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A Note on Minimal Galois Embeddings of Abelian Surfaces

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ABSTRACT - We show that the least number N such that an abelian surface has a Galois embedding in $\mathbb{P}^N$ is seven and then we give examples of such surfaces.

1. Introduction.

This is a continuation of our previous paper [7]. The least number N such that an abelian surface can be embedded in $\mathbb{P}^N$ is four, and in that case the abelian surface has a special structure, see for example [3]. Similarly it might have some interest to study the least number N such that an abelian surface A can be Galois-embedded in $\mathbb{P}^N$. Moreover, in that case we want to know the structure of A. In this short note we give the answer to the problem. Before stating it, we recall from [7] some of the definitions and properties of Galois embeddings of algebraic varieties.

Let k be the ground field of our discussions, which is assumed to be algebraic closed. Let V be a nonsingular projective algebraic variety of dimension n and D a very ample divisor. We denote this by a pair (V, D). Let $f = f_D : V \hookrightarrow \mathbb{P}^N$ be the embedding of V associated with the complete linear system $|D|$, where $N + 1 = \dim H^0(V, \mathcal{O}(D))$. Suppose that W is a linear subvariety of $\mathbb{P}^N$ such that $\dim W = N - n - 1$ and $W \cap f(V) = \emptyset$. Then consider the projection W, $\pi_W : \mathbb{P}^N \dashrightarrow W_0$ with center W, where W_0 is an n-dimensional linear subvariety not meeting W. The composition $\pi = \pi_W \cdot f$ is a surjective morphism from V to $W_0 \cong \mathbb{P}^n$. Let $K = k(V)$ and $K_0 = k(W_0)$ be the function fields of V and W_0 respectively. The covering map π induces a finite extension of fields $\pi^* : K_0 \hookrightarrow K$ of degree $d = \deg f(V) = D^n$, which is the self-intersection number of D. It is easy to see that the structure of this extension does not depend on the choice of W_0

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but only on W, hence we denote by K_W the Galois closure of this extension and by $G_W = Gal(K_W/K_0)$ the Galois group of K_W/K_0. Note that G_W is isomorphic to the monodromy group of the covering $\pi : V \longrightarrow W_0$.

DEFINITION 1. In the above situation we call G_W the Galois group at W. If the extension K/K_0 is Galois, we call f and W a Galois embedding and a Galois subspace for the embedding, respectively.

DEFINITION 2. A nonsingular projective algebraic variety V is said to have a Galois embedding if there exist a very ample divisor D such that the embedding associated with the complete linear system $|D|$ has a Galois subspace. In this case the pair (V, D) is said to define a Galois embedding.

In this note we use the following notation.

- Z_m : the cyclic group of order m
- D_m : the dihedral group of order $2m$
- $|G|$: the order of a group G
- ρ : $\exp(2\pi\sqrt{-1}/6)$
- $Aut(V)$: the automorphism group of a variety V
- $\langle a_1, \dots, a_m \rangle$: the subgroup generated by $a_1, \dots, a_m$
- $\mathbf{1}_2$: the unit matrix of degree two

We shall make use of the following criterion (cf. [7, Theorem 2.2]).

THEOREM A. *Let V and D be as above. The pair (V, D) defines a Galois embedding if and only if the following conditions hold:*

(1) *There exists a subgroup G of $Aut(V)$ such that $|G| = D^n$.*

(2) *There exists a G-invariant linear subspace L of $H^0(V, \mathcal{O}(D))$ of dimension $n+1$ such that, for any $\sigma \in G$, the restriction $\sigma^*|_L$ is a multiple of the identity.*

(3) *The linear system L has no base points.*

The original form of the study of the Galois embedding is given in [5] or [6]. We have applied the above method to abelian surfaces A over $k = \mathbb{C}$ and obtained some results.

2. Statement of Theorem.

Let A be an abelian surface defined over $k = \mathbb{C}$ and G be a finite subgroup of $Aut(A)$. Fix a covering morphism $\mathbb{C}^2 \longrightarrow A$. An element $g \in G$ has a representation $\widetilde{g}$ on the universal covering $\mathbb{C}^2$ such that $\widetilde{g}z = M(g)z + t(g)$, where $M(g) \in GL(2, \mathbb{C})$, $z \in \mathbb{C}^2$ and $t(g) \in \mathbb{C}^2$. We call $M(g)$ and $t(g)$ the matrix and translation part of the representation $\widetilde{g}$, respectively. Put $G_0 = \{g \in G \mid M(g) = \mathbf{1}_2\}$ and $H = \{M(g) \mid g \in G\}$. Then, we have the following exact sequence of groups

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow H \longrightarrow 1.$$

Clearly $B = A/G_0$ is also an abelian surface and $H \cong G/G_0$ is a subgroup of $Aut(B)$.

Hereafter we assume that (A, D) defines a Galois embedding in $\mathbb{P}^N$ and let G be the Galois group. Then G is a subgroup of $Aut(A)$ and B/H is isomorphic to $A/G \cong \mathbb{P}^2$. With the notation above, we have:

THEOREM B. [7, Theorem 3.7] *If an abelian surface A has a Galois embedding, then H is isomorphic to D_3, D_4 or a semidirect product $Z_2 \ltimes K$, where $K \cong D_4$ or $Z_m \times Z_m$ $(m = 3,\ 4,\ 6)$*

Now, the answer to the question in the introduction is given as follows:

THEOREM C. *Suppose that (A, D) defines a Galois embedding. Then*

(1) *The least number N is seven, i.e., there exists an abelian surface A such that it can be embedded into $\mathbb{P}^7$ with a Galois subspace, and no abelian surface embedded in $\mathbb{P}^N$ $(N \leq 6)$ has a Galois subspace.*

(2) *The abelian surface $B = A/G_0$ is isomorphic to the self-product $E \times E$ of an elliptic curve, such that $H = G/G_0$ acts on B and $H \cong D_4$ or $Z_2 \ltimes D_4$.*

REMARK 1. The minimal Galois embedding for an elliptic curve E is given as follows. If E can be Galois-embedded in $\mathbb{P}^2$, then E must have an automorphism of order three with a fixed point. In fact, the elliptic curve is unique and is defined by $Y^2Z = 4X^3 + Z^3$. The centers of the projections are $(1:0:0), (0:\sqrt{-3}:1)$ and $(0:-\sqrt{-3}:1)$. However, note that every elliptic curve has a Galois embedding in $\mathbb{P}^3$, where the group is isomorphic to $Z_2 \times Z_2$ (further, if the j-invariant is 1728, then it has another projection center whose Galois group is isomorphic to Z_4).

In what follows we shall give the proof of Theorem C and some examples. In particular, there is an example where B is the Jacobian of a curve. Indeed, let $J(C)$ be the Jacobian of the normalization C of the curve $y^2 = x(x^4 + ax^2 + 1)$, $a \neq \pm 2$, and let A be an abelian surface which is an etale double covering $q : A \longrightarrow J(C)$. Then we shall show that $(A,\ q^*(C + C'))$ gives a minimal Galois embedding, where C' is a translated of C.

3. Proof.

By Theorem B we have $|H| = 2^a 3^b$, where $(a, b) = (3, 0), (4, 0), (5, 0), (1, 1), (1, 2)\ (3, 2)$. We consider the possible values of $|G| = D^2 = 2m$. Since A can be embedded in $\mathbb{P}^{m-1}$ by the complete linear system $|D|$, we have $m - 1 \geq 4$ by [3]. In view of Theorem B we have $m \neq 5$, hence $m \geq 6$. Suppose that $m = 6$. Then, from Theorem B again we infer that $H \cong D_3$ and $|G_0| = 2$. Every two dimensional complex crystallographic group G with $X/G \cong \mathbb{P}^2$ has been classified in [4]. Referring to it, we see that A can be expressed as $A = \mathbb{C}^2/\Omega$ such that Ω is the period matrix

$$\Omega = \begin{pmatrix} -1 & \rho^2 & -\omega & \rho^2\omega \\ 1 & \rho & \omega & \rho\omega \end{pmatrix} = \begin{pmatrix} -1 & \rho^2 \\ 1 & \rho \end{pmatrix} \begin{pmatrix} 1 & 0 & \omega & 0 \\ 0 & 1 & 0 & \omega \end{pmatrix},$$

where ω is a complex number with $\Im\omega > 0$.

Define four vectors as follows:

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} \rho^2 \\ \rho \end{pmatrix},\ v_3 = \begin{pmatrix} -\omega \\ \omega \end{pmatrix} = \omega v_1,\ v_4 = \begin{pmatrix} \rho^2\omega \\ \rho\omega \end{pmatrix} = \omega v_2.$$

Let $\mathcal{L}_A$ be the lattice in $\mathbb{C}^2$ generated by v_1, v_2, v_3 and v_4. Let g_1 be a generator of G_0 whose representation is $\widetilde{g}_1 z = z + e$, where $z,\ e \in \mathbb{C}^2$. Since ${g_1}^2$ is the identity on A, we have $2e \in \mathcal{L}_A$. Let $\mathcal{L}_B$ be the lattice generated by $\mathcal{L}_A$ and e. Then $B = \mathbb{C}^2/\mathcal{L}_B = A/\langle g_1 \rangle$ is also an abelian surface on which the group H acts. As shown in [4], H is generated by g_2 and g_3, whose matrix parts are

$$M_2 = M(g_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } M_3 = M(g_3) = \begin{pmatrix} -\rho & 0 \\ 0 & \rho^2 \end{pmatrix}$$

respectively. Since $2e \in \mathcal{L}_A$, the vector e can be expressed as

$$e = \frac{1}{2} \sum_{i=1}^{4} n_i v_i,$$

where $n_i = 0$ or 1 $(1 \leq i \leq 4)$. Since G_0 is a normal subgroup of G and $|G_0| = 2$, g_1 commutes with each element of G. Therefore, we infer that

$M_i e - e \in \mathcal{L}_A$ $(i = 2, 3)$. Indeed we have

$$M_2 e - e = \{-(2n_1 + n_2)v_1 - (2n_3 + n_4)v_3\}/2 \ \in \ \mathcal{L}_A.$$

Thus we have

$$(-n_2 v_1 - n_4 v_3)/2 \in \mathcal{L}_A. \tag{1}$$

Similarly, considering $M_3 e - e$, we have

$$\{-n_2 v_1 + (n_1 - n_2)v_2 - n_4 v_3 + (n_3 - n_4)v_4\}/2 \in \mathcal{L}_A. \tag{2}$$

From (1) we get $n_2 = n_4 = 0$, hence from (2) we get $n_1 = n_3 = 0$. Since $e \notin \mathcal{L}_A$, this is a contradiction. Thus we have $m \geq 7$. From Theorem B we infer $|G| \neq 14$, hence we have $m \geq 8$. We conclude that the least number m is 8 by the examples in the next section.

REMARK 2. Referring to [4, Theorem 1], we see that in the case (i) $H \cong D_4$ the dimension of the moduli space is 1, but in the case (ii) $H \cong Z_2 \ltimes D_4$ the dimension is zero.

4. Examples.

When we make examples, the following lemma is useful.

LEMMA 3. *If $M(g) - \mathbf{1}_2$ is a nonsingular matrix for $g \in G$, then we can assume $t(g) = 0$.*

PROOF. We consider $\tau G \tau^{-1}$ instead of G, where τ is a translation $\widetilde{\tau} z = z + \ell$. If $M(g) - \mathbf{1}_2$ is nonsingular, then by putting $\ell = (M(g) - \mathbf{1}_2)^{-1} t(g)$, we get $t(\tau g \tau^{-1}) = 0$. □

In case $m = 8$ we have $H \cong D_4$ or $Z_2 \ltimes D_4$ by Theorem B. We shall give examples of both. If (i) $H \cong D_4$, then $|G_0| = 2$. Since G_0 is a normal subgroup of G and $|G_0| = 2$, we infer that $G = G_0 \times H$. Such examples are given in Examples 4 and 5. On the other hand, if $H \cong Z_2 \ltimes D_4$, we have $G \cong H$. Such an example is given in Example 7.

EXAMPLE 4. Let A be the abelian surface with the period matrix

$$\begin{pmatrix} 1 & 0 & \omega & 0 \\ 0 & 1 & 0 & \omega \end{pmatrix} \text{ such that } \Im \omega > 0.$$

Let $\mathcal{L}_A$ be the lattice generated by the column vectors of the period matrix. Let us consider the automorphisms g_1, g_2 and g_3 of A, whose representations on $\mathbb{C}^2$ are as follows:

$$\begin{aligned}
\widetilde{g_1}z &= z+\tfrac{1}{2}\begin{pmatrix} n_1+n_3\omega \\ n_2+n_4\omega \end{pmatrix}, \\
\widetilde{g_2}z &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \\
\widetilde{g_3}z &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z
\end{aligned}$$

where $(n_1, n_2, n_3, n_4) = (0,0,1,1), (1,1,0,0), (1,1,1,1)$,

$$\begin{pmatrix} \alpha_1+\alpha_2 \\ \alpha_1+\alpha_2 \end{pmatrix} \in \mathcal{L}_A \text{ and } \begin{pmatrix} 2\alpha_1 \\ 0 \end{pmatrix} \in \mathcal{L}_A.$$

We have ${g_1}^2 = {g_2}^2 = {g_3}^4 = id$, $g_2g_3g_2 = {g_3}^{-1}$ and $g_ig_1 = g_1g_i$ $(i = 2, 3)$ on A. Putting $G = \langle g_1, g_2, g_3\rangle$, we have $G_0 = \langle g_1\rangle$ and $G = G_0 \times H$ where $H = \langle M(g_2), M(g_3)\rangle$. Clearly $H \cong D_4$. The group G is a subgroup of $Aut(A)$ and $A/G \cong \mathbb{P}^2$. The very ample divisor D is given by $\pi^*(L)$, where $\pi : A \longrightarrow A/G \cong \mathbb{P}^2$ and L is a line in $\mathbb{P}^2$ (cf. [7, Lemma 3.5]). We infer from Theorem A that (A, D) defines a Galois embedding.

By [7, Corollary 3.8], if A has a Galois embedding, then the abelian surface $B = A/G_0$ is isomorphic to $E \times E$ for some elliptic curve E. On the other hand, $E \times E$ can be a Jacobian of a curve for some E (cf. [2]). So one may ask what type of genus 2 curve can give the Jacobian whose double covering has a minimal Galois embedding. Let us consider this question in the next example.

EXAMPLE 5. Let Γ be the curve defined by $y^2 = x(x^4 + ax^2 + 1)$, where we assume $a \neq \pm 2$ (cf. [1, Theorem 4.8]). This curve has a singular point at ∞. Let C be the normalization of Γ. The genus of C is two. Let σ and τ be the birational transformations of Γ defined by

$$\sigma(x) = -x,\ \sigma(y) = iy \quad \text{and} \quad \tau(x) = 1/x,\ \tau(y) = y/x^3$$

respectively. Clearly we have $\sigma^4 = \tau^2 = id$ and $\tau\sigma\tau = \sigma^{-1}$. Let $\mathcal{H}$ be the group generated by σ and τ. Then we have $\mathcal{H} \cong D_4$. This group acts on C. Let $\mathbb{C}(x, y)$ be the function field of C, where $y^2 = x(x^4 + ax^2 + 1)$. Clearly the invariant field of $\mathbb{C}(x, y)$ by σ is $\mathbb{C}(x^2)$. Let ϑ_1 and ϑ_2 be a basis of holomorphic 1-forms on C induced from dx/y and xdx/y respectively.

We have

$$\sigma^*(\vartheta_1) = i\vartheta_1,\ \sigma^*(\vartheta_2) = -i\vartheta_2 \ \text{ and } \ \tau^*(\vartheta_1) = -\vartheta_2,\ \tau^*(\vartheta_2) = -\vartheta_1.$$

Let $J(C)$ be the Jacobian of C. Taking a base point $P \in C$, the Abel-Jacobi map j_P is given as

$$j_P : C \longrightarrow J(C) \ \text{ such that } \ j_P(Q) \equiv \left(\int_P^Q \vartheta_1, \int_P^Q \vartheta_2 \right) \ \text{(modulo the lattice)},$$

where $Q \in C$. If P is a fixed point of σ, then we have

$$\int_P^{\sigma(Q)} \vartheta = \int_P^Q \sigma^*(\vartheta),$$

for $Q \in C$, where $\vartheta = \vartheta_1$ or ϑ_2. Similarly if P' is a fixed point of τ, then we have

$$\int_{P'}^{\tau(Q')} \vartheta = \int_{P'}^{Q'} \tau^*(\vartheta),$$

where $Q' \in C$. Note that if $j_P : C \longrightarrow J(C)$ is defined with a base point P, and $j_{P'} : C \longrightarrow J(C)$ is defined with a base point P', then $j_{P'} = t \cdot j_P$, where t is a translation in $J(C)$. Assume that P is a fixed point of σ. Then, letting $\widetilde{\sigma}$ and $\widetilde{\tau}$ be the representations of σ and τ on $\mathbb{C}^2$ respectively, we obtain that they can be expressed as $\widetilde{\sigma}z = M(\sigma)z$ and $\widetilde{\tau}z = M(\tau)z + v$, where $z \in \mathbb{C}^2$, $v \in \mathbb{C}^2$ and

$$M(\sigma) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } M(\tau) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Put $H = \langle M(\sigma), M(\tau) \rangle$. Then we have $H \cong D_4$. Note that the curve $j_P(C)$ is fixed by σ and $C/\langle\sigma\rangle$ is isomorphic to a smooth rational curve. We infer from the above arguments that $J(C)/H$ is isomorphic to $\mathbb{P}^2$ and $C/\langle\sigma\rangle$ is a line. Let $p : J(C) \longrightarrow J(C)/H \cong \mathbb{P}^2$ be the quotient morphism. Then $p^*(L)$ can be expressed as $C + C'$, where L is the line and C' is a translation of C on $J(C)$. Let A be an abelian surface such that $q : A \longrightarrow J(C)$ is an etale double covering given as follows. Express $J(C) = \mathbb{C}^2/\mathcal{L}$ and $A = \mathbb{C}^2/\mathcal{L}_0$, where $\mathcal{L}$ and $\mathcal{L}_0$ are lattices satisfying $|\mathcal{L} : \mathcal{L}_0| = 2$. Take an element $\ell \in \mathcal{L} \setminus \mathcal{L}_0$ so that $\rho(z) = z + \ell$ is a translation of order two on A. Then we have $\mathcal{L} = \langle \mathcal{L}_0,\ \ell \rangle$. Since $2\ell \in \mathcal{L}_0$, we have $M(2\ell) = 2M(\ell) \in \mathcal{L}_0$, where $M \in H$. Hence we infer that σ and τ induce automorphisms on A. We use the same letter H to denote the group consisting of the elements which are induced from H. Let G be the automorphism group on A generated by H and ρ. Then put $\pi = p \cdot q : A \longrightarrow A/G \cong \mathbb{P}^2$. Since $\deg(\pi) = 16 \geq 10$, we see that $\pi^*(L)$ is very ample (cf. [7, Lemma 3.5]). From Theorem A we infer that $(A,\ \pi^*(L))$ defines a Galois embedding in $\mathbb{P}^7$.

REMARK 6. Note that $q^*(C) = \widetilde{C}$ is irreducible. Because, if not so, then $q^*(C)$ can be written as $C_1 + C_2$. We have $(q^*(C))^2 = 4$ and $C_1^2 = C_2^2 = 2$. Since C_1 is ample, we have $(C_1, C_2) \geq 1$. This is a contradiction. Similarly $q^*(C') = \widetilde{C'}$ is also irreducible. The divisor $\widetilde{C} + \widetilde{C'}$ gives the minimal Galois embedding of A. If L is the image of C, then $\widetilde{C} + \widetilde{C'} = \pi^*(L)$.

EXAMPLE 7. Let A be the abelian surface with period matrix

$$\begin{pmatrix} 1 & 0 & i & (1+i)/2 \\ 0 & 1 & 0 & (1+i)/2 \end{pmatrix}.$$

This abelian surface has the automorphisms g_1, g_2 and g_3, whose representations on $\mathbb{C}^2$ are as follows:

$$\begin{aligned} \widetilde{g_1} z &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix}, \\ \widetilde{g_2} z &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix}, \\ \widetilde{g_3} z &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} z, \end{aligned}$$

where the following vectors belong to the lattice generated by the column vectors of the period matrix:

$$\begin{pmatrix} 0 \\ 2e_{12} \end{pmatrix}, \quad \begin{pmatrix} e_{21} + e_{22} \\ e_{21} + e_{22} \end{pmatrix}, \quad \begin{pmatrix} e_{11} - e_{12} - 2e_{21} \\ e_{11} + e_{12} \end{pmatrix},$$

$$\begin{pmatrix} (1-i)e_{11} \\ (1-i)e_{12} \end{pmatrix}, \quad \begin{pmatrix} e_{21} - ie_{22} \\ e_{22} + ie_{21} \end{pmatrix}.$$

We have ${g_1}^2 = {g_2}^2 = {g_3}^4 = id$, $g_1 g_2 g_1 = g_2 g_3^2$, $g_1 g_3 g_1 = g_3$ and $g_2 g_3 g_2 = {g_3}^{-1}$. Putting $G = \langle g_1, g_2, g_3 \rangle$, we see that G is isomorphic to the semidirect product $Z_2 \ltimes D_4$ and G is a subgroup of $Aut(A)$ and $A/G \cong \mathbb{P}^2$. The very ample divisor D is given by $\pi^*(L)$, where $\pi : A \longrightarrow A/G \cong \mathbb{P}^2$ and L is a line in $\mathbb{P}^2$. We infer from Theorem A that (A, D) defines a Galois embedding.

REMARK 8. The abelian surface A in Example 7 is isogenous to $E_i \times E_i$, where $E_i = \mathbb{C}/(1,\ i)$. (In fact, we can show that A is isomorphic to $E_i \times E_i$.) Thanks to [2], this abelian surface cannot be a Jacobian of a curve.

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