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On Groups of Odd Order Admitting an Elementary 2-Group of Automorphisms

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ABSTRACT - Let G be a finite group of odd order with derived length k. We show that if G is acted on by an elementary abelian group A of order 2^n and $C_G(A)$ has exponent e, then G has a normal series $G = G_0 \geq T_0 \geq G_1 \geq T_1 \geq \cdots \geq G_n \geq T_n = 1$ such that the quotients G_i/T_i have $\{k, e, n\}$-bounded exponent and the quotients T_i/G_{i+1} are nilpotent of $\{k, e, n\}$-bounded class.

Dedicated to Professor Said Sidki on the occasion of his 70th birthday

1. Introduction

Let G be a group and A a group of automorphisms of G. The subgroup of all elements of G fixed by A is usually denoted by $C_G(A)$. It is well-known that very often the structure of $C_G(A)$ has strong influence over the structure of the whole group G. The influence seems especially strong in the case where G is a finite group of odd order and A is an elementary abelian 2-group. It was shown in [5] that if G is a finite group of derived length k on which an elementary abelian group A of order 2^n acts fixed-point-freely, then G has a normal series $G = N_0 \geq \cdots \geq N_{n-1} \geq N_n = 1$ all of whose quotients are nilpotent and the class of N_{i-1}/N_i is bounded with a

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function of k and i (see also [7] for a short proof of this result). In the present paper we further exploit the techniques developed in [7]. One of the obtained results is the following theorem.

THEOREM 1.1. *Let G be a finite group of odd order and of derived length k. Suppose that G admits an elementary abelian group A of automorphisms of order 2^n such that $C_G(A)$ has exponent e. Then G has a normal series*

$$G = G_0 \geq T_0 \geq G_1 \geq T_1 \geq \cdots \geq G_n \geq T_n = 1$$

such that the quotients G_i/T_i have $\{k, e, n\}$-bounded exponent and the quotients T_i/G_{i+1} are nilpotent of $\{k, e, n\}$-bounded class.

The other result deals with the situation where $\gamma_c(C_G(a))$ has exponent e for all automorphisms $a \in A^{\#}$. Here $A^{\#}$ denotes the set of non-trivial elements of A and $\gamma_i(H)$ stands for the i-th term of the lower central series.

THEOREM 1.2. *Let G be a finite group of odd order and of derived length k. Suppose that a four-group A acts on G in such way that $\gamma_c(C_G(a))$ has exponent e for all $a \in A^{\#}$. Then G has a normal series*

$$G = T_4 \geq T_3 \geq T_2 \geq T_1 \geq T_0 = 1$$

such that the quotients T_4/T_3 and T_2/T_1 are nilpotent of $\{e, c, k\}$-bounded class and the quotients T_3/T_2 and T_1 have $\{e, c, k\}$-bounded exponent.

Throughout the paper we say that a group G is nilpotent of class c meaning that the class of G is at most c and we say that G has exponent e meaning that the exponent of G divides e. The expression "$(a, b, \dots)$-bounded" stands for "bounded from above by a function depending only on the parameters a, b, $\dots$".

2. Auxiliary Results

The following lemmas are well-known (see for example Theorem 6.2.2, Theorem 6.2.4 and Theorem 10.4.1 in [1]).

LEMMA 2.1. *Let G be a finite group admitting a coprime group of automorphisms A. Then we have*

a) $C_{G/N}(A) = C_G(A)N/N$ *for any A-invariant normal subgroup N of G;*
b) $G = C_G(A)[G, A]$;
c) $[G, A] = [G, A, A]$;
d) *If G is abelian, then* $G = C_G(A) \times [G, A]$.

LEMMA 2.2. *Let G be a finite group admitting an abelian non-cyclic coprime group of automorphisms A. Then* $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.

LEMMA 2.3. *Let G be a finite group of odd order admitting an automorphism a of order 2 such that* $G = [G, a]$. *Suppose that N is an a-invariant normal subgroup of G such that* $C_N(a) = 1$. *Then* $N \leq Z(G)$.

Our next lemma is immediate from [3, Lemma 2.6].

LEMMA 2.4. *Let G be a finite group admitting a coprime group of automorphisms A. Suppose that G is generated by a family* $\{H_i \mid i \in I\}$ *of normal A-invariant subgroups. Then*

$$C_G(A) = \langle C_{H_i}(A) \mid i \in I \rangle.$$

LEMMA 2.5. *Let G be a metabelian finite group of odd order admitting an automorphism a of order 2 such that* $C_G(a)$ *has exponent e and* $G = [G, a]$. *Then* G' *has exponent e.*

PROOF. Since G/G' is abelian and $G = [G, a]$, by Lemma 2.1 $C_{G/G'}(a) = 1$ and so $C_G(a) \leq G'$. Let N be the normal closure of $C_G(a)$ in G. Since G' is abelian we conclude that N has exponent e. On the other hand, G/N is abelian because a acts on G/N fixed-point-freely. Therefore $G' = N$ and the result follows. □

PROPOSITION 2.6. *Let G be a finite group of odd order and of derived length k admitting an automorphism a of order 2 such that* $C_G(a)$ *has exponent e and* $G = [G, a]$. *Then* G' *has* $\{e, k\}$*-bounded exponent.*

PROOF. In view of Lemma 2.5 the result is obvious if $k \leq 2$. Suppose that $k \geq 3$ and let $M = G^{(k-1)}$. By induction we conclude that G'/M has $\{e, k\}$-bounded exponent. Hence, it is enough to show that M has $\{e, k\}$-bounded exponent. Working with the quotient $G/\langle C_M(a)^G \rangle$ we can simply assume that $C_M(a) = 1$. Lemma 2.3 now shows that $M \leq Z(G)$. Therefore $G^{(k-2)}$ is nilpotent of class 2. Since $G^{(k-2)}/M$ has $\{e, k\}$-bounded exponent, it follows that $G^{(k-2)}$ has $\{e, k\}$-bounded exponent [4, Theorem 2.5.2]. In particular, M has $\{e, k\}$-bounded exponent. □

A well-known theorem of Hall says that if G is a soluble group of derived length k and all metabelian sections of G are nilpotent of class at most c, then G is nilpotent of $\{k, c\}$-bounded class [2]. We will require the following related result obtained in [6]. We denote by $f(g, c)$ the expression $(g-1)\dfrac{c(c+1)}{2} + c$.

THEOREM 2.7. *Let G be a group and N a nilpotent normal subgroup of G of class g such that $\gamma_{c+1}(G/N')$ has exponent e. Suppose that $\gamma_{c+1}(G)$ is soluble of derived length d. Then $\gamma_{f(g,c)+1}(G)$ has finite $\{g, e, c, d\}$-bounded exponent.*

The next proposition was obtained in [7, Corollary 3.2]. It plays a crucial role in the subsequent proofs.

PROPOSITION 2.8. *There exists a number $s = s(k, n)$ depending only on k and n with the following property. Suppose that G is a finite group of odd order that is soluble with derived length at most k. Assume that an elementary group A of order 2^n acts on G and let R be a normal A-invariant subgroup of G such that $C_R(A) = 1$. Set $N = \bigcap_{a \in A^\#} [G, a]$. Then $[R, \underbrace{N, \dots, N}_{s}] = 1$.*

3. Main Results

PROPOSITION 3.1. *Let G be a finite group of odd order and of derived length k. Suppose that G admits an elementary abelian group of automorphisms A of order 2^n such that $C_G(A)$ has exponent e. Then $\bigcap_{a \in A^\#} [G, a]$ is an extension of a group of $\{k, e, n\}$-bounded exponent by a nilpotent group of $\{k, e, n\}$-bounded class.*

PROOF. Set $N = \bigcap_{a \in A^\#} [G, a]$ and use induction on the derived length of N. If N is abelian, the result is trivial. Suppose that the derived length of N is greater than or equal to 2 and let L be the metabelian term of the derived series of N. Put $M = L' \cap C_G(A)$ and let D be the normal closure of M in G. Since D is abelian with generators of order e, we see that D has exponent e. Considering the quotient G/D we can simply suppose that $C_{L'}(A) = 1$. Proposition 2.8 shows that $L' \subseteq Z_s(L)$ and so L is nilpotent of class $s + 1$. By induction, N/L' is an extension of a group of $\{k, e, n\}$-bounded exponent by a nilpotent group of $\{k, e, n\}$-bounded class, say c. Since the derived length of $\gamma_{c+1}(N)$ is at most k, by Theorem 2.7 we conclude that $\gamma_{f(s+1,c)+1}(N)$ has $\{k, e, n\}$-bounded exponent, as required. $\square$

Given a group H and positive integers e and c, we denote by $V(H)$ the verbal subgroup of H corresponding to the group-word $[x_1, \dots, x_c]^e$ and by $W(H)$ the verbal subgroup of H corresponding to the group-word $[x_1, \dots, x_{3c-2}]^{e^3}$.

LEMMA 3.2. *Let G be a finite metabelian group of odd order admitting a four-group of automorphisms A such that $\gamma_c(C_G(a))$ has exponent e for all $a \in A^\#$. Then there exists a normal subgroup M of G such that $C_M(A) = 1$ and $\gamma_{3c-2}(G/M)$ has exponent e^3.*

PROOF. For every $a \in A^\#$ put $G_a = G'C_G(a)$. It is easy to see that $[G_a, a] = [G', a]$ is normal in G_a. Since $G_a/[G', a]$ is isomorphic to a quotient of $C_G(a)$, it follows that $V(G_a)$ is contained in $[G', a]$. Since G' is abelian, by Lemma 2.1 we have $[G', a] \cap C_G(a) = 1$. It follows that $V(G_a) \cap C_G(a) = 1$. Let $M = \prod_{a \in A^\#} V(G_a)$, obviously M is a normal subgroup of G. By Lemma 2.4, $C_M(A) = \langle C_{V(G_a)}(A) \mid a \in A^\# \rangle$ and so $C_M(A) = 1$. By Lemma 2.2 we have $G = \langle G_a \mid a \in A^\# \rangle$. Thus, G/M is product of three normal subgroups each of which is an extension of a group of exponent e by a nilpotent group of class $c - 1$. Thus G/M is an extension of a group of exponent e^3 by a nilpotent group of class $3c - 3$. $\square$

LEMMA 3.3. *Let G be a metabelian group such that $\gamma_c(G/Z(G))$ has exponent e. Then $\gamma_{c+1}(G)$ has exponent e.*

PROOF. Let $E = \gamma_c(G)$. Then $E/(Z(G) \cap E)$ has exponent e. For arbitrary elements $x_1, x_2, \dots, x_c, x_{c+1} \in G$ we have

$$[x_1, x_2, \dots, x_c] \in E.$$

Therefore

$$[x_1, x_2, \dots, x_c]^e \in Z(G) \cap E,$$

whence

$$[[x_1, x_2, \dots, x_c]^e, x_{c+1}] = 1.$$

Since $[x_1, x_2, \dots, x_c] \in G'$, which is a normal abelian subgroup of G, we obtain that

$$[x_1, x_2, \dots, x_c, x_{c+1}]^e = [[x_1, x_2, \dots, x_c]^e, x_{c+1}] = 1.$$

Hence, $\gamma_{c+1}(G)$ has exponent e. □

PROPOSITION 3.4. *Let G be a finite group of odd order and of derived length k admitting a four-group of automorphisms A such that $\gamma_c(C_G(a))$ has exponent e for all $a \in A^{\#}$. Then $\bigcap_{a \in A^{\#}} [G, a]$ is an extension of a group of $\{e, c, k\}$-bounded exponent by a nilpotent group of $\{e, c, k\}$-bounded class.*

PROOF. Let $N = \bigcap_{a \in A^{\#}} [G, a]$. We use induction on the derived length of N. If N is abelian the result is trivial. Suppose that the derived length of N is greater than or equal to 2 and let L be the metabelian term of the derived series of N. First we will show that L is an extension of a group of $\{e, c, k\}$-bounded exponent by a nilpotent group of $\{e, c, k\}$-bounded class. By Lemma 3.2 there exists a normal subgroup M of L such that $C_M(A) = 1$ and $\gamma_{3c-2}(L/M)$ has exponent e^3. Thus $W(L) \subseteq M$ and so $W(L) \cap C_L(A) = 1$. If $W(L) = 1$, then $\gamma_{3c-2}(L)$ has exponent e^3. Suppose that $W(L) \neq 1$. Since $W(L) \cap C_G(A) = 1$ it follows from Proposition 2.8 that there exists a bounded number s such that

$$[W(L), \underbrace{N, \dots, N}_{s}] = 1.$$

Let t be the smallest number such that $W(L) \subseteq Z_t(N)$. We know that $t \leq s$. It will be shown by induction on t that L is an extension of a group of $\{e, c, k\}$-bounded exponent by a nilpotent group of $\{e, c, k\}$-bounded class. For $t = 1$ we have the inclusion $W(L) \subseteq Z(N)$. In this case $L/(L \cap Z(N))$ is an extension of a group of exponent e^3 by a nilpotent group of class $3c - 3$. By Lemma 3.3 we deduce that $\gamma_{3c-1}(L)$ has exponent e^3, as required.

Suppose now that $t \geq 2$ and the result is valid for $t - 1$. Let $K = [W(L), \underbrace{N, \dots, N}_{t-1}]$. It is clear that $K \leq Z(N)$. By induction it follows that L/K is an extension of a group of $\{e, c, k\}$-bounded exponent by a nilpotent group of $\{e, c, k\}$-bounded class and by Lemma 3.3 also L is an extension of a group of $\{e, c, k\}$-bounded exponent by a nilpotent group of $\{e, c, k\}$-bounded class, say c_1. Put $H = \gamma_{c_1+1}(L)$. So H has $\{e, c, k\}$-bounded exponent. Passing to the quotient G/H we can simply assume that $H = 1$. By induction on the derived length of N, we deduce that N/L' is an extension of a group of $\{e, c, k\}$-bounded exponent by a nilpotent group of $\{e, c, k\}$-bounded class, say c_2. By Theorem 2.7 we conclude that $\gamma_{f(c_1,c_2)+1}(N)$ has $\{e, c, k\}$-bounded exponent, as required. □

Now we are in a position to prove our main results.

Proof of Theorem 1.1. If $n = 1$, Proposition 2.6 guarantees that $[G, a]'$ has $\{e, k\}$-bounded exponent and so

$$G = G_0 \geq [G, a] = T_0 \geq [G, a]' = G_1 \geq T_1 = 1$$

is the required series. Suppose that $n \geq 2$ and use induction on n. For every normal A-invariant subgroup N of G the group A induces a group of automorphisms of G/N. In particular A induces a group of automorphisms B_a of $G/[G, a]$ for every $a \in A^\#$ with $|B_a| \leq 2^{n-1}$ and $C_{G/[G,a]}(B_a)$ of exponent e. Let B be the elementary abelian group of order 2^{n-1}. We can define an action of B on $G/[G, a]$ as follows. For all $a \in A^\#$ we consider B_a as a subgroup of B and we can write $B = B_a \times D_a$ for a suitable subgroup D_a of B. For any $b \in B$ there exists a unique pair $(b_1, b_2) \in B_a \times D_a$ such that $b = b_1 b_2$. Then for any $x \in G/[G, a]$ we put $x^b = x^{b_1}$. This action is well defined and $C_{G/[G,a]}(B)$ has exponent e. Let $K = \prod_{a \in A^\#} G/[G, a]$, we have an action of B on K that extends the action of B on every factor $G/[G, a]$ and $C_K(B)$ has exponent e.

By induction, K has a series of length $2(n - 1)$ with the required properties. By Proposition 3.1 the subgroup $N = \bigcap_{a \in A^\#} [G, a]$ is an extension of a group of $\{k, e, n\}$-bounded exponent by a nilpotent group of $\{k, e, n\}$-bounded class. Since G/N embeds in K the result follows. □

Proof of Theorem 1.2. Each factor $G/[G, a]$ is isomorphic to a quotient of $C_G(a)$, so it is an extension of a group of exponent e by a nilpotent group of class $c - 1$. Therefore $K = \prod_{a \in A^\#} G/[G, a]$ is an extension of a group of exponent e^3 by a nilpotent group of class $3c - 3$. Let $N = \bigcap_{a \in A^\#} [G, a]$. Since G/N embeds in K, we conclude that $\gamma_{3c-2}(G/N)$ has exponent e^3. Proposition 3.4 ensures that N is an extension of a group of $\{e, c, k\}$-bounded exponent by a nilpotent group of $\{e, c, k\}$-bounded class. The proof is complete. □

REFERENCES

[1] D. Gorenstein, *Finite Groups*, Harper and Row, New York, Evanston, London, 1968.

[2] P. Hall, *Some sufficient conditions for a group to be nilpotent*, Illinois J. Math., **2** (1958), pp. 787–801.

[3] B. Hartley, *Periodic locally soluble groups containing an element of prime order with Černikov centralizer*, Quart. J. Math. Oxford Ser., (2), **33**, n. 131 (1982), pp. 309–323.

[4] E. Khukhro, *Nilpotent groups and their automorphisms*, Berlin, New York, De Gruyter expositions in mathematics; **8** (1993).

[5] P. Shumyatsky, *Groups with regular elementary* 2*-groups of automorphisms*, Algebra and Logic, **27**, no. 6 (1988), pp. 447–457.

[6] P. Shumyatsky, *On extensions of groups of finite exponent*, Glasgow Math J., **45**, n. 3 (2003), pp. 535–538.

[7] P. Shumyatsky - C. Sica, *On groups admitting a fixed-point-free elementary* 2*-group of automorphisms*, Comm. Algebra, **38**, n. 11 (2010), pp. 4188–4192.

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