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# On Quasi-Polarized Manifolds Whose Sectional Genus is Equal to the Irregularity

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ABSTRACT - Let $(X, L)$ be a quasi-polarized manifold of dimension $n$. In our previous paper, we proved that if $\dim X = 3$ and $h^0(L) \geq 2$, then $g(X, L) \geq h^1(\mathcal{O}_X)$ holds. Here $g(X, L)$ denotes the sectional genus of $(X, L)$. In this paper, we give the classification of quasi-polarized 3-folds $(X, L)$ with $h^0(L) \geq 3$ and $g(X, L) = h^1(\mathcal{O}_X)$. Moreover as an application of this result, we also give the classification of polarized manifolds $(X, L)$ with $\dim \mathrm{Bs}|L| = 1$, $h^0(L) \geq n$ and $g(X, L) = h^1(\mathcal{O}_X)$.

## 1. Introduction.

Let $(X, L)$ be a quasi-polarized manifold with $\dim X = n$. For this pair $(X, L)$, the *sectional genus* $g(X, L)$ is defined by the following formula:

$$g(X, L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where $K_X$ is the canonical bundle of $X$. Then there is the following conjecture which was proposed by Fujita [7, (13.7) Remark].

CONJECTURE 1.1 (Fujita). *Let $(X, L)$ be a quasi-polarized manifold. Then $g(X, L) \geq q(X)$, where $q(X) := \dim H^1(\mathcal{O}_X)$ is the irregularity of $X$.*

For this conjecture, there are some results (see [9], [10], [12] and so on). But it is unknown whether this conjecture is true or not even for the case of $\dim X = 2$. If $\dim X = 2$, then this conjecture is true if $h^0(L) > 0$ (see [9]). Moreover the classification of quasi-polarized surfaces $(X, L)$ with $g(X, L) = q(X)$ and $h^0(L) \geq 1$ was obtained (see [8], [9]).

If $\dim X = 3$ and $h^0(L) \geq 2$, it is known that this conjecture is true, and

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the classification of *polarized* 3-folds $(X,L)$ with $g(X,L)=q(X)$ and $h^0(L)\geq 3$ was given (see [12]).

In this paper, we will give the classification of *quasi*-polarized 3-folds with $g(X,L)=q(X)$ and $h^0(L)\geq 3$. As an application of this result, we are able to give the classification of polarized $n$-fold $(X,L)$ with $g(X,L)=q(X)$, $\dim \mathrm{Bs}|L|=1$ and $h^0(L)\geq n$. (Here we note that $g(X,L)\geq q(X)$ holds if $\dim \mathrm{Bs}|L|=1$.)

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## 2. Preliminaries.

DEFINITION 2.1. Let $X$ and $Y$ be projective varieties with $\dim X>\dim Y\geq 1$, and let $f:X\to Y$ be a surjective morphism with connected fibers. Then $(f,X,Y)$ is called a fiber space. Moreover if $L$ is a nef and big (resp. an ample) line bundle on $X$, then $(f,X,Y,L)$ is called a quasi-polarized (resp. polarized) fiber space.

LEMMA 2.1. *Let $X$ and $C$ be smooth projective varieties with* $\dim X=n$ *and* $\dim C=1$, *and let $L$ be a nef and big line bundle on $X$. Assume that there exists a fiber space $f:X\to C$ such that $h^0(K_F+L_F)\neq 0$ for a general fiber $F$ of $f$. Then $f_*(K_{X/C}+L)$ is ample.*

PROOF. First we note that there exists a natural number $m$ such that $(mL)^n-n(mL)^{n-1}F>0$. Then by [3, Lemma 4.1], there exists a natural number $k$ such that $\mathcal{O}_X(k(mL-F))$ has a nontrivial global section. Hence we have an injective map $\mathcal{O}_X(kF)\to\mathcal{O}(kmL)$. On the other hand, there exists a line bundle $\mathcal{N}$ on $C$ such that $\mathcal{O}(kF)=f^*(\mathcal{N})$. Hence by [4, Corollary 1.9] we see that $f_*(K_{X/C}+L)$ is ample and we get the assertion. □

DEFINITION 2.2. (i) Let $(X_1,L_1)$ and $(X_2,L_2)$ be quasi-polarized varieties. Then $(X_1,L_1)$ and $(X_2,L_2)$ are said to be *birationally equivalent* if there is another variety $G$ with birational morphisms $g_i:G\to X_i$ $(i=1,2)$ such that $g_1^*L_1=g_2^*L_2$.

(ii) Let $(f_1,X_1,Y,L_1)$ and $(f_2,X_2,Y,L_2)$ be quasi-polarized fiber spaces. Then $(f_1,X_1,Y,L_1)$ and $(f_2,X_2,Y,L_2)$ are said to be *birationally equivalent* if there is another variety $G$ with birational morphisms $g_i:G\to X_i$ $(i=1,2)$ such that $g_1^*L_1=g_2^*L_2$ and $f_1\circ g_1=f_2\circ g_2$.

DEFINITION 2.3. Let $X$ be a normal projective variety of dimension $n$ and let $D$ be a $\mathbb{Q}$-divisor on $X$. Then $D$ is said to be *generically nef* if $DL_1 \cdots L_{n-1} \geq 0$ for any collection of ample Cartier divisors $L_1, \ldots, L_{n-1}$ on $X$.

DEFINITION 2.4. Let $(X, L)$ be a quasi-polarized variety of dimension $n$. Then the *Δ-genus* $\Delta(X, L)$ of $(X, L)$ is defined by the following:

$$\Delta(X, L) = n + L^n - h^0(L).$$

PROPOSITION 2.1. *Let $(X, L)$ be a quasi-polarized manifold of dimension n. If $K_X + (n-1)L$ is not generically nef, then $\Delta(X, L) = 0$ or $(X, L)$ is birationally equivalent to a scroll over a smooth curve.*

PROOF. See [16, Proposition 1.3]. □

## 3. Main results.

First we will prove the following theorem.

THEOREM 3.1. *Let $(f, X, C, L)$ be a quasi-polarized fiber space such that $X$ and $C$ are smooth with* $\dim X = n$ *and* $\dim C = 1$. *Then $g(X, L) \geq g(C)$. Moreover if $g(X, L) = g(C)$, then $(X, L)$ is one of the following two types.*

(a) $\Delta(X, L) = 0$.

(b) *The pair $(X, L)$ is birationally equivalent to a scroll over $C$.*

PROOF. (1) If $g(C) = 0$, then $g(X, L) \geq 0 = g(C)$ by [16, Theorem 1.1]. Moreover if $g(X, L) = 0 = g(C)$, then by [16, Theorem 1.2] we have $\Delta(X, L) = 0$.

(2) Next we assume that $g(C) \geq 1$.

(2.1) First we assume that $K_X + (n-1)L$ is generically nef. Then by [15, 1.2 Theorem] we see that there exists a natural number $j$ with $1 \leq j \leq n-1$ such that $h^0(K_X + jL) > 0$. Hence $h^0(K_F + jL_F) > 0$ for any general fiber $F$ of $f$. Then $f_*(K_{X/C} + jL) \neq 0$. By Lemma 2.1 we see that $f_*(K_{X/C} + jL)$ is ample. By the same argument as [10, Lemma 1.4.1], we get $(K_{X/C} + jL)L^{n-1} > 0$. Since $1 \leq j \leq n-1$, we have $(K_{X/C} + (n-1)L)L^{n-1} > 0$. Then

$$\begin{aligned} g(X, L) &= g(C) + \frac{1}{2}(K_{X/C} + (n-1)L)L^{n-1} + (g(C)-1)((L_F)^{n-1} - 1) \\ &> g(C). \end{aligned}$$

(2.2) Next we assume that $K_X + (n-1)L$ is not generically nef. Then by Proposition 2.1 we see that $\Delta(X, L) = 0$ or there exist a quasi-polarized variety $(X', L')$, a smooth projective variety $M$ and birational morphisms $\mu_1 : M \to X$ and $\mu_2 : M \to X'$ such that $(X', L')$ is a scroll over a smooth curve.

If $\Delta(X, L) = 0$, then we infer that $h^1(\mathcal{O}_{X'}) = h^1(\mathcal{O}_X) = 0$ (see [12, Lemma 1.15]). Hence $g(C) = 0$ and this contradicts the assumption of $g(C) > 0$. So we may assume that $(X', L')$ is a scroll over a smooth curve $B$. Let $f' : X' \to B$ be its fibration and let $h := f' \circ \mu_2 : M \to B$. Then for any general fiber $F_h$ of $h$, we have $h^1(\mathcal{O}_{F_h}) = 0$. Since $g(C) > 0$, we see that $f \circ \mu_1(F_h)$ is a point. Therefore by [2, Lemma 4.1.13] there exists a surjective morphism $\delta : B \to C$ such that $f \circ \mu_1 = \delta \circ h$. But since $f$ and $f'$ have connected fibers, we see that $\delta$ is an isomorphism. On the other hand, we can easily check that $g(X', L') = g(B)$. So we get $g(X, L) = g(X', L') = g(B) = g(C)$. Therefore we get the assertion. □

Remark 3.1. There exists an example of a quasi-polarized fiber space $(f, X, C, L)$ such that $g(X, L) = g(C)$ and $(X, L)$ is birationally equivalent to $(V, H)$ with $\Delta(V, H) = 0$. For example, let $(V, H) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Then we can easily see that $\Delta(V, H) = 0$. We take two general members $H_1$ and $H_2$ in $|H|$ and let $\Lambda$ be a pencil which is generated by $H_1$ and $H_2$. By using this pencil, we can make a fiber space over a smooth curve. Namely, there exist a smooth projective variety $X$, a birational morphism $\mu : X \to \mathbb{P}^n$ and a fiber space $f : X \to C$ over a smooth curve $C$. We set $L := \mu^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Since $q(X) = 0$, we see that $C \cong \mathbb{P}^1$. Moreover $g(X, L) = g(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = 0 = g(C)$ and $(X, L)$ is birationally equivalent to $(V, H)$.

Next we consider quasi-polarized manifolds $(X, L)$ with $\dim X = 3$, $h^0(L) \geq 3$ and $g(X, L) = q(X)$.

Theorem 3.2. *Let $(X, L)$ be a quasi-polarized 3-fold. Assume that $h^0(L) \geq 3$. If $g(X, L) = q(X)$, then $(X, L)$ satisfies one of the following two types.*

(a) $\Delta(X, L) = 0$.

(b) *The pair $(X, L)$ is birationally equivalent to a scroll over a smooth curve $C$.*

Proof. By [6, Theorem 4.2], there exists a quasi-polarized variety $(X', L')$ which is birationally equivalent to $(X, L)$ and satisfies one of the following conditions:

(i) $K_{X'} + 2L'$ is nef for the canonical $\mathbb{Q}$-bundle $K_{X'}$;
(ii) $\Delta(X, L) = \Delta(X', L') = 0$;
(iii) $(X', L')$ is a scroll over a curve,

where $X'$ is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities. Since $g(X, L) = g(X', L')$ and $q(X) = q(X')$, we may assume that $X$ has only $\mathbb{Q}$-factorial terminal singularities and $(X, L)$ satisfies one of the above conditions.

If $(X, L)$ is the type (ii), then $g(X, L) = 0$ by [6, (1.7) Corollary] and $q(X) = 0$ by [12, Lemma 1.15]. Hence we obtain $g(X, L) = q(X)$ in this case.

If $(X, L)$ is the type (iii), then we can check $g(X, L) = q(X)$ by easy calculation.

So we may assume that $K_X + 2L$ is nef. Let $\pi : \widetilde{X} \to X$ be a resolution of $X$ such that $\widetilde{X} \backslash \pi^{-1}(\mathrm{Sing}(X)) \cong X \backslash \mathrm{Sing}(X)$, and $\widetilde{L} = \pi^*(L)$. Then $h^0(\widetilde{L}) = = h^0(L) \geq 3$. Let $\Lambda$ be a linear pencil which is contained in $|\widetilde{L}|$ such that $\Lambda = \Lambda_M + Z$, where $\Lambda_M$ is the movable part of $\Lambda$ and $Z$ is the fixed part of $|\widetilde{L}|$. We will make a fiber space by using this $\Lambda$. Let $\varphi : \widetilde{X} \dashrightarrow \mathbb{P}^1$ be the rational map associated with $\Lambda_M$, and $\theta : \widetilde{X}' \to \widetilde{X}$ an elimination of indeterminacy of $\varphi$. So we obtain a surjective morphism $\varphi' : \widetilde{X}' \to \mathbb{P}^1$. If necessary, we take the Stein factorization $\delta : C \to \mathbb{P}^1$ of $\varphi'$. Then we have a fiber space $f' : \widetilde{X}' \to C$ such that $\varphi' = \delta \circ f'$. Let $F'$ be a general fiber of $f'$ and let $a := \deg \delta$. We consider this quasi-polarized fiber space $(f', \widetilde{X}', C, \theta^*(\widetilde{L}))$. By the proof of [12, Theorem 2.1], we see that there exists a quasi-polarized fiber space $(f_1, X_1, C, L_1)$ which is birationally equivalent to $(f', \widetilde{X}', C, \theta^*(\widetilde{L}))$ such that $(f_1, X_1, C, L_1)$ satisfies one of the following conditions.

- $K_{X_1} + 2L_1$ is $f_1$-nef.
- $(f_1, X_1, C, L_1)$ is a scroll.

If $(f_1, X_1, C, L_1)$ is a scroll, then we see that $g(X, L) = q(X)$ and this is the type (b) in Theorem 3.2. So we may assume that $K_{X_1} + 2L_1$ is $f_1$-nef. In this case, by [14, Lemma 0.2], we see that $K_{X_1/C} + 2L_1$ is nef.

(a) The case of $g(C) \geq 1$. Then $\theta$ is the identity map. So we have $\widetilde{X}' = \widetilde{X}$ and $\theta^*(\widetilde{L}) = \widetilde{L}$. By the construction of the fiber space $(f', \widetilde{X}', C, \theta^*(\widetilde{L}))$, we get $\widetilde{L} = \sum_{i=1}^{a} F_i + Z$, where each $F_i$ is a fiber of $f'$ and $Z$ is the fixed part of $|\widetilde{L}|$. Then there exists an ample line bundle $P \in \mathrm{Pic}(C)$ such that $\sum_{i=1}^{a} F_i = (f')^*(P)$. In particular $\deg P = a$.

Claim 3.1. $a \geq 3$.

Proof. First we note that $h^0(L) = h^0(\widetilde{L}) = h^0\Big(\sum_{i=1}^{a} F_i + Z\Big) = h^0\Big(\sum_{i=1}^{a} F_i\Big) = h^0(P)$. Since $h^0(L) \geq 3$, we have $h^0(P) \geq 3$. If $a \leq 2$, then $\Delta(C, P) = 1 + \deg P - h^0(P) = 1 + a - h^0(P) \leq 0$. On the other hand, since $P$ is an ample line bundle on $C$, we have $\Delta(C, P) \geq 0$ by [5, Corollary 1.10] or [7, (4.2) Theorem]. Therefore $\Delta(C, P) = 0$. But then $C \cong \mathbb{P}^1$ (see [5, Lemma 3.1]) and this contradicts the assumption that $g(C) \geq 1$. Hence we have $a \geq 3$. □

Here we note that $\widetilde{L}$ is numerically equivalent to $aF' + Z$ by the construction above. By the same argument as in the proof of [12, Claim 2.2], we have

$$(K_{\widetilde{X}/C} + 2\widetilde{L})(\widetilde{L})^2 \geq t(K_{\widetilde{X}/C} + 2\widetilde{L})(\widetilde{L})F'$$

for any natural number $t$ with $t \leq a$. Hence $(K_{\widetilde{X}/C} + 2\widetilde{L})(\widetilde{L})^2 \geq 3(K_{\widetilde{X}/C} + 2\widetilde{L})(\widetilde{L})F'$ holds because $a \geq 3$. Since $g(C) \geq 1$ and $(\widetilde{L}_{F'})^2 \geq 1$, we get

$$\begin{aligned} g(\widetilde{X}, \widetilde{L}) &= 1 + \frac{1}{2}(K_{\widetilde{X}} + 2\widetilde{L})(\widetilde{L})^2 \\ &= g(C) + \frac{1}{2}(K_{\widetilde{X}/C} + 2\widetilde{L})(\widetilde{L})^2 + (g(C) - 1)\big((\widetilde{L}_{F'})^2 - 1\big) \\ &\geq g(C) + \frac{3}{2}(K_{\widetilde{X}/C} + 2\widetilde{L})(\widetilde{L})F' \\ &= g(C) + 3g(F', \widetilde{L}|_{F'}) + \frac{3}{2}(\widetilde{L})^2F' - 3. \end{aligned}$$

Since $h^0(\widetilde{L}|_{F'}) > 0$ and $\dim F' = 2$ we have $g(F', \widetilde{L}|_{F'}) \geq q(F')$ by [9, Lemma 1.2 (2)]. Because $g(C) + q(F') \geq q(\widetilde{X})$, we have

$$g(\widetilde{X}, \widetilde{L}) \geq q(\widetilde{X}) + 2g(F', \widetilde{L}|_{F'}) + \frac{3}{2}(\widetilde{L})^2F' - 3.$$

Since $g(\widetilde{X}, \widetilde{L}) = g(X, L) = q(X) = q(\widetilde{X})$ holds, we get $2g(F', \widetilde{L}|_{F'}) + \frac{3}{2}(\widetilde{L})^2F' - 3 \leq 0$. Hence we have $g(F', \widetilde{L}|_{F'}) = 0$. Therefore $\kappa(F') = -\infty$ and by [9, Theorem 2.1] we have $q(F') = 0$. So $q(\widetilde{X}) = g(C)$ because $g(C) = q(F') + g(C) \geq q(\widetilde{X}) \geq g(C)$. Hence we obtain $g(\widetilde{X}, \widetilde{L}) = g(C)$, and by Theorem 3.1 we get the assertion in this case.

(b) The case of $g(C) = 0$. Let $\gamma := \pi \circ \theta$.

(b.1) If $a \geq 2$, then

$$
\begin{aligned}
g(X,L) &= g(\widetilde{X},\widetilde{L})\\
&= g(\widetilde{X}',\theta^*(\widetilde{L}))\\
&= 1+\frac{1}{2}\gamma^*(K_X+2L)(\theta^*(\widetilde{L}))^2\\
&\geq 1+\gamma^*(K_X+2L)(\theta^*(\widetilde{L}))F'
\end{aligned}
$$

because $K_X+2L$ is nef and $a\geq 2$. Let $\widetilde{D}:=\theta(F')$. By [12, Claims 2.3 and 2.4], we have

$$
\begin{aligned}
g(X,L) &\geq 1+\gamma^*(K_X+2L)(\theta^*(\widetilde{L}))F'\\
&= 1+\theta^*(\pi^*(K_X)+2\widetilde{L})(\theta^*(\widetilde{L}))F'\\
&= 1+\theta^*(K_{\widetilde{X}}+2\widetilde{L})(\theta^*(\widetilde{L}))F'\\
&\geq 1+(\theta^*(K_{\widetilde{X}}+\widetilde{D})+\theta^*(\widetilde{L}))(\theta^*(\widetilde{L}))F'\\
&\geq 1+(K_{\widetilde{X}'}+F'+\theta^*(\widetilde{L}))(\theta^*(\widetilde{L}))F'\\
&= 2g(F',\theta^*(\widetilde{L})|_{F'})-1.
\end{aligned}
$$

Since $\dim F'=2$ and $h^0(\theta^*(\widetilde{L})|_{F'})>0$, we have $g(F',\theta^*(\widetilde{L})|_{F'})\geq q(F')$ by [9, Lemma 1.2 (2)]. Moreover since $q(F')=q(F')+g(C)\geq q(\widetilde{X}')=q(X)$, we get $g(X,L)\geq 2q(X)-1$. Therefore $q(X)\leq 1$ because $g(X,L)=q(X)$. In particular $g(X,L)\leq 1$. From [6, Corollaries (4.8) and (4.9)], we see that $(X,L)$ is birationally equivalent to one of the types (a) and (b) in Theorem 3.2. (Here we use the assumption that $g(X,L)=q(X)$.)

(b.2) Here we assume that $a=1$. Then $h^0(\theta^*(\widetilde{L})|_{F'})\geq 2$. By the same argument as in Case (2) in the proof of [12, Theorem 2.1] we have

$$q(X)=g(X,L)\geq g(F',\theta^*(\widetilde{L})|_{F'})\geq q(F')\geq q(X).$$

Hence we have $\kappa(F')=-\infty$ by [9, Theorem 3.1] since $g(F',\theta^*(\widetilde{L})|_{F'})=q(F')$ and $h^0(\theta^*(\widetilde{L})|_{F'})\geq 2$. Moreover we get

$$q(\widetilde{X}')=q(X)=q(F'). \tag{1}$$

Here we apply the relatively minimal model theory for the fibration $f':\widetilde{X}'\to C\cong\mathbb{P}^1$. Since $\kappa(F')=-\infty$, we see that there exist smooth projective varieties $X^\sharp$ and $T$ with $\dim X^\sharp=3$ and $1=\dim C\leq\dim T\leq 2$, a birational morphism $\delta^\sharp:X^\sharp\to\widetilde{X}'$ and surjective morphisms $\delta_1:X^\sharp\to T$ and $\delta_2:T\to C$ with connected fibers such that $f'\circ\delta^\sharp=\delta_2\circ\delta_1$ and $F_{\delta_1}$ is birationally equivalent to a Fano manifold, where $F_{\delta_1}$ is a general fiber of $\delta_1$. In particular $q(F_{\delta_1})=0$. We put $f^\sharp:=f'\circ\delta^\sharp$.

(b.2.1) Assume that $\dim T = 1$. Then $\delta_2$ is an isomorphism. Hence $q(T) = 0$ because $C \cong \mathbb{P}^1$. On the other hand, since $q(F_{\delta_1}) = 0$, we have $q(X) = q(X^\sharp) = q(T) = 0$. Thus we get $g(X, L) = 0$ from the assumption that $g(X, L) = q(X)$. Therefore by [6, (4.8) Corollary] we get the assertion.

(b.2.2) Next we assume that $\dim T = 2$. If $q(X) \leq 1$, then we have $g(X, L) \leq 1$ and by [6, Corollaries (4.8) and (4.9)] we get the assertion. So we may assume that $q(X) \geq 2$. Let $F_{\delta_2}$ (resp. $F^\sharp$) be a general fiber of $\delta_2$ (resp. $f^\sharp$). Then

$$\delta_1|_{F^\sharp} : F^\sharp \to F_{\delta_2}$$

is a surjective morphism with connected fibers. Since a general fiber of $\delta_1|_{F^\sharp}$ is $\mathbb{P}^1$, we have $q(F^\sharp) = q(F_{\delta_2})$. On the other hand, we have $q(X^\sharp) = q(T)$ because any general fiber of $\delta_1$ is $\mathbb{P}^1$, and we have $q(F^\sharp) = q(X^\sharp)$ by (1). So we get $q(T) = q(F^\sharp) = q(F_{\delta_2}) = q(F_{\delta_2}) + q(\mathbb{P}^1)$. Now we are assuming that $q(X^\sharp) = q(X) \geq 2$, so we have $q(F_{\delta_2}) \geq 2$. Therefore, considering the fiber space $\delta_2 : T \to C \cong \mathbb{P}^1$, we see from [1, Lemme] or [9, Lemma 1.5] that $T$ is birationally equivalent to $F_{\delta_2} \times \mathbb{P}^1$. In particular $\kappa(T) = -\infty$. So, taking the Albanese map of $T$, there exists a morphism $\alpha : T \to B$, where $B$ is a smooth projective curve with $g(B) = q(T) = q(F_{\delta_2})$. Then $\alpha \circ \delta_1 : X^\sharp \to B$ has connected fibers. Moreover since $q(X^\sharp) = q(F^\sharp) = q(F_{\delta_2}) = g(B)$ we obtain $g(X^\sharp, (\theta \circ \delta^\sharp)^*(\widetilde{L})) = g(X, L) = q(X) = q(X^\sharp) = g(B)$. Since $(X^\sharp, (\theta \circ \delta^\sharp)^*(\widetilde{L}))$ is a quasi-polarized 3-fold, we get the assertion by Theorem 3.1. □

Here we want to propose the following conjecture which is a quasi-polarized manifolds' version of [12, Conjecture 2.15].

CONJECTURE 3.1. *Let $(X, L)$ be a quasi-polarized $n$-fold. Assume that $h^0(L) \geq n$. If $g(X, L) = q(X)$, then $(X, L)$ is one of the following.*

(a) $\Delta(X, L) = 0$.

(b) *The pair $(X, L)$ is birationally equivalent to a scroll over a smooth curve.*

REMARK 3.2. If $n = 2$ (resp. $n = 3$), then this conjecture is true by [9, Theorem 3.1] (resp. Theorem 3.2 above).

Let $(X, L)$ be a polarized manifold of dimension $n$. If $\mathrm{Bs}|L| = \emptyset$ (resp. $\dim \mathrm{Bs}|L| = 0$), then by [2, Theorem 7.2.10] (resp. [11, Theorem 3.2]) we see that $g(X, L) \geq q(X)$. Moreover, we can get a classification of $(X, L)$ with $g(X, L) = q(X)$ and $\dim \mathrm{Bs}|L| \leq 0$ (see [17, (3.6) Theorem] and [11, Theorem 3.2]). So, as the next step, we consider the case where $\dim \mathrm{Bs}|L| = 1$.

THEOREM 3.3. *Let $(X, L)$ be a polarized manifold of dimension $n \geq 3$. Assume that* $\dim \mathrm{Bs}|L| = 1$.

(i) *The inequality $g(X, L) \geq q(X)$ holds.*

(ii) *Furthermore we assume that $h^0(L) \geq n$. If $g(X, L) = q(X)$, then $(X, L)$ is one of the following.*

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(c) *A scroll over a smooth curve.*

PROOF. From the assumption, we see that there exist an $(n-3)$-ladder $X \supset X_1 \supset \cdots \supset X_{n-3}$ such that each $X_j$ is a normal and Gorenstein projective variety of dimension $n - j$ (see [13, Proposition 1.12 (2)]). Let $L_j = L_{X_j}$ for every $j$ with $1 \leq j \leq n-3$. Then we see that

$$h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{X_1}) = \cdots = h^1(\mathcal{O}_{X_{n-3}}) \tag{2}$$

and

$$g(X, L) = g(X_1, L_1) = \cdots = g(X_{n-3}, L_{n-3}). \tag{3}$$

Let $\pi : M_{n-3} \to X_{n-3}$ be a resolution of $X_{n-3}$. Then

$$g(M_{n-3}, \pi^*(L_{n-3})) = g(X_{n-3}, L_{n-3}) \tag{4}$$

and

$$h^1(\mathcal{O}_{M_{n-3}}) \geq h^1(\mathcal{O}_{X_{n-3}}). \tag{5}$$

(i) Here we note that $h^0(L_{n-3}) \geq 2$. Hence by [12, Theorem 2.1] we have

$$g(M_{n-3}, \pi^*(L_{n-3})) \geq q(M_{n-3}). \tag{6}$$

Therefore by (2), (3), (4), (5) and (6), we get $g(X, L) \geq q(X)$.

(ii) Assume that $h^0(L) \geq n$. Then $h^0(L_{n-3}) \geq 3$. If $g(X, L) = q(X)$, then by (2), (3), (4), (5) and (6) we have $g(M_{n-3}, \pi^*(L_{n-3})) = q(M_{n-3})$ and $q(M_{n-3}) = = q(X_{n-3})$. In particular, $X_{n-3}$ has the Albanese map (see [2, Remark 2.4.2]). Let $\alpha : X_{n-3} \to \mathrm{Alb}(X_{n-3})$ be its Albanese map, where $\mathrm{Alb}(X_{n-3})$ is the Albanese variety of $X_{n-3}$. Then $\alpha \circ \pi : M_{n-3} \to \mathrm{Alb}(X_{n-3})$ is the Albanese map of $M_{n-3}$. Since $(M_{n-3}, \pi^*(L_{n-3}))$ is a quasi-polarized 3-fold with $h^0(\pi^*(L_{n-3})) \geq 3$ and $g(M_{n-3}, \pi^*(L_{n-3})) = q(M_{n-3})$, we can apply Theorem 3.2. Then $(M_{n-3}, \pi^*(L_{n-3}))$ satisfies one of the following types:

- $\Delta(M_{n-3}, \pi^*(L_{n-3})) = 0$.
- $(M_{n-3}, \pi^*(L_{n-3}))$ is birationally equivalent to a scroll over a smooth curve.

If $\Delta(M_{n-3}, \pi^*(L_{n-3})) = 0$, then $g(X, L) = g(M_{n-3}, \pi^*(L_{n-3})) = 0$. Therefore we get the assertion from Fujita's results (see [7, (12.1) Theorem and (5.10) Theorem]).

Next we assume that $(M_{n-3}, \pi^*(L_{n-3}))$ is birationally equivalent to a scroll over a smooth curve. Let $(V, H)$ be its scroll. If $h^1(\mathcal{O}_V) = 0$, then we see that $g(X, L) = 0$ and we get the assertion. So we may asssume that $h^1(\mathcal{O}_V) \geq 1$. Then the dimension of the image of Albanese map of $V$ is one because $(V, H)$ is a scroll over a smooth curve. Since $M_{n-3}$ and $V$ are birationally equivalent each other, we see that the dimension of the image of $\alpha \circ \pi$ is also one. Hence the dimension of the image of $\alpha$ is also one. Since $h^1(\mathcal{O}_V) > 0$ implies $h^1(\mathcal{O}_X) > 0$, we can take the Albanese map $\beta : X \to \mathrm{Alb}(X)$ of $X$.

Claim 3.2. $\dim \beta(X) = 1$.

Proof. First we consider a map $b : X_{n-3} \hookrightarrow X \to \mathrm{Alb}(X)$. By the universality of the Albanese map, there exists a morphism $c : \mathrm{Alb}(X_{n-3}) \to$ $\to \mathrm{Alb}(X)$ such that $c \circ \alpha = b$. On the other hand, since $\dim \alpha(X_{n-3}) = 1$, we have $\dim b(X_{n-3}) = \dim (c \circ \alpha)(X_{n-3}) \leq \dim \alpha(X_{n-3}) = 1$. But by [2, Propositions 5.1.1 and 5.1.2] we have $\dim b(X_{n-3}) \geq 1$ because $\dim \beta(X) \geq 1$. Hence $\dim b(X_{n-3}) = 1$. Furthermore by using [2, Propositions 5.1.1 and 5.1.2], we also see $\dim \beta(X) = 1$. □

Since $\dim \beta(X) = 1$, we find that $\beta(X)$ is smooth and $\beta : X \to \beta(X)$ is a fiber space over a smooth curve $\beta(X)$. Let $C = \beta(X)$. Since $h^1(\mathcal{O}_X) = g(C)$, we get $g(X, L) = g(C)$. By [10, Theorem 1.4.2] we see that $(X, L)$ is a scroll over $C$. So we get the assertion. □

Remark 3.3. (i) Theorem 3.3 shows that [12, Conjecture 2.15] is true for the case of $\dim \mathrm{Bs}|L| = 1$.

(ii) If $\dim \mathrm{Bs}|L| \leq 0$, then we see that $h^0(L) \geq n$. Hence by [17, (3.6) Theorem] and [11, Theorem 3.2] we infer that [12, Conjecture 2.15] is true for the case of $\dim \mathrm{Bs}|L| \leq 0$.

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