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Some Results on the Partitions of Groups

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ABSTRACT - We investigate some properties of partitions of groups. Some information on the structure of nilpotent groups as well the embedding of groups with nontrivial partitions into a larger one are discussed. Moreover, we consider the structure of groups admitting a nontrivial partition with 1, 2 or 3 subgroups (considering the whole group) with nontrivial partition and the relation between the number of subgroups with nontrivial partition and the number of primes dividing the order of the group. Finally we analyze the relation between the number of components of a nontrivial partition of a group and its order.

Introduction.

Let G be a nontrivial group. A collection Π of nontrivial subgroups of G is said to be a *partition* of G if every nontrivial element of G belongs to a unique subgroup in Π. If $|\Pi| = 1$, then Π is said to be the trivial partition. The subgroups in Π are the component of the partition.

The study of groups partitions was first considered by Miller in 1906 and have been continued by many author's such as Young, Hughes, Thompson, Khukhro, Isaacs, Herzer, Schulz, Pannone, Kontorovich, Wall etc. Baer [2,3], Kegel [10] and Suzuki [16] in 1960-1961 obtained the classification of those finite groups, other than p-groups, admitting a nontrivial partition. We refer the reader to [21] for a history of partitions.

THEOREM (The Classification Theorem). *Let G be a finite group admitting a nontrivial partition. Then G is isomorphic with exactly one of the following groups*

(1) S_4;
(2) *a p-group with $H_p(G) \neq G$;*

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(3) *a group of Hughes-Thompson type;*
(4) *a Frobenius groups;*
(5) $PSL(2, p^n)$ *with* $p^n \geq 4$*;*
(6) $PGL(2, p^n)$ *with* $p^n \geq 5$ *and* p *odd;*
(7) $Sz(2^{2n+1})$,

where p *is a prime and* n *is a natural number.*

This paper is organized in four different sections investigating several problems concerning partitions of groups. In section 1 we obtain some results on the structure of nilpotent groups with partition, in particular we prove that a p-group of nilpotent class less than p admitting a nontrivial partition has exponent p. In section 2, we use a group admitting a nontrivial partition with a component containing its derived subgroup to generate an infinite sequence of groups with the same property containing the first group as subgroup. This result can be applied to show that all finite groups admitting a nontrivial partition can be embedded in larger groups with nontrivial partitions. In section 3, we classify all finite groups with a nontrivial partition, which admit 1, 2 or 3 subgroups (considering the whole group) with a nontrivial partition. Also we give an upper bound for the number of subgroups admitting a nontrivial partition of a group with a nontrivial partition in terms of the order of group. Finally, in section 4, we use the classification theorem of finite groups admitting a nontrivial partition to show that if Π is a partition of a finite group G, then the greatest common divisor between the order of G and the number $|\Pi| - 1$ is greater than one. Finally we show that a finite group G with nontrivial partition has a partition Π such that $|\Pi| - 1$ divides $|G|$.

1. Nilpotent groups with a nontrivial partition.

We start with some elementary lemmas, the first of which is well known (see for instance [2]).

Lemma 1.1. *Let* G *be a group and let* Π *be a partition of* G*. If* $x, y \in G \setminus \{1\}$ *with* $xy = yx$ *and either*

(1) $|x| \neq |y|$, *or*
(2) $|x| = |y|$ *is a composite number,*

then x, y *belong to the same component of* G*.*

In order to prove Theorem 1.3 we need the following lemma of which we give a proof (see also [11])

LEMMA 1.2. *Let G be a group which admits a nontrivial partition. Then $Z(G)$ is isomorphic with one of the following groups:*

(1) *trivial group;*
(2) *an elementary abelian p-group; or*
(3) *a torsion-free group,*

where p a is prime number.

PROOF. Let Π be a nontrivial partition of G and suppose $Z(G) \neq 1$. Let $x, y \in Z(G) \setminus \{1\}$ and suppose that X and Y are components of Π with $x, y \in X \cup Y$. Let $g \in G \setminus (X \cup Y)$. Since $xg = gx$ and $yg = gy$ we have, by Lemma 1.1, that $|x| = |g| = |y|$ is a prime or $|x| = |g| = |y|$ is infinite. Therefore $Z(G)$ is either an elementary abelian p-group or a torsion-free group. □

THEOREM 1.3. *If G is a nilpotent group admitting a nontrivial partition, then either G is a p-group for some prime p, or G is a torsion-free group.*

PROOF. Since G is nilpotent $Z(G) \neq 1$ and so, by Lemma 1.2, either there exists a prime p such that $Z(G)$ is an elementary abelian p-group or $Z(G)$ is a torsion-free abelian group. If $Z(G)$ is torsion-free, then by a well-known result of Mal'cev [15, 5.2.19] G is torsion-free too. Also if $Z(G)$ is an elementary abelian p-group, then by [15, 5.2.22] G is a p-group. □

Concerning the p-groups with small nilpotent classes we have the following structural result, which will be used in sections 3 and 4. We refer the interested reader to [13, 20] for related topics. Recall that if G is a group and p is a prime, then the Hughes subgroup $H_p(G)$ is the subgroup of G generated by all the elements of order different from p. As in [12, p. 183] it can be proved that a p-group of nilpotent class n has a nontrivial partition if and only if $H_p(G) \neq G$.

THEOREM 1.4. *Let G be a p-group of nilpotent class less than p admitting a nontrivial partition. Then G is a p-group of exponent p.*

PROOF. Let G be a minimal counterexample with respect to the nilpotent class n which is less than p. Observe that $1 \neq Z(G) \neq G$ because G is not abelian by Lemma 1.2. Moreover we have that $H_p(G) \neq G$ by a previous observation and $Z(G) \leq H_p(G)$ by Lemma 1.1 and Lemma 1.2. Note also that $H_p(G/Z(G)) \leq H_p(G)/Z(G)$. So $H_p(G/Z(G)) \neq G/Z(G)$ and $G/Z(G)$ has a nontrivial partition. Now the class of $G/Z(G)$ is $n-1$. Then, by minimality of G, we have that $G/Z(G)$ has exponent p and $x^p \in Z(G)$, for every $x \in G$. Moreover $H = \langle x, y\rangle$ is finite for all $x, y \in G$ (see [15, 5.2.18]). Since $n < p$ we have that H is regular. So for any couple $a, b \in H$ we have $1 = [a^p, b] = [a, b]^p$ (see [6, Satz 10.6]). Also, by the regularity of H we have that $\Omega_1(H)$ is the set of all the element of order p. Then H' has exponent p. Since $x^p y^p = (xy)^p c$ (see [6, p. 322]) where $c \in \mho_1(H')$, we have that $c = 1$ and so $x^p y^p = (xy)^p$. Now since $G = \langle G \setminus H_p(G)\rangle$ the group G has exponent p, which is a contradiction. □

2. Embedding of groups with a nontrivial partition into a larger one.

It is straightforward to show, by using Lemma 1.1, that a nontrivial direct product of finite groups has a nontrivial partition if and only if there exists a prime p such that each of the groups in the product, except probably one group, is a p-group of exponent p and the subgroup generated by all elements of order $\neq p$ is proper. But the situation is not too clear for subgroups of direct products. Here we consider groups G admitting a nontrivial partition with a component containing G' to show that all direct products of G have subgroups containing G properly as an isomorphic copy with the same property as G.

Definition. Let X be a subgroup of G and $n \leq m$. Then

$$\Gamma_{m,n}(G;X) := \{(g_1, \ldots, g_m) \in G^m : (g_1 \cdots g_n, \ldots, g_{m-n+1} \cdots g_m) \in X^{m-n+1}\}$$

Lemma 2.1. *If $X \leq G$ and $2 \leq n \leq m$, then $\Gamma_{m,n}(G;X)$ is a subgroup of G^m if and only if $G' \subseteq X$.*

Proof. First assume that $\Gamma_{m,n}(G;X)$ is a subgroup of G^m. If $x, y \in G$, then by the definition

$$(x^{-1}, x, \overbrace{1, \ldots, 1}^{n-2 \text{ times}}, x^{-1}, x, \overbrace{1, \ldots, 1}^{n-2 \text{ times}}, \ldots) \in \Gamma_{m,n}(G;X)$$

and

$$(y^{-1}, y, \overbrace{1, \ldots, 1}^{n-2 \text{ times}}, y^{-1}, y, \overbrace{1, \ldots, 1}^{n-2 \text{ times}}, \ldots) \in \Gamma_{m,n}(G;X).$$

Hence

$$(x^{-1}y^{-1}, xy, \overbrace{1, \ldots, 1}^{n-2 \text{ times}}, \ldots) = (x^{-1}, x, \overbrace{1, \ldots, 1}^{n-2 \text{ times}}, \ldots)(y^{-1}, y, \overbrace{1, \ldots, 1}^{n-2 \text{ times}}, \ldots)$$

belongs to $\Gamma_{m,n}(G;X)$, whence

$$[x,y] = (x^{-1}y^{-1})(xy)1 \cdots 1 \in X.$$

Hence $G' \subseteq X$.

Now suppose that $G' \subseteq X$ and $(x_1, \ldots, x_m), (y_1, \ldots, y_m) \in \Gamma_{m,n}(G;X)$. Then $x_{i+1} \cdots x_{i+n}, y_{i+1} \cdots y_{i+n} \in X$ so that

$$x_{i+1} \cdots x_{i+n} y_{i+n}^{-1} \cdots y_{i+1}^{-1} \in X,$$

for each $i = 0, 1, \ldots, m-n$. Now since $G' \subseteq X$ we conclude that

$$(x_{i+1}y_{i+1}^{-1}) \cdots (x_{i+n}y_{i+n}^{-1}) \in X$$

that is

$$(x_1, \ldots, x_m)(y_1, \ldots, y_m)^{-1} = (x_1y_1^{-1}, \ldots, x_my_m^{-1}) \in \Gamma_{m,n}(G;X).$$

Therefore $\Gamma_{m,n}(G;X)$ is a subgroup of G^m. □

Utilizing the above result we can prove the main result of this section.

THEOREM 2.2. *Let G be a group admitting a nontrivial partition Π. If Π has a component X containing G', then also $\Gamma_{m,2}(G;X)$ admits a nontrivial partition with X^m as a component, in which $\Gamma_{m,2}(G;X)' \subseteq X^m$.*

PROOF. By the definition $X^m \subseteq \Gamma_{m,2}(G;X)$ and $(g, g^{-1}, \ldots) \in \Gamma_{m,2}(G;X)$. Let Ω be the set of all m-tuples $(X_1, \ldots, X_m) \in \Pi^m$ such that $X_1, \ldots, X_m \neq X$ and

$$\Gamma_{m,2}(G;X) \cap (X_1 \times \cdots \times X_m) \neq 1.$$

Then

$$\Gamma_{m,2}(G;X) = X^m \cup \bigcup_{(X_1,\ldots,X_m)\in\Omega} (\Gamma_{m,2}(G;X) \cap (X_1 \times \cdots \times X_m)),$$

for if $X_i = X$ for some i and $(x_1, \ldots, x_m) \in \Gamma_{m,2}(G;X) \cap (X_1 \times \cdots \times X_m)$, then

$$x_1x_2, \ldots, x_{i-1}x_i, x_ix_{i+1}, \ldots, x_{m-1}x_m \in X$$

and we have in turn

$$\begin{gathered} x_{i-1}, x_{i+1} \in X \\ x_{i-2}, x_{i+2} \in X \\ \vdots, \end{gathered}$$

hence we deduce that $\Gamma_{m,2}(G;X) \cap (X_1 \times \cdots \times X_m) \subseteq X^m$. Now assume that

$$(x_1, \ldots, x_m) \in (\Gamma_{m,2}(G;X) \cap (X_1 \times \cdots \times X_m)) \cap (\Gamma_{m,2}(G;X) \cap (Y_1 \times \cdots \times Y_m)),$$

in which $(X_1, \ldots, X_m), (Y_1, \ldots, Y_m) \in \Omega$ are distinct m-tuples. Then $X_i \neq Y_i$ for some i and $x_i \in X_i \cap Y_i = 1$. By the same reason which has been considered before it follows that $(x_1, \ldots, x_m) \in X^m$ that is $(x_1, \ldots, x_m) = (1, \ldots, 1)$. Hence

$$\Pi_m = \{X^m\} \cup \{\Gamma_{m,2}(G;X) \cap (X_1 \times \cdots \times X_m) : (X_1, \ldots, X_m) \in \Omega\}$$

is a nontrivial partition of $\Gamma_{m,2}(G;X)$. Moreover, $\Gamma_{m,2}(G,X)' \subseteq X^m$ as $G' \subseteq X$. The proof is complete. □

LEMMA 2.3. *If G is a finite group and X is a subgroup of G, then $|\Gamma_{m,2}(G;X)| = |X|^{m-1}|G|$ for each natural number m. In particular, for every finite group G admitting a nontrivial partition with a component X containing G' and each natural number m, there exists a group of order $|G||X|^{m-1}$, which admits a nontrivial partition.*

PROOF. Let $(g_1, \ldots, g_m) \in \Gamma_{m,2}(G;X)$. Then there exists $x_i \in X$ such that $g_i g_{i+1} = x_i$, for each $i = 1, \ldots, m-1$. Hence every element of $\Gamma_{m,2}(G;X)$ is of the form

$$(g_1, g_1^{-1}x_1, g_2^{-1}x_2, \ldots, g_{m-1}^{-1}x_{m-1}),$$

where $g_1 \in G$, $x_i \in X$ and $g_{i+1} = g_i^{-1}x_i$, for each $i = 1, \ldots, m-1$. On the other hand, we observe that different choices of m-tuples $(g_1, x_1, \ldots, x_{m-1}) \in G \times X^{m-1}$ produces different elements of $\Gamma_{m,2}(G;X)$. Therefore $\Gamma_{m,2}(G;X)$ has exactly $|G|\,|X|^{m-1}$ elements, as required. □

REMARK. An easy observation shows that if we replace $\Gamma_{m,2}(G;X)$ in Theorem 2.2 by

$$\Gamma_1(G;X) = \{(g_1, g_2, \ldots) : g_i \in G, g_i g_{i+1} \in X, \forall i \in \mathbb{N}\}$$

or

$$\Gamma_2(G;X) = \{(\ldots, g_{-1}, g_0, g_1, \ldots) : g_i \in G, g_i g_{i+1} \in X, \forall i \in \mathbb{Z}\},$$

then Theorem 2.2 still remains true and we can obtain infinite groups with a nontrivial partition.

PROPOSITION 2.4. *Let $m, n \geq 2$, G be a finite group admitting a nontrivial partition with a component X containing G' and let X^n be the corresponding component of $\Gamma_{n,2}(G;X)$ containing $\Gamma_{n,2}(G;X)'$. Then*

$$\Gamma_{mn,2}(G;X) \cong \Gamma_{m,2}(\Gamma_{n,2}(G;X), X^n).$$

PROOF. If $((g_{1,1},\dots,g_{1,n}),\dots,(g_{m,1},\dots,g_{m,n})) \in \Gamma_{m,2}(\Gamma_{n,2}(G;X),X^n)$, then we know from the definition that $g_{i,j}g_{i+1,j}$ and $g_{i,j}g_{i,j+1} \in X$, when $i+1$ in the former and $j+1$ in the latter does not exceed m and n, respectively. Let T be an $m \times n$ grid and let ϕ be a space-filling curve of T, which is a bijective map from $A = \{1,\dots,mn\}$ to T with the property that the image of every two consecutive members of A are adjacent blanks of T. It is evident that ϕ induces a map from $\Gamma_{m,2}(\Gamma_{n,2}(G;X),X^m)$ to $\Gamma_{mn,2}(G;X)$, which sends an element $((g_{1,1},\dots,g_{1,n}),\dots,(g_{m,1},\dots,g_{m,n}))$ to $(g_{\phi_1},\dots,g_{\phi_{mn}})$. Clearly, this map is an injective homomorphism. What remains is to show that this is indeed an epimorphism. For this let $(g'_1,\dots,g'_{mn}) \in \Gamma_{mn,2}(G;X)$ and put g'_i for $\phi(i)$-blank of T. Now if g'_i, g'_j are placed in two adjacent blanks of T, then the path defined by ϕ connecting g'_i and g'_j has odd length. From this and the fact that the product of each two consecutive elements on the path belongs to X, we conclude that $g'_ig'_j$ belongs to X. Hence $((g_{1,1},\dots,g_{1,n}),\dots,(g_{m,1},\dots,g_{m,n}))$, where $g_{i,j}$ is the element located at (i,j)-blank, belongs to $\Gamma_{m,2}(\Gamma_{n,2}(G;X),X^m)$ and will be sent to $(g'_1,\dots,g'_{mn})$ by the map. □

REMARK. According to the above statements, if G is a group admitting a nontrivial partition with a component X containing G', then the groups

$$\Gamma_{2,2}(G;X), \Gamma_{3,2}(G;X), \dots$$

are the only groups which can be obtained in this way up to isomorphism.

Now we attempt to obtain the structure of all finite groups G admitting a nontrivial partition with a component containing G'. Recall that if G is a finite group, then G is a group of *Hughes-Thompson type*, if G is not a p-group and that $G \neq H_p(G)$ for some prime divisor p of $|G|$.

PROPOSITION 2.5. *A finite group G has a nontrivial partition with a component containing G' if and only if G is isomorphic with one of the following groups:*

(1) *a p-group with $H_p(G) \neq G$;*
(2) *a group of Hughes-Thompson type; or*
(3) *a Frobenius group with cyclic complement.*

PROOF. It is easy to see that all of the groups in (1), (2) and (3) have the desired property. Thus we assume that G is a finite group, Π is a nontrivial partition of G and X is a component of Π such that $G' \subseteq X$. Let $Y \neq X$ be a component of Π and let $x, y \in Y \setminus \{1\}$ such that x, y are conjugate in G. Then $y = x^g$ for some $g \in G$ and so $x^{-1}y = [x, g] \in X \cap Y = 1$ that is $x = y$. Now if $g \in N_G(Y)$ and $x \in Y$, then $x^g \in Y$ so that $x^g = x$ and consequently $g \in C_G(Y)$. Thus $N_G(Y) = C_G(Y)$. If $Y = N_G(Y)$ for some $Y \in \Pi \setminus \{X\}$, then $Y = C_G(Y)$ is abelian and so $Y \cap Y^g = 1$, for all $g \in G \setminus Y$, since Y has no conjugate elements. Hence G is a Frobenius group with complement Y. By [15, 10.5.6], the Sylow p-subgroups of Y are cyclic, which implies that Y is cyclic. Now suppose that G is not a Frobenius group. Then for each component $Y \neq X$ of Π, we have $C_G(Y) = N_G(Y) \supset Y$ and so by Lemma 1.1, it follows that every component of Π other than X is an elementary abelian p-group for some prime p. Let $Y, Z \neq X$ be distinct components of Π and let $y \in Y$ and $z \in Z$ be of orders p and q, respectively. Then $|yG'| = p$ and $|zG'| = q$ and if $p \neq q$, then $|yzG'| = pq$ so that $pq \| |yz|$. Hence $yz \in G \setminus X$ is of composite order, which is impossible. Therefore there exists a prime p such that each element of $G \setminus X$ has order p so that $H_p(G) \subseteq X \neq G$. Consequently G is either a p-group with a nontrivial partition or a group of Hughes-Thompson type, the desired conclusion. □

EXAMPLE. Let p, q be primes such that $q | p-1$. Then there exists a non-cyclic group G of order pq such that its nontrivial subgroups form a nontrivial partition for G with a component $X = H_q(G)$ of order p containing G'. Let $M = G \times \mathbb{Z}_q^m$. Then $H_q(M) \neq M$ and so M has a nontrivial partition. Moreover $|H_q(M)| = pq^m$ and $H_q(M) \geq M'$. Hence, by Lemma 2.3, we can construct a group of order $p^{n+1}q^{mn+m+1}$, which admits a nontrivial partition for each $n \geq 0$.

Utilizing the above results we obtain the following embedding theorem.

THEOREM 2.6. *Let G be a finite group admitting a nontrivial partition. Then G can be embedded in a group of larger size with a nontrivial partition.*

PROOF. By a theorem of Dickson [6, p. 214] and a theorem of Suzuki [17], each of the groups $PSL(2,p^n)$, $PGL(2,p^n)$ and $Sz(2^{2n+1})$, where p is a prime and n is a natural number can be embedded into a finite group of larger size with a nontrivial partition. Also, by Theorem 2.2 and Proposition 2.5, every p-group with a nontrivial partition and every group of Hughes-Thompson type can be embedded in a finite group with larger size admitting a nontrivial partition. Now if G is a Frobenius group with kernel N and complement H, then G can be embedded into a larger Frobenius group with kernel $N \times N$ and complement H, where H acts on each component of $N \times N$ as H acts on N. □

3. Number of subgroups with a nontrivial partition.

If G is an arbitrary finite group containing a subgroup with a nontrivial partition, then G has minimal subgroups with the same property. In this section, we attempt to describe the structure of these subgroups and to interrelate the number of subgroups with a nontrivial partition with the order of group G, when G admits a nontrivial partition.

We first show the existence of finite groups with arbitrary number of subgroups admitting a nontrivial partition.

LEMMA 3.1. *For each natural number n, there exists a finite group G with exactly n subgroups admitting a nontrivial partition.*

PROOF. Let q be a prime number and let n be a natural number. Using a famous theorem of Dirichlet on arithmetic progressions [1, Theorem 7.9], there exists a prime number p such that $q^n | p-1$. Now since $U(\mathbb{Z}_p)$, the group of units of $\mathbb{Z}_p$ under multiplication is cyclic, there exists a natural number r such that r is of order q^n modulo p. Let

$$G = \langle x, y : x^p = y^{q^n} = 1, x^y = x^r \rangle.$$

We can see that G is a Frobenius group with kernel $K = \langle x \rangle$ and complement $H = \langle y \rangle$ and the subgroups KH_i, in which $H_i = \langle y^{q^i} \rangle$, for $i = 0, 1, \ldots, n-1$ are distinguished as the subgroups of G, which admit a nontrivial partition, as required. □

Now we determine the structure of all finite groups admitting a nontrivial partition, which have one, two or three subgroups with a nontrivial partition, respectively. The first of which is the most important because it gives a criterion for a group to have only trivial partition. We use this criterion in section 4 to show that the complements of a Frobenius group have only trivial partition.

PROPOSITION 3.2. *Let G be a finite group admitting a nontrivial partition. Then the maximal subgroups of G have no nontrivial partitions if and only if G is a non-cyclic group of order p^2 or pq, where p, q are distinct primes.*

PROOF. Let G be a finite group admitting a nontrivial partition. Clearly, if $|G| = p^2$ or pq, where p, q are distinct primes, then all of the maximal subgroups of G have only trivial partitions. Now assume that G has no maximal subgroups with a nontrivial partition and let Π be a nontrivial partition of G. If M is a maximal subgroup of G, then $M \subseteq X$ for some component X of Π and so $M = X$, otherwise Π induces a nontrivial partition on M, which is a contradiction. On the other hand, if X is a component of Π, then $X \subseteq M$ for some maximal subgroup M of G. So in conjunction with the previous result we deduce that $X = M$. Hence Π is exactly the set of all maximal subgroups of G. Suppose that $X = N_G(X)$ for all components X. If $X \in \Pi$, then $G \neq \cup_{g\in G} X^g$ and consequently there exists a component Y of Π, which is not conjugate to X. Hence

$$\begin{aligned}|G| &\geq \left|\bigcup_{g\in G} X^g \cup \bigcup_{g\in G} Y^g\right| \\ &= [G:X](|X|-1) + [G:Y](|Y|-1) + 1 \\ &> |G|,\end{aligned}$$

which is a contradiction. Thus there is a component X of Π such that $X \subset N_G(X)$ so that $X \trianglelefteq G$ since X is a maximal subgroup of G. If $g \in G \setminus X$ is of prime order p, then $X\langle g\rangle = G$ that is $|G| = p|X|$ and consequently every component $Y \neq X$ of Π has order p. If X is a p-group of order p^m, then $m = 1$ since G is a p-group with maximal subgroups of order p. Now suppose that X is not a p-group and $|X|$ has a prime divisor $q \neq p$. If $p||X|$, then Sylow p-subgroups of G are of order at least p^2, which contradicts the maximality of components of Π different from X. Let Q be a Sylow q-subgroup of X. Then $G = N_G(Q)X$. As p does not divide the order of X, we should have $p||N_G(Q)|$ so that $N_G(Q) = G$, for $|N_G(Q)|$ is divisible by pq but no maximal subgroups of G has this property. Hence $Q \trianglelefteq G$ and so $G = QY$ for all components $Y \neq X$, as QY does not lie in a maximal subgroup. Then it follows $X = Q$, i.e., X is a q-group. If $Z \neq 1$ is a characteristic subgroup of X, then $Z \trianglelefteq G$ and similarly $G = ZY$ for each $Y \in \Pi \setminus \{X\}$. So it follows that $X = Z$. In particular, $\Phi(X)$ the Frattini subgroup of X is trivial and X is an elementary abelian q-group. Since X has no nontrivial partitions we obtain $|X| = q$ so that $|G| = pq$ and the proof is complete. □

We observe that if the maximal subgroups of a finite group G have no nontrivial partition, then by Proposition 3.2, G has no subgroup with nontrivial partition. So we can deduce the following

COROLLARY 3.3. *A finite group G with nontrivial partition has only one subgroup with nontrivial partition if and only if G is non-cyclic of order p^2 or pq, where p and q are different primes.*

If G is a group and Π is a partition of G, then Π is said to be a *maximal partition*, if each component of Π has only trivial partition. In fact, our maximal partitions are just the primitive partitions of Young [19]. But since the primitive partitions are the maximal elements in the set of all partitions ordered by $\preceq$, where $\Pi \preceq \Pi'$ if and only if every component of Π is Π'-admissible, in what follows we shall use the name maximal partition instead of primitive partition. Clearly, maximal partitions have the maximal cardinal among all partitions of a group. Note that, Young [19] and also Zorn's lemma guarantee the existence of maximal partitions for any group and the definition assures the uniqueness of maximal partition.

PROPOSITION 3.4. *Let G be a finite group admitting a nontrivial partition. Then G has exactly two subgroups with a nontrivial partition if and only if G is a Frobenius group with kernel K and complement H, in which H, K satisfy the following properties:*

(1) *K is non-cyclic of order p^2, which is the normalizer of its nontrivial subgroups and H is cyclic of order q; or*
(2) *K is cyclic of order p and H is cyclic of order q^2,*

where p, q are distinct primes.

PROOF. Let G be a group admitting a nontrivial partition and let M be the unique proper subgroup of G, which admits a nontrivial partition. Also let Π and Π' be the maximal partitions of G and M, respectively and let $\mathcal{A}$ be the set of all components of Π intersecting M nontrivially. By Proposition 3.2, M is a maximal subgroup of G and since $M \trianglelefteq^c G$ we have $|G| = r|M|$ for some prime r. If $M \subseteq X$ for some component X of Π, then $M = X$, which is impossible as X has only trivial partition. Then it follows that $\Pi \cap M$ is a nontrivial partition of M and by Proposition 3.2, it is exactly Π'. In particular, there exist primes p, q and components $X, Y_1, \ldots, Y_p \in \Pi$ such that

$$M = (M \cap X) \cup \bigcup_{i=1}^{p} (M \cap Y_i),$$

where $|M \cap X| = p$, $|M \cap Y_1| = \cdots = |M \cap Y_p| = q$ and $M \cap X \trianglelefteq M$. First suppose that $p = q = r$. Then G is a p-group of order p^3 of nilpotent class at most 2. If p is odd, then by Theorem 1.4, G is of exponent p so that every maximal subgroup of G admits a nontrivial partition, which is impossible. Also if $p = 2$, then as G cannot be of exponent 2 we should have

$$G \cong D_8 = \langle x, y : x^4 = y^2 = 1, x^y = x^{-1} \rangle.$$

But D_8 has two different subgroups

$$\langle x^2, y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \ \langle x^2, yx \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

which have nontrivial partitions. Hence G is not p-group. Now we investigating the possibilities that $p = q$ or $p \neq q$. First suppose that $p = q$ that is $|M| = p^2$. We observe that M is a completely reducible $F_p - \langle x \rangle$-module. If M is not irreducible then there is a submodule N of order p on which $\langle x \rangle$ acts nontrivially. In this case G has at least two proper subgroups which have nontrivial partition, namely M and $N\langle x \rangle$. So the action of $\langle x \rangle$ on M must be irreducible. In particular G is a Frobenius group and r divides $p^2 - 1$ but it does not divide $p - 1$.

Finally, assume that $p \neq q$. In this case $M \cap X$ is a characteristic subgroup of M and hence normal in G. If $M \cap X \subset X$, then $X \trianglelefteq G$ and since $|G|$ is the product of three primes, we have $G = XZ$ for each component $Z \neq X$ of Π. In particular, $G = XY_1$, which implies that $|Z| = |Y_1| = q$ is of prime order for all components $Z \neq X$ of Π. Clearly, X is cyclic and if $|X| = pq$ or pr, where $r \neq p, q$, then X has a characteristic subgroup $\langle x \rangle$ of order q or r, which is normal in G and hence $\langle x \rangle Y_1$ is a proper subgroup with a nontrivial partition of order q^2 or qr, which is different from M, a contradiction. Thus $|X| = p^2$ and since $|\Pi| = p^2 + 1 > p + 1 = |\Pi'|$ there exists a component Z of Π such that $(M \cap X)Z$ is a proper subgroup with a nontrivial partition different from M, which is impossible. Therefore we must have $M \cap X = X$ that is $X \subseteq M$. If $r \neq q$, then there exists an element $x \in G \setminus X$ of order r such that $X\langle x \rangle$ is a proper subgroup with a nontrivial partition different from M. Hence every component of Π other than X is a cyclic q-subgroup and moreover $|G| = pq^2$. If $|\Pi| > |\Pi'|$ and $Z \in \Pi \setminus \Pi'$, then $X\Omega_1(Z)$ is a proper subgroup with a nontrivial partition other than M, which is impossible. Thus $|\Pi| = |\Pi'|$. Now since $|Z| \leq q^2$ for each component $Z \neq X$ we get

$$pq^2 = |G| = |X| + (|Y_1| + \cdots + |Y_p|) - p \leq p + pq^2 - p = pq^2,$$

which is possible only if $|Y_1| = \cdots = |Y_p| = q^2$. Therefore G is a Frobenius group with cyclic kernel X of order p and cyclic complements Y_i of order q^2. Conversely, each of the groups in parts (1) and (2) satisfies the hypothesis of the theorem. □

PROPOSITION 3.5. *Let G be a finite group admitting a nontrivial partition. Then G has exactly three subgroups with a nontrivial partition if and only if either $G \cong D_8$ or G is a Frobenius group with kernel K and complement H, in which H, K satisfy the following properties:*

(1) *K is non-cyclic of order p^2, H is cyclic of order q^2 and K is the normalizer of its nontrivial subgroups;*
(2) *K is cyclic of order p and H is cyclic of order q^3. In this case,*

$$G = \langle x, y : x^p = y^{q^3} = 1, x^y = x^i, i^{q^2} \overset{p}{\not\equiv} 1, i^{q^3} \overset{p}{\equiv} 1 \rangle; \text{or}$$

(3) *K is cyclic of order p and H is cyclic of order qr. In this case,*

$$G = \langle x, y : x^p = y^{qr} = 1, x^y = x^i, i^q \overset{p}{\not\equiv} 1, i^r \overset{p}{\not\equiv} 1, i^{qr} \overset{p}{\equiv} 1 \rangle,$$

where p, q, r are distinct primes and $1 < i < p$.

PROOF. Let G be a finite group admitting a nontrivial partition and let M, N be the proper subgroups of G, which admit a nontrivial partition. Also let Π, Π' and Π'' be the maximal partitions of G, M and N, respectively. Since M, N are the only proper subgroups of G admitting a nontrivial partition, either M, N are normal in G or M, N are conjugate. First suppose that M, N are conjugate and that $N = M^g$ for some $g \in G$. Now since M, N have only two conjugates $[G : N_G(M)] = [G : N_G(N)] = 2$ so that $N_G(M), N_G(N) \trianglelefteq G$ and consequently

$$N_G(N) = N_G(M^g) = N_G(M)^g = N_G(M).$$

This implies that $M, N \subset N_G(M) \subset G$, which is impossible by invoking Proposition 3.2. Hence M, N are normal subgroups of G. Now we consider two possibilities:

(i) One of M, N is contained in the other. Without loss of generality we may assume that $M \subset N$ so that N is a maximal subgroup of G and $[G : N] = r$ for some prime r. Also, by Proposition 3.4, N is of order pq^2 and M is of order pq or q^2, for some distinct primes p, q. Since N is a maximal subgroup of G admitting a nontrivial partition it does not lie in a component of Π. Also we should have $|\Pi| = |\Pi \cap N|$, otherwise we can choose a component $X \in \Pi$ such that $N \cap X = 1$. Then it follows that MX is a proper subgroup with a nontrivial partition other than M, N, which is impossible. Suppose that $M \subset X$ for some component X of Π. Thus X is a normal subgroup of G and its order is a product of three primes. Moreover, $G = XY$ for each component $Y \in \Pi \setminus \{X\}$. Hence there exists a prime s such that $|Y| = s$ for each component $Y \in \Pi \setminus \{X\}$ and $|\Pi| = |X| + 1$. On the other hand, $|\Pi \cap N| = |M| + 1$ and since $N \cap X = M$, each component of $\Pi \cap N$ other than M is of order s and $|N| = |M|s$. Then it follows that $|\Pi| > |\Pi \cap N|$, which is impossible as it has been observed before. Moreover, as for N we have $M \neq X$ for each $X \in \Pi$. Now invoking Proposition 3.4, we obtain

$$\Pi \cap M = \Pi' \text{ and } \Pi \cap N = \Pi''.$$

If $r \neq p, q$, then G has an element x of order r and $M\langle x \rangle$ is a proper subgroup with a nontrivial partition different from M, N, which is impossible. Thus $r = p$ or q. In this case, we have two possibilities by Proposition 3.2. First assume that $|M| = pq$. If $X' \in \Pi'$ is the Sylow p-subgroup of M as in Proposition 3.4, then X' is a characteristic subgroup of M and consequently $X' \trianglelefteq G$. Also $X' = X'' \in \Pi''$ and for each component $Y'' \in \Pi'' \setminus \{X''\}$, the subgroups $M \cap Y''$ and Y'' are of orders q and q^2, respectively. Let X be a component of Π containing X'. If $p \| Z|$, for some component $Z \in \Pi \setminus \{X\}$, then G has a proper subgroup with a nontrivial partition of order p^2, which is impossible. Hence the components of Π other than X are q-subgroups. Also $X \trianglelefteq G$ for $X' \trianglelefteq G$, from which XY' is a proper subgroup with a nontrivial partition different from N for each $Y' \in \Pi' \setminus \{X'\}$. Thus $XY' = M$ and we deduce that $X = X'$ is of order p. Since N is a Frobenius group with complements $Y'' \neq X''$, the components of Π'' other than X'' are conjugate pairwise, which induces the same property for components of Π different from X. Consequently $|Y| = q^3$ for each $Y \in \Pi \setminus \{X\}$. If the component $Y \in \Pi \setminus \{X\}$ is not cyclic and $y \in Y_1 \neq Y \cap N$ is nontrivial, then y is of order q or q^2 and $M\langle y \rangle$ is a proper subgroup with a nontrivial partition different from M, N, which is impossible. Hence the components of Π other than X are cyclic. Therefore G is a Frobenius group with kernel X of order p and complements

$Y \in \Pi \setminus \{X\}$ of order q^3 and we have

$$G = \langle x, y : x^p = y^{q^3} = 1, x^y = x^i, i^{q^2} \overset{p}{\not\equiv} 1, i^{q^3} \overset{p}{\equiv} 1 \rangle,$$

for some $1 < i < p$. In the second case, we have $|M| = q^2$, the components of Π' are of order q and $|\Pi'| = q + 1$. From this $\Pi' \subset \Pi''$, the components in $\Pi'' \setminus \Pi'$ are of order p and $|\Pi''| = q^2 + q + 1$. If $X'' \in \Pi'' \setminus \Pi'$ and $q \| X|$ for some component X of Π containing X'', then X has an element x of order q and $M\langle x \rangle$ is a proper subgroup with a nontrivial partition of order q^3, which is impossible. Moreover, since the components of $\Pi'' \setminus \Pi'$ are pairwise conjugate, the same property is valid for the components of Π, which intersect M trivially. Hence these components are p-subgroups of the same order. If there is a component $X \in \Pi$ such that $X \cap M \neq 1$ and $q^2 \| X|$, then X has a subgroup H of order q^2 possessing $M \cap X$, which implies that HM is a proper subgroup with a nontrivial partition of order q^3, a contradiction. On the other hand, if there is a component $X \in \Pi$ such that $X \cap M \neq 1$ and $p \| X|$, then X has an element x of order p, which does not belong to N and thus $M\langle x \rangle$ is a proper subgroup with a nontrivial partition different from M, N, which is impossible. Hence $\Pi' \subset \Pi$ and the components in $\Pi \setminus \Pi'$ are cyclic of order p^2. Therefore G is a Frobenius group with non-cyclic kernel M of order q^2 and cyclic complements $Y \in \Pi \setminus \Pi'$ of order p^2. Moreover, M is the normalizer of its nontrivial subgroups and p^2 divides $q^2 - 1$ but it does not divide $q - 1$.

(ii) None of the M, N is contained in the other. In this case, both of M, N are maximal subgroups of G. Thus the components of Π are cyclic subgroups of orders, which are the products of at most two primes. Also $M \cap N \neq 1$ is a subgroup of order a prime p. Let $M \cap N = \langle x \rangle$ and X be the component of Π containing x. Then $\langle x \rangle \trianglelefteq G$ and consequently $X \trianglelefteq G$. If G is a p-group, then $|G| = p^3$ and G has nilpotent class at most 2. G is not of exponent p, otherwise every nontrivial element of G lies in a proper subgroup with a nontrivial partition so that $G = M \cup N$, which is impossible. Hence, by Theorem 1.4, $G \cong D_8$. Now suppose that G is not a p-group. Then at least one between M, N is non-abelian. Assume that M is non-abelian of order pq. If $M \subseteq Y$ for some component $Y \in \Pi$, then $M = Y$, which is impossible as Y is cyclic. The same statement is also true for N. If $\langle x \rangle \subset X$, then for each component $Y \in \Pi \setminus \{X\}$ we have $G = XY$, for X is a maximal subgroup of G. Thus every component of Π other than X has order q as M has elements of order q outside X. Now since $\langle x \rangle \trianglelefteq M, N$, we have $|\Pi'| = |\Pi''| = p + 1$ and $|\Pi| \geq 2p + 1$. But $|\Pi|$ cannot exceed $2p + 1$, otherwise $M \cap Y = N \cap Y = 1$ for some component Y of Π and $\langle x \rangle Y$ is a proper subgroup with a nontrivial partition different from M, N, which is a contradiction. Now let $|X| = pr$. Then

$$pqr = |G| = |X| + (|\Pi| - 1)(q-1) = pr + (|\Pi| - 1)(q-1)$$

and we have that $|\Pi| = pr + 1$. Thus we should have $r = 2$. If p is odd, then X has a characteristic subgroup $\langle y \rangle$ of order 2. Hence $\langle y \rangle \trianglelefteq G$ and $\langle y \rangle Y$ is a proper subgroup with a nontrivial partition for each component $Y \in \Pi \setminus \{X\}$, which is impossible because $|M| = |N| = pq \neq 2q = |\langle y \rangle Y|$. Thus $p = 2$ and so q is odd, as G is not a 2-group. In this term, G has a unique element x of order 2, which implies that $x \in Z(G)$ and this is in contradiction with Lemma 1.1. Hence $M \cap N = \langle x \rangle = X$. Clearly, the components of Π' other than X are conjugate by powers of x. Now if $Y, Z \subseteq W$ for some component $W \in \Pi$, $Y \in \Pi'$ and $Z \in \Pi''$ such that $Y, Z \neq X$, then $Y^{x^i}, Z^{x^i} \subseteq W^{x^i}$ for $i = 0, 1, \ldots, p-1$. In particular, we deduce that N is also non-abelian, because the conjugates of Z by powers of x should be different. If $|Z| = r$, then $|W| = qr$ and we have $|\Pi| = p+1$. In this case, G is a Frobenius group with cyclic kernel X of order p and cyclic complement W of order qr. We note that, $q \neq r$ otherwise $Y = Z$, and so we would have $M = N$. Hence

$$G = \langle x, y : x^p = y^{qr} = 1, x^y = x^i, i^q \overset{p}{\not\equiv} 1, i^r \overset{p}{\not\equiv} 1, i^{qr} \overset{p}{\equiv} 1 \rangle,$$

for some $1 < i < p$. Now assume that no components of Π' and Π'' other than X lie in a component of Π simultaneously. Then $|\Pi| = 2p + 1$ and we can separate the components of Π other than X into two classes $Y_1, \ldots, Y_p$ and $Z_1, \ldots, Z_p$ in such a way that $M \cap Y_i \neq 1$ and $N \cap Z_i \neq 1$ for $i = 1, \ldots, p$. Also we have $|Y_1| = \cdots = |Y_p|$ and $|Z_1| = \cdots = |Z_p|$. If $M \cap Y_i \subset Y_i$, then $|Y_i|$ is a product of two primes, and we would have $G = X \cup Y_1 \cup \cdots \cup Y_p$, a contradiction. Hence $M \cap Y_i = Y_i$ and $Y_i \subseteq M$. Similarly $Z_1, \ldots, Z_p \subseteq N$. Let $|Z_1| = \cdots = |Z_p| = r$. Then

$$\begin{aligned} pqr &= |G| \\ &= |X| + (|Y_1| - 1) + \cdots + (|Y_p| - 1) + (|Z_1| - 1) + \cdots + (|Z_p| - 1) \\ &= p + p(q-1) + p(r-1) \\ &< pqr, \end{aligned}$$

which is a contradiction. The proof is complete. □

As it is shown in Propositions 3.2, 3.4 and 3.5, the number of subgroups with a nontrivial partition depends on the number of prime divisors of the order of G. In the sequel, we are investigating this relation more precisely.

Recall that for each natural number $n = p_1^{a_1} \cdots p_m^{a_m}$, where $n > 1$ and $p_1, \ldots, p_m$ are distinct primes, the function $\Omega(n)$ determines the number of prime divisors of n considering repetitive primes, i.e.,

$$\Omega(n) = a_1 + \cdots + a_m.$$

THEOREM 3.6. *Let G be a finite group admitting a nontrivial partition and let n be the number of subgroups of G with a nontrivial partition. Then $\Omega(|G|) \leq n + 1$ with equality if and only if the subgroups of G with a nontrivial partition form a chain.*

PROOF. For an arbitrary group G let $\nu(G)$ denotes the number of subgroups of G admitting a nontrivial partition. Now let G be a finite group, which admits a nontrivial partition. If $\nu(G) = 1$, then by Proposition 3.2, $\Omega(|G|) = 2 \leq \nu(G) + 1$. Assume that the result holds for all groups with ν-value less than that of G. According to Proposition 3.2, G possesses a maximal subgroup M with a nontrivial partition. Hence $\nu(M) \leq \nu(G) - 1$ and consequently $\Omega(|M|) \leq \nu(M) + 1$. If $M \trianglelefteq G$, then $[G : M]$ is a prime so that $\Omega(|G|) = \Omega(|M|) + 1 \leq \nu(G) + 1$. Now suppose that $M \ntrianglelefteq G$, then $N_G(M) = M$. Since G, the conjugates of M and proper subgroups of M admitting a nontrivial partition are all distinct, we obtain

$$\nu(G) \geq 1 + [G : M] + \nu(M) - 1 = [G : M] + \nu(M).$$

Let $[G : M] = p_1^{a_1} \cdots p_n^{a_n}$ be the canonical decomposition of $[G : M]$ into primes. Then $[G : M] \geq 2^{a_1 + \cdots + a_n} = 2^{\Omega([G:M])}$ that is $\Omega([G : M]) \leq \log_2([G : M])$. Therefore

$$\begin{aligned} \Omega(|G|) &= \Omega(|M|) + \Omega([G : M]) \\ &\leq \nu(G) - [G : M] + 1 + \log_2([G : M]) \\ &< \nu(G) + 1. \end{aligned}$$

Also the last inequality implies that for a finite group G with a nontrivial partition $\Omega(|G|) = \nu(G) + 1$ if and only if the subgroups of G admitting a nontrivial partition form a characteristic chain

$$H_1 \trianglelefteq^c H_2 \trianglelefteq^c \cdots \trianglelefteq^c H_{\nu(G)} = G,$$

where H_i is a maximal subgroup of H_{i+1}. □

We close this section by determining the structure of those finite groups G admitting a nontrivial partition such that the subgroups of G with a nontrivial partition form a chain. To do this we first need the following lemma. Recall that a partition Π of a group G is normal if $\Pi^g = \Pi$ for each $g \in G$.

LEMMA 3.7. *Let G be a finite group admitting a nontrivial partition. Then G has exactly one maximal subgroup with a nontrivial partition if and only if G is a Frobenius group with elementary abelian kernel K and cyclic complement H of prime power order. Moreover, G has no nontrivial normal subgroups properly contained in K.*

PROOF. Let Π be a nontrivial normal partition of G and let M be the unique maximal subgroup of G, which admits a nontrivial partition. Then $M \trianglelefteq G$ and $[G : M] = q$ is a prime. If no component of Π lies in M, then $|X| = |X \cap M|q$ for each component X of Π. Moreover, $\Pi \cap M$ form a partition for M and we have

$$\begin{aligned} q|M| &= |G| \\ &= \sum_{X\in\Pi} |X| - (|\Pi| - 1) \\ &= \sum_{X\in\Pi} q|X \cap M| - (|\Pi| - 1) \\ &= q(|M| + (|\Pi| - 1)) - (|\Pi| - 1). \end{aligned}$$

Then it follows that $q = 1$, a contradiction. Let $\mathcal{A}$ be the set of all components of Π, which are contained in M. If $X \in \Pi \setminus \mathcal{A}$, then $X \nsubseteq M$ and we conclude that X is a maximal subgroup of G as M is the only maximal subgroup of G admitting a nontrivial partition. Now we consider two possibilities:

(1) $N_G(X) = X$, for some component $X \in \Pi \setminus \mathcal{A}$. In this term G is a Frobenius group with complement X and since

$$\left|\bigcup_{g\in G} (X \setminus X \cap M)^g\right| = [G : X]|X \setminus X \cap M| = \frac{|G|}{|X|} \cdot \left(|X| - \frac{|X|}{q}\right) = |G| - |M|$$

the components of $\Pi \setminus \mathcal{A}$ are pairwise conjugate. This means $K = \cup_{Y\in\mathcal{A}} Y$ is the Frobenius kernel of G, which is nilpotent by [15, 10.5.6]. If K has a nontrivial characteristic subgroup H, then $H \trianglelefteq G$ and for each $Y \in \Pi \setminus \mathcal{A}$, HY is contained in a maximal subgroup of G with a nontrivial partition different from M, a contradiction. Thus K is an elementary abelian p-group, which is the normalizer of its nontrivial subgroups. Let $x \in G \setminus M$ be of the least possible order. Then $|x| = q^t$ for some $t \geq 1$. Since $K\langle x\rangle \nsubseteq M$ we have $G = K\langle x\rangle$, otherwise G has a maximal subgroup with a nontrivial partition containing $K\langle x\rangle$ different from M, which is impossible. Therefore the components of $\Pi \setminus \mathcal{A}$, as the Frobenius complements of G are cyclic of order

q^t. Conversely, every group with the aforementioned properties has a unique maximal subgroup, which admits a nontrivial partition.

(2) $X \trianglelefteq G$ for every component $X \in \Pi \setminus \mathcal{A}$. Let $X \in \Pi \setminus \mathcal{A}$ and let $x \in G \setminus X$. Then $G = X\langle x\rangle$ as X is a maximal subgroup of G and hence there is a prime p such that $|x| = p$ for each $x \in G \setminus X$. If $|\Pi| > |\mathcal{A}| + 1$, then there exists a component $Y \in \Pi \setminus \mathcal{A}$ different from X and prime r such that $|x| = r$ for each $x \in G \setminus Y$. But since $G \setminus X \cup Y$ is nonempty we deduce that $p = r$. Therefore G is a p-group of exponent p and we can see that G has maximal subgroups admitting a nontrivial partition different from M. Therefore $|\Pi| = |\mathcal{A}| + 1$ and consequently $G = M \cup X$, where $\Pi \setminus \mathcal{A} = \{X\}$, which is impossible. □

THEOREM 3.8. *Let G be a finite group admitting a nontrivial partition. Then the subgroups of G, which admit a nontrivial partition form a chain if and only if G is a Frobenius group with kernel K and complement H satisfying the following conditions:*

(1) *K is cyclic of order p and H is cyclic of order q^n. In this case*

$$G = \langle x, y : x^p = y^{q^n} = 1, x^y = x^i, i^{q^{n-1}} \overset{p}{\not\equiv} 1, i^{q^n} \overset{p}{\equiv} 1\rangle; \text{ or}$$

(2) *K is non-cyclic of order p^2 and H is cyclic of order q^n. In this case*

$$G = \langle x, y, z : x^p = y^p = z^{q^n} = 1, xy = yx, x^z = y, y^z = x^i y^j\rangle,$$

where

$$A = \begin{bmatrix} 0 & i \\ 1 & j \end{bmatrix} \in GL(2,p),\ A^{q^{n-1}} \neq I,\ A^{q^n} = I,\ \det(A^m - kI) \neq 0,$$

in which $m = 1, 2, \ldots, q^n - 1$, $k = 1, 2, \ldots, p-1$ and p, q are distinct primes.

PROOF. By Lemma 3.7 and the fact that the subgroups of G, which admit a nontrivial partition form a chain, G is a Frobenius group with elementary abelian kernel K of order p or p^2 and cyclic complement H of order q^n for some distinct primes p, q and natural number n. For, if $|K| \geq p^3$, then it has subgroups with nontrivial partition which are not members of a chain. Also in the case where $|K| = p^2$, K is the normalizer of its subgroups of order p in G. Considering the order of K we have two possibilities:

(i) K is cyclic of order p. Let $K = \langle x\rangle$ and $H = \langle y\rangle$. Since $K \trianglelefteq G$, $x^y = x^i$ for some i and consequently $x^{y^t} = x^{i^t}$ for each $t \in \mathbb{N}$. Then it follows $x = x^{y^{q^n}} = x^{i^{q^n}}$ that is $i^{q^n} \overset{p}{\equiv} 1$. Also $x^{y^{q^{n-1}}} = x^{i^{q^{n-1}}}$. Then it follows that

$i^{q^{n-1}} \overset{p}{\not\equiv} 1$, because $xy^{q^{n-1}} \neq y^{q^{n-1}}x$. On the other hand, every group with these properties is a Frobenius group with cyclic kernel of order p and cyclic complement of order q^n satisfying the hypothesis of theorem. Therefore

$$G = \langle x, y : x^p = y^{q^n} = 1, x^y = x^i, i^{q^{n-1}} \overset{p}{\not\equiv} 1, i^{q^n} \overset{p}{\equiv} 1\rangle.$$

(ii) K is an elementary abelian p-group of order p^2. Let $H = \langle z\rangle$, $x \in K \setminus \{1\}$ and $y = x^z$, then $y \notin \langle x\rangle$ and hence $K = \langle x, y\rangle$. Let $y^z = x^i y^j$, where i, j are assumed to be in $\mathbb{Z}_p$ and let A be the matrix, which is introduced in (ii). Then

$$(x^u y^v)^{z^m} = x^{u_m} y^{v_m} \text{ and } \begin{bmatrix} u_m \\ v_m \end{bmatrix} = A^m \begin{bmatrix} u \\ v \end{bmatrix},$$

for all $u, v \in \mathbb{Z}_p$ and integers m. Now since K is the normalizer of its nontrivial subgroups, for each $u, v \in \mathbb{Z}_p$ such that $k \neq 0$, $(u, v) \neq (0, 0)$ and $1 \leq m < q^n$

$$(x^u y^v)^{z^m} \neq (x^u y^v)^k,$$

which is equivalent to

$$A^m \begin{bmatrix} u \\ v \end{bmatrix} \neq k \begin{bmatrix} u \\ v \end{bmatrix},$$

i.e., $\det(A^m - kI) \neq 0$. On the other hand, $A^{q^{n-1}} \neq I$ and $A^{q^n} = I$, for the same is true for z. Conversely, we can verify that every group with these properties is a Frobenius group with elementary abelian kernel of order p^2 and cyclic complement of order q^n satisfying the hypothesis of the theorem. Therefore

$$G = \langle x, y, z : x^p = y^p = z^{q^n} = 1, xy = yx, x^z = y, y^z = x^i y^j\rangle,$$

where $A^{q^{n-1}} \neq I$, $A^{q^n} = I$ and A^m has no eigenvalues in $\mathbb{Z}_p$ for each $m = 1, 2, \ldots, q^n - 1$. □

Remark. Note that the structure of groups in Proposition 3.4(1) and Proposition 3.5(1) can be obtained from Theorem 3.8(2) by putting $n = 1$ and 2, respectively.

The case of infinite group is more difficult. In the spirit of Proposition 3.2 we have the following result, but a precise classification of those infinite groups admitting a nontrivial partition without proper subgroups admitting a nontrivial partition remains open.

Theorem 3.9. *Let G be an infinite group, which admits a nontrivial partition. If G has no proper subgroups admitting a nontrivial partition, then either G is a simple group or $G' = H_p(G)$, $[G : G'] = p$ and $G' = G''$ for some prime p.*

Proof. Let Π be a nontrivial partition of G. If $X \in \Pi$ is not maximal, then there exist a subgroup H of G containing X properly. But then $\Pi \cap H$ is a nontrivial partition for H, which is impossible by hypothesis. Thus G has a unique nontrivial partition with maximal subgroups as its components. Suppose that G is not simple. Then G has a nontrivial normal subgroup N, which lies in a component X of Π. If $x \in G \setminus X$, then $N\langle x\rangle$ has a nontrivial partition so that $G = N\langle x\rangle$ and $N = X$. Hence there exists a prime p such that $|x| = p$ for each $x \in G \setminus X$, from which $N = H_p(G)$. As N has no characteristic subgroups $N = N'$. Moreover, since $G' \subseteq N$, we should have $N = G'$, as required. □

4. Number of components of a partition.

Schulz [16] had shown by using the classification theorem of finite groups admitting a nontrivial partition that a finite group with a linear partition is either an elementary abelian p-group or a Frobenius group. In this section, we shall invoke a similar procedure to prove the following result concerning the number of components of a nontrivial partition and the order of group itself.

Theorem 4.1. *Let G be a finite group admitting a nontrivial partition Π.*

(1) *If $G \not\cong PSL(2, 2^n)$, then* $\gcd(|\Pi| - 1, |G|) \neq 1$ *and if $G \cong PSL(2, 2^n)$, then there exists a nontrivial partition Π' such that* $\gcd(|\Pi'| - 1, |G|) = 1$;
(2) *If Π is normal, then* $\gcd(|\Pi| - 1, |G|) \neq 1$; *and*
(3) *There exists a nontrivial partition Π^* of G such that $|\Pi^*| - 1$ divides $|G|$.*

Proof. Using the classification theorem of finite groups admitting a nontrivial partition it is enough to deal with each case separately. So, let G be a finite group with a nontrivial partition. Then

(i) $G = S_4$. Let $\Pi^\dagger$ be the maximal partition of G. Then it is easy to see that the components of $\Pi^\dagger$ are cyclic subgroups of G. Let P be a Sylow 2-

subgroup of G. Then $P \cong D_8$ and P contains exactly one cyclic subgroup of order 4. If P were $\Pi^{\dagger}$-admissible, then considering an element $x \in G \setminus P$ with $|\langle x \rangle| = 4$ we would have $P \cap \langle x \rangle = 1$ and $|P\langle x \rangle| = 32$, a contradiction. Since there is no partition of G containing two different components of order 6, the length of a nontrivial partition of G is either 10 or 13, which satisfy $\gcd(10-1, |G|) \neq 1$ and $13 - 1 \| G|$, respectively.

(ii) G is a p-group. Let Π be a nontrivial partition of G and let $\delta(H)$ be the number of cyclic subgroups of order p for an arbitrary subgroup H of G. Then, by [6, Satz I.7.2], $\delta(H) \overset{p}{\equiv} 1$ and so

$$|\Pi| = \sum_{X \in \Pi} 1 \overset{p}{\equiv} \sum_{X \in \Pi} \delta(X) = \delta(G) \overset{p}{\equiv} 1$$

that is $\gcd(|\Pi| - 1, |G|) \neq 1$. Also if M is a maximal subgroup of G containing $H_p(G)$, then M together with the cyclic subgroups of order p generated by elements of $G \setminus M$ form a nontrivial partition Π^* of G and we have $p|M| = |G| = |M| + (|\Pi^*| - 1)(p-1)$. Hence $|\Pi^*| - 1 = |M|$ and so $|\Pi^*| - 1$ divides $|G|$.

(iii) G is a group of Hughes-Thompson type. Let p be a prime such that $H_p(G) \subset G$, then $[G : H_p(G)] = p$, for G is not a p-group. Also, by [9], $H_p(G)$ is a nilpotent group. If $p \nmid |H_p(G)|$, then G is a Frobenius group and we do this in part (iv). Hence we can assume that $H_p(G)$ is not a group of prime power order, whence by Theorem 1.3, $H_p(G)$ has only the trivial partition. Thus $H_p(G)$ together with the cyclic subgroups of order p generated by elements of $G \setminus H_p(G)$ form the only nontrivial partition Π of G and we have $p|H_p(G)| = |G| = |H_p(G)| + (|\Pi| - 1)(p-1)$. Then it follows that $|\Pi| - 1 = |H_p(G)|$ and so $|\Pi| - 1$ divides $|G|$.

(iv) Let G be a Frobenius group and suppose that K and H are respectively the Frobenius kernel and a complement of G. By [15, 10.5.5] and Corollary 3.3, H has no nontrivial partitions. In addition, by [15, 10.5.6], K is nilpotent and if K has a nontrivial partition, then by Theorem 1.3, K is a p-group. Clearly, K together with the conjugates of H form a nontrivial partition Π^* for G and we have $|\Pi^*| = |K| + 1$ so that $|\Pi^*| - 1$ divides $|G|$. Also if K has no nontrivial partitions, then Π^* is the only nontrivial partition of G. Now assume that K is a p-group with a nontrivial partition. Thus the maximal partition of K together with the conjugates of H make the maximal partition $\Pi^{\dagger}$ of G. If H is a proper subgroup of a Π^*-admissible subgroup M of G, then M is also a Frobenius group with kernel $K_M \subset K$. Thereby every nontrivial partition Π of G has the form

$$\Pi = \{K_1, \ldots, K_m, M_1, \ldots, M_n, H^{g_1}, \ldots, H^{g_k}\},$$

where K_i are subgroups of K and M_i are Frobenius subgroups of G with kernel K_{M_i} as subgroups of K and complements H_{M_i} as conjugates of H. As M_i contains exactly $|K_{M_i}|$ conjugates of H and $K_1, \dots, K_m, K_{M_1}, \dots, K_{M_n}$ form a partition for K, we get

$$\begin{aligned} |\Pi| &= m + n + |K| - |K_{M_1}| - \cdots - |K_{M_n}| \\ &\overset{p}{\equiv} m + n \\ &\overset{p}{\equiv} \delta(K_1) + \cdots + \delta(K_m) + \delta(K_{M_1}) + \cdots + \delta(K_{M_n}) \\ &\overset{p}{\equiv} \delta(K) \\ &\overset{p}{\equiv} 1. \end{aligned}$$

Therefore $\gcd(|\Pi| - 1, |G|) \neq 1$.

(v) $G = PSL(2, p^n)$ with $p^n \geq 4$. By [6, Satz II.8.5], there are subgroups $\mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ of G of orders p^n, $(p^n - 1)/d$ and $(p^n + 1)/d$, respectively, where $d = \gcd(p^n - 1, 2)$ such that the conjugates of $\mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ makes a nontrivial partition for G. Moreover, $\mathcal{H}$ is an elementary abelian p-group, $\mathcal{K}$ and $\mathcal{L}$ are cyclic groups, and $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ have $p^n + 1$, $p^n(p^n + 1)/d$ and $p^n(p^n - 1)/d$ conjugates, respectively. Let $\Pi^{\dagger}$ be the maximal partition of G and let H be a $\Pi^{\dagger}$-admissible proper subgroup of G, which is not a component of $\Pi^{\dagger}$. We shall use [6, Hauptsatz II.8.27] of Dickson to determine the structure of H. If $H \cong A_4$, then as $\mathcal{K}$ and $\mathcal{L}$ are cyclic Hall subgroups of G and A_4 has no elements of order 4, we have that 4 divides $|\mathcal{H}| = p^n$. Then it follows that $p = 2$ and $d = 1$. Since H is $\Pi^{\dagger}$-admissible we have $p^n \pm 1 = 3$. Then it follows that $n = 2$ and $G = PSL(2, 4) \cong A_5$. If $H \cong A_5$, then similarly we can show that $p = 2$ and $d = 1$. Since A_5 has no elements of order 15 and H is $\Pi^{\dagger}$-admissible, we should have $p^n - 1 = 3$ and $p^n + 1 = 5$. Then we have $G \cong A_5 \cong H$, a contradiction. If $H \cong S_4$, then as the Sylow 2-subgroups of H are not cyclic and $\mathcal{K}$, $\mathcal{L}$ are cyclic Hall subgroups we should have $8|p^n$. But then the Sylow 2-subgroups of H should be elementary abelian p-groups, which is impossible. If H is a dihedral group of order $2m$ with $m|(p^n \pm 1)/d$, then $m = (p^n \pm 1)/d$ as $m \neq 1$. Also if p is odd, then $(p^n \pm 1)/d = 2$. Therefore we have $p^n = 5$ and $G \cong PSL(2, 5) \cong PSL(2, 4) \cong A_5$. If H is the semidirect product of an elementary abelian p-group of order p^m with a cyclic subgroup of order k such that k divides both $p^m - 1$ and $p^n - 1$, then $k|(p^n - 1)/d$ and hence $k = (p^n - 1)/d$. On the other hand, $k|p^{\gcd(m,n)} - 1$ so that $p^n - 1|d(p^{\gcd(m,n)} - 1)$. Since $d \leq 2$ we have $\gcd(m, n) = n$, i.e., $m = n$. If $H \cong PSL(2, p^m)$, in which $m|n$, then $|PSL(2, p^m)| = \dfrac{p^m(p^{2m} - 1)}{d}$ and $p^m - 1|p^n - 1$. If $p^m - 1 \neq 1$, then we should have $p^m - 1 = p^n - 1$. Then we have $m = n$, a contradiction. Thus $p^m = 2$ and so $H \cong PSL(2,2) \cong S_3$. Therefore $p^n \pm 1 = 3$ and so $n = 2$ and $G = PSL(2,4)$. Finally, if $H \cong PGL(2,p^m)$ then we must have that $2m$ divides n. On the other hand, H contains cyclic subgroups of order $p^m - 1$ and $p^m + 1$. So, if H were $\Pi^{\dagger}$-admissible then we would have $\frac{p^n - 1}{\gcd(2, p-1)} \leq p^m + 1$. This is possible if and only if $n = 2m$, $p = 3$ or $p = 2$ and $m = 1$. In the case $p = 3$ we have that $G = PSL(2,9)$ and $H \cong PGL(2,3) \cong S_4$. In such a case H is not $\Pi^{\dagger}$-admissible, because all the involutions of H cannot be covered by the 3 cyclic subgroups of H of order 4. Therefore we must have $p = 2$ and so $G = PSL(2,4)$ and $H \cong PGL(2,2) \cong PSL(2,2) \cong S_3$ and this case has been considered before.

Now assume that p is odd and $G \ncong PSL(2,4)$. Let Π be a nontrivial partition of G and let X be a component of Π, which is neither a conjugate of $\mathcal{K}$ and $\mathcal{L}$ nor a Sylow p-subgroup. Then X is a semidirect product of a p-subgroup $\mathcal{H}_X$ of order p^n and a cyclic subgroup of order $(p^n - 1)/2$, which is a conjugate of $\mathcal{K}$. Hence X is a Frobenius group with kernel $\mathcal{H}_X$. Suppose that $X_1, \ldots, X_m$ are such components and $\mathcal{H}_{X_i}$ and $\mathcal{K}_{X_i,1}, \ldots, \mathcal{K}_{X_i,p^n}$ are their Frobenius kernel and complements, respectively. Let k be the number of p-components of Π. Then

$$\begin{aligned}
|\Pi| &= k + m + ([G : N_G(\mathcal{K})] - mp^n) + [G : N_G(\mathcal{L})] \\
&= k + m + \left(\frac{p^n(p^n+1)}{2} - mp^n\right) + \frac{p^n(p^n-1)}{2} \\
&\overset{p}{\equiv} k + m.
\end{aligned}$$

But $k + m$ is the number of p-components of Π', where Π' is obtained from Π by omitting X_i from Π and putting the partition of X_i instead. As Sylow p-subgroups of G are pairwise disjoint we can classify p-components of Π' in such a way that the union over components of each class is a Sylow p-subgroup of G. Now if $\mathcal{A}$ is the set of all p-components of Π', then

$$\begin{aligned}
|\Pi| &\overset{p}{\equiv} \sum_{X \in \mathcal{A}} \delta(X) \\
&\overset{p}{\equiv} \sum_{g \in G} \delta(\mathcal{H}^g) \\
&\overset{p}{\equiv} [G : N_G(\mathcal{H})]\delta(\mathcal{H}) \\
&\overset{p}{\equiv} p^n + 1 \\
&\overset{p}{\equiv} 1.
\end{aligned}$$

Therefore $\gcd(|\Pi|-1,|G|)\neq 1$. Now let $G=PSL(2,2^n)$, where $n\geq 2$. Then G has a dihedral subgroup M of order $2(2^n-1)$, which is a $\Pi^{\dagger}$-admissible subgroup, where $\Pi^{\dagger}$ is the maximal partition of G. Moreover, since $[G:N_G(\mathcal{H})]=2^n+1$, there exist Sylow 2-subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ of G such that $\mathcal{H}_1\cap M=\mathcal{H}_2\cap M=1$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are elementary abelian 2-groups, if $x_1,y_1\in\mathcal{H}_1$ and $x_2,y_2\in\mathcal{H}_2$ are distinct elements such that $\langle x_1\rangle\cap\langle y_1\rangle=\langle x_2\rangle\cap\langle y_2\rangle=1$, then $\langle x_1,y_1\rangle=\langle x_1\rangle\cup\langle y_1\rangle\cup\langle x_1y_1\rangle$ and $\langle x_2,y_2\rangle=\langle x_2\rangle\cup\langle y_2\rangle\cup\langle x_2y_2\rangle$. Thus we can construct a partition Π' from $\Pi^{\dagger}$ by omitting the components of M, $\langle x_1,y_1\rangle$ and $\langle x_2,y_2\rangle$ and replacing M, $\langle x_1,y_1\rangle$ and $\langle x_2,y_2\rangle$ instead. Therefore

$$\begin{aligned}|\Pi'|&=|\Pi|-(2^n-1)-2-2\\&=[G:N_G(\mathcal{H})](2^n-1)+[G:N_G(\mathcal{K})]+[G:N_G(\mathcal{L})]-2^n-3\\&=2^{2n}-1+\frac{2^n(2^n+1)}{2}+\frac{2^n(2^n-1)}{2}-2^n-3\\&=2^{2n+1}-2^n-4\end{aligned}$$

and we can see that $\gcd(|\Pi'|-1,|G|)=1$. Now let Π be a normal partition of G. If H is both a dihedral group of order $2(2^n\pm 1)$ and a component of Π, then the conjugates of H are disjoint as the components of Π. Then it follows that G has at least $2^n(2^{2n}-1)/2$ involutions while G has only $2^{2n}-1$ involutions and this is possible only if $n=1$. Also if H is both a semidirect product of a Sylow 2-subgroup with a cyclic subgroup of order 2^n-1 and a component of Π, then $\mathcal{K}$ has at least $[G:N_G(\mathcal{H})]|\mathcal{H}|=2^n(2^n+1)$ different conjugates while the number of conjugates of $\mathcal{K}$ is $[G:N_G(\mathcal{K})]=2^n(2^n+1)/2$, which is impossible. Therefore every component of Π is either a p-subgroup or a conjugate of $\mathcal{K}$ or $\mathcal{L}$. Now if $\mathcal{A}$ is the set of all p-components of Π, then

$$\begin{aligned}|\Pi|&=|\mathcal{A}|+[G:N_G(\mathcal{K})]+[G:N_G(\mathcal{L})]\\&\overset{2}{\equiv}\sum_{X\in\mathcal{A}}\delta(X)+\frac{2^n(2^n+1)}{2}+\frac{2^n(2^n-1)}{2}\\&\overset{2}{\equiv}\sum_{g\in G}\delta(\mathcal{H}^g)\\&\overset{2}{\equiv}(2^n+1)\delta(\mathcal{H})\\&\overset{2}{\equiv}1.\end{aligned}$$

So it follows that $\gcd(|\Pi|-1,|G|)\neq 1$. Finally, assume that $G=PSL(2,p^n)$

and Π^* is the partition of G consisting of all conjugates of $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$. Then

$$
\begin{aligned}
|\Pi^*| &= [G : N_G(\mathcal{H})] + [G : N_G(\mathcal{K})] + [G : N_G(\mathcal{L})] \\
&= p^n + 1 + \frac{p^n(p^n+1)}{2} + \frac{p^n(p^n-1)}{2} \\
&= p^{2n} + p^n + 1.
\end{aligned}
$$

Hence $|\Pi^*| - 1$ divides $|G|$.

(vi) $G = PGL(2, p^n)$ with p odd and $p^n \geq 5$. Let Π be a partition of G and let M be the subgroup of G, which is isomorphic to $PSL(2, p^n)$. If X is a component of Π, then $[X : X \cap M] \leq 2$. Now if $X \cap M = 1$, then $|X| = 2$. But X contains either a p-subgroup of G or a conjugate of $\mathcal{K}$ or $\mathcal{L}$. By this we conclude that $|\mathcal{K}| | 2$ or $|\mathcal{L}| | 2$ and consequently $p^n = 2$ or 3, a contradiction. Therefore $X \cap M \neq 1$ and so $|\Pi| = |\Pi \cap M|$. Now utilizing part (v), $\gcd(|\Pi \cap M| - 1, |PSL(2, p^n)|) \neq 1$, which implies that $\gcd(|\Pi| - 1, |G|) \neq 1$. On the other hand, by [6, Satz II.8.5], G has a partition Π^* consisting of all conjugates of $\mathcal{H}'$, $\mathcal{K}'$ and $\mathcal{L}'$ of orders p^n, $p^n - 1$ and $p^n + 1$, respectively, in such a way that

$$
\begin{aligned}
|\Pi^*| &= [G : N_G(\mathcal{H}')] + [G : N_G(\mathcal{K}')] + [G : N_G(\mathcal{L}')] \\
&= p^n + 1 + \frac{p^n(p^n+1)}{2} + \frac{p^n(p^n-1)}{2} \\
&= p^{2n} + p^n + 1.
\end{aligned}
$$

Hence $|\Pi^*| - 1$ divides $|G|$.

(vii) $G = Sz(2^{2n+1})$. Let $\Pi^\dagger$ be the maximal partition of G and let H be a $\Pi^\dagger$-admissible subgroup of G, which is not a component of $\Pi^\dagger$. By [7, Theorem 3.10] and [8], $\Pi^\dagger$ consists of all conjugates of subgroups $\mathcal{H}$, $\mathcal{K}$, $\mathcal{U}_1$ and $\mathcal{U}_2$ of orders q^2, $q-1$, $q+2r+1$ and $q-2r+1$ with q^2+1, $q^2(q^2+1)/2$, $4^{-1}|G|/|\mathcal{V}_1|$ and $4^{-1}|G|/|\mathcal{V}_2|$ conjugates, respectively, where $q = 2^{2n+1}$, $r = 2^n$, $\mathcal{V}_i = N_G(\mathcal{U}_i)$ and $[\mathcal{V}_i : \mathcal{U}_i] = 4$, for $i = 1, 2$. We shall use [17, Theorem 9] of Suzuki to determine the structure of H. If $H \cong Sz(s)$ such that q is a power of s, then the order of Sylow 2-subgroups of H is less than that of $\mathcal{H}$, which is impossible. Also if H is either a conjugate to a subgroup of $\mathcal{V}_i$ $(i = 1, 2)$ or a conjugate to a subgroup of $N_G(\mathcal{K})$, then the Sylow 2-subgroups of H are of order 2 or 4 while the order of Sylow 2-subgroups of G are at least 64. Moreover, if H is a conjugate to a subgroup of $\mathcal{H}\mathcal{K}$, then H is a conjugate of $\mathcal{H}\mathcal{K}$. Hence a $\Pi^\dagger$-admissible subgroup of G is either a component of $\Pi^\dagger$ or a conjugate of $\mathcal{H}\mathcal{K}$. Now let Π be a nontrivial partition of G and let m be the number of components conjugate to $\mathcal{H}\mathcal{K}$.

Then

$$\begin{aligned}|\Pi| &= ([G : N_G(\mathcal{H})] - m) + ([G : N_G(\mathcal{K})] - mq) \\ &\quad + [G : N_G(\mathcal{U}_1)] + [G : N_G(\mathcal{U}_1)] + m \\ &= (q^2+1) + \frac{q^2(q^2+1)}{2} - mq + \frac{1}{4}\cdot\frac{|G|}{|\mathcal{U}_1|} + \frac{1}{4}\cdot\frac{|G|}{|\mathcal{U}_2|} \\ &\overset{4}{\equiv} 1 + \frac{1}{4}\cdot\frac{|\mathcal{U}_1| + |\mathcal{U}_2|}{|\mathcal{U}_1||\mathcal{U}_2|} \\ &\overset{4}{\equiv} 1 + \frac{1}{4}q^2(q^2+1)(q-1)\frac{2(q+1)}{q^2+1} \\ &\overset{4}{\equiv} 1\end{aligned}$$

and consequently $\gcd(|\Pi| - 1, |G|) \neq 1$. Moreover, if Π^* is the partition containing the conjugates of $\mathcal{H}$, $\mathcal{K}$, $\mathcal{U}_1$ and $\mathcal{U}_2$, then

$$\begin{aligned}|\Pi^*| &= [G : N_G(\mathcal{H})] + [G : N_G(\mathcal{K})] + [G : N_G(\mathcal{U}_1)] + [G : N_G(\mathcal{U}_2)] \\ &= q^4 + q^2 + 1.\end{aligned}$$

Then it follows that $|\Pi^*| - 1$ divides $|G|$.

Now the result follows from parts (i) - (vii) and the proof is complete. □

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