathbb{R}^n)} \langle f, \varphi_Q \rangle \psi_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$. For $L \in \mathbb{N}$, set

$$g_L \equiv \sum_{|j|\leq L} \sum_{\substack{Q\in\mathcal{Q}_j(\mathbb{R}^n) \\ Q\subset[-2^L, 2^L]^n}} \langle f, \varphi_Q \rangle \psi_Q \equiv \sum_{|j|\leq L} \sum_{\substack{Q\in\mathcal{Q}_j(\mathbb{R}^n) \\ Q\subset[-2^L, 2^L]^n}} \lambda_Q \psi_Q.$$

Obviously, $g_L \in \mathcal{S}_\infty(\mathbb{R}^n)$. From the φ -transform characterization of $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$

(see [23, Theorem 3.1]), we deduce that

$$\begin{aligned}
& \|f - g_L\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \\
& \leq C \left(\sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{\substack{j=j_P \\ |j|>L}}^{\infty} \sum_{\ell(Q)=2^{-j}} 2^{j(s+\frac{n}{2})q} |\lambda_Q|^q \chi_Q(x) \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \right. \\
& \quad \left. + \sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{\substack{j=j_P \\ |j| \leq L}}^{\infty} \sum_{\substack{\ell(Q)=2^{-j} \\ Q \notin [-2^L, 2^L]^n}} 2^{j(s+\frac{n}{2})q} |\lambda_Q|^q \chi_Q(x) \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \right) \\
& \equiv C \left(\sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \tilde{I}_P + \sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \tilde{J}_P \right),
\end{aligned}$$

where C is a positive constant independent of f and L .

The estimate of \tilde{I}_P is similar to the estimate of I_P in (i). In fact, since $f \in C_{c,M}^\infty(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$, by Lemma 2.3, we see that for $Q = Q_{jk}$, $|\lambda_Q| \lesssim 2^{-\frac{jn}{2}-jK}(1+|2^{-j}k|)^{-n-K-1}$ when $j \geq 0$, where K is the same as in (6), and $|\lambda_Q| \lesssim 2^{-\frac{jn}{2}} 2^{j(n+M+1)}(1+|k|)^{-n-M-1}$ when $j < 0$.

If $j_P \geq -L$, similarly to the estimate of I_P , we have

$$\tilde{I}_P \lesssim \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=(j_P \vee L)}^{\infty} \sum_{k \in \mathbb{Z}^n} \frac{2^{j(s-K)q} \chi_{Q_{jk}}(x)}{(1+|2^{-j}k|)^{(n+K+1)q}} \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \lesssim 2^{L(s-K+n[\tau+(\frac{1}{q}-\frac{1}{p}) \vee 0])}.$$

If $j_P < -L$, we see that

$$\begin{aligned}
\tilde{I}_P & \leq C \left(\frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{-L-1} \sum_{k \in \mathbb{Z}^n} \frac{2^{j(s+n+M+1)q} \chi_{Q_{jk}}(x)}{(1+|k|)^{(n+M+1)q}} \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \right. \\
& \quad \left. + \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=L+1}^{\infty} \sum_{k \in \mathbb{Z}^n} \frac{2^{j(s-K)q} \chi_{Q_{jk}}(x)}{(1+|2^{-j}k|)^{(n+K+1)q}} \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} \right) \\
& \equiv C(\tilde{I}_1 + \tilde{I}_2),
\end{aligned}$$

where C is a positive constant independent of L . Similarly to the estimate of I_2 in (i), we obtain $\tilde{I}_2 \lesssim 2^{L(s-K+n[\tau+(\frac{1}{q}-\frac{1}{p}) \vee 0])}$. For \tilde{I}_1 , by $|P|^{-\tau} \leq 1$ and $M > \max\{n[\frac{1}{(p \wedge q)} - 1] - 1, -s + n(\frac{1}{p} - 1) - 1\}$, similarly to the estimate of I_1 in (i), we see that $\tilde{I}_1 \lesssim 2^{-L(s+n(1-\frac{1}{p})+M+1)}$. Thus,

$$\tilde{I}_P \lesssim \max \left\{ 2^{L(s-K+n[\tau+(\frac{1}{q}-\frac{1}{p}) \vee 0])}, 2^{-L(s+n(1-\frac{1}{p})+M+1)} \right\}.$$

The estimate of \tilde{J}_P is similar to that of J_P . Indeed, if $j_P > L$, then $\tilde{J}_P = 0$. If $0 \leq j_P \leq L$, then $\tilde{J}_P \lesssim 2^{-L\{K+1+n[1-\frac{1}{p\wedge q}]\}}$. If $J_P < 0$, then

$$\tilde{J}_P \lesssim \max \left\{ 2^{-L(s+n(1-\frac{1}{p})+M+1)}, L2^{-L\{M+1+n[1-\frac{1}{p\wedge q}]\}} \right\},$$

which together with the estimate of \tilde{I}_P yields that

$$\|f - g_L\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C \max \left\{ 2^{L(s-K+n[\tau+(\frac{1}{q}-\frac{1}{p})\vee 0])}, 2^{-L\{K+1+n[1-\frac{1}{p\wedge q}]\}}, \right. \\ \left. 2^{-L[s+M+n(1-\frac{1}{p})+1]}, L2^{-L\{M+1+n[1-\frac{1}{p\wedge q}]\}} \right\},$$

where C is a positive constant independent of L . For any given $\varepsilon > 0$, choosing L sufficiently large, we then have $\|f - g_L\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \varepsilon$, which completes the proof of Theorem 2.1. \square

Finally, we point out that Theorem 2.1 is also true for Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Since the proof is similar, we omit the details.

Theorem 2.6. *Let $s \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0, \infty]$, $M \in \mathbb{Z}_+ \cup \{-1\}$, $\tilde{J} \equiv \frac{1}{\min\{1,p\}}$ and $\tilde{N} \equiv \max\{\lfloor \tilde{J} - n - s \rfloor, -1\}$.*

- (i) *Let $\tau \in [0, \frac{1}{p} + \frac{1+|\tilde{J}-s|-\tilde{J}+s}{n}]$ when $\tilde{N} \geq 0$ or $\tau \in [0, \frac{1}{p} + \frac{s+n-\tilde{J}}{n}]$ when $\tilde{N} < 0$. Then ${}_V\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset {}_M\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$.*
- (ii) *Let $\tau \in [0, \infty)$. Then ${}_M\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset {}_V\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ if $M > \max\{n(\frac{1}{p}-1) - 1, -s + n(\frac{1}{p}-1) - 1\}$.*

3. Dual properties of tent spaces

In this section, we focus on the tent spaces $F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $F\dot{W}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$. These tent spaces are originally introduced in [22, Definition 4.2] and applied therein to establish the dual relation between $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We first recall some results on $F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $F\dot{W}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ in [22], and then establish the duality that for s, τ, p as in Theorem 1.3, $({}_CF\dot{W}_{p,p}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1}))^* = F\dot{T}_{p',p'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$, where ${}_CF\dot{W}_{p,p}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ denotes the closure of the set of all functions in $F\dot{W}_{p,p}^{s,\tau}(\mathbb{R}_{\mathbb{Z}_+}^{n+1})$ with compact support.

We begin with recalling the notions of $F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $F\dot{W}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$. For all functions F on $\mathbb{R}_{\mathbb{Z}}^{n+1}$ or \mathbb{R}_+^{n+1} and $j \in \mathbb{Z}$, we set $F^j(x) \equiv F(x, 2^{-j})$ for all $x \in \mathbb{R}^n$. For any set $A \subset \mathbb{R}^n$, define $T(A) \equiv \{(x, t) \in \mathbb{R}_{\mathbb{Z}}^{n+1} : B(x, t) \subset A\}$.

Definition 3.1 ([22, Definition 4.2]). Let $s \in \mathbb{R}$.

- (i) Let $p, q \in (1, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)'}]$. The *tent space* $F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ is defined to be the set of all functions F on $\mathbb{R}_{\mathbb{Z}}^{n+1}$ such that $\{F^j\}_{j \in \mathbb{Z}}$ are Lebesgue measurable and $\|F\|_{F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} < \infty$, where

$$\|F\|_{F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} \equiv \inf_{\omega} \left\| \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} |F^j|^q [\omega^j]^{-q} \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)},$$

where the infimum is taken over all nonnegative Borel measurable functions ω on \mathbb{R}_+^{n+1} satisfying (3) and with the restriction that ω is allowed to vanish only where F vanishes.

- (ii) Let $p \in (1, \infty)$, $q \in (1, \infty]$ and $\tau \in [0, \infty)$. The *tent space* $F\dot{W}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ is defined to be the set of all functions F on $\mathbb{R}_{\mathbb{Z}}^{n+1}$ such that $\{F^j\}_{j \in \mathbb{Z}}$ are Lebesgue measurable and $\|F\|_{F\dot{W}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} < \infty$, where

$$\|F\|_{F\dot{W}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} \equiv \sup_B \frac{1}{|B|^\tau} \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left[\int_{\mathbb{R}^n} |F^j(x)|^p \chi_{T(B)}(x, 2^{-j}) dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}},$$

where B runs over all balls in \mathbb{R}^n .

Also, for simplicity, we use $F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $F\dot{W}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ to denote the spaces $F\dot{T}_{p,p}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $F\dot{W}_{p,p}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$, respectively.

Remark 3.2. (i) We recall that when $s = \frac{n-d}{2}$, $\tau = \frac{d}{2n}$ and $p = q = 2$, the tent spaces $F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $F\dot{W}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ are, respectively, the discrete variants of $T_d^1(\mathbb{R}_+^{n+1})$ and $T_d^\infty(\mathbb{R}_+^{n+1})$ in [7, p.391]; see [22, p.2786]. In particular, when $s = 0$, $\tau = \frac{1}{2}$ and $p = q = 2$, $F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $F\dot{W}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ are, respectively, the discrete variants of T_2^1 and T_2^∞ , the well-known tent spaces introduced by Coifman, Meyer and Stein in [5].

(ii) We also remark that the space $F\dot{W}_{q'}^{0,1/p-1/q}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ is a discrete variant of $T_{q'}^{p,\infty}$, where $T_{q'}^{p,\infty}$ is introduced in [18, Definition 1.3].

(iii) It was pointed out in [22, p.2786, (4.6)] that $\|\cdot\|_{F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})}$ is a quasi-norm, namely, there exists a positive constant ρ such that for all functions F and G on $\mathbb{R}_{\mathbb{Z}}^{n+1}$,

$$\|F + G\|_{F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} \leq 2^\rho \left\{ \|F\|_{F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} + \|G\|_{F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} \right\}.$$

Let φ be as in Definition 1.1. Define an operator ρ_φ by setting, for all $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $(x, t) \in \mathbb{R}_+^{n+1}$, $\rho_\varphi(f)(x, t) \equiv \varphi_t * f(x)$. It is easy to check that $\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim \|\rho_\varphi(f)\|_{F\dot{W}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})}$ and $\|f\|_{F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim \|\rho_\varphi(f)\|_{F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})}$. We also recall the dual relation between $F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $F\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ in [22] as follows.

Theorem 3.3 ([22, Theorem 4.1(iii)]). *Let $s \in \mathbb{R}$, $p, q \in (1, \infty)$ and $\tau \in (0, \frac{1}{(p \vee q)'}]$. Then the dual space of $F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ is $F\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ under the following pairing:*

$$\langle F, G \rangle \equiv \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} F^j(x) G^j(x) dx, \quad (8)$$

where $F \in F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $G \in F\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$.

Now we turn to the main result of this section.

Theorem 3.4. *Let $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in (0, \frac{1}{p}]$. The dual space of ${}_CF\dot{W}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ is $F\dot{T}_{p'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ under the pairing (8).*

To prove this theorem, we need the atomic decomposition characterization of the space $F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ established in [22, Theorem 4.1(i)]. We first recall the notion of $F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ -atoms; see [22, Definition 4.3].

Definition 3.5. Let $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in (0, \frac{1}{p}]$. A function a on $\mathbb{R}_{\mathbb{Z}}^{n+1}$ is called an $F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ -atom associated to a ball B , if a is supported in $T(B) \equiv \{(x, t) \in \mathbb{R}_{\mathbb{Z}}^{n+1} : B(x, t) \subset B\}$ and satisfies that

$$\int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} 2^{jsp} |a^j(x)|^p \chi_{T(B)}(x, 2^{-j}) dx \leq |B|^{-\tau p}.$$

Proposition 3.6. *Let $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in (0, \frac{1}{p}]$. If $F \in F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$, then there exist a sequence $\{a_j\}_j$ of $F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ -atoms and a sequence $\{\lambda_j\}_j \subset \mathbb{C}$ such that $F = \sum_j \lambda_j a_j$ in $F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $\sum_j |\lambda_j| \leq C \|F\|_{F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})}$.*

Conversely, for a sequence $\{a_j\}_j$ of $F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ -atoms and an l^1 -sequence $\{\lambda_j\}_j \subset \mathbb{C}$, $F \equiv \sum_j \lambda_j a_j$ converges in $F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $\|F\|_{F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} \leq C \sum_j |\lambda_j|$, where C is a positive constant independent of F .

For all $F \in F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$, set

$$\| \| F \| \|_{F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})} \equiv \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\},$$

where the infimum is taken over all possible atomic decompositions of F as in Proposition 3.6. It is easy to see that $\| \| \cdot \| \|_{F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})}$ is a norm of $F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$. Furthermore, by Proposition 3.6, we know that $\| \| \cdot \| \|_{F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})}$ is equivalent to $\| \cdot \|_{F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})}$ and hence $(F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1}), \| \| \cdot \| \|_{F\dot{T}_p^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})})$ is a Banach space.

Lemma 3.7. *Let $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in (0, \frac{1}{p}]$. Then there exists a positive constant C such that for all $F \in \dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})$,*

$$C^{-1} \|F\|_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})} \leq \sup_{\substack{\|G\|_{F\dot{W}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})} \leq 1 \\ G \text{ has compact support}}} \left\{ \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} F^j(x) G^j(x) dx \right\} \leq C \|F\|_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})}.$$

Proof. The second inequality is an immediate consequence of Theorem 3.3. To finish the proof of Lemma 3.7, we still need to show the first inequality.

Recall that $\|\cdot\|_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})}$ is equivalent to the norm $|||\cdot|||_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})}$ and the space $(\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1}), |||\cdot|||_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})})$ is a Banach space. For each $F \in \dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})$, by Theorem 3.3 and the Hahn-Banach theorem, there exists a function $H \in F\dot{W}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})$ with $\|H\|_{F\dot{W}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})} \leq 1$ such that

$$\|F\|_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})} \sim |||F|||_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})} \sim \left| \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} F^j(x) H^j(x) dx \right|.$$

For any $M \in \mathbb{N}$, $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, set

$$H_M(x, 2^{-j}) \equiv H(x, 2^{-j}) \chi_{\{(x, 2^{-j}) \in \mathbb{R}_Z^{n+1} : |x| \leq M, M^{-1} \leq 2^{-j} \leq M\}}(x, 2^{-j}).$$

Then $\|H_M\|_{F\dot{W}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})} \leq \|H\|_{F\dot{W}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})} \leq 1$ and H_M has compact support. Notice that

$$\int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |F^j(x)| |H^j(x)| dx \lesssim \|F\|_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})} \|H\|_{F\dot{W}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})} \lesssim \|F\|_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})}.$$

Lebesgue's dominated convergence theorem implies that if M is large enough,

$$\|F\|_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})} \sim |||F|||_{\dot{F}T_p^{s,\tau}(\mathbb{R}_Z^{n+1})} \sim \left| \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} F^j(x) H_M^j(x) dx \right|,$$

which completes the proof of Lemma 3.7. \square

The following lemma is a variation of [6, Lemma 4.2] for tent spaces.

Lemma 3.8. *Let $p \in (1, \infty)$, $\tau \in (0, \frac{1}{p}]$ and $\{F_m\}_{m \in \mathbb{N}}$ be a uniformly bounded sequence in $\dot{F}T_p^{0,\tau}(\mathbb{R}_Z^{n+1})$. Then there exist a function $F \in \dot{F}T_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ and a subsequence $\{F_{m_i}\}_{i \in \mathbb{N}}$ of $\{F_m\}_{m \in \mathbb{N}}$ such that for all $G \in F\dot{W}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ with compact support, $\langle F_{m_i}, G \rangle \rightarrow \langle F, G \rangle$ as $i \rightarrow \infty$, where $\langle F, G \rangle$ is defined as in (8), and $\|F\|_{\dot{F}T_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \leq C \sup_{m \in \mathbb{N}} \|F_m\|_{\dot{F}T_p^{0,\tau}(\mathbb{R}_Z^{n+1})}$ with C being a positive constant independent of F .*

Proof. Without loss of generality, we may assume that $\|F_m\|_{F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \leq 1$ for all $m \in \mathbb{N}$.

By [22, Theorem 4.1] and its proof (see [22, pp. 2792–2793]), each F_m has an atomic decomposition representation $F_m = \sum_{j \in \mathbb{Z}} \sum_{Q \in I_j^{(m)}} \lambda_{m,j,Q} a_{m,j,Q}$ in $F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$, where $I_j^{(m)} \subset \mathcal{Q}(\mathbb{R}^n)$, $\lambda_m \equiv \{\lambda_{m,j,Q}\}_{j \in \mathbb{Z}, Q \in I_j^{(m)}} \subset \mathbb{C}$ satisfies that $\sum_{j \in \mathbb{Z}} \sum_{Q \in I_j^{(m)}} |\lambda_{m,j,Q}| \lesssim 1$ and each $a_{m,j,Q}$ is an $F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ -atom supported in $T(B_Q)$, where and in what follows, for all $Q \in \mathcal{Q}(\mathbb{R}^n)$, $B_Q \equiv B(c_Q, \frac{5}{2}\sqrt{nl}(Q))$.

For all $m \in \mathbb{N}$, define a sequence $\tilde{\lambda}_m \equiv \{\tilde{\lambda}_{m,j,Q}\}_{j \in \mathbb{Z}, Q \in \mathcal{Q}(\mathbb{R}^n)} \subset \mathbb{C}$ by setting, for all $j \in \mathbb{Z}$, $\tilde{\lambda}_{m,j,Q} \equiv \lambda_{m,j,Q}$ when $Q \in I_j^{(m)}$ and $\tilde{\lambda}_{m,j,Q} \equiv 0$ otherwise, and a set $\{\tilde{a}_{m,j,Q}\}_{j \in \mathbb{Z}, Q \in \mathcal{Q}(\mathbb{R}^n)}$ of functions on \mathbb{R}_Z^{n+1} by setting, for all $j \in \mathbb{Z}$, $\tilde{a}_{m,j,Q} \equiv a_{m,j,Q}$ when $Q \in I_j^{(m)}$ and $\tilde{a}_{m,j,Q} \equiv 0$ otherwise. We see that for each $m \in \mathbb{N}$,

$$\|\tilde{\lambda}_m\|_{l^1} = \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} |\tilde{\lambda}_{m,j,Q}| = \sum_{j \in \mathbb{Z}} \sum_{Q \in I_j^{(m)}} |\lambda_{m,j,Q}| \lesssim 1 \quad (9)$$

and each $\tilde{a}_{m,j,Q}$ is still an $F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ -atom supported in $T(B_Q)$. Moreover, we have $F_m = \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} \tilde{\lambda}_{m,j,Q} \tilde{a}_{m,j,Q}$ in $F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$.

Since (9) holds for all $m \in \mathbb{N}$, a diagonalization argument yields that there exist a sequence $\lambda \equiv \{\lambda_{j,Q}\}_{j \in \mathbb{Z}, Q \in \mathcal{Q}(\mathbb{R}^n)} \in l^1$ and a subsequence $\{\tilde{\lambda}_{m_i}\}_{i \in \mathbb{N}}$ of $\{\tilde{\lambda}_m\}_{m \in \mathbb{N}}$ such that $\tilde{\lambda}_{m_i,j,Q} \rightarrow \lambda_{j,Q}$ as $i \rightarrow \infty$ for all $j \in \mathbb{Z}$ and $Q \in \mathcal{Q}(\mathbb{R}^n)$, and $\|\lambda\|_{l^1} \lesssim 1$.

On the other hand, recall that $\text{supp } \tilde{a}_{m,j,Q} \subset T(B_Q)$ for all $m \in \mathbb{N}$ and $j \in \mathbb{Z}$. From Definition 3.5, it follows that $\{\|\tilde{a}_{m,j,Q}\|_{L^p(l^p(T(B_Q)))}\}_{m \in \mathbb{N}}$ is a uniformly bounded sequence in $L^p(l^p(T(B_Q)))$, where $L^p(l^p(T(B_Q)))$ consists of all functions on $T(B_Q)$ equipped with the norm that

$$\|F\|_{L^p(l^p(T(B_Q)))} \equiv \left\{ \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |F(x, 2^{-j})|^p \chi_{T(B_Q)}(x, 2^{-j}) dx \right\}^{\frac{1}{p}}.$$

Then by the Alaoglu theorem, there are a unique function $a_{j,Q} \in L^p(l^p(T(B_Q)))$ and a subsequence of $\{\tilde{a}_{m_i,j,Q}\}_{i \in \mathbb{N}}$, denoted by $\{\tilde{a}_{m_i,j,Q}\}_{i \in \mathbb{N}}$ again, such that for all functions $G \in L^{p'}(l^{p'}(T(B_Q)))$, $\langle \tilde{a}_{m_i,j,Q}, G \rangle \rightarrow \langle a_{j,Q}, G \rangle$ as $i \rightarrow \infty$ and each $a_{j,Q}$ is also a constant multiple of an $F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ -atom supported in $T(2B_Q)$ with the constant independent of j and Q . Applying a diagonalization argument again, we conclude that there exists a subsequence, denoted by $\{\tilde{a}_{m_i,j,Q}\}_{i \in \mathbb{N}}$ again, such that for all $G \in L^{p'}(l^{p'}(T(B_Q)))$, $\langle \tilde{a}_{m_i,j,Q}, G \rangle \rightarrow \langle a_{j,Q}, G \rangle$ as $i \rightarrow \infty$ for all $j \in \mathbb{Z}$ and $Q \in \mathcal{Q}(\mathbb{R}^n)$. Let $F \equiv \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} \lambda_{j,Q} a_{j,Q}$. By Proposition 3.6, we see that $F \in F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ and $\|F\|_{F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \lesssim 1$.

Let $G \in F\dot{W}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ such that $\text{supp } G \subset B(0, 2^M) \times \{2^{-M}, \dots, 2^M\}$ for some $M \in \mathbb{N}$. Without loss of generality, we may assume that $\|G\|_{F\dot{W}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})} = 1$. We need to show that $\langle F_{m_i}, G \rangle \rightarrow \langle F, G \rangle$ as $i \rightarrow \infty$. It is easy to see that $\|G\|_{L^{p'}(l^{p'}(T(B(0,2^M))))} \lesssim \|G\|_{F\dot{W}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})} \sim 1$. Therefore, by the above argument, $\langle \tilde{a}_{m_i,j,Q}, G \rangle \rightarrow \langle a_{j,Q}, G \rangle$ as $i \rightarrow \infty$ for all $j \in \mathbb{Z}$ and $Q \in \mathcal{Q}(\mathbb{R}^n)$.

Recall that $\|a\|_{F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \leq \tilde{C}$ for all $F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ -atoms a , where \tilde{C} is a positive constant independent of a . By $\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} |\tilde{\lambda}_{m_i,j,Q}| \lesssim 1$, we see that for any $\varepsilon > 0$, there exists an $L \in \mathbb{N}$ such that

$$\sum_{\{j \in \mathbb{Z}: |j| > L\}} \sum_{\{Q \in \mathcal{Q}(\mathbb{R}^n): |j_Q| > L \text{ or } Q \not\subset [-2^L, 2^L]^n\}} |\tilde{\lambda}_{m_i,j,Q}| < \frac{\varepsilon}{\tilde{C}}$$

and hence

$$\begin{aligned} & \sum_{\substack{j \in \mathbb{Z} \\ |j| > L}} \sum_{\substack{Q \in \mathcal{Q}(\mathbb{R}^n) \\ |j_Q| > L \text{ or } Q \not\subset [-2^L, 2^L]^n}} |\tilde{\lambda}_{m_i,j,Q}| \cdot |\langle \tilde{a}_{m_i,j,Q}, G \rangle| \\ & \leq \sum_{\substack{j \in \mathbb{Z} \\ |j| > L}} \sum_{\substack{Q \in \mathcal{Q}(\mathbb{R}^n) \\ |j_Q| > L \text{ or } Q \not\subset [-2^L, 2^L]^n}} |\tilde{\lambda}_{m_i,j,Q}| \cdot \|\tilde{a}_{m_i,j,Q}\|_{F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \|G\|_{F\dot{W}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})} \\ & \leq \tilde{C} \sum_{\substack{j \in \mathbb{Z} \\ |j| > L}} \sum_{\substack{Q \in \mathcal{Q}(\mathbb{R}^n) \\ |j_Q| > L \text{ or } Q \not\subset [-2^L, 2^L]^n}} |\tilde{\lambda}_{m_i,j,Q}| < \varepsilon. \end{aligned}$$

Similarly, by $\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} |\lambda_{j,Q}| \lesssim 1$, there exists an $L \in \mathbb{N}$ such that

$$\sum_{\substack{j \in \mathbb{Z} \\ |j| > L}} \sum_{\substack{Q \in \mathcal{Q}(\mathbb{R}^n) \\ |j_Q| > L \text{ or } Q \not\subset [-2^L, 2^L]^n}} |\lambda_{j,Q}| \cdot |\langle a_{j,Q}, G \rangle| < \varepsilon,$$

which further implies that $\lim_{i \rightarrow \infty} \langle F_{m_i}, G \rangle = \langle F, G \rangle$ and completes the proof of Lemma 3.8. \square

Now we turn to prove Theorem 3.4

Proof of Theorem 3.4. By Theorem 3.3 and the definition of ${}_C F\dot{W}_p^{s,\tau}(\mathbb{R}_Z^{n+1})$, we have that ${}_C F\dot{W}_p^{s,\tau}(\mathbb{R}_Z^{n+1}) \subset F\dot{W}_p^{s,\tau}(\mathbb{R}_Z^{n+1}) = (F\dot{T}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1}))^*$, which implies that $F\dot{T}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1}) \subset (F\dot{T}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1}))^{**} \subset ({}_C F\dot{W}_p^{s,\tau}(\mathbb{R}_Z^{n+1}))^*$.

To show $({}_C F\dot{W}_p^{s,\tau}(\mathbb{R}_Z^{n+1}))^* \subset F\dot{T}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})$, we first claim that if this is true when $s = 0$, then it is also true for all $s \in \mathbb{R}$. To see this, for all $u \in \mathbb{R}$, define an operator A_u by setting, for all functions F on \mathbb{R}_Z^{n+1} , $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, $(A_u F)(x, 2^{-j}) \equiv 2^{ju} F(x, 2^{-j})$. Obviously, A_u is an isometric isomorphism from $F\dot{W}_p^{s,\tau}(\mathbb{R}_Z^{n+1})$ to $F\dot{W}_p^{s+u,\tau}(\mathbb{R}_Z^{n+1})$ and from $F\dot{T}_{p'}^{s,\tau}(\mathbb{R}_Z^{n+1})$ to

$F\dot{T}_p^{s+u,\tau}(\mathbb{R}_Z^{n+1})$. If $L \in ({}_C F\dot{W}_p^{s,\tau}(\mathbb{R}_Z^{n+1}))^*$, then $L \circ A_s \in ({}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1}))^*$ and hence, by the above assumption, there exists a function $G \in F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ such that $L \circ A_s(F) = \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} F^j(x) G^j(x) dx$ for all $F \in {}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$. Notice that $A_s \circ A_{-s}$ is the identity on ${}_C F\dot{W}_p^{s,\tau}(\mathbb{R}_Z^{n+1})$ and A_{-s} is an isometric isomorphism from ${}_C F\dot{W}_p^{s,\tau}(\mathbb{R}_Z^{n+1})$ onto ${}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$. Therefore,

$$\begin{aligned} L(F) &= L \circ A_s \circ A_{-s}(F) = \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} (A_{-s}F)^j(x) G^j(x) dx \\ &= \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} F^j(x) (A_{-s}G)^j(x) dx \end{aligned}$$

for all $F \in {}_C F\dot{W}_p^{s,\tau}(\mathbb{R}_Z^{n+1})$. Since $G \in F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$, we obtain that $A_{-s}G \in F\dot{T}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})$ and $\|A_{-s}G\|_{F\dot{T}_{p'}^{-s,\tau}(\mathbb{R}_Z^{n+1})} = \|G\|_{F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})}$. Thus, the above claim is true.

Next we prove that $({}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1}))^* \subset F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$. To this end, we chose $L \in ({}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1}))^*$. It suffices to show that there exists a $G \in F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ such that for all $F \in {}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ with compact support, L has a form as in (8). In fact, for $F \in {}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ with compact support, if $\langle H, F \rangle = 0$ holds for all $H \in F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$, then Theorem 3.3 implies that F must be the zero element of ${}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$. Thus, $F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ is a total set of linear functionals on ${}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$.

To complete the proof of Theorem 3.4, we need the following functional analysis result (see [8, p. 439, Exercise 41]): *Let \mathcal{X} be a locally convex linear topological space and \mathcal{Y} be a linear subspace of \mathcal{X}^* . Then \mathcal{Y} is \mathcal{X} -dense in \mathcal{X}^* if and only if \mathcal{Y} is a total set of functionals on \mathcal{X} .* From this functional result and the fact that $F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ is a total set of linear functionals on ${}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$, we deduce that $F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ is weak $*$ -dense in $({}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1}))^*$. Then there exists a sequence $\{G^{(m)}\}_{m \in \mathbb{N}}$ in $F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ such that $\langle G^{(m)}, F \rangle \rightarrow L(F)$ as $m \rightarrow \infty$ for all F in ${}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$. Applying the Banach–Steinhaus theorem, we conclude that the sequence $\{\|G^{(m)}\|_{F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})}\}_{m \in \mathbb{N}}$ is uniformly bounded. Then by Lemmas 3.8 and 3.7, we obtain a subsequence $\{G^{(m_i)}\}_{i \in \mathbb{N}}$ and $G \in F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ such that $L(F) = \lim_{i \rightarrow \infty} \langle G^{(m_i)}, F \rangle = \langle G, F \rangle$ for all $F \in {}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ with compact support and

$$\|G\|_{F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})} \lesssim \sup_{\substack{\|F\|_{{}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \leq 1 \\ F \text{ has compact support}}} |\langle G, F \rangle| \lesssim \sup_{\substack{\|F\|_{{}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \leq 1 \\ F \text{ has compact support}}} |L(F)| \lesssim \|L\|_{({}_C F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1}))^*},$$

which completes the proof of Theorem 3.4. \square

Remark 3.9. It is still unclear whether Theorem 3.4 is true for the spaces ${}_C F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ and $F\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ when $p \neq q$ or not. The difficulty lies in the fact that the space $F\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_{\mathbb{Z}}^{n+1})$ when $p \neq q$ is only known to be a quasi-Banach space so far. Thus, Lemma 3.7 in the case that $p \neq q$ seems not available, due to the Hahn-Banach theorem is not valid for these spaces.

4. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. We begin with recalling the notion of the sequence space corresponding to $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$; see [24, Definition 2.1].

Definition 4.1. Let $s \in \mathbb{R}$, $p, q \in (1, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)'}]$. The space $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is then defined to be the set of all $t \equiv \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \subset \mathbb{C}$ such that $\|t\|_{f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|t\|_{f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \inf_{\omega} \left\| \left\{ \sum_{j \in \mathbb{Z}} 2^{j(s + \frac{n}{2})q} \left(\sum_{\ell(Q)=2^{-j}} |t_Q| \chi_Q[\omega(\cdot, 2^{-j})]^{-1} \right)^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}$$

and the infimum is taken over all nonnegative Borel measurable functions ω on \mathbb{R}_+^{n+1} such that ω satisfies (3) and with the restriction that for any $j \in \mathbb{Z}$, $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\sum_{\ell(Q)=2^{-j}} |t_Q| \chi_Q$ vanishes.

Recall that for s, τ, p, q as in Theorem 3.3, it was established in [24, Proposition 2.1] that the dual space of $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is $f\dot{H}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$. In what follows, for simplicity, we write $f\dot{H}_p^{s,\tau}(\mathbb{R}^n) \equiv f\dot{H}_{p,p}^{s,\tau}(\mathbb{R}^n)$ and $f\dot{H}_p^{-s,\tau}(\mathbb{R}^n) \equiv f\dot{H}_{p,p}^{-s,\tau}(\mathbb{R}^n)$.

Let ${}_V f\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ be the set of all sequences with finite non-vanishing elements, which is obviously a subspace of $f\dot{H}_p^{s,\tau}(\mathbb{R}^n)$. We have the following conclusion.

Proposition 4.2. Let $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in [0, \frac{1}{p}]$. Then

$$({}_V f\dot{H}_p^{s,\tau}(\mathbb{R}^n))^* = f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)$$

in the following sense: for each $t \equiv \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)$, the map

$$\lambda \equiv \{\lambda_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \mapsto \langle \lambda, t \rangle \equiv \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} \lambda_Q \overline{t_Q} \quad (10)$$

induces a continuous linear functional on ${}_V f\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ with the operator norm no more than a positive constant multiple of $\|t\|_{f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)}$.

Conversely, every $L \in ({}_V f\dot{H}_p^{s,\tau}(\mathbb{R}^n))^*$ is of the form (10) for a certain $t \in f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)$ and $\|t\|_{f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)}$ is no more than a positive constant multiple of $\|L\|$.

Proof. Since Proposition 4.2 when $\tau = 0$ is just the classic result on $\dot{f}_{p,p}^{s,\tau}(\mathbb{R}^n)$ in [10, Remark 5.11], we only need consider the case that $\tau > 0$. By [24, Proposition 2.1] and the definition of ${}_V\dot{f}_p^{s,\tau}(\mathbb{R}^n)$, we have that ${}_V\dot{f}_p^{s,\tau}(\mathbb{R}^n) \subset \dot{f}_p^{s,\tau}(\mathbb{R}^n) = (f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n))^*$, which implies that $f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n) \subset (f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n))^{**} \subset ({}_V\dot{f}_p^{s,\tau}(\mathbb{R}^n))^*$.

To show $({}_V\dot{f}_p^{s,\tau}(\mathbb{R}^n))^* \subset f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)$, we first claim that if this is true when $s = 0$, then it is also true for all $s \in \mathbb{R}$. In fact, for all $u \in \mathbb{R}$, define an operator T_u by setting, for all sequences $t \equiv \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \subset \mathbb{C}$ and $Q \in \mathcal{Q}(\mathbb{R}^n)$, $(T_u t)_Q \equiv |Q|^{-\frac{u}{n}} t_Q$. Then T_u is an isometric isomorphism from $\dot{f}_p^{s,\tau}(\mathbb{R}^n)$ to $\dot{f}_p^{s+u,\tau}(\mathbb{R}^n)$ and from $f\dot{H}_{p'}^{s,\tau}(\mathbb{R}^n)$ to $f\dot{H}_{p'}^{s+u,\tau}(\mathbb{R}^n)$. If $L \in ({}_V\dot{f}_p^{s,\tau}(\mathbb{R}^n))^*$, then $L \circ T_s \in ({}_V\dot{f}_p^{0,\tau}(\mathbb{R}^n))^*$ and hence there exists a sequence $\lambda \equiv \{\lambda_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in f\dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)$ such that $L \circ T_s(t) = \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} t_Q \overline{\lambda_Q}$ for all $t \in {}_V\dot{f}_p^{0,\tau}(\mathbb{R}^n)$. Since $T_s \circ T_{-s}$ is the identity on ${}_V\dot{f}_p^{s,\tau}(\mathbb{R}^n)$ and T_{-s} is an isometric isomorphism from ${}_V\dot{f}_p^{s,\tau}(\mathbb{R}^n)$ onto ${}_V\dot{f}_p^{0,\tau}(\mathbb{R}^n)$, then

$$L(t) = L \circ T_s \circ T_{-s}(t) = \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} (T_{-s} t)_Q \overline{\lambda_Q} = \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} t_Q \overline{(T_{-s} \lambda)_Q}$$

for all $t \in {}_V\dot{f}_p^{s,\tau}(\mathbb{R}^n)$. Since $\lambda \in f\dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)$, we see that $T_{-s} \lambda \in f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)$ and $\|T_{-s} \lambda\|_{f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)} = \|\lambda\|_{f\dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)}$. Thus, the above claim is true.

Next we prove that $({}_V\dot{f}_p^{0,\tau}(\mathbb{R}^n))^* \subset f\dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)$. Notice that ${}_V\dot{f}_p^{0,\tau}(\mathbb{R}^n)$ consists of all sequences in $\dot{f}_p^{0,\tau}(\mathbb{R}^n)$ with finite non-vanishing elements. We know that every $L \in ({}_V\dot{f}_p^{0,\tau}(\mathbb{R}^n))^*$ is of the form $\lambda \mapsto \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} \lambda_Q \overline{t_Q}$ for a certain $t \equiv \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \subset \mathbb{C}$. In fact, for any $m \in \mathbb{N}$, let ${}^m_V\dot{f}_p^{0,\tau}(\mathbb{R}^n)$ denote the set of all sequences $\lambda \equiv \{\lambda_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in \dot{f}_p^{0,\tau}(\mathbb{R}^n)$, where $\lambda_Q = 0$ if $Q \cap (-2^m, 2^m]^n = \emptyset$ or $\ell(Q) > 2^m$ or $\ell(Q) < 2^{-m}$. Then $L \in ({}^m_V\dot{f}_p^{0,\tau}(\mathbb{R}^n))^*$. It is easy to see that each linear functional in $({}^m_V\dot{f}_p^{0,\tau}(\mathbb{R}^n))^*$ has the form (10). Thus, there exists $t_m \equiv \{(t_m)_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$, where $(t_m)_Q = 0$ if $Q \cap (-2^m, 2^m]^n = \emptyset$ or $\ell(Q) > 2^m$ or $\ell(Q) < 2^{-m}$, such that $L(\lambda)$ for all $\lambda \in {}^m_V\dot{f}_p^{0,\tau}(\mathbb{R}^n)$ has the form (10) with t replaced by t_m . By this construction, we are easy to see that $(t_{m+1})_Q = (t_m)_Q$ if $Q \subset (-2^m, 2^m]^n$ and $2^{-m} \leq \ell(Q) \leq 2^m$. Thus, if let $t_Q \equiv (t_m)_Q$ when $Q \subset (-2^m, 2^m]^n$ and $2^{-m} \leq \ell(Q) \leq 2^m$, then $t \equiv \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ is the desired sequence. We need to show that

$$\|t\|_{f\dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)} \lesssim \|L\|_{({}_V\dot{f}_p^{0,\tau}(\mathbb{R}^n))^*}.$$

To this end, for all $m \in \mathbb{N}$, define χ_m by setting $\chi_m(Q) \equiv 1$ if $Q \subset (-2^m, 2^m]^n$ and $2^{-m} \leq \ell(Q) \leq 2^m$, and $\chi_m(Q) \equiv 0$ otherwise. For all $\lambda \equiv \{\lambda_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in \dot{f}_p^{0,\tau}(\mathbb{R}^n)$ with $\|\lambda\|_{\dot{f}_p^{0,\tau}(\mathbb{R}^n)} \leq 1$, we see that $\lambda_m \equiv \{\lambda_Q \chi_m(Q)\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in {}_V\dot{f}_p^{0,\tau}(\mathbb{R}^n)$

and $\|\lambda_m\|_{f_p^{0,\tau}(\mathbb{R}^n)} \leq 1$. Thus, using Fatou's lemma yields

$$\begin{aligned}
 \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} |\lambda_Q| |t_Q| &\leq \lim_{m \rightarrow \infty} \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} |\lambda_Q| \chi_m(Q) |t_Q| \\
 &= \lim_{m \rightarrow \infty} \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} \frac{|\lambda_Q| |t_Q|}{|t_Q|} \chi_m(Q) \overline{t_Q} \\
 &\leq \lim_{m \rightarrow \infty} \|L\|_{(V f_p^{0,\tau}(\mathbb{R}^n))^*} \|\lambda_m\|_{f_p^{0,\tau}(\mathbb{R}^n)} \\
 &\leq \|L\|_{(V f_p^{0,\tau}(\mathbb{R}^n))^*}.
 \end{aligned} \tag{11}$$

Notice that for all $m \in \mathbb{N}$, $t_m \equiv \{t_Q \chi_m(Q)\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in f \dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)$. For each m , we define the function $F^{(m)}$ on \mathbb{R}_Z^{n+1} by setting, for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,

$$F^{(m)}(x, 2^{-j}) \equiv \sum_{Q \in \mathcal{Q}_j(\mathbb{R}^n)} |Q|^{-\frac{1}{2}} |t_Q| \chi_m(Q) \chi_Q(x).$$

Then $F^{(m)} \in FT_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ and $\|F^{(m)}\|_{FT_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})} \sim \|t_m\|_{f \dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)}$. Applying Theorem 3.4, we see that

$$\begin{aligned}
 \|F^{(m)}\|_{FT_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})} &\sim \sup \left\{ \left| \int_{\mathbb{R}^n} \sum_{j=0}^{\infty} F^{(m)}(x, 2^{-j}) G(x, 2^{-j}) dx \right| \right\} \\
 &\lesssim \sup \left\{ \left| \sum_{j=0}^{\infty} \sum_{Q \in \mathcal{Q}_j(\mathbb{R}^n)} |t_Q| \chi_m(Q) |Q|^{-\frac{1}{2}} \int_Q G(x, 2^{-j}) dx \right| \right\},
 \end{aligned}$$

where the supremum is taken over all functions $G \in F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ with compact support satisfying $\|G\|_{F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \leq 1$. Define, for all $Q \in \mathcal{Q}_j(\mathbb{R}^n)$, $\lambda_Q \equiv |Q|^{-\frac{1}{2}} \int_Q G(x, 2^{-j}) dx$ and $\lambda \equiv \{\lambda_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$. Hölder's inequality yields that $\|\lambda\|_{f_p^{0,\tau}(\mathbb{R}^n)} \lesssim \|G\|_{F\dot{W}_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \lesssim 1$ and hence

$$\begin{aligned}
 \|t_m\|_{f \dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)} &\sim \|F^{(m)}\|_{FT_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})} \\
 &\lesssim \sup \left\{ \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} |\lambda_Q| |t_Q| : \lambda \in f_p^{0,\tau}(\mathbb{R}^n), \|\lambda\|_{f_p^{0,\tau}(\mathbb{R}^n)} \leq 1 \right\},
 \end{aligned}$$

which together with (11) implies that $\|t_m\|_{f \dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)} \sim \|F^{(m)}\|_{FT_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})} \lesssim \|L\|_{(V f_p^{0,\tau}(\mathbb{R}^n))^*}$.

To show $t \in f \dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)$, let F be the function on \mathbb{R}_Z^{n+1} defined by setting, for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, $F(x, 2^{-j}) \equiv \sum_{Q \in \mathcal{Q}_j(\mathbb{R}^n)} |Q|^{-\frac{1}{2}} |t_Q| \chi_Q(x)$. Obviously, $F^{(m)} \rightarrow F$ pointwise as $m \rightarrow \infty$. Notice that $\|t\|_{f \dot{H}_{p'}^{0,\tau}(\mathbb{R}^n)} \sim \|F\|_{FT_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})}$. It

suffices to prove that $F \in F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$.

Recall that $\|F^{(m)}\|_{F\dot{T}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})} \lesssim \|L\|_{(v\dot{f}_p^{0,\tau}(\mathbb{R}^n))^*}$. By Lemma 3.8, there exist a subsequence $\{F^{(m_i)}\}_{i \in \mathbb{N}}$ and a function $\tilde{F} \in F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ such that for all $G \in F\dot{W}_{p'}^{0,\tau}(\mathbb{R}_Z^{n+1})$ with compact support, $\langle F^{(m_i)}, G \rangle \rightarrow \langle \tilde{F}, G \rangle$ as $i \rightarrow \infty$ and its quasi-norm $\|\tilde{F}\|_{F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \lesssim \|L\|_{(v\dot{f}_p^{0,\tau}(\mathbb{R}^n))^*}$, which together with the uniqueness of the weak limit and the fact that $F^{(m)} \rightarrow F$ pointwise as $m \rightarrow \infty$ yields that $F = \tilde{F}$ in $F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})$ and $\|F\|_{F\dot{T}_p^{0,\tau}(\mathbb{R}_Z^{n+1})} \lesssim \|L\|_{(v\dot{f}_p^{0,\tau}(\mathbb{R}^n))^*}$. This finishes the proof of Proposition 4.2. \square

Now we are ready to prove Theorem 1.3, which is a consequence of Proposition 4.2 and the φ -transform characterizations of $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ obtained in [24, Theorem 2.1] and [23, Theorem 3.1].

Proof of Theorem 1.3. Since the case that $\tau = 0$ is known (see, for example, [17, p. 180] and [10, Remark 5.14]), we only need consider the case that $\tau > 0$. By [22, Theorem 5.1] and the definition of $v\dot{F}_p^{s,\tau}(\mathbb{R}^n)$, we have that $v\dot{F}_p^{s,\tau}(\mathbb{R}^n) \subset \dot{F}_p^{s,\tau}(\mathbb{R}^n) = (F\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n))^*$, which implies that $F\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n) \subset (F\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n))^{**} \subset (v\dot{F}_p^{s,\tau}(\mathbb{R}^n))^*$.

To show $(v\dot{F}_p^{s,\tau}(\mathbb{R}^n))^* \subset F\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)$, let φ satisfy (1) such that for all $\xi \in \mathbb{R}^n \setminus \{0\}$, $\sum_{j \in \mathbb{Z}} |\widehat{\varphi}(2^{-j}\xi)|^2 = 1$. If $L \in (v\dot{F}_p^{s,\tau}(\mathbb{R}^n))^*$, then applying the φ -transform characterization of $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [23, Theorem 3.1], we see that $\tilde{L} \equiv L \circ T_\varphi \in (v\dot{f}_p^{s,\tau}(\mathbb{R}^n))^*$, where T_φ is the inverse φ -transform (see [10, p. 46]). By Proposition 4.2, there exists a $\lambda \equiv \{\lambda_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)$ such that $\tilde{L}(t) = \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} t_Q \overline{\lambda_Q}$ for all $t \equiv \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in v\dot{F}_p^{s,\tau}(\mathbb{R}^n)$ and $\|\lambda\|_{f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)} \lesssim \|\tilde{L}\|_{(v\dot{f}_p^{s,\tau}(\mathbb{R}^n))^*} \lesssim \|L\|_{(v\dot{F}_p^{s,\tau}(\mathbb{R}^n))^*}$. Notice that $\tilde{L} \circ S_\varphi = L \circ T_\varphi \circ S_\varphi = L$, where S_φ is the φ -transform (see [10, p. 46]). Set $g \equiv T_\varphi(\lambda) \equiv \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} \lambda_Q \varphi_Q$. Hence, for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, $L(f) = \tilde{L} \circ S_\varphi(f) = \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} \langle f, \varphi_Q \rangle \overline{\lambda_Q} = \langle f, g \rangle$. Furthermore, by the φ -transform characterization of $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [24, Theorem 2.1], we have $\|g\|_{F\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)} \lesssim \|\lambda\|_{f\dot{H}_{p'}^{-s,\tau}(\mathbb{R}^n)} \lesssim \|L\|_{(v\dot{F}_p^{s,\tau}(\mathbb{R}^n))^*}$, which completes the proof of Theorem 1.3. \square

We end this section by the following interesting remark.

Remark 4.3. (i) We first claim that when $\tau > 0$, the dual property in Theorem 1.3 is not possible to be correct for $v\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $B\dot{H}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ with $p \in (1, \infty)$, $q \in [1, \infty)$ and $q > p$, which is quite different from the case that $\tau = 0$. Recall that when $\tau = 0$, $p \in (1, \infty)$ and $q \in [1, \infty)$, $v\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) = \dot{B}_{p,q}^s(\mathbb{R}^n)$ and $(\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n))^* = B_{p',q'}^{-s}(\mathbb{R}^n)$; see [17, p. 244].

To show the claim, by [24, Propositions 2.2(i) and 2.3(i)], we see that if $1 < p_0 < p_1 < \infty$, $-\infty < s_1 < s_0 < \infty$, $q \in [1, \infty)$ and $\tau \in [0, \min\{\frac{1}{(p_0 \vee q)'} , \frac{1}{(p_1 \vee q)'}\}]$

such that $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$, then $B\dot{H}_{p_0,q}^{s_0,\tau}(\mathbb{R}^n) \subset B\dot{H}_{p_1,q}^{s_1,\tau}(\mathbb{R}^n)$ if and only if $\tau(p_0 \vee q)' = \tau(p_1 \vee q)'$. When $\tau > 0$, the sufficient and necessary condition that $\tau(p_0 \vee q)' = \tau(p_1 \vee q)'$ is equivalent to that $q \geq p_1$. If we assume that Theorem 1.3 is correct for ${}_V\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $B\dot{H}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ with $\tau > 0$ and certain $1 < p < q < \infty$, then by this assumption together with an argument by duality and the embedding $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset \dot{B}_{q,q}^{s-n/p+n/q,\tau}(\mathbb{R}^n)$ (see [23, Proposition 3.3]), we see that $B\dot{H}_{q',q'}^{-s+n/p-n/q,\tau}(\mathbb{R}^n) \subset B\dot{H}_{p',q'}^{s,\tau}(\mathbb{R}^n)$, which is not true since $q' < p'$. Thus, the claim is true.

From the above claim, it follows that if $\tau > 0$ and $p \neq q$, only when $1 \leq q < p < \infty$, the conclusion of Theorem 1.3 may be true for the spaces ${}_V\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $B\dot{H}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$, which is unclear so far to us; see also Remark 3.9.

(ii) Similarly, we claim that when $\tau > 0$, the dual property in Theorem 1.3 is not possible to be correct for all ${}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ with $p, q \in (1, \infty)$ and $q > p$.

In fact, by [24, Propositions 2.2(i) and 2.3(ii)], we know that the embedding $F\dot{H}_{p_0,r}^{s_0,\tau}(\mathbb{R}^n) \subset F\dot{H}_{p_1,q}^{s_1,\tau}(\mathbb{R}^n)$ is true only when $\tau(p_0 \vee r)' \leq \tau(p_1 \vee q)' + \tau(\frac{1}{p_0} - \frac{1}{p_1})(p_0 \vee r)'(p_1 \vee q)'$. If we assume that $({}_V\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n))^* = F\dot{H}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $\tau > 0$ and $1 < p < q < \infty$, then by the embedding $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n) \subset \dot{F}_{q,r}^{s-n/p+n/q,\tau}(\mathbb{R}^n)$ (see [23, Proposition 3.3]) with $r > q$ together with an argument by duality, we have $F\dot{H}_{q',r'}^{-s+n/p-n/q,\tau}(\mathbb{R}^n) \subset F\dot{H}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$, which is not possible by the above conclusion. Thus, the claim is also true.

It is also unclear that when $p \neq q$, for which range of $p, q \in (1, \infty)$, the conclusion of Theorem 1.3 is true.

5. Some applications

We give some applications of Theorem 1.3 in this section. The first one is the following Sobolev-type embedding property of $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$.

Proposition 5.1. *Let $s_0, s_1 \in \mathbb{R}$, $p_0, p_1 \in (1, \infty)$ and $\tau \in [0, \frac{1}{p}]$ such that $p_0 < p_1$ and $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$. Then $F\dot{H}_{p_0}^{s_0,\tau}(\mathbb{R}^n) \subset F\dot{H}_{p_1}^{s_1,\tau}(\mathbb{R}^n)$.*

Proposition 5.1 follows from the corresponding Sobolev-type embedding property of $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [23, Proposition 3.3], Theorem 1.3 and a dual argument. We omit the details.

In [24, Proposition 2.2(ii)], it was proved that for all parameters $s_0, s_1 \in \mathbb{R}$, $p_0, p_1, q, r \in (1, \infty)$ and $\tau \in [0, \min\{\frac{1}{(\max\{p_0,r\})'}, \frac{1}{(\max\{p_1,q\})'}\}]$ such that $p_0 < p_1$, $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$ and $\tau(\max\{p_0,r\})' \leq \tau(\max\{p_1,q\})'$, $F\dot{H}_{p_0,r}^{s_0,\tau}(\mathbb{R}^n) \subset F\dot{H}_{p_1,q}^{s_1,\tau}(\mathbb{R}^n)$. Proposition 5.1 improves [24, Proposition 2.2 (ii)] in the case that $p = q$.

Recall that for all $\varepsilon \in (0, \infty)$, the ε -almost diagonal operators on $f_{p,q}^{s,\tau}(\mathbb{R}^n)$ are bounded when $s \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0, \infty]$ and $\tau \in [0, \frac{1}{p} + \frac{\varepsilon}{2n}]$; see [23, Theorem 3.1]. We now recall the notion of ε -almost diagonal operators on $f\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ in [23, Definition 3.1].

Definition 5.2. Let $\varepsilon \in (0, \infty)$, $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in [0, \frac{1}{p}]$. For all $Q, P \in \mathcal{Q}(\mathbb{R}^n)$, define

$$\omega_{QP}(\varepsilon) \equiv \left[\frac{\ell(Q)}{\ell(P)} \right]^s \left[1 + \frac{|x_P - x_Q|}{\max(\ell(Q), \ell(P))} \right]^{-n-\varepsilon} \min \left\{ \left[\frac{\ell(P)}{\ell(Q)} \right]^{\frac{n+\varepsilon}{2}}, \left[\frac{\ell(Q)}{\ell(P)} \right]^{\frac{n+\varepsilon}{2}} \right\}.$$

An operator A associated with a matrix $\{a_{QP}\}_{Q,P \in \mathcal{Q}(\mathbb{R}^n)}$, namely, for all sequences $t = \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \subset \mathbb{C}$ and $Q \in \mathcal{Q}(\mathbb{R}^n)$, $At \equiv \{(At)_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ with $(At)_Q \equiv \sum_{P \in \mathcal{Q}(\mathbb{R}^n)} a_{QP} t_P$, is called ε -almost diagonal on $f\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, if the matrix $\{a_{QP}\}_{Q,P \in \mathcal{Q}(\mathbb{R}^n)}$ satisfies $\sup_{Q,P \in \mathcal{Q}(\mathbb{R}^n)} \frac{|a_{QP}|}{\omega_{QP}(\varepsilon)} < \infty$.

By [23, Theorem 3.1], Theorem 1.3 and a dual argument, we obtain the following proposition. The details are omitted.

Proposition 5.3. Let $\varepsilon \in (0, \infty)$, $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in [0, \frac{1}{p}]$. Then all the ε -almost diagonal operators are bounded on $f\dot{H}_p^{s,\tau}(\mathbb{R}^n)$.

We remark that Proposition 5.3 improves [24, Theorem 3.1] in the case that $p = q$, since in [24, Theorem 3.1], we need an additional condition that $\varepsilon > 2n\tau$.

Using Proposition 5.3 and repeating the arguments in [24, Sections 3], we can establish the smooth atomic and molecular decomposition characterizations of $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ as follows. We first introduce a slight variant of the smooth molecules in [24].

Definition 5.4. Let $p \in (1, \infty)$, $s \in \mathbb{R}$, $\tau \in [0, \frac{1}{p}]$, $N \equiv \max(\lfloor -s \rfloor, -1)$, $Q \in \mathcal{Q}(\mathbb{R}^n)$, $L \equiv \lfloor s + n\tau \rfloor$ and $s^* \equiv s - \lfloor s \rfloor$.

- (i) A function m_Q is called a *smooth synthesis molecule for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ supported near Q* , if there exist a $\delta \in (\max\{s^*, (s + n\tau)^*\}, 1]$ and $M > n$ such that $\int_{\mathbb{R}^n} x^\gamma m_Q(x) dx = 0$ if $|\gamma| \leq N$,

$$\begin{aligned} |m_Q(x)| &\leq |Q|^{-\frac{1}{2}} (1 + [\ell(Q)]^{-1}|x - x_Q|)^{-\max(M, M-s)} \\ |\partial^\gamma m_Q(x)| &\leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n}} (1 + [\ell(Q)]^{-1}|x - x_Q|)^{-M} \quad \text{if } |\gamma| \leq L \end{aligned}$$

and

$$\begin{aligned} &|\partial^\gamma m_Q(x) - \partial^\gamma m_Q(y)| \\ &\leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n} - \frac{\delta}{n}} |x - y|^\delta \sup_{|z| \leq |x-y|} (1 + [\ell(Q)]^{-1}|x - z - x_Q|)^{-M} \quad \text{if } |\gamma| = L. \end{aligned}$$

A set $\{m_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ of functions is said to be a *family of smooth synthesis molecules* for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ if each m_Q is a smooth synthesis molecule for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ supported near Q .

- (ii) A function b_Q is called a *smooth analysis molecule* for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ supported near Q , if there exist a $\rho \in ((n-s)^*, 1]$ and $M > n$ such that $\int_{\mathbb{R}^n} x^\gamma b_Q(x) dx = 0$ for all $|\gamma| \leq L$,

$$\begin{aligned} |b_Q(x)| &\leq |Q|^{-\frac{1}{2}} (1 + [\ell(Q)]^{-1}|x - x_Q|)^{-\max(M, M+s+n\tau)} \\ |\partial^\gamma b_Q(x)| &\leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n}} (1 + [\ell(Q)]^{-1}|x - x_Q|)^{-M} \quad \text{if } |\gamma| \leq N \end{aligned}$$

and

$$\begin{aligned} &|\partial^\gamma b_Q(x) - \partial^\gamma b_Q(y)| \\ &\leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n} - \frac{\delta}{n}} |x - y|^\delta \sup_{|z| \leq |x-y|} (1 + [\ell(Q)]^{-1}|x - z - x_Q|)^{-M} \quad \text{if } |\gamma| = N. \end{aligned}$$

A set $\{b_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ of functions is said to be a *family of smooth analysis molecules* for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, if each b_Q is a smooth analysis molecule for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ supported near Q .

We remark that the molecules in Definition 5.4 are “weaker” than those in [24, Definition 3.2] in the case that $p = q$, since in [24, Definition 3.2], the corresponding numbers $N \equiv \max(\lfloor -s + 2n\tau \rfloor, -1)$, $L \equiv \lfloor s + 3n\tau \rfloor$ and $M > n + 2n\tau$.

Following the arguments in [24, Section 3], we obtain the following conclusion, which improves [24, Theorem 3.2] in the case that $p = q$. We also omit the details.

Theorem 5.5. *Let $s \in \mathbb{R}$, $p \in (1, \infty)$ and $\tau \in [0, 1/p']$.*

- (i) *If $\{m_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ is a family of smooth synthesis molecules for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, then there exists a positive constant C such that*

$$\left\| \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} t_Q m_Q \right\|_{F\dot{H}_p^{s,\tau}(\mathbb{R}^n)} \leq C \|t\|_{f\dot{H}_p^{s,\tau}(\mathbb{R}^n)}$$

for all $t = \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in f\dot{H}_p^{s,\tau}(\mathbb{R}^n)$.

- (ii) *If $\{b_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ is a family of smooth analysis molecules for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, then there exists a positive constant C such that*

$$\| \{ \langle f, b_Q \rangle \}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \|_{f\dot{H}_p^{s,\tau}(\mathbb{R}^n)} \leq C \|f\|_{F\dot{H}_p^{s,\tau}(\mathbb{R}^n)}$$

for all $f \in F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$.

We now introduce the following smooth atoms for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$.

Definition 5.6. Let $s \in \mathbb{R}$, $p \in (1, \infty)$, τ and N be as in Definition 5.2. A function a_Q is called a *smooth atom* for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ supported near a dyadic cube Q , if there exist \tilde{K} and \tilde{N} with $\tilde{K} \geq \max(\lfloor s + n\tau + 1 \rfloor, 0)$ and $\tilde{N} \geq N$ such that a_Q satisfies the following support, regularity and moment conditions: $\text{supp } a_Q \subset 3Q$, $\|\partial^\gamma a_Q\|_{L^\infty(\mathbb{R}^n)} \leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n}}$ if $|\gamma| \leq \tilde{K}$, and $\int_{\mathbb{R}^n} x^\gamma a_Q(x) dx = 0$ if $|\gamma| \leq \tilde{N}$.

A set $\{a_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ of functions is called a *family of smooth atoms* for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, if each a_Q is a smooth atom for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ supported near Q .

Recall that in [24, Definition 3.3], the number $N \equiv \max(\lfloor -s + 2n\tau \rfloor, -1)$ and the condition on \tilde{K} is that $\tilde{K} \geq \max(\lfloor s + 3n\tau + 1 \rfloor, 0)$. In this sense, the smooth atoms in Definition 5.6 are “weaker” than those in [24, Definition 3.3]. Similarly to the proof of [24, Theorem 3.3], we obtain the following conclusion, which improves [24, Theorem 3.3] in the case that $p = q$. The details are omitted.

Theorem 5.7. Let s, p, τ be as in Theorem 5.5. Then for each $f \in F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, there exist a family $\{a_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ of smooth atoms for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, a coefficient sequence $t \equiv \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in f\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, and a positive constant C such that $f = \sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} t_Q a_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $\|t\|_{f\dot{H}_p^{s,\tau}(\mathbb{R}^n)} \leq C\|f\|_{F\dot{H}_p^{s,\tau}(\mathbb{R}^n)}$.

Conversely, there exists a positive constant C such that for all families $\{a_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)}$ of smooth atoms for $F\dot{H}_p^{s,\tau}(\mathbb{R}^n)$ and coefficient sequences $t \equiv \{t_Q\}_{Q \in \mathcal{Q}(\mathbb{R}^n)} \in f\dot{H}_p^{s,\tau}(\mathbb{R}^n)$, $\|\sum_{Q \in \mathcal{Q}(\mathbb{R}^n)} t_Q a_Q\|_{F\dot{H}_p^{s,\tau}(\mathbb{R}^n)} \leq C\|t\|_{f\dot{H}_p^{s,\tau}(\mathbb{R}^n)}$.

By Theorems 5.5, 5.7 and the arguments in [24, Section 4], the results in [24, Theorems 4.1 and 4.2] about the mapping properties of pseudo-differential operators and the trace properties on $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ can be improved when $p = q$.

Let $m \in \mathbb{Z}$. The class $\dot{\mathcal{S}}_{1,1}^m(\mathbb{R}^n)$ is the collection of all smooth functions a on $\mathbb{R}_x^n \times (\mathbb{R}_\xi^n \setminus \{0\})$ satisfying that, for all $\alpha, \beta \in \mathbb{Z}_+^n$,

$$\sup_{x \in \mathbb{R}^n, \xi \in (\mathbb{R}_\xi^n \setminus \{0\})} |\xi|^{-m - |\alpha| + |\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < \infty;$$

see, for example, [12] or [24, Definition 4.1]. For $a \in \dot{\mathcal{S}}_{1,1}^m(\mathbb{R}^n)$, the corresponding pseudo-differential operator $a(x, D)$ is defined by setting, for all $x \in \mathbb{R}^n$ and smooth synthesis molecules for $F\dot{H}_p^{s+m,\tau}(\mathbb{R}^n)$,

$$a(x, D)f(x) \equiv \int_{\mathbb{R}^n} a(x, \xi) \widehat{f}(\xi) e^{ix\xi} d\xi.$$

Similarly to the proof of [24, Theorem 4.1], we obtain the following result.

Proposition 5.8. *Let $m \in \mathbb{Z}$, $s \in \mathbb{R}$, $p \in (1, \infty)$, $\tau \in [0, \frac{1}{p}]$ and $a \in \mathcal{S}'_{1,1}(\mathbb{R}^n)$. Assume that $a(x, D)$ is the corresponding pseudo-differential operator to a and its formal adjoint $a(x, D)^*$ satisfies $a(x, D)^*(x^\beta) = 0$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ for all $\beta \in \mathbb{Z}_+^n$ with $|\beta| \leq \max\{-s, -1\}$. Then $a(x, D)$ is a bounded linear operator from $F\dot{H}_p^{s+m, \tau}(\mathbb{R}^n)$ to $F\dot{H}_p^{s, \tau}(\mathbb{R}^n)$.*

Recall that in [24, Theorem 4.1], the condition on β is $|\beta| \leq \max\{-s + 2n\tau, -1\}$. Thus, Proposition 5.8 improves [24, Theorem 4.1] in the case that $p = q$.

Similarly to the proof of [24, Theorem 4.2], we have the following trace property of $F\dot{H}_p^{s, \tau}(\mathbb{R}^n)$.

Proposition 5.9. *Let $n \geq 2$, $p \in (1, \infty)$, $\tau \in [0, \frac{n-1}{np}]$ and $s \in (\frac{1}{p}, \infty)$. Then there exists a surjective and continuous operator*

$$\text{Tr} : f \in F\dot{H}_p^{s, \tau}(\mathbb{R}^n) \mapsto \text{Tr}(f) \in F\dot{H}_p^{s-1/p, n/(n-1)\tau}(\mathbb{R}^{n-1})$$

such that $\text{Tr}(a)(x') = a(x', 0)$ holds for all $x' \in \mathbb{R}^{n-1}$ and smooth atoms a for $F\dot{H}_p^{s, \tau}(\mathbb{R}^n)$.

Notice that the condition $s \in (\frac{1}{p} + 2n\tau, \infty)$ in [24, Theorem 4.2] is replaced by $s \in (\frac{1}{p}, \infty)$ in Proposition 5.9. Thus, Proposition 5.9 improves [24, Theorem 4.2] in the case that $p = q$. The proofs of Propositions 5.8 and 5.9 are also omitted.

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