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$\mathscr{A}$-Schemes and Zariski-Riemann Spaces

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ABSTRACT - In this paper, we will investigate further properties of $\mathscr{A}$-schemes introduced in [Tak]. The category of $\mathscr{A}$-schemes possesses many properties of the category of coherent schemes, and in addition, it is co-complete and complete. There is the universal compactification, namely, the Zariski-Riemann space in the category of $\mathscr{A}$-schemes. We compare it with the classical Zariski-Riemann space, and characterize the latter by a left adjoint.

Contents

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0. Introduction

In this paper, we will investigate further properties of $\mathscr{A}$-schemes introduced in [Tak]. The first motivation of introducing $\mathscr{A}$-schemes was to construct a scheme-like geometrical object from various kinds of algebraic systems, such as commutative monoids, semirings, and etc. However, it happens to have advantages even in the case the algebraic system is that of rings. The category of $\mathscr{A}$-schemes is much more flexible than that of ordinary schemes: let us list up the properties of $\mathscr{A}$-schemes, and compare with ordinary schemes:

(1) Let (**Coh**.**Sch**) be the category of coherent schemes and quasi-compact morphisms. Then, the category ($\mathscr{A}$-**Sch**) of $\mathscr{A}$-schemes contains (**Coh**.**Sch**) as a full subcategory. Also, ($\mathscr{A}$-**Sch**) is a full subcategory of the category (**LRCoh**) of locally ringed coherent spaces and quasi-compact morphisms (Proposition 2.1.1).

(2) There is a spectrum functor, and is the left adjoint of the global section functor $\Gamma : (\mathscr{A}\text{-}\mathbf{Sch})^{\mathrm{op}} \to (\mathbf{Rng})$ ([Tak]).

(3) The inclusion functor (**Coh**.**Sch**) $\to$ ($\mathscr{A}$-**Sch**) preserves fiber products (Corollary 2.3.4), and patchings via quasi-compact opens (Proposition 2.1.6).

(4) There is a valuative criterion of separatedness (Proposition 3.4.3) and that of properness (Proposition 3.4.4).

These imply that $\mathscr{A}$-schemes behave much like ordinary schemes. By contrast, they have more virtues than ordinary schemes:

(5) The category ($\mathscr{A}$-**Sch**) is small co-complete (Proposition 2.1.6) and small complete (Proposition 2.3.3). *We don't need filteredness.*

(6) There is a functorial epi-monic decomposition of morphisms of $\mathscr{A}$-schemes (Theorem 2.2.12). In particular, we have the 'image scheme' for each morphism.

Therefore, we need not distinguish pro-schemes and ind-schemes from schemes anymore, if we work out on this category of $\mathscr{A}$-schemes. These properties give us various profits:

(7) We can consider quotient $\mathscr{A}$-schemes whenever there is a group action on an $\mathscr{A}$-scheme. We don't need any additional condition.

(8) Formal schemes can be treated on the same platform, as $\mathscr{A}$-schemes (Example 3.1.7).

(9) We can think of universal 'separation' of $\mathscr{A}$-schemes (Proposition 3.3.2).

(10) We can think of universal 'compactification' of $\mathscr{A}$-schemes, namely, the Zariski-Riemann space (Theorem 4.1.2). The construction is the analog of the Stone-Čech compactification.

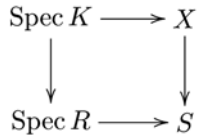
The key of this extension of the category of ordinary schemes is simple: *to abandon the principal property of schemes*, namely, 'locally being a spectrum of a ring'. It is because this condition forces us only to use finite categorical operation in the category of ordinary schemes, and makes it very inconvenient. In particular, we have to give up Zariski-Riemann spaces in the category of ordinary schemes, although it is a fairly nice locally ringed space and there are various applications. On the other hand, Zariski-Riemann spaces can be treated equally with ordinary schemes, when we extend our perspective to the category of $\mathscr{A}$-schemes. Moreover, the construction of Zariski-Riemann spaces appears to be as natural as the spectrum of a ring.

Let us describe the contents of this paper. In § 1, we will prove some properties of coherent spaces, which we will need later. In particular, a quasi-compact morphism of coherent spaces is epic if and only if it is surjective, and its image is closed if and only if it is specialization closed.

In § 2, we will discuss the properties of the category of $\mathscr{A}$-schemes, namely we will prove the co-completeness and completeness. The key lemma is the functorical decomposition of morphisms. This gives us the upper bound of the cardinality of the set of morphisms with fixed targets, and hence enables us to construct limits. This is the analog of the construction of co-limits in the category of algebras of various kinds.

In § 3, we define separated and proper morphisms of $\mathscr{A}$-schemes, and give valuative criteria for separatedness and properness. Unlike ordinary schemes, we don't have a canonical morphism $\operatorname{Spec} \mathscr{O}_{X,x} \to X$ in the category of $\mathscr{A}$-schemes, where x is a point of an $\mathscr{A}$-scheme X. Hence, we had to modify the testing morphisms. The valuative criteria in the category of ordinary schemes check the right lifting properties of the com-

mutative square



where K is an arbitrary field and R is its valuation field. The left vertical arrow is the 'testing morphism'. In the category of $\mathscr{A}$-schemes, we must replace the testing morphisms to formulate the valuative criteria: namely, just take the set $\{\xi, \eta\}$ of the generic point and the closed point of Spec R with a natural induced $\mathscr{A}$-scheme structure. Once we have the valuative criteria, we can construct the universal separation and the universal compactification; the latter is treated in § 4. We note here, that we will not include 'of finite type' condition in the definition of proper morphisms, for we want to take limits. We emphasize the fact that separated morphisms and universally closed morphisms are closed under taking infinite limits, while morphisms of finite types are not.

In § 4, we construct the Zariski-Riemann space as the universal compactification using the adjoint functor theorem. Later, we will also consider 'classical Zariski-Riemann space' on irreducible, reduced $\mathscr{A}$-schemes, as it happens to be more easier to analyze than the previous Zariski-Riemann space. This construction is the analog of the conventional one, but with different flavor: *we tried not to use valuation rings when defining it*. This reveals the naturalness of the concept of valuation rings— that it is as natural as the concept of 'local rings' of spectra of rings. Also, note that what localization is to the spectrum is what separated, of finite type morphism is to the Zariski-Riemann space. These also imply that the way of constructing a topological object from rings is not at all unique—we can consider another 'algebraic geometry'.

Here, we also compare our previous Zariski-Riemann space with the classical one. Actually, the topology of the classical Zariski-Riemann space happens to be coarser: it is, in a sense, the coarsest possible topology. This property, which we will call 'of profinite type', characterizes the classical Zariski-Riemann space. Though the classical Zariski-Riemann space has a weaker universality, it is valuable since it gives us concrete descriptions of its structure and morphisms. Any separated dominant morphism of ordinary integral schemes is of profinite type, so that they can be embedded into a universal proper, of profinite type $\mathscr{A}$-scheme.

In this paper, we only constructed Zariski-Riemann space for irreducible, reduced $\mathscr{A}$-schemes, since this assumption makes the argument

much simpler, and it will be sufficient for most of the applications. We believe that it is possible to extend it to arbitrary $\mathscr{A}$-schemes with a little more effort. We have proved a variant of the Nagata embedding (Corollary 4.6.6). The original version of the Nagata embedding can also be proven, and will be shown in the forthcoming paper. We decided not to prove it here, since there are various proofs published already ([Nag], [Con], [Tem]), and it takes a little more detailed work which will make this paper more longer if we include it. However, the proof is rather natural and intuitive than the former ones.

We summarized the definition of $\mathscr{A}$-schemes at the end of this paper, as an appendix. This will be sufficient for the reader to go through this paper, though he hasn't looked over [Tak].

0.1 – *Notation and conventions*

The reader is assumed to have standard knowledge of categorical theories; see for example, [CWM], [KS]. We fix a universe, and all sets are assumed to be small. The category of small sets (resp. sober spaces and continuous maps) is denoted by (**Set**) (resp. (**Sob**)).

When we talk of an algebraic system, all the operators are finitary, and all the axioms are identities. Any ring and any monoid is commutative, and unital. For a ring (or, other algebras with a structure of a multiplicative monoid) R, we denote by R_S the localization of R along the multiplicative system S of R.

For any set $\mathcal{S}$, $\mathscr{P}(\mathcal{S})$ is the power set of $\mathcal{S}$, and $\mathscr{P}^f(\mathcal{S})$ is the set of finite subsets of $\mathcal{S}$.

When given an $\mathscr{A}$-scheme (or, any topological space with its structure sheaf) X, we denote the underlying space by $|X|$.

We frequently denote finite summations by $\sum^{<\infty}$, when the range of the index is not crucial. The same thing can be said for the notation $\cup^{<\infty}$ for finite unions.

1. Properties of coherent spaces

In this section, we will investigate some properties of sober and coherent spaces. Recall that a topological space X is *sober*, if every irreducible closed subset of X has a unique generic point. For a sober space X, $C(X)$ is the set of closed subsets of X. This becomes a complete II-ring; see appendix § 5 for

further details. A topological space X is *coherent*, if it is sober, quasi-compact, quasi-separated, and has a quasi-compact open basis.

1.1 – *Monic and epic maps*

LEMMA 1.1.1. *Let σ be any algebraic system, and $f : A \to B$ be a homomorphism of σ-algebras. Then, f is monic if and only if f is injective.*

PROOF. The 'if' part is clear.

Suppose f is not injective. Then there are two distinct elements $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Let R_0 be the initial object in (σ-**alg**). Define two homomorphisms $g_i : R_0[X] \to A$ by $X \mapsto a_i$ for $i = 1, 2$. Then, $fg_1 = fg_2$ but $g_1 \neq g_2$, a contradiction. □

In the sequel, let $f : X \to Y$ be a morphism of sober spaces, and $f^{\#} : C(Y) \to C(X)$ the corresponding homomorphism of complete idealic rings (for complete idealic rings, see Appendix).

PROPOSITION 1.1.2. *The followings are equivalent:*

(i) *f is injective.*
(ii) *f is monic.*
(iii) *Any prime element of $C(X)$ is in the image of $f^{\#}$.*

PROOF. (i)⇔(ii) follows from Lemma 1.1.1.

(i)⇒(iii): Let x be a point of X. It suffices to show that $f^{-1}(\overline{f(x)}) = \overline{\{x\}}$. Let $w \in f^{-1}(\overline{f(x)})$ be the generic point of an irreducible component of $f^{-1}(\overline{f(x)})$. Then we have $f(w) = f(x)$. Since f is injective, $w = x$. This shows that $f^{-1}(\overline{f(x)}) = x$.

(iii)⇒(i): Note that $\operatorname{Spec} C(X) \subset \operatorname{Im} f^{\#}$ shows that $\overline{\{x\}} = f^{-1}(\overline{f(x)})$ for any point x of X. If $f(x) = f(x')$ for two points of X, then

$$\overline{\{x\}} = f^{-1}(\overline{f(x)}) = f^{-1}(\overline{f(x')}) = \overline{\{x'\}}.$$

Since X is sober, x coincides x'. □

PROPOSITION 1.1.3. *The followings are equivalent:*

(i) *$f^{\#}$ is surjective.*
(ii) *$\operatorname{Im} f$ is homeomorphic to X.*

PROOF. (i)⇒(ii): Since $f^{\#}$ is surjective, it is epic, hence f is monic. This shows that f is injective. The surjectivity of $f^{\#}$ shows that $f^{-1}(\overline{f(z)} \cap \operatorname{Im} f) = z$ for any closed subset z of X. Hence $\overline{f(z)} \cap \operatorname{Im} f = f(z)$, which implies $f(z)$ is closed in $\operatorname{Im} f$.

(ii)⇒(i): Since f is injective, $f^{-1}(f(z)) = z$ for any closed subset z of X. On the other hand, $f(z)$ is closed in $\operatorname{Im} f$ since f is homeomorphic onto the image. Therefore, $f^{-1}(\overline{f(z)}) = z$, which implies $f^{\#}$ is surjective. □

Next, we will investigate the condition when a morphism $f : X \to Y$ of sober spaces becomes epic. Let (**IIRng**†) be the catogory of complete idealic semirings (see Appendix). First, note that the functor $C : (\mathbf{Sob})^{\mathrm{op}} \to (\mathbf{IIRng}^{\dagger})$ is fully faithful, since it is the right adjoint and left inverse of Spec ([Tak]). An object R of (**IIRng**†) is *spatial*, if R is isomorphic to $C(X)$ for some sober space X.

REMARK 1.1.4. There exist some non-spatial complete II-rings. Let $C(\mathbb{R})$ be the complete II-rings of closed subsets of the real line with the standard topology. Let $R = C(\mathbb{R})/\equiv$ be the quotient complete II-ring, where $\equiv$ is the congruence generated by $Z = \overline{Z^o}$ for any Z, where Z^o is the open kernel of Z. Then, R is a non-trivial II-ring, but has no points; see [Ste].

LEMMA 1.1.5. *Let R be an object of* (**IIRng**†)*, and $R[t]$ be a polynomial complete idealic semiring with idempotent multiplication.*

(1) *An element of $R[t]$ can be expressed by $a + bt$, where a, b are elements of R and $a \leq b$.*

(2) *A prime element p of $R[t]$ is either of the following:*

(a) *$p = a + bt$, where a, b are prime elements of R and $a \leq b$.*

(b) *$p = a + t$, where a is a prime element.*

(3) *If an object R of* (**IIRng**†) *is spatial, then so is $R[t]$.*

(4) *In particular, a morphism $f : X \to Y$ of sober spaces is epic if and only if $f^{\#} : C(Y) \to C(X)$ is monic.*

PROOF. (1) Easy.

(2) Let $p = a + bt$ be a prime element. It is easy to see that $a \in R$ must be prime. Also, b must be prime or 1, since $xt \cdot yt = xyt$.

(3) Easy.

(4) The 'if' part is obvious, since $(\mathbf{Sob})^{\mathrm{op}}$ is can be regarded as a full subcategory of (**IIRng**†) via C. If $f^{\#}$ is not monic, we have two distinct

morphisms $g, h : \mathbb{F}_1[t] \to C(Y)$ such that $f^\# g = f^\# h$, where $\mathbb{F}_1$ is the initial object of (**IIRng**†). But since $\mathbb{F}_1[t]$ is spatial, we conclude that f is not epic.

PROPOSITION 1.1.6. *The followings are equivalent:*

(1) *f is epic.*
(2) *$f^\#$ is injective.*
(3) *$f^\#$ is monic (in* (**IIRng**†)*).*
(4) *$\mathrm{Im}\, f \cap z$ is dense in z, for any closed subset z of Y.*

PROOF. (i)⇔(ii)⇔(iii) is a consequence of Lemma 1.1.1 and 1.1.5.

(ii)⇒(iv): For any closed subset z of X, let z' be the closure of $\mathrm{Im}\, f \cap z$. Then we have $f^\#(z') = f^\#(z)$. Since $f^\#$ is injective, we have $z' = z$, hence the result follows.

(iv)⇒(ii): Suppose $f^\#(z) = f^\#(z')$ for some closed subsets z, z' of Y. The equation $ff^{-1}(z) = \mathrm{Im}\, f \cap z$ induces

$$z = \overline{\mathrm{Im}\, f \cap z} = \overline{\mathrm{Im}\, f \cap z'} = z'.$$

□

1.2 – *Coherent spaces*

PROPOSITION 1.2.1. *Let $f : X \to Y$ be an epimorphism of sober spaces, and X be noetherian. Then f is surjective.*

PROOF. Let y be any point of Y, and $Z = \overline{\{y\}}$ be the irreducible subset corresponding to y. Since X is noetherian, $f^{-1}(Z)$ can be covered by a finite number of irreducible closed subsets: $f^{-1}(Z) = \cup_i^{<\infty} W_i$. Let ξ_i be the generic point of W_i for each i. Since the image of f is dense in Z and Z is irreducible, at least one of the ξ_i's must be mapped to y. □

The proof of the next theorem requires some preliminaries on ultrafilters (see [CN], for example). The reader who knows well may skip and go on to the next theorem.

DEFINITION 1.2.2. *Let $\mathcal{S}$ be a non-empty set.*

(1) A *filter* $\mathscr{F}$ on $\mathcal{S}$ is a non-empty subset of $\mathscr{P}(\mathcal{S})$ satisfying:
  (i) $\emptyset \notin \mathscr{F}$.
  (ii) If $A \in \mathscr{F}$ and $A \subset B$, then $B \in \mathscr{F}$.
  (iii) If A and B are elements of $\mathscr{F}$, then $A \cap B \in \mathscr{F}$.

The set of filters becomes a poset by inclusions.

(2) *A maximal filter with respect to inclusions is called an* ultrafilter.

A filter $\mathscr{U}$ is an ultrafilter if and only if $a \in \mathscr{U}$ or $a^c \in \mathscr{U}$ for any subset a of $\mathcal{S}$. Also, note that exactly one of a or a^c is in $\mathscr{U}$.

For any $s \in \mathcal{S}$,

$$\mathscr{U}_s = \{a \subset \mathcal{S} \mid s \in a\}$$

becomes an ultrafilter, which is called *principal*. An ultrafilter is principal, if and only if it contains a finite subset of $\mathcal{S}$.

Let $\mathscr{S}$ be a subset of $\mathscr{P}(\mathcal{S})$ satisfying

(i) $\emptyset \notin \mathscr{S}$.

(ii) If A and B are elements of $\mathscr{S}$, then $A \cap B \in \mathscr{S}$.

Then there exists a ultrafilter containing $\mathscr{S}$, using the axiom of choice.

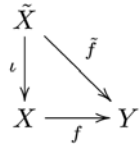
A filter $\mathscr{F}$ on a non-empty set is *prime*, if $a \cup b \in \mathscr{F}$ implies either $a \in \mathscr{F}$ or $b \in \mathscr{F}$. One can easily prove that the notion of prime filters is equivalent to that of ultrafilters.

LEMMA 1.2.3. *Let $X = \{x_\lambda\}_\lambda$ be a set, and $X_\lambda = \{x_\lambda\}$ be one-pointed spaces, regarded as coherent spaces. Let $X_\infty = \coprod_\lambda X_\lambda$ be the coproduct of X_λ's in the category of coherent spaces. Then, any point of X_∞ corresponds to a ultrafilter on X.*

PROOF. First, note that X is isomorphic to $\mathrm{Spec}\,(\prod_\lambda \mathbb{F}_1)$, where $\mathbb{F}_1$ is the initial object in the category of (**IIRng**). Hence, a closed subset of X_∞ corresponds to a filter on X, and any point of X_∞ corresponds to a prime filter, in other words, a ultrafilter on X. □

THEOREM 1.2.4. *Let $f : X \to Y$ be an epimorphism of coherent spaces. Then f is surjective.*

PROOF. Let $y_0 \in Y$ be any point of Y, and Y_0 be the closure of $\{y_0\}$ in Y. Since f is epic, $\{y_\lambda\}_\lambda = \mathrm{Im}\, f \cap Y_0$ is dense in Y_0. Assume that $y_0 \notin \mathrm{Im}\, f \cap Y_0$. Then, $\mathrm{Im}\, f \cap Y_0$ must be an infinite set, since if it is finite, then its closure is equal to $\cup_\lambda^{<\infty} \overline{\{y_\lambda\}}$, which is a proper closed subset of Y_0. Choose $x_\lambda \in X$ such that $f(x_\lambda) = y_\lambda$ for each λ, and set $S = \{x_\lambda\}_\lambda$. Also, let $\tilde{X} = \coprod_\lambda \{x_\lambda\}$ be the coproduct of $\{x_\lambda\}$'s in the category of coherent spaces. By Lemma 1.2.3, the points of $\tilde{X}$ correspond to the ultrafilters on S. We have the natural commutative diagram



Next, we define a map $\varphi : C(Y_0) \setminus \{Y_0\} \to \mathscr{P}S \setminus \{\emptyset\}$ by sending Z to $\{x \in S \mid f(x) \notin Z\}$. This is well defined, since $f(S)$ is dense in Y_0. Also, note that $\operatorname{Im}\varphi$ is stable under taking finite intersections: $\varphi(Z_1) \cap \varphi(Z_2) = \varphi(Z_1 \cup Z_2)$. It follows that there is an ultrafilter $\mathscr{U}$ on S, containing the image of φ. Let x_0 be the point of $\tilde{X}$, corresponding to $\mathscr{U}$. We claim that $\tilde{f}(x_0) = y_0$, from which $f(\iota(x_0)) = y_0$ follows. Indeed, the image of x_0 is the generic point of the intersections of $Z \in C(X)_{\mathrm{cpt}}$ such that $x_0 \in \tilde{f}^{-1}(Z)$. Hence, it suffices to show that Y_0 is the only closed subset satisfying the condition. If $Z \neq Y_0$, then Z is a proper closed subset of Y_0, and $x_0 \in \tilde{f}^{-1}(Z)$ implies $\varphi(Z)^c \in \mathscr{U}$. But on the other hand, $\varphi(Z) \in \mathscr{U}$, which is a contradiction to $\mathscr{U}$ being a ultrafilter. □

COROLLARY 1.2.5. *Let $f : X \to Y$ be a morphism of coherent spaces. Then, the image of f with its induced topology is also coherent.*

PROOF. The morphism f corresponds to the homomorphism $f^{\#} : C(Y)_{\mathrm{cpt}} \to C(X)_{\mathrm{cpt}}$ of II-rings. Let R be the image of $f^{\#}$. Then, $C(Y)_{\mathrm{cpt}} \to R$ is surjective, and $R \to C(X)_{\mathrm{cpt}}$ is injective. Set $W = \operatorname{Spec} R$. Then, f factors through W, with $X \to W$ epic (hence surjective by Theorem 1.2.4) and $W \subset Y$ an immersion by Proposition 1.1.3. This tells that W coincides with the image of f. □

EXAMPLE 1.2.6. (1) Let $Y = |\mathbb{A}^1_{\mathbb{C}}|$ be the underlying space of the affine line over $\mathbb{C}$, and $X = Y(\mathbb{C})$ be the set of closed points of Y, endowed with a discrete topology. Then, the natural map $X \to Y$ is a morphism of sober spaces. This is an epimorphism by Proposition 1.1.6, but not surjective, since the image does not contain the generic point of Y.

On the other hand, we have a morphism $\underline{\mathrm{alg}}(X) \to Y$ of coherent spaces: here, $\underline{\mathrm{alg}}$ is the left adjoint of the underlying functor (**Coh**) $\to$ (**Sob**) ([Tak]). In this case, $\underline{\mathrm{alg}}(X)$ is just the coproduct $\coprod_{x \in X} \{x\}$ in the category of coherent spaces. This is also epic, hence surjective by 1.2.4. The non-principal points of $\underline{\mathrm{alg}}(X)$ maps onto the generic point of Y.

(2) Let X be as above, and set $V = |\mathbb{A}^2_{\mathbb{C}}|$, the affine plane over $\mathbb{C}$. Since there is a non-canonical bijection $\varphi : \mathbb{C} \to \mathbb{C}^2$, there exists a map

$\varphi : X \to V$, the image of which is the set of closed points of V. When we algebraize X, we again obtain a surjective map $\underline{\mathrm{alg}}(X) \to V$. Some of the non-principal points of X map to a generic point of a curve on V, others map to the generic point of V. These two examples tell us that the non-principal points of $\underline{\mathrm{alg}}(X)$ behave like 'universal generic points' of X, although the Krull dimension of $\underline{\mathrm{alg}}(X)$ is zero.

The next theorem is important when we consider valuative criteria.

THEOREM 1.2.7. *Let $f : X \to Y$ be a dominant morphism of coherent spaces. Then, any minimal point of Y (that is, the generic point of an irreducible component of Y) is contained in the image of f.*

PROOF. Let y_0 be any minimal point of Y, and $\{U^\lambda\}_\lambda$ be the filtered system of quasi-compact open neighborhood of y_0. For each λ, $f^{-1}(U^\lambda) \to U^\lambda$ is dominant since f is so. Set $U^\infty = \varprojlim_\lambda U^\lambda$. This is a pointed space $\{y_0\}$, since the underlying functor (**Coh**) $\to$ (**Set**) preserves limits ([Tak]). We claim that $f^{-1}(U^\infty) \to U^\infty$ is dominant (in particular, $f^{-1}(U^\infty) \neq \emptyset$). Since $C(U^\infty)_{\mathrm{cpt}}$ and $C(f^{-1}(U^\infty))_{\mathrm{cpt}}$ are naturally isomorphic to $\varinjlim C(U^\lambda)_{\mathrm{cpt}}$ and $\varinjlim C(f^{-1}(U^\lambda))_{\mathrm{cpt}}$ respectively, it suffices to show that the homomorphism

$$f^\# : \varinjlim C(U^\lambda)_{\mathrm{cpt}} \to \varinjlim C(f^{-1}(U^\lambda))_{\mathrm{cpt}}$$

satisfies $f^\#(Z) = 0 \Rightarrow Z = 0$. Let Z be an element satisfying $f^\#(Z) = 0$. Since $\{C(U^\lambda)_{\mathrm{cpt}}\}_\lambda$ is a filtered inductive system, Z and $f^\#(Z)$ can be represented by an element of $C(U^\lambda)_{\mathrm{cpt}}$ and $C(f^{-1}(U^\lambda))_{\mathrm{cpt}}$ for some λ, respectively. Since $C(U^\lambda)_{\mathrm{cpt}} \to C(f^{-1}(U^\lambda))_{\mathrm{cpt}}$ is dominant, Z must be zero. □

COROLLARY 1.2.8. *Let X be a coherent subspace of a coherent space Y. Then, the closure of X consists of all points which are specializations of points on X.*

PROOF. Let Z be the set of points which are specializations of points on X. It is clear that $Z \subset \overline{X}$, so we will show the converse. Let y be any point in the closure of X in Y. Since $\overline{X}$ is also a coherent subspace, there exists a minimal point y_0 of $\overline{X}$ which is a generalization of y. Since $X \to \overline{X}$ is dominant, y_0 is contained in X by Theorem 1.2.7. Since Z is stable under specializations, y must be in Z. □

2. The category of $\mathscr{A}$-Schemes

In [Tak], we introduced the definition of $\mathscr{A}$-schemes. The advantage of the notion of $\mathscr{A}$-schemes is not only generalizing the concept of schemes to other algebraic systems, but also giving the way to infinite categorical operations: in fact, the category of $\mathscr{A}$-schemes is small complete and co-complete. A finite patching over quasi-compact open sets, and fiber products commute with those of $\mathscr{Q}$-schemes, namely, ordinary schemes.

Notations: from now on, the homomorphism $C(Y)_{\mathrm{cpt}} \to C(X)_{\mathrm{cpt}}$ associated to a morphism $f : X \to Y$ of $\mathscr{A}$-schemes is denoted by f^{-1}, or $|f|^{-1}$. We use the notation $f^{\#}$ for the morphism of structure sheaves $\mathscr{O}_Y \to f_*\mathscr{O}_X$.

2.1 – *Co-completeness*

First, we begin with describing what $\mathscr{A}$-schemes is like when the algebraic system is that of rings.

PROPOSITION 2.1.1. *Let σ be an algebraic system of rings, and* (**LRCoh**) *be the category of locally ringed coherent spaces and quasi-compact morphisms. Then, there is a natural underlying functor* ($\mathscr{A}$**-Sch**) $\to$ (**LRCoh**) *defined by* $(X, \mathscr{O}_X, \beta_X) \mapsto (X, \mathscr{O}_X)$. *Further, this functor is fully faithful.*

PROOF. Let $X = (X, \mathscr{O}_X, \beta_X)$ be an $\mathscr{A}$-scheme. What we have to show first is that X is a locally ringed space, i.e. $\mathscr{O}_{X,x}$ is a local ring for any $x \in X$. Let $\mathfrak{M}_x$ be a subset of $\mathscr{O}_{X,x}$, consisting of germs a such that $\beta_X\alpha_2(a) \ni x$. We will show that this is the unique maximal ideal of $\mathscr{O}_{X,x}$.

Let a, b be two elements of $\mathfrak{M}_x$. We have $\alpha_2(a+b) \leq \alpha_2(a) + \alpha_2(b)$; recall that $\alpha_2(a)$ is the principal ideal generated by a. Hence,

$$\beta_X\alpha_2(a+b) \leq \beta_X\alpha_2(a) + \beta_X\alpha_2(b) \leq x + x = x,$$

which shows that $a + b \in \mathfrak{M}_X$. It is easier to show that $ca \in \mathfrak{M}_x$ for any $c \in \mathscr{O}_{X,x}$ and $a \in \mathfrak{M}_x$. Hence, $\mathfrak{M}_x$ is an ideal of $\mathscr{O}_{X,x}$.

Suppose $\langle U, a\rangle \in \mathscr{O}_{X,x}$ is not contained in $\mathfrak{M}_x$. Set $Z = \beta_X\alpha_2(a)$. This does not contain x. Set $V = U \setminus Z$. Since restriction morphisms reflect localizations (see appendix § 5 for the terminology), $a|_V$ is invertible, hence a is invertible in $\mathscr{O}_{X,x}$. This shows that $\mathscr{O}_{X,x}$ is local, the maximal ideal of which is $\mathfrak{M}_x$.

Let $f : X \to Y$ be a morphism of $\mathscr{A}$-schemes. It suffices to show that $f^{\#} : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local homomorphism for any $x \in X$. Let a be an element of $\mathfrak{M}_y$, where $y = f(x)$. Then,

$$\beta_X \alpha_2(f^{\#}(a)) = \beta_X(\alpha_1 f)(\alpha_2(a)) = |f|^{-1}\beta_Y \alpha_2(a).$$

Since $\beta_Y \alpha_2(a) \leq y$, we have $|f|^{-1}\beta_X \alpha_2(a) \leq x$, which shows that $\mathfrak{M}_y \subset (f^{\#})^{-1}\mathfrak{M}_x$.

Hence, we have a functor $U : (\mathscr{A}\textbf{-Sch}) \to (\textbf{LRCoh})$. It remains to show that this functor is fully faithful. Let X, Y be two $\mathscr{A}$-schemes, and $f = (|f|, f^{\#}) : UX \to UY$ be a morphism of locally ringed spaces. It suffices to show that the following diagram

$$\begin{array}{ccc} \alpha_1\mathscr{O}_Y & \xrightarrow{\alpha_1 f^{\#}} & |f|_*\alpha_1\mathscr{O}_X \\ {\scriptstyle \beta_Y}\downarrow & & \downarrow{\scriptstyle |f|_*\beta_X} \\ \tau_Y & \xrightarrow[|f|^{-1}]{} & |f|_*\tau_X \end{array}$$

is commutative. Let $\mathfrak{a}$ be a section of $\alpha_1\mathscr{O}_Y$. Then

$$|f|^{-1} \circ \beta_Y(\mathfrak{a}) = \{x \mid \mathfrak{M}_{Y,f(x)} \supset \mathfrak{a}\}, \quad f_*\beta_X \circ \alpha_1 f^{\#}(\mathfrak{a}) = \{x \mid \mathfrak{M}_{X,x} \supset f^{\#}\mathfrak{a}\},$$

where $\mathfrak{M}_{Y,f(x)}$ and $\mathfrak{M}_{X,x}$ are the maximal ideals of $\mathscr{O}_{Y,f(x)}$, $\mathscr{O}_{X,x}$, respectively. But the right-hand sides of the both equations coincide, since $f^{\#} : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local homomorphism for any x. □

In the sequel, we fix a schematizable algebraic type $\mathscr{A} = (\sigma, \alpha_1, \alpha_2, \gamma)$ ([Tak]).

DEFINITION 2.1.2. (1) *An $\mathscr{A}$-scheme X is a $\mathscr{Q}$-*scheme*, if it is locally isomorphic to* $\mathrm{Spec}^{\mathscr{A}} R$*, for some σ-algebra R.*

(2) *Let $(\mathscr{Q}\textbf{-Sch})$ be the full subcategory of $(\mathscr{A}\textbf{-Sch})$, consisting of $\mathscr{Q}$-schemes.*

Proposition 2.1.1 tells that, if σ is the algebraic system of rings, then $(\mathscr{Q}\textbf{-Sch})$ is the category of coherent schemes and quasi-compact morphisms, since a morphism of schemes is a morphism of locally ringed spaces.

PROPOSITION 2.1.3. *The category $(\mathscr{Q}\textbf{-Sch})$ admits finite patching via quasi-compact opens, namely: let $X_1, \cdots, X_n$ be $\mathscr{Q}$-schemes, $U_{ij} \subset X_i$ be*

quasi-compact open subsets, and $\varphi_{ji} : U_{ij} \to U_{ji}$ *be isomorphisms satisfying* $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ *on* $U_{ij} \cap U_{jk}$. *Then, there exists a co-equalizer* X *of* $\coprod_{ij} U_{ij} \rightrightarrows \coprod_i X_i$, *such that* $\coprod_i X_i \to X$ *is a quasi-compact open covering.*

Note that, when we speak of a covering, it must be always surjective.

PROOF. By induction on n, it suffices to prove for $n = 2$: let X_1, X_2 be two $\mathcal{O}$-schemes, and $X_1 \hookleftarrow U \hookrightarrow X_2$ be the intersection quasi-compact open subscheme of X_1 and X_2. Let X be the amalgamation $X_1 \coprod_U X_2$ of X_1 and X_2 along U. This is well defined, and coincides with the usual topology, since X is coherent, thanks to U being quasi-compact. This also shows that $\{X_i \to X\}_{i=1,2}$ is indeed a covering. □

The next lemma is peculiar to II-rings.

LEMMA 2.1.4. *Let* $C^\bullet = \{C^\lambda\}$ *be a projective system of II-rings. Let* C *be the limit of* $C^\bullet$, *and* $\pi_\lambda : C \to C^\lambda$ *be the natural morphisms. Then, for any* $Z \in C$, *the localization* C_Z *along* Z *is naturally isomorphic to* $\varinjlim_\lambda (C^\lambda)_{\pi_\lambda Z}$.

PROOF. Since Z maps to 1 by the morphism $C \to (C_\lambda)_{\pi_\lambda Z}$, We have a natural morphism $C_Z \to (C^\lambda)_{\pi_\lambda Z}$. This in turn gives a natural morphism $C_Z \to \varinjlim_\lambda (C^\lambda)_{\pi_\lambda Z}$.

Next, we see that if $\varphi : C_Z \to C_Z$ is a endomorphism such that $\pi_\lambda \varphi = \pi_\lambda$ for any λ, then φ is the identity. $\pi_\lambda \varphi(x) = \pi_\lambda x$ in $(C^\lambda)_{\pi_\lambda Z}$ is equivalent to $\pi_\lambda(Z\varphi(x)) = \pi_\lambda(Zx)$ in C^λ. Since this holds for any λ, we have $Z\varphi(x) = Zx$, which is equivalent to $\varphi(x) = x$ in C_Z.

Finally, we construct $\varinjlim_\lambda (C^\lambda)_{\pi_\lambda Z} \to C_Z$. The map $f : \varinjlim_\lambda C^\lambda \to C_Z$ is already defined, hence we only need to verify that $f(x) = f(y)$ implies $x = y$ in $\varinjlim_\lambda (C^\lambda)_{\pi_\lambda Z}$, but this is obvious.

Combining all the arguments, we see that C_Z coincides with $\varinjlim_\lambda (C^\lambda)_{\pi_\lambda Z}$. □

PROPOSITION 2.1.5. (1) *Let* $\{X_\lambda\}$ *be an inductive system of coherent spaces, and* $X = \varprojlim_\lambda X_\lambda$. *Then, there is a natural isomorphism* $\tau_X \simeq \varprojlim_\lambda \iota_{\lambda *} \tau_{X_\lambda}$, *where* $\iota_\lambda : X_\lambda \to X$ *are the induced morphisms.*

(2) *Let* $\{X^\lambda\}$ *be a filtered projective system of coherent spaces, and* $X = \varinjlim_\lambda X^\lambda$. *Then, there is a natural isomorphism* $\tau_X \simeq \varprojlim_\lambda \pi_\lambda^{-1} \tau_{X^\lambda}$, *where* $\pi_\lambda : X \to X^\lambda$ *are the induced morphisms.*

PROOF. (1) This follows from the above lemma.

(2) First, there is a natural morphism $\tau_{X^\lambda} \to \pi_{\lambda *} \tau_X$. Taking the adjoint, we obtain $f_\lambda : \pi_\lambda^{-1} \tau_{X^\lambda} \to \tau_X$. Taking the limit, we obtain $\varprojlim_\lambda \pi_\lambda^{-1} \tau_{X^\lambda} \to \tau_X$.

Next, we show that if $\varphi : \tau_X \to \tau_X$ is an endomorphism with $\varphi f_\lambda = f_\lambda$ for any λ, then φ is the identity. But this follows from the fact that the projective system is filtered. We can also construct a natural morphism $\tau_X \to \varprojlim \pi_\lambda^{-1}\tau_{X^\lambda}$, using the filteredness. Combining all these, we obtain the required isomorphism. □

PROPOSITION 2.1.6. (1) *The category* ($\mathscr{A}$-**Sch**) *is small co-complete.*

(2) *The category* ($\mathscr{A}$-**Sch**) *admits finite patchings via quasi-compact opens. Moreover, the inclusion functor* $I : (\mathscr{Q}$-**Sch**$) \to (\mathscr{A}$-**Sch**$)$ *preserves finite patchings via quasi-compact opens.*

PROOF. (1) Let $\{X_\lambda\}$ be a small inductive system of $\mathscr{A}$-schemes. First, we will construct the $\mathscr{A}$-scheme $X = (|X|, \mathscr{O}_X, \beta_X)$. The underlying space $|X|$ is given by the colimit of the underlying spaces $|X_\lambda|$. Set $|\iota_\lambda| : |X| \to |X_\lambda|$ be the associated morphisms. The structure sheaf $\mathscr{O}_X$ is defined by the limit of $|\iota_\lambda|_*\mathscr{O}_{X_\lambda}$, as a ($\sigma$-**alg**)-valued sheaf on X.

We have a morphism

$$\alpha_1|\iota_\lambda|_*\mathscr{O}_{X_\lambda} \simeq |\iota_\lambda|_*\alpha_1\mathscr{O}_{X_\lambda} \overset{|\iota_\lambda|_*\beta_{X_\lambda}}{\longrightarrow} |\iota_\lambda|_*\tau_{X_\lambda},$$

which extends to give $\alpha_1\mathscr{O}_X \to |\iota_\lambda|_*\tau_{X_\lambda}$. Taking the limit and using Proposition 2.1.5, we obtain

$$\beta_X : \alpha_1\mathscr{O}_X \to \varprojlim_\lambda |\iota_\lambda|_*\tau_{X_\lambda} \simeq \tau_X.$$

We will verify that restriction maps reflect localizations. Let $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ be a restriction map, and let $Z = U \setminus V$ be the closed subset of U. Let Z_λ be the inverse image of Z by $X_\lambda \to X$, so that we will denote Z by $(Z_\lambda)_\lambda$. It suffices to show that if $a = (a_\lambda)_\lambda \in \mathscr{O}_X(U)$ satisfies $\beta_X\alpha_2(a) \geq Z$, then a is invertible in $\mathscr{O}_X(V)$. Since $\beta_{X_\lambda}\alpha_2(a_\lambda) \geq Z_\lambda$, a_λ is invertible in $\mathscr{O}_{X_\lambda}(V_\lambda)$, where $V = (V_\lambda)_\lambda$. The uniqueness of the inverse element shows that a is also invertible in $\mathscr{O}_X(V)$. Thus, we have defined an $\mathscr{A}$-scheme X. We also have natural morphisms $\iota_\lambda : X_\lambda \to X$.

We will show that X is actually the colimit. Let $j_\lambda : X_\lambda \to Y$ be morphisms, compatible with the transition morphisms. There is a unique natural morphism $|j| : |X| \to |Y|$ between the underlying spaces. The morphisms $\mathscr{O}_Y \to |j_\lambda|_*\mathscr{O}_{X_\lambda}$ give

$$j^\# : \mathscr{O}_Y \to \varprojlim_\lambda |j_\lambda|_*\mathscr{O}_{X_\lambda} \simeq \varprojlim_\lambda |j|_*|\iota_\lambda|_*\mathscr{O}_{X_\lambda} \simeq |j|_*\varprojlim_\lambda |\iota_\lambda|_*\mathscr{O}_{X_\lambda} \simeq |j|_*\mathscr{O}_X.$$

This is the unique morphism which satisfies $|j|_*\iota_\lambda^\# \circ j^\# = j_\lambda^\#$. It is easy to see that $j = (|j|, j^\#)$ commutes with the support morphisms β_X and β_Y.

Thus, we obtained a unique morphism $j : X \to Y$ of $\mathscr{A}$-schemes, hence X is indeed the co-limit.

(2) We only have to show that if X is obtained by patching $X_1, \cdots, X_n$ by quasi-compact opens, then $\{X_i \to X\}$ is a covering. By induction, it suffices to prove for $n = 2$. There is a surjective morphism $X_1 \coprod X_2 \to X$, hence it remains to show that:

If R_1 and R_2 are two II-rings, then any element of the spectrum of $R = R_1 \times R_2$ are in the image of $\operatorname{Spec} R_1 \to \operatorname{Spec} R$ or that of $\operatorname{Spec} R_2 \to \operatorname{Spec} R$.

This is easy to prove, so we will skip.

It is clear from the construction that the functor I preserves finite patching via quasi-compact opens. □

REMARK 2.1.7. Even though small coproducts exist in the category of $\mathscr{A}$-schemes, their behavior is somewhat different from those in schemes. For example, a point of an infinite coproduct $X = \coprod_\lambda X_\lambda$ does not necessarily come from a point of some X_λ, i.e strictly speaking, $\{X_\lambda \to X\}$ is not a covering of X.

EXAMPLE 2.1.8. The spectrum functor is mal-behaved, when we consider infinite product of rings: the underlying space $\operatorname{Spec} \prod_n R_n$ does not coincide with the co-product $\coprod_n \operatorname{Spec} R_n$, even in the category of coherent spaces.

Here is a typical counterexample: Let k be a field, and set $R = \prod_{n\in\mathbb{N}} R_n$, where $R_n = k[x]/(x^n)$. Then, the spectrum of R_n is a point for any R_n, hence $\operatorname{Spec} \prod \alpha_1 R_n$ must be the set of all ultrafilters over $\mathbb{N}$, in particular, its Krull dimension is zero.

On the other hand, the Krull dimension of $\operatorname{Spec} R$ is not zero: fix a non-principal ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, and define an ideal $\mathfrak{M}$ of R as

$$f = (f_n)_n \in \mathfrak{M} \Leftrightarrow f_n \notin R_n^\times \quad \text{a.e. } \mathfrak{U}.$$

Here, $P(s)$ a.e $\mathfrak{U}$ for a condition $P(s)$ of s means that the set $\{s \mid P(s)\}$ belongs to $\mathfrak{U}$. This is a maximal ideal of R (in fact, any maximal ideal of R is of this form). On the other hand, define an ideal $\mathfrak{p}$ of R as

$$f = (f_n)_n \in \mathfrak{p} \Leftrightarrow \text{For any } c > 0,\ f_n \in (x^{\lceil cn \rceil}) \text{a.e. } \mathfrak{U}.$$

This is also a prime ideal, and obviously smaller than $\mathfrak{M}$. Hence, the Krull dimension of R is not 0. In fact, one can prove similarly that the Krull dimension of R is infinite.

It is obvious that $\{\operatorname{Spec} R_n \to \operatorname{Spec} R\}_n$ is not a covering of $\operatorname{Spec} R$.

2.2 – *Decomposition of morphisms*

In this subsection, we prove that there is a functorial decomposition of morphisms in the category of $\mathscr{A}$-schemes. This decomposition plays an important role in the proof of completeness of ($\mathscr{A}$-**Sch**), since it gives an upper bound of the cardinality of a family of morphisms. Also, note that this decomposition is peculiar to the category of $\mathscr{A}$-schemes.

DEFINITION 2.2.1. *A morphism $f : X \to Y$ of $\mathscr{A}$-schemes is a* P-morphism *if:*

(1) $|f|$ *is epic, i.e.* $f^{-1} : C(Y)_{\mathrm{cpt}} \to C(X)_{\mathrm{cpt}}$ *is injective,*
(2) $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$ *is injective.*

Let us mention some trivial facts:

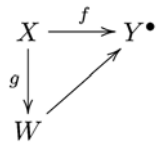
PROPOSITION 2.2.2. (1) *A P-morphism is epic.*
(2) *P-morphisms are stable under compositions.*
(3) *If gf is a P-morphism, then so is g.*

These are all obvious, so we will skip the proof.

PROPOSITION 2.2.3. *Let $\{f_\lambda : X_\lambda \to Y_\lambda\}_\lambda$ be an inductive system of P-morphisms of $\mathscr{A}$-schemes, and set $X_\infty = \varinjlim_\lambda X_\lambda$ and $Y_\infty = \varinjlim_\lambda Y_\lambda$. Then, the natural morphism $f : X_\infty \to Y_\infty$ is also a P-morphism.*

PROOF. First, we will see that $C(Y_\infty)_{\mathrm{cpt}} \to C(X_\infty)_{\mathrm{cpt}}$ is injective. Let $Z = (Z_\lambda)_\lambda$ and $W = (W_\lambda)_\lambda$ be two elements of $C(Y_\infty)_{\mathrm{cpt}}$ with $f^{-1}(Z) = f^{-1}(W)$. Since f^{-1} is defined by $(Z_\lambda)_\lambda \mapsto (f_\lambda^{-1}Z_\lambda)_\lambda$, this implies that $f_\lambda^{-1}Z_\lambda = f_\lambda^{-1}W_\lambda$. Since f_λ is a P-morphism, Z_λ and W_λ must coincide for all λ, which is equivalent to $Z = W$. Hence $C(Y_\infty)_{\mathrm{cpt}} \to C(X_\infty)_{\mathrm{cpt}}$ is injective. A similar argument shows that $\mathscr{O}_{Y_\infty} \to f_*\mathscr{O}_{X_\infty}$ is also injective, so that f is a P-morphism. □

DEFINITION 2.2.4. (1) *Fix a small index category I. Let $Y^\bullet : I \to$ ($\mathscr{A}$-**Sch**) be a small projective system of $\mathscr{A}$-schemes, and $f : \Delta(X) \to Y^\bullet$ be a morphism in ($\mathscr{A}$-**Sch**)^I, where $\Delta : (\mathscr{A}$-**Sch**$) \to (\mathscr{A}$-**Sch**$)^I$ is the diagonal functor. (In the sequel, we simply denote $\Delta(X)$ by X for brevity.) Let $\mathscr{S}$ be the set of isomorphism classes of the commutative diagram*



where g is a P-morphism. $\mathscr{S}$ is small, from the property of P-morphisms. Set $I(X, Y^\bullet) = \varinjlim_{W \in \mathscr{S}} W$. Then, by Proposition 2.2.3, *$X \to I(X, Y^\bullet)$ becomes a P-morphism.*

(2) *Let $f : X \to Y^\bullet$ be as above. f is a* Q-morphism, *if $X \to I(X, Y^\bullet)$ is an isomorphism.*

Roughly speaking, a P-morphism can be regarded as a schematic surjection and a Q-morphism as a schematic immersion. Thus, if $X \to I(X, Y) \to Y$ is the PQ-decomposition of a morphism $f : X \to Y$, $I(X, Y)$ can be regarded as the 'image scheme' of f.

The next proposition is purely category-theoretical.

PROPOSITION 2.2.5. (1) *Let $f : X \to Y^\bullet$ be a morphism from an $\mathscr{A}$-scheme X to a projective system $Y^\bullet$ of $\mathscr{A}$-schemes. Then, the morphism $h : I(X, Y^\bullet) \to Y^\bullet$ is a Q-morphism.*

(2) *A morphism $f : X \to Y$ of $\mathscr{A}$-schemes is an isomorphism if and only if f is a P-morphism and Q-morphism.*

PROOF. (1) Let $g : X \to I(X, Y^\bullet)$ be the induced P-morphism. Set $W = I(I(X, Y^\bullet), Y^\bullet)$, and let $\tilde{h} : W \to Y^\bullet$ be the induced morphism. Since $\pi : I(X, Y) \to W$ is a P-morphism, $X \to W$ is also a P-morphism by (1). Hence, there is a morphism $\imath : W \to I(X, Y^\bullet)$ such that $\imath \circ \pi \circ g = g$. This implies that $\imath \circ \pi$ is the identity, since g is epic. Hence, $\pi \circ \imath \circ \pi = \pi$, and this shows that $\pi \circ \imath$ is the identity since π is also epic. This shows that $I(X, Y^\bullet) \to W$ is an isomorphism.

(2) The proof is similar to (1). □

COROLLARY 2.2.6. (1) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a series of morphisms of $\mathscr{A}$-schemes. If gf is a Q-morphism, then so is f.*

(2) *Let $f : X \to Y^\bullet$ be a morphism of $\mathscr{A}$-schemes. Then, the decomposition $f = hg$ of f into a P-morphism g and a Q-morphism h is unique up to unique isomorphism.*

We refer to the decomposition of (2) as *PQ-decomposition*.

PROOF. (1) Let $h : X \to I(X, Y)$ and $\tilde{f} : X \to I(X, Z)$ be the induced morphisms. By universality, we have a morphism $\pi : I(X, Y) \to I(X, Z)$ such that $\pi h = \tilde{f}$. $\tilde{f}$ is an isomorphism since gf is a Q-morphism. Hence, $\tilde{f}^{-1}\pi h$ is the identity. Also, $h\tilde{f}^{-1}\pi h = h$ and h epic implies that $h\tilde{f}^{-1}\pi$ is also the identity. Hence, h is an isomorphism.

(2) Let $X \xrightarrow{g} W \xrightarrow{h} Y^{\bullet}$ be a decomposition of f into a P-morphism and a Q-morphism. Then, there is a morphism $\pi : W \to I(X, Y^{\bullet})$ by universality. Since $X \to I(X, Y^{\bullet})$ is a P-morphism, π is also a P-morphism. Since h is a Q-morphism, (1) tells that π is a Q-morphism, hence an isomorphism. □

On the other hand, it seems to be impossible to prove that Q-morphisms are stable under compositions, using only categorical operations.

LEMMA 2.2.7. *Let $\mathscr{A} = (\sigma, \alpha_1, \alpha_2, \gamma)$ be the schematizable algebraic type. Then:*

(1) *α_1 preserves filtered colimits.*

(2) *α_1 preserves images: namely, if $f : A \to B$ is a homomorphism of σ-algebras, then $\alpha_1(\mathrm{Im}\, f) = \mathrm{Im}\,(\alpha_1 f)$.*

PROOF. (1) Let $\{R_\lambda\}_\lambda$ be a filtered inductive system of σ-algebras, and set $R_\infty = \varinjlim_\lambda R_\lambda$. Then we have a natural homomorphism $\varphi : \varinjlim_\lambda \alpha_1 R_\lambda \to \alpha_1(R_\infty)$. We will show that φ is bijective.

First, we will prove the surjectivity. For any $\mathfrak{a} \in \alpha_1 R_\infty$, $\mathfrak{a}$ can be written as $\sum_i^n \alpha_2(a_i)$ for some $a_i \in R_\infty$, since $\alpha_2(R) \subset \alpha_1 R$ generates $\alpha_1 R$. Since the inductive system is filtered, there exists λ_0 such that $\{a_i\}_i \subset R_{\lambda_0}$. Then, $\mathfrak{a}$ is contained in the image of

$$\alpha_1 R_{\lambda_0} \to \varinjlim_\lambda \alpha_1 R_\lambda \xrightarrow{\varphi} \alpha_1 R_\infty,$$

hence in the image of φ.

Next, we prove the injectivity. Suppose $\varphi(\mathfrak{a}) = \varphi(\mathfrak{b})$ for some $\mathfrak{a} = \sum_i \alpha_2(a_i), \mathfrak{b} = \sum_j \alpha_2(b_j) \in \varinjlim_\lambda \alpha_1 R_\lambda$. Then, $\mathfrak{a}$ and $\mathfrak{b}$ must coincide in $\alpha_1 R_{\lambda_0}$ for some λ_0, which shows that $\mathfrak{a} = \mathfrak{b}$ in $\varinjlim_\lambda \alpha_1 R_\lambda$.

(2) Let $\mathfrak{a}$ be an element of $\alpha_1(\mathrm{Im}\, f)$. Then,

$$\mathfrak{a} = \sum_i \alpha_2 f(a_i) = (\alpha_1 f) \sum_i \alpha_2(a_i)$$

for some $a_i \in A$. This shows that $\alpha_1(\mathrm{Im}\, f) \subset \mathrm{Im}\,(\alpha_1 f)$. The converse is similar. □

PROPOSITION 2.2.8. *Let $f : X \to Y$ be a morphism of $\mathscr{A}$-schemes. Then, there exists a decomposition $X \xrightarrow{g} W \xrightarrow{h} Y$ of f, where g is a P-morphism, and h satisfies*

(i) $h^{-1} : C(Y)_{\mathrm{cpt}} \to C(W)_{\mathrm{cpt}}$ *is surjective, and*

(ii) $h^{\#} : \mathscr{O}_Y \to h_*\mathscr{O}_W$ *is* stalkwise surjective, *that is,* $\mathscr{O}_{Y,h(w)} \to \mathscr{O}_{W,w}$ *is surjective for any* $w \in W$.

PROOF. Let R be the image of $f^{-1} : C(Y)_{\mathrm{cpt}} \to C(X)_{\mathrm{cpt}}$, and set $|W| = \operatorname{Spec} R^{\dagger}$. The structure sheaf $\mathscr{O}_W : R \simeq C(W)_{\mathrm{cpt}} \to (\sigma\text{–}\mathbf{alg})$ is defined by the sheafification of

$$R \ni Z \mapsto \operatorname{Im}[f^{-1}\mathscr{O}_Y(Z) \to \mathscr{O}_X(Z)]_S,$$

where $S = \{a \mid \beta_X\alpha_2(a) = 1 \text{ in } R_Z\}$ is a multiplicative system of $\operatorname{Im}[f^{-1}\mathscr{O}_Y(Z) \to \mathscr{O}_X(Z)]$.

The support morphism $\beta_W : \alpha_1\mathscr{O}_W \to \tau_W$ is defined as follows: for any $Z \in C(W)_{\mathrm{cpt}}$, $\alpha_1\mathscr{O}_W(Z)$ is locally isomorphic to

$$\begin{aligned}\alpha_1 \operatorname{Im}[f^{-1}\mathscr{O}_Y(Z) \to \mathscr{O}_X(Z)]_S &\simeq \alpha_1 \varinjlim_{f^{-1}V=Z} \operatorname{Im}[\mathscr{O}_Y(V) \to \mathscr{O}_X(Z)]_S \\ &\simeq \varinjlim_{f^{-1}V=Z} \operatorname{Im}[\alpha_1\mathscr{O}_Y(V) \to \alpha_1\mathscr{O}_X(Z)]_S\end{aligned}$$

by Lemma 2.2.7. Since we have a commutative square

$$\begin{array}{ccc} \alpha_1\mathscr{O}_Y(V) & \longrightarrow & \alpha_1\mathscr{O}_X(Z) \\ \downarrow{\scriptstyle \beta_Y} & & \downarrow{\scriptstyle \beta_X} \\ (C(Y)_{\mathrm{cpt}})_V & \longrightarrow & (C(X)_{\mathrm{cpt}})_Z \end{array}$$

and the lower horizontal arrow factors through $R_Z = \tau_W(Z)$, we obtain a homomorphism $\alpha_1\mathscr{O}_W(Z) \to \tau_W(Z)$. Note that the localization by S does not affect. It is obvious that the restrictions reflect localizations, hence $W = (|W|, \mathscr{O}_W, \beta_W)$ is well defined as an $\mathscr{A}$-scheme. $g : X \to W$ is defined by the injections $g^{-1} : R \to C(X)_{\mathrm{cpt}}$ and

$$g^{\#} : \mathscr{O}_W(Z) = \operatorname{Im}[f^{-1}\mathscr{O}_Y(Z) \to \mathscr{O}_X(Z)]_S \to \mathscr{O}_X(Z).$$

It is obvious that g is a P-morphism. $h : W \to Y$ is defined by $h^{-1} : C(Y)_{\mathrm{cpt}} \to R$ and

$$h^{\#} : \mathscr{O}_Y(Z) \to \mathscr{O}_W(g^{-1}Z) = \operatorname{Im}[\mathscr{O}_Y(Z) \to \mathscr{O}_X(f^{-1}Z)]_S.$$

Let us verify that $h^{\#}$ is stalkwise surjective, namely, $\mathscr{O}_{Y,h(w)} \to \mathscr{O}_{W,w}$ is surjective for any $w \in W$. Let $\langle U, a\rangle$ be any element of $\mathscr{O}_{W,w}$. Since h^{-1} is

surjective, $U = h^{-1}V$ for some quasi-compact open $V \subset Y$. The germ a can be expressed as b/c, where $\beta_X\alpha_2(c) = 1$, and b, c is in the image of $\mathscr{O}_Y(V) \to \mathscr{O}_W(U)$. This implies that c is a unit in $\mathscr{O}_{W,w}$, hence also a unit in $\mathscr{O}_{Y,f(w)}$. Hence $\mathscr{O}_{Y,h(w)} \to \mathscr{O}_{W,w}$ is surjective. □

COROLLARY 2.2.9. (1) *Let $f : X \to Y$ be a morphism of $\mathscr{A}$-schemes. Then, the W constructed in* 2.2.8 *is naturally isomorphic to $I(X, Y)$. In particular, the followings are equivalent:*

(i) *f is a Q-morphism.*

(ii) *$f^{-1} : C(Y)_{\mathrm{cpt}} \to C(X)_{\mathrm{cpt}}$ is surjective, and $f^{\#} : \mathscr{O}_Y \to h_*\mathscr{O}_X$ is stalkwise surjective.*

(2) *Q-morphisms are stable under compositions.*

(3) *Q-morphisms are monic.*

PROOF. (1) By Proposition 2.2.8, there exists a decomposition $X \xrightarrow{g} W \xrightarrow{h} Y$ of f, where g is a P-morphism, $h^{-1} : C(Y)_{\mathrm{cpt}} \to C(W)_{\mathrm{cpt}}$ is surjective, and $h^{\#} : \mathscr{O}_Y \to h_*\mathscr{O}_W$ is stalkwise surjective. By the universal property, there is a P-morphism $u : W \to I(X, Y)$. Note that $I(X, Y)$ is isomorphic to X, since f is a Q-morphism. Since h^{-1} is surjective, $u^{-1} : C(X)_{\mathrm{cpt}} \to C(W)_{\mathrm{cpt}}$ is an isomorphism. Also, the stalkwise surjectivity of $h^{\#}$ implies that $u^{\#}$ is also an isomorphism.

(2) It is clear from (1).

(3) Obvious. □

LEMMA 2.2.10. *If $X \to Y \to Z$ is a series of morphisms of $\mathscr{A}$-schemes, then we have a natural isomorphism*

$$I(X, Z) \simeq I(I(X, Y), I(Y, Z)).$$

PROOF. Since $W = I(I(X, Y), I(Y, Z)) \to I(Y, Z)$ and $I(Y, Z) \to Z$ are Q-morphisms, the composition $W \to Z$ is also a Q-morphism. Hence, $X \to W \to Z$ is the PQ-decomposition of $X \to Z$. □

PROPOSITION 2.2.11. *The PQ-decomposition is functorial.*

PROOF. Suppose given a commutative square

$$\begin{array}{ccc} X_1 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & Y_2 \end{array}$$

Then, we will show that there is a natural morphism $I(X_1, Y_1) \to I(X_2, Y_2)$. By the universality, we have a unique morphism $I(X_1, Y_1) \to I(X_1, Y_2)$. On the other hand, we have again a unique morphism $I(X_1, Y_2) \to I(X_2, Y_2)$ by Lemma 2.2.10, hence combining them gives the required morphism $I(X_1, Y_1) \to I(X_2, Y_2)$. □

Let us summarize what we have obtained in this subsection:

THEOREM 2.2.12. *For any morphism $f : X \to Y$ of $\mathscr{A}$-schemes, we have a functorial decomposition $X \to I(X, Y) \to Y$ of f, where*

(1) *$X \to I(X, Y)$ is a P-morphism (in particular, epic), and*
(2) *$I(X, Y) \to Y$ is a Q-morphism (in particular, monic).*

Moreover, the decomposition of the given morphism f into a P-morphism and a Q-morphism is unique up to unique isomorphism. Also, this decomposition is universal: if f is factors as $X \to W \to Y$ where $X \to W$ is a P-morphism (resp. $W \to Y$ is a Q-morphism), then there is a unique morphism $W \to I(X, Y)$ (resp. $I(X, Y) \to W$) making the whole diagram commutative.

REMARK 2.2.13. We know that this decomposition is impossible in the category of schemes. For example, let k be a field, $X = \operatorname{Spec} k[x, y/x]$, $Y = \operatorname{Spec} k[x, y]$ where x and y are indeterminants. The image of the natural morphism $f : X \to Y$ cannot be a scheme: the origin has no affine neighborhood in the image.

2.3 – *Completeness*

PROPOSITION 2.3.1. *The category ($\mathscr{O}$-**Sch**) is finite complete, i.e. there are fiber products.*

PROOF. The construction of fiber products are similar to that of general schemes. Note that we use the fact that quasi-compact open immersions are stable under base changes by quasi-compact morphisms. □

REMARK 2.3.2. Let σ be the algebraic system of rings. Then, the natural inclusion functor ($\mathscr{O}$-**Sch**) $\to$ (**Sch**) preserves fiber products. This is clear from the construction.

We already know that the category of ordinary schemes is not complete. However:

PROPOSITION 2.3.3. *The category* ($\mathscr{A}$-**Sch**) *is small complete.*

PROOF. Let $X^\bullet$ be a small projective system of $\mathscr{A}$-schemes. Let $\mathscr{S}$ be the set of all isomorphism classes of Q-morphisms $Y \to X^\bullet$. $\mathscr{S}$ is indeed a small set: if $Y \to X^\bullet$ is a Q-morphism, then the underlying space of Y and its structure sheaf is generated by those of $X^\bullet$, hence $\mathscr{S}$ is small.

Let X be the co-limit of $\mathscr{S}$. Then, X is the limit of $X^\bullet$. □

COROLLARY 2.3.4. *The natural inclusion functor* ($\mathscr{Q}$-**Sch**) $\to$ ($\mathscr{A}$-**Sch**) *preserves fiber products.*

PROOF. Let X, Y be $\mathscr{Q}$-schemes over a $\mathscr{Q}$-scheme S. We will show that the fiber product $V = X \times_S Y$ in the category of $\mathscr{Q}$-schemes is indeed that in the category of $\mathscr{A}$-schemes.

STEP 1. If X, Y, S are all affine, then V is the fiber product in ($\mathscr{A}$-**Sch**), by the adjunction $\mathrm{Spec}^{\mathscr{A}} : (\sigma-\mathbf{alg}) \rightleftarrows (\mathscr{A}\text{-}\mathbf{Sch})^{\mathrm{op}} : \Gamma$.

STEP 2. Suppose Y, S is affine. Let $X = \cup_i X_i$ be an open affine cover of X. Suppose given the following commutative square:

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

Then, there is a unique morphism $f^{-1}(X_i) \to X_i \times_S Y$ by Step 1 for each i, which patches up to give the morphism $Z \to X \times_S Y$.

STEP 3. Same arguments as in Step 2 shows that if S is affine, then V is the fiber product in ($\mathscr{A}$-**Sch**).

STEP 4. Suppose $S = \cup_i S_i$ is an open affine cover of S. Set $X_i = X \times_S S_i$, $Y_i = Y \times_S S_i$. Then, $V_i = X_i \times_{S_i} Y_i$ is also the fiber product in ($\mathscr{A}$-**Sch**) by Step 3. Note that $V = \cup_i V_i$ is an quasi-compact open covering. Suppose given a commutative diagram as in Step 2, and set $Z_i = f^{-1}(X_i)$. This coincides with $g^{-1}(Y_i)$. Then we have a unique morphism $Z_i \to V_i$ for each i, which patches up to give a morphism $Z \to V$. □

The next proposition is purely category-theoretical.

PROPOSITION 2.3.5. (1) *Q-morphisms are stable under pullbacks.*
(2) *P-morphisms are stable under pushouts.*

PROOF. We will only prove (1): the proof of (2) can be proven by the dual argument. Consider the following pullback diagram:

$$\begin{array}{ccc} X_T & \xrightarrow{\tilde{f}} & X \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ T & \xrightarrow[f]{} & S \end{array}$$

Suppose f is a Q-morphism. Then the PQ-decomposition

$$X_T \overset{\alpha(\tilde{f})}{\to} \mathrm{Im}\tilde{f} = I(X_T, X) \overset{\beta(\tilde{f})}{\to} X$$

of $\tilde{f}$ gives a morphism $w : \mathrm{Im}\tilde{f} \to T$ such that $w \circ \alpha(\tilde{f}) = \tilde{g}$ and $f \circ w = g \circ \beta(\tilde{f})$. By the universal property of the pullback, there exists a unique morphism $u : \mathrm{Im}\tilde{f} \to X_T$ such that $\tilde{g} \circ u = w$ and $\tilde{f} \circ u = \beta(\tilde{f})$. The second equality shows that u is a Q-morphism, hence it suffices to show that u is a P-morphism. To prove this, we will show that $u \circ \alpha(\tilde{f})$ is the identity. Since

$$\tilde{g} \circ u \circ \alpha(\tilde{f}) = w \circ \alpha(\tilde{f}) = \tilde{g}, \quad \text{and} \quad \tilde{f} \circ u \circ \alpha(\tilde{f}) = \beta(\tilde{f}) \circ \alpha(\tilde{f}) = \tilde{f},$$

the universal property of the pullback shows that $u \circ \alpha(\tilde{f})$ is the identity. □

2.4 – *Filtered limits*

PROPOSITION 2.4.1. *Let $X^\bullet = \{X^\lambda\}$ be a small filtered projective system of $\mathscr{A}$-schemes, and Y be the limit of $X^\bullet$. Then, the underlying space of Y coincides with the limit $\varprojlim X^\lambda$ in the category of coherent spaces. The structure sheaf $\mathscr{O}_Y$ coincides with the colimit $\varinjlim p_\lambda^{-1}\mathscr{O}_{X^\lambda}$, where $p_\lambda : Y \to X^\lambda$ are the natural morphisms.*

PROOF. Let X^∞ be the limit of the $X^\bullet$ in the category of coherent spaces, and set $\mathscr{O}_{X^\infty} = \varinjlim \pi_\lambda^{-1}\mathscr{O}_{X^\lambda}$, where $\pi_\lambda : X^\infty \to X^\lambda$ is the natural morphism of coherent spaces. First, we will construct the support morphism β_{X^∞}. By Proposition 2.1.5, we have a natural map

$$\alpha_1\pi_\lambda^{-1}\mathscr{O}_{X^\lambda} \to \pi_\lambda^{-1}\tau_{X^\lambda} \to \varinjlim_\lambda \pi_\lambda^{-1}\tau_{X^\lambda} \simeq \tau_{X^\infty}.$$

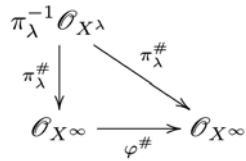
This gives a natural map $\varinjlim_\lambda(\alpha_1\pi_\lambda^{-1}\mathscr{O}_{X^\lambda}) \to \tau_{X^\infty}$. The left-hand side is isomorphic to $\alpha_1\mathscr{O}_{X^\infty}$ since α_1 is filtered co-continuous by Lemma 2.2.7. Therefore, we obtain the required morphism $\beta_{X^\infty} : \alpha_1\mathscr{O}_{X^\infty} \to \tau_{X^\infty}$. It is obvious that restrictions reflects localizations. Hence, we have constructed an $\mathscr{A}$-scheme $X^\infty = (X^\infty, \mathscr{O}_{X^\infty}, \beta_{X^\infty})$. There are also natural morphisms $\pi_\lambda : X^\infty \to X^\lambda$ of $\mathscr{A}$-schemes, compatible with the transitions.

We will show that X^∞ is naturally isomorphic to Y. Since we already have a morphism $X^\infty \to Y$ by the universal property of Y, it suffices to show that:

(i) If $\varphi : X^\infty \to X^\infty$ is a endomorphism with $\pi_\lambda\varphi = \pi_\lambda$ for any λ, then φ is the identity.

(ii) There exists a morphism $\psi : Y \to X^\infty$ with $\pi_\lambda\psi = p_\lambda$ for any λ.

First, we prove (i). It is obvious that φ is the identity on the underlying space. For the structure sheaves, we have the commutative diagram:



which shows that $\varphi^\#$ is the identity, since $\mathscr{O}_{X^\infty} = \varinjlim_\lambda \pi_\lambda^{-1}\mathscr{O}_{X^\lambda}$.

It remains to prove (ii). There is a natural morphism $|\psi| : |Y| \to |X^\infty|$ between the underlying spaces. Since there are morphisms

$$p_\lambda^\# : \mathscr{O}_{X^\lambda} \to p_{\lambda*}\mathscr{O}_Y \simeq \pi_{\lambda*}|\psi|_*\mathscr{O}_Y,$$

these give morphisms $\pi_\lambda^{-1}\mathscr{O}_{X^\lambda} \to |\psi|_*\mathscr{O}_Y$. It is obvious that these are compatible with the transition morphisms, hence we obtain $\mathscr{O}_{X^\infty} \to \psi_*\mathscr{O}_Y$. Also, this morphism commutes with β_{X^∞} and β_Y, hence we have a morphism of $\mathscr{A}$-schemes. It is obvious that $\pi_\lambda\psi = p_\lambda$. □

PROPOSITION 2.4.2. *Let $\{g_\lambda : X^\lambda \to Y^\lambda\}_\lambda$ be a filtered projective system of $\mathscr{Q}$-morphisms of $\mathscr{A}$-schemes, and set $X^\infty = \varprojlim_\lambda X^\lambda$ and $Y^\infty = \varprojlim_\lambda Y^\lambda$. Then, the natural morphism $g : X^\infty \to Y^\infty$ is also a $\mathscr{Q}$-morphism.*

PROOF. First, we will see that $g^{-1} : C(Y^\infty)_{\mathrm{cpt}} \to C(X^\infty)_{\mathrm{cpt}}$ is surjective. Since $C(X^\infty)_{\mathrm{cpt}} = \varinjlim_\lambda C(X^\lambda)$ is a filtered colimit, any element Z of $C(X^\infty)_{\mathrm{cpt}}$ is in the image of $C(X^\lambda)_{\mathrm{cpt}}$ for some λ. Since $C(Y^\lambda)_{\mathrm{cpt}} \to C(X^\lambda)_{\mathrm{cpt}}$ is surjective, there is an element $W \in C(Y^\lambda)_{\mathrm{cpt}}$ such that $g_\lambda^{-1}W = Z$. Hence, $g^{-1}\pi_\lambda^{-1}W = g_\lambda^{-1}W = Z$. This shows that g^{-1} is surjective. A similar argument shows that $\mathscr{O}_{Y^\infty} \to g_*\mathscr{O}_{X^\infty}$ is also stalkwise surjective. □

3. Separated and Proper morphisms

3.1 – *Reduced schemes*

In the sequel, the algebraic system is that of rings.

DEFINITION 3.1.1. *an $\mathscr{A}$-scheme X is* reduced, *if $\beta_X\alpha_2(a) = 0$ implies $a = 0$ for any section $a \in \mathscr{O}_X$.*

Note that if X is reduced, then the radical of any ring of sections become 0.

PROPOSITION 3.1.2. *Let X be an $\mathscr{A}$-scheme and Z be a closed subset of the underlying space of X. Then, there is a reduced $\mathscr{A}$-scheme structure $(Z, \mathscr{O}_Z, \beta_Z)$ on Z, referred to as the* reduced induced subscheme structure *of Z. Also, there is a $\mathbb{Q}$-morphism $Z \to X$, satisfying the following universal property:*

If $Y \to X$ is a morphism of $\mathscr{A}$-schemes, with Y reduced and the set-theoretic image contained in Z, then it factors through Z.

PROOF. The structure sheaf $\mathscr{O}_Z$ of Z is defined by the sheafification of the presheaf

$$W \mapsto \varinjlim_{V+Z\geq W} \mathscr{O}_X(V)/\{a \mid \beta_X\alpha_2(a) \cdot W \leq Z\},$$

where the colimit runs through all $V \in C(X)_{\mathrm{cpt}}$ in X such that $V + Z \geq W$. For any closed W in Z, and closed V in X satisfying $V + Z \geq W$, the morphism $\alpha_1\mathscr{O}_X(V) \to \tau_X(W)/Z$ induced from β_X factors through $\alpha_1\mathscr{O}_X(V)/\{a \mid \beta_X\alpha_2(a) \cdot W \leq Z\}$. Hence, we can define the support morphism $\beta_Z : \alpha_1\mathscr{O}_Z \to \tau_Z$. These give the reduced $\mathscr{A}$-scheme structure $(Z, \mathscr{O}_Z, \beta_Z)$ on Z.

We will give a morphism $\iota : Z \to X$ of $\mathscr{A}$-schemes. The map between the underlying spaces is obvious. For any closed W in X, we have a natural morphism $\mathscr{O}_X(W) \to \iota_*\mathscr{O}_Z(W) = \mathscr{O}_Z(W + Z)$, which gives a stalkwise surjective morphism $\mathscr{O}_X \to \iota_*\mathscr{O}_Z$. It is clear that this gives a $\mathbb{Q}$-morphism.

Suppose we are given a morphism $f : Y \to X$ with Y reduced and $\operatorname{Im} f \subset Z$. Then, the morphism $f^\# : \mathscr{O}_X \to f_*\mathscr{O}_Y$ factors through $\iota_*\mathscr{O}_Z$, since Y is reduced. Thus, f factors through Z. □

THEOREM 3.1.3. *Let* (**red**$_{\mathscr{A}}$**-Sch**) *be the full subcategory of $\mathscr{A}$-schemes, which consists of reduced $\mathscr{A}$-schemes. Then, the underlying*

functor $U : (\mathbf{red}_{\mathscr{A}}\text{-}\mathbf{Sch}) \to (\mathscr{A}\text{-}\mathbf{Sch})$ *has a right adjoint, and the counit morphism is a Q-morphism.*

PROOF. Proposition 3.1.2 tells that for any $\mathscr{A}$-scheme X, there exists a Q-morphism $\eta : X^{\mathrm{red}} \to X$ from a reduced $\mathscr{A}$-scheme X^{red}, the underlying space of which coincides with X. Any morphism $X \to Y$ of $\mathscr{A}$-schemes gives rise to a morphism $X^{\mathrm{red}} \to Y^{\mathrm{red}}$ of reduced $\mathscr{A}$-schemes, by the universal property. Hence, we have a functor $\mathrm{red} : (\mathscr{A}\text{-}\mathbf{Sch}) \to (\mathbf{red}_{\mathscr{A}}\text{-}\mathbf{Sch})$. We see that this is the right adjoint of U. The unit $\varepsilon : \mathrm{Id} \to \mathrm{red} \circ U$ is the identity. The counit η is already given. □

PROPOSITION 3.1.4. *Let $f : X \to Y$ be a morphism of $\mathscr{A}$-schemes, with X reduced. Let $X \to I(X,Y) \to Y$ be the PQ-decomposition. Then, $I(X,Y)$ is also reduced.*

PROOF. Since X is reduced, the P-morphism $X \to I(X,Y)$ factors through $I(X,Y)^{\mathrm{red}}$. Therefore $I(X,Y)^{\mathrm{red}} \to I(X,Y)$ is also a P-morphism, hence an isomorphism. □

DEFINITION 3.1.5. *An $\mathscr{A}$-scheme X is* integral, *if $\mathscr{O}_X(Z)$ is integral for any Z.*

Note that, in the category of $\mathscr{A}$-schemes, integrality is a weaker condition than 'irreducible and reduced'.

PROPOSITION 3.1.6. (1) *Let X be a reduced irreducible $\mathscr{A}$-scheme, and $x_0 \rightsquigarrow x_1$ a specialization. Then, the restriction map $\mathscr{O}_{X,x_1} \to \mathscr{O}_{X,x_0}$ is an injection. Also, $\mathscr{O}_{X,\xi}$ is a field, where ξ is the generic point of X.*

(2) *An $\mathscr{A}$-scheme X is integral if X is reduced and irreducible.*

(3) *Let $f : X \to Y$ be a dominant morphism of $\mathscr{A}$-schemes, with Y reduced. Then, $f^{\#} : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is injective for any $x \in X$.*

PROOF. (1) Let $\langle U, a\rangle$ be a germ of $\mathscr{O}_{X,x_1}$ which is in the kernel of $\mathscr{O}_{X,x_1} \to \mathscr{O}_{X,x_0}$. Then, $a|_V = 0$ for some neighborhood V of x_0. Since X is irreducible, V is dense in X, hence also in U. This implies that $\beta_X\alpha_2(a) = 0$. Since X is reduced, a must be 0. Therefore, the map $\mathscr{O}_{X,x_1} \to \mathscr{O}_{X,x_0}$ is injective.

Let a be a non-zero element of $\mathscr{O}_{X,\xi}$. Then, a is invertible, since $\beta_X\alpha_2(a) \neq 0$ and the restriction maps reflect localizations.

(2) Let $a, b \in \mathscr{O}_X(Z)$ be two sections with $ab = 0$. Then, $\beta_X\alpha_2(a) \cdot$

$\beta_X\alpha_2(b) = \beta_X\alpha_2(ab) = 0$. Since X is irreducible, we may assume that $\beta_X\alpha_2(a) = 0$. Since X is reduced, a must be 0.

(3) Suppose $a \in \mathscr{O}_{Y,f(x)}$ is in the kernel of $f^{\#}$. Then,

$$|f^{-1}|\beta_Y\alpha_2(a) = \beta_X f^{\#}\alpha_2(a) = 0.$$

Since f is dominant, we have $\beta_Y\alpha_2(a) = 0$. Y is reduced, hence $a = 0$. □

EXAMPLE 3.1.7. Let A be a noetherian ring, and $I \subset A$ be a non-trivial ideal. We can consider a colimit $\mathfrak{X} = \varinjlim_n \operatorname{Spec}^{\mathscr{A}} A/I^n$ in the category of $\mathscr{A}$-schemes. This becomes an integral $\mathscr{A}$-scheme if $\hat{A} = \varprojlim_n A/I^n$ is a domain. On the other hand, the underlying space of $\mathfrak{X}$ coincides with the support of I, hence $\mathfrak{X}$ is not reduced. In fact, $\mathfrak{X}$ can be regarded as a noetherian formal scheme.

3.2 – *Right lifting properties*

DEFINITION 3.2.1. *Let $\mathscr{C}$ be a category, and $\mathcal{I}$ be a non-empty family of morphisms in $\mathscr{C}$. Fix a morphism $f : X \to Y$ of $\mathscr{C}$. Given a morphism $g : A \to B$ in $\mathcal{I}$, we have a natural map $\varphi_{f,g} : \operatorname{Hom}_{\mathscr{C}}(B, X) \to \operatorname{Hom}_{\operatorname{Mor}(\mathscr{C})}(g, f)$, where $\operatorname{Mor}(\mathscr{C})$ is the category of morphisms in $\mathscr{C}$. We say that f is $\mathcal{I}$-*separated *(resp. $\mathcal{I}$-*universally closed, $\mathcal{I}$-*proper*) if $\varphi_{f,g}$ is injective (resp. surjective, bijective) for any $g \in \mathcal{I}$.*

REMARK 3.2.2. The conventional definition of properness includes the condition 'of finite type'. However, we dropped this condition here, since it does not seem to be essential when we discuss about valuative criteria. Moreover, note that morphisms of finite type are not stable under taking limits, while the other conditions do.

Here, we list up some properties of $\mathcal{I}$-separated morphisms, etc. The proofs are all straightforward.

PROPOSITION 3.2.3. *Let $\mathscr{C}$ be a category, and $\mathcal{I}$ be a non-empty family of morphisms in $\mathscr{C}$.*

(1) *Isomorphisms are $\mathcal{I}$-proper.*

(2) *Monics are $\mathcal{I}$-separated.*

(3) *The class of $\mathcal{I}$-separated (resp. $\mathcal{I}$-universally closed, $\mathcal{I}$-proper) morphisms are stable under compositions. Thus, we can think of the subcategory $\mathscr{C}(\mathcal{I})_i$ (resp. $\mathscr{C}(\mathcal{I})_s$, $\mathscr{C}(\mathcal{I})_b$) of $\mathscr{C}$ consisting of $\mathcal{I}$-separated (resp. $\mathcal{I}$-universally closed, $\mathcal{I}$-proper) morphisms.*

(4) *If $\mathscr{C}$ has fiber products, then $\mathscr{C}(\mathcal{I})_i$, $\mathscr{C}(\mathcal{I})_s$ and $\mathscr{C}(\mathcal{I})_b$ are stable under pull backs.*

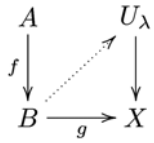
(5) *If $\mathscr{C}$ is small complete, then so is $\mathscr{C}(\mathcal{I})_i$, $\mathscr{C}(\mathcal{I})_s$ and $\mathscr{C}(\mathcal{I})_b$. Also, the inclusion functor $\mathscr{C}(\mathcal{I})_* \to \mathscr{C}$ is small continuous for $* = i, s, b$.*

(6) *If gf is $\mathcal{I}$-separated, then f is $\mathcal{I}$-separated.*

(7) *If gf is $\mathcal{I}$-universally closed (resp. $\mathcal{I}$-proper) and g is $\mathcal{I}$-separated, then f is $\mathcal{I}$-universally closed (resp. $\mathcal{I}$-proper).*

DEFINITION 3.2.4. *Let $\mathscr{C}$ be a category, and $\mathcal{I}$ be a non-empty family of morphisms in $\mathscr{C}$. Let X be an object of $\mathscr{C}$.*

(1) *A family $\{U_\lambda \to X\}_\lambda$ of morphisms with target X is an* $\mathcal{I}$-covering *of X, if for any morphism $f : A \to B$ in $\mathcal{I}$ and any morphism $g : B \to X$, g lifts to $B \to U_\lambda$ for some λ:*



(2) *Suppose $\mathscr{C}$ has fiber products. Let $\mathcal{J}$ be a family of morphisms in $\mathscr{C}$. We say $\mathcal{J}$ is* local on the base *with respect to $\mathcal{I}$, if the following holds: let $f : X \to Y$ be a morphism, and $\{U_\lambda \to Y\}_\lambda$ be an* $\mathcal{I}$-covering of *Y. Set $X_\lambda = X \times_Y U_\lambda$. Then, f is contained in $\mathcal{J}$ if $f_\lambda : X_\lambda \to U_\lambda$ is contained in $\mathcal{J}$ for any λ.*

Then, we also have:

PROPOSITION 3.2.5. *Let $\mathscr{C}$ be a category, and $\mathcal{I}$ be a non-empty family of morphisms in $\mathscr{C}$. Then, $\mathscr{C}(\mathcal{I})_i$ (resp. $\mathscr{C}(\mathcal{I})_s$, $\mathscr{C}(\mathcal{I})_b$) is local on the base.*

3.3 – *Separatedness*

Throughout this subsection, we fix a base $\mathscr{A}$-scheme S. The category of $\mathscr{A}$-schemes over S is denoted by $(\mathscr{A}\textbf{-Sch}/S)$. Also, fix a family of morphisms $\mathcal{I}$.

DEFINITION 3.3.1. *Let $(\mathcal{I}-\textbf{sep}.\mathscr{A}\textbf{-Sch}/S)$ be a full subcategory of $(\mathscr{A}\textbf{-Sch}/S)$ consisting of $\mathscr{A}$-schemes, which is $\mathcal{I}$-separated over S.*

PROPOSITION 3.3.2. *The underlying functor*

$$U : (\mathcal{I}\text{-}\mathbf{sep}_{\mathcal{A}}\text{-}\mathbf{Sch}/S) \to (\mathcal{A}\text{-}\mathbf{Sch}/S)$$

has a left adjoint, and the unit morphism is a P-morphism.

PROOF. Let X be an $\mathcal{A}$-scheme over S. Let $\mathcal{S}$ be the set of isomorphism classes of P-morphisms $X \to Y$, where Y is an $\mathcal{A}$-scheme which is $\mathcal{I}$-separated over S. We see that $\mathcal{S}$ is a small set, since the elements are represented by P-morphisms with the source fixed. Suppose given a morphism $f : X \to Z$, where Z is $\mathcal{I}$-separated over S. The PQ-decomposition $X \to Z' \to Z$ gives a P-morphism $X \to Z'$, where Z' is $\mathcal{I}$-separated over S, since the Q-morphism $Z' \to Z$ is monic. Therefore, f factors through a morphism in $\mathcal{S}$. Using Freyd's adjoint functor theorem ([CWM], p. 121), we obtain the result. □

3.4 – *Valuative criteria*

DEFINITION 3.4.1. *A Q-morphism is a* closed immersion *if its image is closed.*

DEFINITION 3.4.2. *Let $\mathcal{I}$ be a family of morphisms in the category of $\mathcal{A}$-schemes. We say that $\mathcal{I}$* parametrizes specializations *if the following conditions hold:*

(1) *For any morphism $f : U \to V$ in $\mathcal{I}$, V is irreducible, reduced and local: the generic point will be denoted by ξ, and the closed point by η. U is a one-point reduced $\mathcal{A}$-scheme (hence, a spectrum of a field) with its image onto ξ.*

(2) *Let $f : U \to V$ be a morphism in $\mathcal{I}$, and $g, h : V \to X$ a pair of morphisms with $gf = hf$ and $g(\eta) = h(\eta)$. Then, $g = h$.*

(3) *Let $g : X \to Y$ be a dominant morphism between two reduced irreducible $\mathcal{A}$-schemes, and $y \in Y$. Then, there exists a morphism $f : U \to V$ in $\mathcal{I}$ and a commutative square*

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ V & \longrightarrow & Y \end{array}$$

such that U maps onto the generic point of X, and $\eta \in V$ maps onto y.

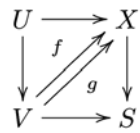
(4) *Let $f : U \to V$ be a morphism in $\mathcal{I}$, and Z be a reduced irreducible $\mathscr{A}$-scheme. If f factors as $U \xrightarrow{g} Z \xrightarrow{h} V$ where $g : U \to Z$ is dominant, and $h : Z \to V$ is surjective on the underlying space, then there is a section $V \to Z$ of h.*

PROPOSITION 3.4.3. *Suppose $\mathcal{I}$ is a family of morphisms, parametrizing specializations. Let X be an $\mathscr{A}$-schemes over S. Then, the followings are equivalent:*

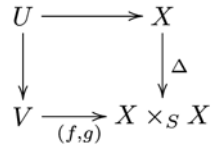
(i) *X is separated over S, i.e. the diagonal morphism $\Delta : X \to X \times_S X$ is a closed immersion.*
(ii) *X is $\mathcal{I}$-separated over S.*

Note that the diagonal morphism is monic, since it is the equalizer of $\pi_1, \pi_2 : X \times_S X \rightrightarrows X$, where π_i is the i-th projection for $i = 1, 2$.

PROOF. (i)⇒(ii): Suppose there is a commutative diagram



with $U \to V$ a morphism in $\mathcal{I}$. Then we obtain a commutative diagram



Since Δ is a closed immersion and $(f,g)(\xi) \rightsquigarrow (f,g)(\eta)$ is a specialization, $(f,g)(\eta)$ is in the image of Δ. This shows that $\pi_1 \circ (f,g) = \pi_2 \circ (f,g)$ from condition (2) of 3.4.2. Hence, (f,g) lifts to give a morphism $h : V \to X$ since $\Delta : X \to X \times_S X$ is the equalizer of π_1, π_2, and h coincides with $f = \pi_1 \circ (f,g)$ and $g = \pi_2 \circ (f,g)$. Therefore, f and g must coincide.

(ii)⇒(i): It suffices to show that the image of Δ is stable under specializations, by Corollary 1.2.8.

So let $\Delta(x) \rightsquigarrow y$ be a specialization, and $Z = \overline{\{x\}}$ be the closed subset of X, with the reduced induced subscheme structure. Also, let W be the closure of $\Delta(Z)$ in Y, with the reduced induced subscheme structure. Then, by condition (3) of 3.4.2, we have a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & Z & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta} \\ V & \longrightarrow & W & \longrightarrow & X \times_S X \end{array}$$

where $U \to V$ is a morphism in $\mathcal{I}$. Let u be the image of $U \to X$, and y be the image of the closed point η of V by the morphism $h : V \to X \times_S X$. Set $f = \pi_1 h$ and $g = \pi_2 h$, where $\pi_i : X \times_S X \to X$ is the i-th projection. Since X is $\mathcal{I}$-separated over S, f and g must coincide. Therefore, h factors through X, since $\Delta : X \to X \times_S X$ is the equalizer of π_1, π_2. This shows that y is in the image of Δ. □

PROPOSITION 3.4.4. *Suppose $\mathcal{I}$ is a family of morphisms, parametrizing specializations. Let $g : X \to S$ be a morphism of $\mathscr{A}$-schemes. Then, the followings are equivalent:*

(i) *X is universally closed, i.e. $X \times_S T \to T$ is closed for any $\mathscr{A}$-scheme T over S.*
(ii) *X is $\mathcal{I}$-universally closed over S.*

PROOF. (i)⇒(ii): Suppose the following commutative square is given:

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & S \end{array}$$

where $U \to V$ is a morphism in $\mathcal{I}$. Let $U \to V \times_S X$ be the induced morphism, $p_0 \in V \times_S X$ be the image of U, and $Z = \overline{\{p_0\}}$ be the closed subset with the reduced induced subscheme structure. Since X is universally closed, the image of $Z \to V$ is closed, hence there is a section $\iota : V \to Z$ by condition (4) of 3.4.2. Composing ι with $Z \to X$ gives the required morphism.

(ii)⇒(i): Since universally-closedness is stable under pullbacks, it suffices to show that $f : X \to S$ is closed, i.e. the image of f is stable under specializations. Let $f(x) \rightsquigarrow s$ be a specialization on S. Set $Z = \overline{\{x\}} \subset X$ and $W = \overline{\{f(x)\}} \subset S$ be closed subsets, with the reduced induced subscheme structures. Then, condition (3) of 3.4.2 implies that there exists a morphism $U \to V$ in $\mathcal{I}$ and a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & Z & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & W & \longrightarrow & S \end{array}$$

with s in the image of $V \to S$. Since X is $\mathcal{I}$-universally closed, there exists a morphism $V \to X$ making the whole diagram commutative. Since the generic point ξ of V is contained in Z and V is reduced, this morphism factors through Z. This shows that s is in the image of $Z \to S$. □

Now, we will give a family of morphisms which parametrizes specializations.

DEFINITION 3.4.5. *Let $\mathcal{I}_0$ be a family of morphisms $f : U \to V$ such that:*

(1) *There is a valuation ring R, and $V = \tilde{\operatorname{Spec}} R = \{\xi, \eta\} \subset \operatorname{Spec} R$ with the induced topology, where ξ and η are the generic point and the closed point of* $\operatorname{Spec} R$, *respectively.*

(2) *The structure sheaf $\mathscr{O}_V$ of V is defined as follows: the ring of global sections is R, and $\mathscr{O}_{V,\xi} = K$, where K is the fractional field of R. $\beta_V : \alpha_1 \mathscr{O}_V \to \tau_V$ is defined by*

$$\mathfrak{a} \mapsto \begin{cases} 1 & (\mathfrak{a} = R) \\ \{\eta\} & (0 \neq \mathfrak{a} \leq \mathfrak{M}_\eta) \\ 0 & (0 = \mathfrak{a}) \end{cases}$$

(3) *U is the spectrum of K, and $f : U \to V$ is the canonical inclusion.*

It is obvious that quasi-compact open coverings are $\mathcal{I}_0$-coverings.

PROPOSITION 3.4.6. *The above $\mathcal{I}_0$ parametrizes specializations.*

PROOF. We will verify the condition of Definition 3.4.2.

(1) Obvious from the definition.

(2) The maps between the underlying spaces obviously coincide, hence we only have to show that the two maps $g^\#, h^\# : \mathscr{O}_{X,x} \rightrightarrows \mathscr{O}_{V,\eta}$ coincide, where x is the image of η. Set $\iota : \mathscr{O}_{V,\eta} \to \mathscr{O}_{V,\xi} \simeq K$. This is injective, hence $\iota g^\# = \iota h^\#$ shows that $g^\# = h^\#$.

(3) Let $g : X \to Y$ be a dominant morphism of two reduced irreducible $\mathscr{A}$-schemes, and x_0, y_0 be the generic points of X, Y, respectively. x_0 maps to y_0 by this morphism, since it is dominant. Let y_1 be any point of Y. We have a injective morphism

$$\mathscr{O}_{Y,y_1} \hookrightarrow \mathscr{O}_{Y,y_0} \hookrightarrow \mathscr{O}_{X,x_0}.$$

Set $K = \mathscr{O}_{X,x_0}$. Then, there is a valuation ring R of K, dominating $\mathscr{O}_{Y,y_1}$.

This gives a morphism $u : U = \operatorname{Spec} K \to X$ and $v : V = \tilde{\operatorname{Spec}} R \to Y$ making the following diagram commutative:

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ \downarrow & & \downarrow \\ V & \xrightarrow[v]{} & Y \end{array}$$

satisfying $u(\xi) = x_0$ and $v(\eta) = y_1$.

(4) Let $V = \tilde{\operatorname{Spec}} R$, and suppose $U \to V$ factors through a reduced irreducible $\mathscr{A}$-scheme Z, with $U \to Z$ dominant and $g : Z \to V$ surjective. Let z_0 be the generic point of Z, and z_1 be a point in Z such that $g(z_1) = \eta$. Then, we have a commutative diagram of dominating morphisms

$$\begin{array}{ccccc} \mathscr{O}_{V,\eta} & \longrightarrow & \mathscr{O}_{Z,z_1} & & \\ \downarrow & & \downarrow & & \\ \mathscr{O}_{V,\xi} & \longrightarrow & \mathscr{O}_{Z,z_0} & \longrightarrow & \mathscr{O}_{V,\xi} \end{array}$$

All arrows are injective. Since the composition of the second row is the identity, we have $\mathscr{O}_{V,\xi} \simeq \mathscr{O}_{Z,z_0}$. Since a valuation ring is maximal among dominating morphisms, the arrow of the upper row must be an isomorphism. This gives a section $V \to Z$ of g. □

REMARK 3.4.7. Suppose X is irreducible and reduced, and K is the function field of X. In this case, we can strengthen the valuative criteria as follows: let $\mathscr{I}_1 = \{\operatorname{Spec} K \to V\}$ be the subfamily of $\mathscr{I}_0$, consisting of all morphisms the sources of which are $\operatorname{Spec} K$. Then,

(1) X is separated over S if and only if X is $\mathscr{I}_1$-separated over S.

(2) X is universally closed over S if and only if X is $\mathscr{I}_1$-universally closed over S.

This is easily seen, by taking x in the proofs of Proposition 3.4.3 and Proposition 3.4.4 as the generic point of X. We will make use of this observation in Subsection 4.4.

REMARK 3.4.8. In this article, we only use $\mathscr{I}$-separatedness and $\mathscr{I}$-properness for describing the valuative criteria. However, the reader may know that other properties of morphisms can also be formulated by the right lifting properties with respect to other families $\mathscr{I}$ of morphisms, even

in the classical algebraic geometry. For example, let $\mathcal{I}$ be a family of morphisms $X_0 \to X$ of affine S-schemes, where X_0 is a closed subscheme of X defined by a nilpotent ideal. Then, a S-scheme Y is *formally unramified* (resp. *formally smooth*, *formally étale* if it is $\mathcal{I}$-separated (resp. *$\mathcal{I}$-universally closed*, *$\mathcal{I}$-proper*). Y is *unramified* if it is formally unramified and of finite type over S. Y is *smooth* (resp. *étale*) if it is formally smooth (resp. formally étale) and of finite presentation over S ([EGA4], § 17). We will treat these subjects in the future, and the category-theoretical arguments in this paper will be its base.

4. Zariski-Riemann spaces

In this section, we will construct a universal proper $\mathscr{A}$-scheme for a given $\mathscr{A}$-scheme, which is known as the Riemann-Zariski space. The construction is somewhat difficult than the universal separated scheme constructed previously, since we cannot use the PQ-decomposition to bound the cardinality of the morphisms.

4.1 – *Zariski-Riemann spaces*

PROPOSITION 4.1.1. *Let $f : X \to Y$ be a morphism of $\mathscr{A}$-schemes. Then there exists a decomposition $X \to Z \to Y$ of f, such that:*

(1) *$Z \to Y$ is a closed immersion.*

(2) Universality*: if $X \to \tilde{Z} \to Y$ is another decomposition of f with $\tilde{Z} \to Y$ a closed immersion, then there is a unique morphism $Z \to \tilde{Z}$ making the whole diagram commutative:*

Further, the image of f is dense in Z, and $\mathscr{O}_Z \to \mathscr{O}_X$ is injective.

PROOF. Let $\mathscr{S}$ be the set of all isomorphism classes of series of morphisms $\{X \to Z_\lambda \to Y\}$ of $\mathscr{A}$-schemes, where $Z_\lambda \to Y$ is a closed immersion. Then $\mathscr{S}$ is small, since closed immersions are Q-morphisms. Let Z be the limit of $\{Z_\lambda\}$. Then $Z \to Y$ is also a closed immersion, since proper

morphisms and Q-morphisms are stable under taking limits. The universality is clear from the construction. It remains to show that $X \to Z$ is dominant. We may assume that $X \to Y$ is a Q-morphism. To see this, it is enough to show that there is an $\mathscr{A}$-scheme structure on the closure $\overline{X}$ of X in Y. We define the structure sheaf $\mathscr{O}_{\overline{X}}$ by the sheafification of

$$W \mapsto \varinjlim_{V+\overline{X} \geq W} \mathscr{O}_Y(V)/\ker[f^{\#} : \mathscr{O}_Y(V) \to \mathscr{O}_X(f^{-1}V)].$$

Since $\alpha_1\mathscr{O}_Y(V) \to \tau_Y(W)/\overline{X} = \tau_{\overline{X}}(W)$ factors through $\alpha_1\mathscr{O}_Y(V)/\ker f^{\#}$, we obtain the support morphism $\beta_{\overline{X}} : \alpha_1\mathscr{O}_{\overline{X}} \to \tau_{\overline{X}}$. We also have the natural morphisms $X \to \overline{X}$ and $\overline{X} \to Y$, which shows that $X \to Z$ is indeed dominant. We also see that $\mathscr{O}_{\overline{X}} \to \mathscr{O}_X$ is injective, hence $\mathscr{O}_{\overline{X}} \to \mathscr{O}_Z$ is injective. On the other hand, $\mathscr{O}_Y \to \mathscr{O}_Z$ is stalkwise surjective, which shows that Z is actually isomorphic to $\overline{X}$ as an $\mathscr{A}$-scheme. □

THEOREM 4.1.2. *Fix an $\mathscr{A}$-scheme S, and Let* (**prop**.$\mathscr{A}$-**Sch**/**S**) *be the full subcategory of* ($\mathscr{A}$-**Sch**/**S**)*, consisting of $\mathscr{A}$-schemes proper over S. Then, the underlying functor* (**prop**.$\mathscr{A}$-**Sch**/**S**) $\to$ ($\mathscr{A}$-**Sch**/**S**) *has a left adjoint.*

PROOF. Let X be an $\mathscr{A}$-scheme over S, and $\mathscr{S}$ be a set of isomorphism classes of dominant S-morphisms $f : X \to Y$, with Y proper over S, and $\mathscr{O}_Y \to f_*\mathscr{O}_X$ injective. From Proposition 4.1.1 and Freyd's adjoint functor theorem ([CWM], p. 121), it suffices to show $\mathscr{S}$ is small.

Let $f : X \to Y$ be a dominant S-morphism, with Y proper over S.

STEP 1. The points y of Y are parametrized by the commutative squares

$$\begin{array}{ccc} \operatorname{Spec}\kappa(x) & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \xrightarrow[\pi]{} & S \end{array}$$

where $\operatorname{Spec}\kappa(x) \to V$ is a morphism in $\mathcal{I}_0$ which is described in Definition 3.4.5. y is given by the image of the closed point by the unique map $V \to Y$: this is true, since the set of points in Y given by the above diagram is stable under specialization, hence closed. On the other hand, f is a dominant map, hence any point of Y must be given by the above diagram.

STEP 2. Note that the set $\{\kappa(x)\}_{x \in X}$ is a small set. This implies that the set of isomorphism classes of morphisms in $\mathcal{I}_0$ of the form $\kappa(x) \to V$, where

$x \in X$, is also small. Let η be the closed point of V. Then, $\mathscr{O}_{V,\eta}$ is also a small set, and the morphism $V \to S$ is determined by the map $\mathscr{O}_{S,\pi(x)} \to \mathscr{O}_{V,\eta}$. Summing up, we see that the set of isomorphism classes of the above commutative squares are small. Since the points of Y are parametrized by these morphisms, the set of isomorphism classes of the underlying spaces of Y's are small.

STEP 3. Since $\mathscr{O}_Y \to f_*\mathscr{O}_X$ is injective, the set of isomorphism classes of Y's (as $\mathscr{A}$-schemes) is also small, ditto for the set of morphisms $X \to Y$. This shows that $\mathscr{S}$ is a small set, and we have finished the proof. □

We will denote the above left adjoint functor by ZR_S.

REMARK 4.1.3. The above functor and its construction is known as the Stone-Čech compactification.

4.2 – *Embedding into the Zariski-Riemann spaces*

In the sequel, fix a base $\mathscr{A}$-scheme S.
First, we confirm basic facts.

PROPOSITION 4.2.1. *Let X, Y be a scheme over S.*

(1) *If $X \to \mathrm{ZR}_S(X)$ is a Q-morphism, then X is separated.*
(2) *If there is a Q-morphism $X \to Y$, with Y proper over S, then $X \to \mathrm{ZR}_S(X)$ is a Q-morphism.*
(3) *If $X \to \mathrm{ZR}_S(X)$ is a Q-morphism and $Y \to X$ is a Q-morphism, then $Y \to \mathrm{ZR}_S(Y)$ is a Q-morphism.*
(4) *Let $\{X^\lambda\}$ be a filtered projective system of $\mathscr{A}$-schemes over S such that $X^\lambda \to \mathrm{ZR}_S(X^\lambda)$ is a Q-morphism for any λ. If $X = \varprojlim_\lambda X^\lambda$, then $X \to \mathrm{ZR}_S(X)$ is also a Q-morphism.*

PROOF. (1)-(3) are straightforward. We will only prove (4). Since $X^\lambda \to \mathrm{ZR}_S(X^\lambda)$ is a Q-morphism, $X \to \varprojlim_\lambda \mathrm{ZR}_S(X^\lambda)$ is also a Q-morphism, by Proposition 2.4.2. This morphism factors through $\mathrm{ZR}_S(X)$ since $\varprojlim_\lambda \mathrm{ZR}_S(X^\lambda)$ is proper, hence $X \to \mathrm{ZR}_S(X)$ is a Q-morphism. □

PROPOSITION 4.2.2. *Let X, Y be an $\mathscr{A}$-scheme over S. Let $f : X \to Y$ be a dominant Q-morphism of $\mathscr{A}$-schemes over S, with Y proper. Then, f is an open immersion if and only if $\imath : X \to \mathrm{ZR}_S(X)$ is.*

PROOF. Let $\pi : \mathrm{ZR}_S(X) \to Y$ be the canonical morphism. Note that π is proper, since Y is proper over S. Also, since f is dominant, π must be surjective.

First, we will show (*): $\mathrm{ZR}_S(X) \setminus \imath(X) = \pi^{-1}(Y \setminus f(X))$. It is obvious that the left-hand side contains the right-hand side, so we will show the converse. Assume that there exists $u \in \mathrm{ZR}_S(X) \setminus \imath(X)$ such that $\pi(u)$ is in the image of f, say $\pi(u) = f(x)$. Since $\imath$ is dominant, there is a point $\xi \in X$ such that $\imath(\xi)$ specializes to u. Also, since f is a Q-morphism and

$$f(\xi) = \pi\imath(\xi) \rightsquigarrow \pi(u) = f(x),$$

we see that ξ specializes to x. Let W, W', W'' be the closure of $\{\xi\}$, $\{\imath(\xi)\}$, $\{f(\xi)\}$, respectively, with induced reduced subscheme structures. We have a series of local homomorphisms

$$\mathscr{O}_{W'',f(x)} \to \mathscr{O}_{W',\imath(x)} \to \mathscr{O}_{W,x},$$

and these are injective since $W \to W''$ is dominant. Also, these are surjective since f is a Q-morphism, hence isomorphisms. Therefore, the homomorphism

$$\mathscr{O}_{W',\imath(x)} \simeq \mathscr{O}_{W'',\pi(u)} \to \mathscr{O}_{W',u}$$

implies that $\mathscr{O}_{W',u}$ dominates $\mathscr{O}_{W',\imath(x)}$. Let R be a valuation ring of $K = \mathscr{O}_{W',\imath(\xi)}$ dominating $\mathscr{O}_{W',u}$. Consider the following commutative square:

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathrm{ZR}_S(X) \\ \downarrow & & \downarrow \\ \tilde{\operatorname{Spec}} R & \longrightarrow & S \end{array}$$

Then, there are two morphisms $\tilde{\operatorname{Spec}} R \to \mathrm{ZR}_S(X)$ which send the closed point of $\tilde{\operatorname{Spec}} R$ to $\imath(x)$ and u, respectively. This contradicts to the fact that $\mathrm{ZR}_S(X)$ is separated over S.

Now, suppose f is an open immersion. Since f is a Q-morphism, $\imath$ is also a Q-morphism. Moreover, the right-hand side of (*) is closed, hence $\imath(X)$ is open. This implies that $\imath$ is an open immersion.

Conversely, suppose $\imath$ is an open immersion. Then, the left-hand side of (*) is closed, and π being proper and surjective implies that

$$\pi(\mathrm{ZR}_S(X) \setminus \imath(X)) = Y \setminus f(X)$$

is also closed. Hence, f is an open immersion. □

COROLLARY 4.2.3. *If $Y \to X$ is a closed (resp. open) immersion, and $X \to \mathrm{ZR}_S(X)$ is an open immersion, then $Y \to \mathrm{ZR}_S(Y)$ is an open immersion.*

PROOF. Let $\overline{Y}$ be the closure of Y in $\mathrm{ZR}_S(X)$. Then, $Y \to \overline{Y}$ is an open immersion to a proper $\mathscr{A}$-scheme over S. Then, Proposition 4.2.2 tells that $Y \to \mathrm{ZR}_S(Y)$ is also an open immersion. □

We want to know when $X \to \mathrm{ZR}_S(X)$ is an open immersion for a morphism $X \to S$ of $\mathscr{Q}$-schemes. Note that the condition 'of finite type' is crucial for the open embedding. We will see from now on, what happens if drop off the condition.

PROPOSITION 4.2.4. *Let $X \to S$ be a morphism between affine schemes. Then, $X \to \mathrm{ZR}_S(X)$ is a Q-morphism.*

PROOF. Let $S = \mathrm{Spec}^{\mathscr{A}} A$. It suffices to show when $X = \mathrm{Spec}\, A[x_\lambda]_{\lambda\in\Lambda}$, the spectrum of the polynomial ring of coefficient ring A with infinitely many variables.

For any finite subset Λ' of Λ, set $X^{\Lambda'} = \mathrm{Spec}\, A[x_\lambda]_{\Lambda'}$. These can be embedded into a proper scheme $Y^{\Lambda'}$ over S. Even if $\Lambda_1 \supset \Lambda_2$ is an inclusion between finite subsets of Λ. We need not have a morphism $Y^{\Lambda_1} \to Y^{\Lambda_2}$ extending $X^{\Lambda_1} \to X^{\Lambda_2}$: we only obtain rational maps. However, when given a fixed Λ_1, blow up all the indeterminancy locus of $Y^{\Lambda_1} \to Y^{\Lambda_2}$, where Λ_2 runs through all the subset of Λ_1 and we obtain another proper scheme $\tilde{Y}^{\Lambda_1}$. Replacing Y^{Λ_1} by $\tilde{Y}^{\Lambda_1}$ for each Λ_1 gives a filtered projective system $\{\tilde{Y}^{\Lambda'}\}$ of proper schemes over S, extending the projective system $\{X^{\Lambda'}\}$. The morphisms $X^{\Lambda'} \to \tilde{Y}^{\Lambda'}$ are Q-morphisms, hence

$$X = \varprojlim_{\Lambda'} X^{\Lambda'} \to Y = \varprojlim_{\Lambda'} \tilde{Y}^{\Lambda'}$$

is also a Q-morphism, and Y is proper. □

EXAMPLE 4.2.5. Let $R = \mathbb{Z}[x_n]_{n\in\mathbb{N}}$ be a polynomial ring with infinitely many variables, and set $\mathbb{A}^\infty = \mathrm{Spec}\, R$. We will see that $\mathbb{A}^\infty$ cannot be embedded as an open subscheme of a proper $\mathscr{A}$-scheme. We have a surjection $R \to \mathbb{Q}$, hence there is a closed immersion $\mathrm{Spec}\,\mathbb{Q} \to \mathbb{A}^\infty$. We have a natural dominant immersion $\mathrm{Spec}\,\mathbb{Q} \to \mathrm{Spec}\,\mathbb{Z}$, which is not an open immersion. This shows that $\mathbb{A}^\infty \to \mathrm{ZR}_{\mathbb{Z}}(\mathbb{A}^\infty)$ cannot be an open immersion by Corollary 4.2.3, although it is a Q-morphism. This tells that, we may not

be able to obtain an open embedding if we drop the 'of finite type' condition. The decomposition which Temkin gave does not give the embedding ([Tem]).

As a corollary, we obtain

COROLLARY 4.2.6. *The infinite-dimensional projective space* $\mathbb{P}^\infty = \operatorname{Proj} R$ *is not proper.*

PROOF. We have a natural open immersion $\mathbb{A}^\infty \to \mathbb{P}^\infty$, which shows that $\mathbb{P}^\infty$ cannot be proper. □

4.3 – *Classical Zariski-Riemann space as an* $\mathscr{A}$*-scheme*

So far, we have constructed a universal compactification $\mathrm{ZR}_S(X)$ of a given scheme X. However, since we constructed it by the adjoint functor theorem, it is difficult to understand its structure. Also, the topology may be very different from what we expect; we already have the notion of Zariski-Riemann spaces of a given field K containing a base ring A, but $\mathrm{ZR}_{\operatorname{Spec} A}(\operatorname{Spec} K)$ may not coincide with this conventional one.

Therefore, we would like to construct a more accessible $\mathscr{A}$-object; its topology should be more 'algebraic', so that it coincides with the conventional one in simple cases. These will give the class of $\mathscr{A}$-schemes 'of profinite type', which describe the pro-category of ordinary schemes.

In the sequel, we fix a field K, and any $\mathscr{A}$-scheme X is reduced and has a dominant morphism $\operatorname{Spec} K \to X$. This implies that X is irreducible. Moreover, we consider only dominant morphisms, unless otherwise noticed.

DEFINITION 4.3.1. *Let S be an $\mathscr{A}$-scheme with a dominant morphism* $\operatorname{Spec} K \to S$.

(1) *Set*

$$\mathscr{M}_0^S = \mathscr{P}^f(C(S)_{\mathrm{cpt}} \times (\mathscr{P}^f(K \setminus \{0\}) \setminus \emptyset)).$$

The addition on $\mathscr{M}_0^S$ is defined by taking the union. The multiplication on $\mathscr{M}_0^S$ is defined by

$$\{(Z_{1i}, \alpha_{1i})\}_i \cdot \{(Z_{2j}, \alpha_{2j})\}_j = \{(Z_{1i} \cdot Z_{2j}, \alpha_{1i} \cup \alpha_{2j})\}_{i,j}.$$

Both two operations are associative and commutative, and the addition is idempotent. The distribution law holds, and there is the additive unit $\mathbf{0} = \emptyset$. *This is also the absorbing element with respect to the multiplication. However, there is no multiplicative unit, hence* $\mathscr{M}_0^S$ *fails to be an idempotent semiring.*

(2) *For any* $(Z, \alpha) \in C(S)_{\mathrm{cpt}} \times (\mathscr{P}^f(K \setminus \{0\}) \setminus \emptyset)$, *a set* $Z[\alpha]$ *is defined by the subset of* S, *consisting of all points* $s \in S$ *which satisfies either*

(i) $s \in Z$, *or*

(ii) *The maximal ideal* $\mathfrak{M}_{S,s}$ *is not in the image of* $\operatorname{Spec} \mathscr{O}_{S,s}[\alpha] \to \operatorname{Spec} \mathscr{O}_{S,s}$.

(3) *Let* $\mathfrak{a} = \{(Z_i, \alpha_i)\}_i$ *and* $\mathfrak{b} = \{(W_j, \beta_j)\}_j$ *be two elements of* $\mathscr{M}_0^S$. *We write* $\mathfrak{a} \prec \mathfrak{b}$ *if:*

(a) $\cap_i Z_i[\alpha_i] \supset \cap_j W_j[\beta_j]$, *and*

(b) *For any* i *and any* $s \in S \setminus Z_i[\alpha_i]$, *set* $J_s = \{j \mid s \in S \setminus W_j[\beta_j]\}$. *Then for any map* $\sigma : J_s \to \cup_{j \in J_s} \beta_j$ *such that* $\sigma_j \in \beta_j$, $(\sigma_j^{-1})_j$ *generates the unit ideal in* $\mathscr{O}_{S,s}[\alpha_i][\sigma_j^{-1}]_j$.

This relation $\prec$ *is reflective. It is also true that* $\prec$ *is transitive, but this seems to be difficult to prove it at this moment, so we will not use this fact.*

(3) *Define* $\approx$ *to be the equivalence relation generated be the relation* $\prec$, *namely:* $\mathfrak{a} \approx \mathfrak{b}$ *if and only if there is a sequence* $\mathfrak{a} = \mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_n = \mathfrak{b}$ *of elements of* $\mathscr{M}_0^S$ *such that* $\mathfrak{a}_i \prec \mathfrak{a}_{i+1}$ *and* $\mathfrak{a}_i \succ \mathfrak{a}_{i+1}$ *for each* i. *Let* $\mathscr{M}^S = \mathscr{M}_0^S / \approx$ *be the quotient set.*

PROPOSITION 4.3.2. *The addition and the multiplication on* $\mathscr{M}_0^S$ *descend to* $\mathscr{M}^S$, *and* $\mathscr{M}^S$ *becomes a II-ring with these operations.*

PROOF. We will divide the proof in several steps.

STEP 1. We will show that the addition descends to $\mathscr{M}^S$. To show this, it suffices to show that if $\mathfrak{a}_1 \prec \mathfrak{b}_1$ and $\mathfrak{a}_2 \prec \mathfrak{b}_2$, then $\mathfrak{a}_1 + \mathfrak{a}_2 \prec \mathfrak{b}_1 + \mathfrak{b}_2$. Set $\mathfrak{a}_1 = \{(Z_i, \alpha_i)\}_{i \le m}$, $\mathfrak{a}_2 = \{(Z_i, \alpha_i)\}_{i > m}$, $\mathfrak{b}_1 = \{(W_j, \beta_j)\}_{j \le n}$, and $\mathfrak{b}_2 = \{(W_j, \beta_j)\}_{j > n}$. It is obvious that $\cap_i Z_i[\alpha_i] \supset \cap_j W_j[\beta_j]$. Take arbitrary i and $s \in S \setminus Z[\alpha_i]$. We may assume $i \le m$. For any $\sigma : J_s \to \cup_{j \in J_s} \beta_j$ such that $\sigma_j \in \beta_j$, $(\sigma_j^{-1})_{j \le n}$ generates the unit ideal of $\mathscr{O}_{S,s}[\sigma_j^{-1}]_{j \le n}$, since $\mathfrak{a}_1 \prec \mathfrak{b}_1$. Hence $(\sigma_j^{-1})_j$ generates the unit ideal of $\mathscr{O}_{S,s}[\sigma_j^{-1}]_j$. This shows that $\mathfrak{a}_1 + \mathfrak{a}_2 \prec \mathfrak{b}_1 + \mathfrak{b}_2$.

STEP 2. We will show that the addition descends to $\mathscr{M}^S$. To show this, it suffices to show that if $\mathfrak{a}_1 \prec \mathfrak{a}_2$ and $\mathfrak{b}_1 \prec \mathfrak{b}_2$, then $\mathfrak{a}_1 \cdot \mathfrak{b}_1 \prec \mathfrak{a}_2 \cdot \mathfrak{b}_2$. Set $\mathfrak{a}_1 = \{(Z_i, \alpha_i)\}_{i \le m}$, $\mathfrak{a}_2 = \{(Z_i, \alpha_i)\}_{i > m}$, $\mathfrak{b}_1 = \{(W_j, \beta_j)\}_{j \le n}$, and $\mathfrak{b}_2 = \{(W_j, \beta_j)\}_{j > n}$.

Since $\cap_{i\leq m} Z_i[\alpha_i] \supset \cap_{i>m} Z_i[\alpha_i]$ and $\cap_{j\leq n} W_j[\beta_j] \supset \cap_{j>n} W_j[\beta_j]$, we have

$$\cap_{j\leq n}^{i\leq m} Z_i \cdot W_j[\alpha_i \cup \beta_j] \supset \cap_{j>n}^{i>m} Z_i \cdot W_j[\alpha_i \cup \beta_j]$$

For any $i_0 \leq m, j_0 \leq n$ and any $s \in Z_{i_0} \cdot W_{j_0}[\alpha_{i_0} \cup \beta_{j_0}]$, set

$$J_s = \{(i,j) \mid i > m, j > n, s \in S \setminus Z_i \cdot W_j[\alpha_i \cup \beta_j]\}.$$

Let $\sigma : J_s \to \cup_{j>n}^{i>m}(\alpha_i \cup \beta_j)$ be a map such that $\sigma_{ij} \in \alpha_i \cup \beta_j$. Suppose for any $i > m$, there exists $j = j(i) > n$ such that $\sigma_{ij(i)} \in \alpha_i$. Then, $(\sigma_{ij(i)}^{-1})_i$ generates the unit ideal in $\mathscr{O}_{S,s}[\alpha_{i_0}][\sigma_{ij(i)}^{-1}]_i$, hence $(\sigma_{ij}^{-1})_{ij}$ generates the unit ideal in $\mathscr{O}_{S,s}[\alpha_{i_0} \cup \beta_{j_0}][\sigma_{ij}^{-1}]_{ij}$. On the other hand, if there is a $i_1 > m$ such that $\sigma_{i_1 j} \in \beta_j$ for all $j > n$, then $(\sigma_{i_1 j}^{-1})_j$ generates the unit ideal in $\mathscr{O}_{S,s}[\beta_{j_0}][\sigma_{i_1 j}^{-1}]_j$, which leads us to the same conclusion as above. This shows that $\mathfrak{a}_1 \cdot \mathfrak{b}_1 \prec \mathfrak{a}_2 \cdot \mathfrak{b}_2$.

STEP 3. Set $\mathbf{1} = \{(1, \{1\})\}$. It is obvious that $\mathfrak{a} \prec \mathbf{1}$ for any $\mathfrak{a} \in \mathscr{M}_0^S$. This shows that $\mathbf{1}$ is the absorbing element with respect to the addition.

We will show that $\mathbf{1}$ is the multiplicative unit. It suffices to show that $\mathfrak{a} \prec \mathbf{1} \cdot \mathfrak{a}$. Set $\mathfrak{a} = \{(Z_i, \alpha_i)\}_i$. Then, $\mathbf{1} \cdot \mathfrak{a} = \{(Z_i, \alpha_i \cup \{1\})\}_i$. Note that $Z_i[\alpha_i \cup \{1\}] = Z_i[\alpha_i]$. For any i and $s \in S \setminus Z_i[\alpha_i]$, set $J_s = \{j \mid s \in S \setminus Z_j[\alpha_j]\}$, and let $\sigma : J_s \to \cup_{j \in J_s}(\alpha_j \cup \{1\})$ be any map with $\sigma_j \in \alpha_j \cup \{1\}$. Here, we see that $(\sigma_j^{-1})_j$ generates the unit ideal of $\mathscr{O}_{S,s}[\alpha_i][\sigma_j^{-1}]_j$ in any case.

STEP 4. It remains to prove that the multiplication on $\mathscr{M}^S$ is idempotent. To see this, it suffices to show that $\mathfrak{a} \prec \mathfrak{a}^2$ for any $\mathfrak{a} \in \mathscr{M}_0^S$. Set $\mathfrak{a} = \{(Z_i, \alpha_i)\}_i$. Then

$$\mathfrak{a}^2 = \{(Z_i \cdot Z_j, \alpha_i \cup \alpha_j)\}_{i,j} \supset \{(Z_i, \alpha_i)\}_i = \mathfrak{a}.$$

This shows that $\mathfrak{a} \prec \mathfrak{a}^2$. □

Note that, $\prec$ and $\leq$ coincide in $\mathscr{M}^S$. From now on, we just write (Z, α) instead of $\{(Z, \alpha)\}$ for brevity.

DEFINITION 4.3.3. (1) *There is a natural homomorphism*

$$C(S)_{\mathrm{cpt}} \to \mathscr{M}^S \quad (Z \mapsto (Z, \{1\}))$$

of II-rings. This induces a morphism $|\pi| : \operatorname{Spec} \mathscr{M}^S \to |S|$ *of coherent spaces.*

(2) *Let* p *be an element of* $\operatorname{Spec} \mathscr{M}^S$, *and* $s = |\pi|(p)$. *Set*

$$R_p = \mathscr{O}_{S,s}[a \in K \mid (1, \{a\}) \not\leq p].$$

PROPOSITION 4.3.4. (1) *R_p is a valuation ring of K.*
(2) *For any $a \in K \setminus \{0\}$, $(1, \{a\}) \leq p$ if and only if $a \notin R_p$.*
(3) *R_p dominates $\mathscr{O}_{S,s}$.*

PROOF. (1) Assume that there is an element $a \in K \setminus \{0\}$ such that neither a nor a^{-1} is in R_p. Then this implies $(1, \{a\}), (1, \{a^{-1}\}) \leq p$. Hence,

$$\mathbf{1} = (1, \{a\}) + (1, \{a^{-1}\}) \leq p$$

which contradicts to p being prime.

(2) It suffices to show the 'only if' part. Assume that there is a $a \in R_p$ such that $(1, \{a\}) \leq p$. Then, there are a finite number of x_i's such that $(1, \{x_i\}) \not\leq p$ and $a \in \mathscr{O}_{S,s}[x_i]_i$. This is equivalent to saying that a^{-1} is invertible in $\mathscr{O}_{S,s}[x_i]_i[a^{-1}]$, hence

$$\prod_i (1, \{x_i\}) = (1, \{x_i\}_i) \leq (1, \{a\}) \leq p.$$

Since p is prime, at least one of the $(1, \{x_i\})$'s must be in p, but this is a contradiction.

(3) It suffices to show $\mathfrak{M}_{S,s} \subset \mathscr{O}_{S,s} \cap \mathfrak{M}_p$, where $\mathfrak{M}_{S,s}$ and $\mathfrak{M}_p$ are maximal ideals of $\mathscr{O}_{S,s}$ and R_p, respectively. Assume there is an element $a \in \mathfrak{M}_{S,s} \setminus \mathfrak{M}_p$. Then, $a^{-1} \in R_p$, hence $(1, \{a^{-1}\}) \not\leq p$. On the other hand, $(1, \{a^{-1}\}) = (\beta_S(a), \{1\})$ from the proof of Proposition 2.1.1. Also, $\beta_S(a) \leq s$, since $a \in \mathfrak{M}_{S,s}$. Combining these, we have $(1, \{a^{-1}\}) \leq p$, a contradiction. □

DEFINITION 4.3.5. (1) *Let $\mathrm{ZR}^f(K, S)$ be a set of triples (s, R, ϕ), where $s \in S$, R is a valuation ring of K, and $\phi : \mathscr{O}_{S,s} \to R$ is a dominant homomorphism.*
(2) *The above proposition gives a map $\varphi : \mathrm{Spec}\, \mathscr{M}^S \to \mathrm{ZR}^f(K, S)$ defined by $p \mapsto (\pi(p), R_p, \phi)$, where $\phi : \mathscr{O}_{S,\pi(p)} \to R_p$ is the natural homomorphism.*
(3) *Conversely, if we are given an element $R = (s, R, \phi)$ of $\mathrm{ZR}^f(K, S)$, then set $p_R \in (\mathscr{M}^S)^\dagger$ as the ideal generated by $\{(Z, \{1\})\}_{Z \leq s}$ and $\{(1, \{x\})\}_{x \notin R}$.*

PROPOSITION 4.3.6. (1) *Let Z be a closed subset of S, with a quasi-compact open complement, and α be a non-empty subset of $K \setminus \{0\}$. Then, $(Z, \alpha) \leq p_R$ if and only if $Z \leq s$, or $\alpha \not\subset R$.*
(2) *The ideal p_R is prime. Thus, we have a map $\psi : \mathrm{ZR}^f(K, S) \to \mathrm{Spec}\, (\mathscr{M}^S)^\dagger$.*
(3) *φ is bijective, and the inverse is ψ.*

PROOF. (1) The 'if' part is obvious. Suppose $(Z,\alpha) \leq p_R$ with $Z \not\leq s$ and $\alpha \subset R$. Then, $(Z,\alpha) \prec \{(Z_i,1)\}_{Z_i \not\leq s}^{i \leq m} \cup \{(1,\{b_i\})\}_{b_i \notin R}^{i>m}$. Since $\mathscr{O}_{S,s}[\alpha] \subset R$, we have $s \notin Z[\alpha]$. This implies that $J_s \subset \{i \mid i > m\}$. Hence, $(b_i^{-1})_i$ generate the unit ideal of $\mathscr{O}_{S,s}[\alpha][b_i^{-1}]_i$. But since $b_i \notin R$ for any i, b_i^{-1} must be in the maximal ideal $\mathfrak{M}_R$ of R, a contradiction.

(2) Suppose $(Z,\alpha),(W,\beta) \notin p_R$. Then (1) tells that $Z \not\leq s$ and $W \not\leq s$. Since s is a prime ideal of $C(S)_{\mathrm{cpt}}$, we have $Z \cdot W \not\leq s$. Also, $\alpha \subset R$ and $\beta \subset R$ implies that $\alpha \cup \beta \subset R$. Combining these, we have $(Z,\alpha)\cdot(W,\beta) \notin p_R$. It is obvious that $\mathbf{1} \not\leq p_R$, hence p_R is a prime ideal.

(3) First, we show that $\psi \circ \varphi$ is the identity. Let p be any element of $\mathrm{Spec}\,(\mathscr{M}^S)^\dagger$. Then,

$$(Z,\alpha) \in p \Leftrightarrow Z \leq \ s \quad \text{or} \quad \alpha \not\subset \varphi(p) \Leftrightarrow (Z,\alpha) \in \psi\varphi(p).$$

Next, we show that $\varphi \circ \psi$ is the identity. Let (s,R,ϕ) be any element of $\mathrm{ZR}^f(K,S)$. Then,

$$a \in R \Leftrightarrow (1,\{a\}) \notin \psi(R) \Leftrightarrow a \in R_{\psi(R)}.$$

Also, it is obvious that $\pi(\psi(R)) = s$, so that $\varphi \circ \psi(R) = R$. □

REMARK 4.3.7. By Proposition 4.3.4 and Proposition 4.3.6, we can give a topology on $\mathrm{ZR}^f(K,S)$ induced from $\mathrm{Spec}\,\mathscr{M}^S$. We can see that the topology has an open basis of the form $U(Z,\alpha)$, where $Z \in C(S)_{\mathrm{cpt}}$, $\alpha \in \mathscr{P}^f(K \setminus \{0\}) \setminus \emptyset$, and

$$U(Z,\alpha) = \{(s,R,\phi) \in \mathrm{ZR}^f(K,S) \mid Z \not\leq s, \alpha \subset R\}.$$

By this topology, $\mathrm{ZR}^f(K,S)$ becomes a coherent space. From this, we can also see that $\prec$ in $\mathscr{M}_0^S$ is in fact transitive. Note also that this definition of Zariski-Riemann space coincides with the usual definition when S is an affine $\mathbb{Q}$-scheme.

DEFINITION 4.3.8. *Set* $X = \mathrm{ZR}^f(K,S)$.

(1) *The structure sheaf* $\mathscr{O}_X$ *on* X *is defined by*

$$U \mapsto \{a \in K \mid a \ \in \ R_p \text{ for any } p \ \in \ U\}.$$

It is obvious that this is in fact a sheaf.

(2) *The support morphism* $\beta_X : \alpha_1\mathscr{O}_X \to \tau_X$ *is defined by*

$$\Gamma(U, \alpha_1\mathscr{O}_X) \ni (f_i)_i \mapsto \{(1,\{f_i^{-1}\})\}_i,$$

where f_i*'s are non-zero generators.*

We will verify that $(X, \mathscr{O}_X, \beta_X)$ is an $\mathscr{A}$-scheme.

PROPOSITION 4.3.9. (1) *The support morphism β_X is well defined.*
(2) *The restriction maps reflect localizations.*
(3) *For any $p \in X$, $\mathscr{O}_{X,p} = R_p$.*

PROOF. (1) It suffices to show that $\{(1, \{f_i^{-1}\})\}_i \prec \{(1, \{g_j^{-1}\})\}_j$ if $(f_i)_i \leq (g_j)_j$. Assume $\{(1, \{f_i^{-1}\})\}_i \not\prec \{(1, \{g_j^{-1}\})\}_j$. This implies that there is a valuation ring R of K with $f_i^{-1} \in R$ for some i, and $g_j^{-1} \notin R$ for any j. This is equivalent to $g_j \in \mathfrak{M}_R$. $(f_i)_i \leq (g_j)_j$ tells that $f_i^m = \sum_j a_{ij} g_j$ for some m and some $a_{ij} \in \Gamma(U, \mathscr{O}_X)$. Let $s \in S$ be a point corresponding to R. Then $(g_j)_j$ generates the unit ideal in $\mathscr{O}_{S,s}[f_i^{-1}][g_j]_j$, but this cannot happen since $g_j \in \mathfrak{M}_R$.

(2) Let $V \subset U$ be an inclusion of quasi-open subsets of X, and $Z = U \setminus V$ be the closed subset of U. Let $f \in \Gamma(U, \mathscr{O}_X)$ be a section with $\beta_X(f) \geq Z$. This implies that $f^{-1} \in R_p$ for any $p \in V$, hence f is invertible in $\Gamma(V, \mathscr{O}_X)$.

(3) It is obvious that $\mathscr{O}_{X,p} \subset R_p$. For the converse, let $a \in R_p$ be any element. Then, the closed set Z corresponding to $\{(1, \{a\})\}$ does not contain p. Let U be the complement of Z. Then $a \in \Gamma(U, \mathscr{O}_X) \subset \mathscr{O}_{X,p}$. □

REMARK 4.3.10. (1) There is an alternative way of defining the structure sheaf $\mathscr{O}_X$: namely, $\mathscr{O}_X : \mathscr{M}^S \to (\mathbf{Rng})$ is defined by

$$\{(Z_i, \alpha_i)\}_i \mapsto \cap_i \cap_{s \in S \setminus Z_i} \mathrm{Ic}(K; \mathscr{O}_{S,s}[\alpha_i]),$$

where $\mathrm{Ic}(K; \mathscr{O}_{S,s}[\alpha_i])$ is the integral closure of $\mathscr{O}_{S,s}[\alpha_i]$ in K. We can easily see that this definition is equivalent to the previous one, once we know that the integral closure of a given domain is the intersection of all valuation rings containing it. This implies that we can characterize the Zariski-Riemann space *without using the notion of valuation rings*. However, the arguments get longer when we try to prove other properties, if we start from this definition.

(2) Note that $\mathrm{ZR}^f(\mathrm{Spec}\, K, \mathrm{Spec}\, A)$ coincides with the conventional Zariski-Riemann space, if A is a subring of K; there is a 1-1 correspondence between points of $\mathrm{ZR}^f(\mathrm{Spec}\, K, \mathrm{Spec}\, A)$ and valuations rings of K containing A. Its open basis is given by the form $U(a_1, \cdots, a_n)$, where $a_1, \cdots a_n$ are elements of K and $U(a_1, \cdots, a_n)$ is the set of valuation rings containing $A[a_1, \cdots, a_n]$. See [Mat] for example.

4.4 – *Zariski-Riemann space as a functor*

Now, we focus on the map $\pi : \mathrm{ZR}^f(K,S) \to S$. We have already seen that $|\pi| : |\mathrm{ZR}^f(K,S)| \to |S|$ is well defined as a morphism of coherent spaces. We will see here that π is well defined as a morphism of $\mathscr{A}$-schemes.

PROPOSITION 4.4.1. (1) *The canonical inclusion*

$$\pi^{\#} : \Gamma(U, \mathscr{O}_S) \ni a \mapsto a \in \Gamma(\pi^{-1}U, \mathscr{O}_X)$$

gives a morphism $\pi : \mathrm{ZR}^f(K,S) \to S$ of $\mathscr{A}$-schemes.

(2) *π is a P-morphism: in particular, π is surjective.*

(3) *π is proper.*

PROOF. (1) In order to see that $(\pi, \pi^{\#})$ is a morphism of $\mathscr{A}$-schemes, it suffices to see that the diagram

$$\begin{array}{ccc} \alpha_1\mathscr{O}_{S,s} & \xrightarrow{\pi^{\#}} & \pi_*\alpha_1\mathscr{O}_X \\ {\scriptstyle\beta_S}\downarrow & & \downarrow{\scriptstyle\beta_X} \\ \tau_S & \longrightarrow & \pi_*\tau_X \end{array}$$

is commutative: namely, we must see that $\{(1, \{f_i^{-1}\})\}_i = \{(\beta_S(f_i)_i, 1)\}$ for $(f_i)_i \in \alpha_1\mathscr{O}_S$. Let p be any element in $\cup_i U(1, \{f_i^{-1}\})$. Then there is a dominant morphism $\mathscr{O}_{S,\pi(p)} \to R_p$ with $f_i^{-1} \in R_p$ for some i. Assume that $s = \pi(p) \in \beta_S(f_i)_i$. This is equivalent to saying that f_i' are in the maximal ideal of $\mathscr{O}_{S,s}$. But this contradicts to $\mathscr{O}_{S,\pi(s)} \to R_p$ being dominant. The converse can be proven similarly.

(2) It suffices to show that $C(S)_{\mathrm{cpt}} \to \mathrm{Spec}\,\mathscr{M}^S$ is injective, but this is obvious from the definition.

(3) From Remark 3.4.7, it suffices to show that when we are given a commutative square

$$\begin{array}{ccc} \mathrm{Spec}\,K & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle\pi} \\ \tilde{\mathrm{Spec}}R & \xrightarrow[h]{} & S \end{array}$$

we have a unique morphism $\tilde{\mathrm{Spec}}\,R \to X$ making the whole diagram commutative. Set $s = h(\eta)$, where η is the closed point of $\tilde{\mathrm{Spec}}\,R$. Then, the dominant morphism $\mathscr{O}_{S,s} \to R$ determines a point x of $X = \mathrm{ZR}^f(K,S)$. The

isomorphism $\mathscr{O}_{X,x} \to R$ gives the required morphism $\tilde{\mathrm{Spec}}\, R \to X$. □

REMARK 4.4.2. The reader may notice that (2) follows immediately from (3) and π being dominant. However, we gave a different proof here, since the valuative criterion already uses the fact of (2).

From now on, we refer to this proper morphism $\pi_S : \mathrm{ZR}^f(K,S) \to S$ as the *classical Zariski-Riemann space associated to S*.

DEFINITION 4.4.3. *Let T and S be $\mathscr{A}$-schemes, and $\pi_Y : Y = \mathrm{ZR}^f(K,T) \to T$, $\pi_X : X = \mathrm{ZR}^f(K,S) \to S$ be the associated classical Zariski-Riemann spaces. A morphism $f : T \to S$ of $\mathscr{A}$-schemes induces a morphism $\tilde{f} : Y \to X$ of $\mathscr{A}$-schemes as follows:*

(1) *The morphism $|\tilde{f}| : |Y| \to |X|$ of the underlying spaces are defined by*

$$(t, R, \phi) \mapsto (f(t), R, \phi \circ f^{\#}),$$

where $f^{\#} : \mathscr{O}_{S,s} \to \mathscr{O}_{T,t}$ is the dominant morphism. In terms of II-rings, this can be expressed as

$$\mathscr{M}^S \ni \{(Z,\alpha)\} \mapsto \{(f^{-1}Z,\alpha)\} \in \mathscr{M}^T,$$

which shows that $|\tilde{f}|$ is indeed a quasi-compact morphism.

(2) *The morphism $\tilde{f}^{\#} : \mathscr{O}_X \to |\tilde{f}|_* \mathscr{O}_Y$ is defined by the canonical inclusion. It is easy to see that $|\tilde{f}|_* \beta_Y \circ \tilde{f}^{\#} = \tilde{f}^{-1} \circ \beta_X$.*

Hence, the map $S \mapsto \mathrm{ZR}^f(K,S)$ induces a functor

$$\mathrm{ZR}^f(K,\cdot) : (\mathscr{A}\textbf{-Sch}) \to (\textbf{proper morphism of } \mathscr{A}\textbf{-schemes}).$$

We will call this functor the ZR functor.

4.5 – *Morphisms of profinite type*

Our next aim is to express the ZR functor as a left adjoint, namely to clarify the universal property of the classical Zariski-Riemann space.

PROPOSITION 4.5.1. *Let $f : T \to S$ be a morphism of $\mathscr{A}$-schemes, and $\tilde{f} : \mathrm{ZR}^f(K,T) \to \mathrm{ZR}^f(K,S)$ be the induced morphism. Then, f is separated (resp. universally closed, proper) if and only if $\tilde{f}$ is injective (resp. surjective, bijective) on the underlying spaces.*

This is just the translation of the valuative criteria, so we will omit the proof.

DEFINITION 4.5.2. *Let $f : T \to S$ be a morphism of $\mathscr{A}$-schemes, and $\tilde{f} : \mathrm{ZR}^f(K,T) \to \mathrm{ZR}^f(K,S)$ be the induced morphism.*

(1) *f is* of profinite type, *if $\tilde{f}$ is a Q-morphism.*
(2) *f is* strongly of profinite type, *if $\tilde{f}$ is an open immersion.*

Of course, f is separated if f is of profinite type. The next characterization of morphism of profinite type is obvious.

PROPOSITION 4.5.3. *Let $f : T \to S$ be a morphism of $\mathscr{A}$-schemes. The followings are equivalent:*

(i) *f is of profinite type.*
(ii) *For every $Z \in C(T)_{\mathrm{cpt}}$, there exists $\{(Z_i, \alpha_i)\}_i \in \mathscr{P}^f(C(S)_{\mathrm{cpt}} \times (\mathscr{P}^f(K \setminus \{0\}) \setminus \emptyset))$ such that:*
  (a) $\cap_i (f^{-1}Z_i)[\alpha_i] = Z$.
  (b) *For every $t \in Z$, set $I_t = \{i \mid t \in f^{-1}Z_i\}$. Then for any map $\sigma : I_t \to \cup_{i \in I_t} \alpha_i$ such that $\sigma_i \in \alpha_i$, $(\sigma_i^{-1})_i$ generates the unit ideal of $\mathscr{O}_{T,t}[\sigma_i^{-1}]_i$.*

Roughly speaking, an $\mathscr{A}$-scheme X of profinite type over S has the coarsest topology, which makes the map $X \to S$ quasi-compact, and the domain of meromorphic functions are quasi-compact open: here, a *domain* of a meromorphic function $a \in K$ is $\{x \in X \mid a \in \mathscr{O}_{X,x}\}$.

COROLLARY 4.5.4. *For any $\mathscr{A}$-scheme S, Set $X = \mathrm{ZR}^f(K,S)$. Then, the natural morphism $\pi_X : \mathrm{ZR}^f(K,X) \to X$ is an isomorphism.*

PROOF. It follows from Proposition 4.5.3 that $\pi_S : X \to S$ is of profinite type, which is equivalent to π_X being a Q-morphism. On the other hand, π_X is bijective since π_S is proper. It is obvious that π_X induces isomorphism on each stalks. This implies that π_X is an isomorphism. □

We will verify some basic facts of morphisms of profinite type.

PROPOSITION 4.5.5. (1) *Let A be a ring, and B be a finitely generated A-algebra. Then,* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is strongly profinite.*
(2) *An open immersion is strongly profinite.*

(3) *If $X = \cup_i X_i$ is an quasi-compact open cover of X, then $\{\mathrm{ZR}^f(K, X_i) \to \mathrm{ZR}^f(K, X)\}_i$ is an quasi-compact open cover of $\mathrm{ZR}^f(K, X)$.*

(4) *Morphisms of profinite type (resp. strongly of profinite type) are stable under compositions.*

(5) *Morphisms of profinite type (resp. strongly of profinite type) is local on the base: let $f : T \to S$ be a morphism of $\mathscr{A}$-schemes, $S = \cup_i S_i$ be an quasi-compact open covering of S, and $T_i = S_i \times_S T$. Then, f is of profinite type (resp. strongly of profinite type) if and only if $T_i \to S_i$ is of profinite type (resp. strongly of profinite type) for any i.*

(6) *Let $f : T \to S$ be a separated morphism of $\mathscr{A}$-schemes, and $T = \cup_i T_i$ be a quasi-compact open covering. Then f is of profinite type (resp. strongly of profinite type) if and only if $f|_{T_i}$ is.*

(7) *Let S be an $\mathscr{A}$-scheme, and $\{X^\lambda\}$ be a filtered projective system of $\mathscr{A}$-schemes over S. Set $X = \varprojlim_\lambda X^\lambda$. Then $X \to S$ is of profinite type if $X^\lambda \to S$ is of profinite type for any λ.*

Proof. We will only show (1), (6) and (7); the others are easy.

(1) Set $B = A[x_1, \cdots, x_n]$. Then, $\mathrm{ZR}^f(K, \mathrm{Spec}\, B)$ is isomorphic to the open set $U(1, \{x_1, \cdots, x_n\})$ of $\mathrm{ZR}^f(K, \mathrm{Spec}\, A)$.

(6) The 'only if' part follows from (2) and (4). Suppose $f|_{T_i}$ is of profinite type. Then $\mathrm{ZR}^f(K, T_i) \to \mathrm{ZR}^f(K, S)$ is Q-morphism for any i. Since f is separated, we see that $\mathrm{ZR}^f(K, T) \to \mathrm{ZR}^f(K, S)$ is also a Q-morphism, since $\{\mathrm{ZR}^f(K, T_i)\}_i$ is an quasi-compact open cover of $\mathrm{ZR}^f(K, T)$, from (3).

(7) Since $\mathrm{ZR}^f(K, X^\lambda) \to \mathrm{ZR}^f(K, S)$ is a Q-morphism for any λ, and Q-morphism is stable under taking filtered projective limits by Proposition 2.4.2, it suffices to show that $\mathrm{ZR}^f(K, X) \to \varprojlim_\lambda \mathrm{ZR}^f(K, X^\lambda)$ is an isomorphism.

Since $C(X)_{\mathrm{cpt}} = \varinjlim C(X^\lambda)_{\mathrm{cpt}}$, we have a natural isomorphism

$$\varinjlim_\lambda \mathscr{P}^f(C(X^\lambda)_{\mathrm{cpt}} \times (\mathscr{P}^f(K \setminus 0) \setminus \emptyset)) \simeq \mathscr{P}^f(C(X)_{\mathrm{cpt}} \times (\mathscr{P}^f(K \setminus 0) \setminus \emptyset))$$

which yields $\varinjlim_\lambda \mathscr{M}^{X^\lambda} \simeq \mathscr{M}^X$, namely, $\mathrm{ZR}^f(K, X) \to \varprojlim_\lambda \mathrm{ZR}^f(K, X^\lambda)$ is an isomorphism on the underlying spaces. Since every stalk of both sides is a valuation ring of K, the morphism of structure sheaves is also an isomorphism. □

Corollary 4.5.6. (1) *A separated morphism of $\mathscr{Q}$-schemes is of profinite type.*

(2) *A separated, of finite type morphism of $\mathscr{Q}$-schemes is strongly of profinite type.*

PROOF. We will just prove (1), since the proof of (2) is similar. Let $X \to S$ be a separated morphism of $\mathscr{Q}$-schemes. From (5) and (6) of Proposition 4.5.5, we may assume $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$ for some domains A and B. B is the colimit of all finitely generated sub A-algebras $\{B_\lambda\}$, and $\operatorname{Spec} B_\lambda \to \operatorname{Spec} A$ is of profinite type, by (1). Then, (7) of Proposition 4.5.5 shows that $\operatorname{Spec} B \to \operatorname{Spec} A$ is also of profinite type. □

DEFINITION 4.5.7. (1) *Let* (**K**/**Int**) *be the subcategory of* ($\mathscr{A}$-**Sch**)*, consisting of irreducible reduced $\mathscr{A}$-schemes X with a dominant morphism* $\operatorname{Spec} K \to X$*. The morphisms in* (**K**/**Int**) *are dominant morphisms, under* $\operatorname{Spec} K$*.*

(2) *Let* (**PrPf**) *be the category of proper, of profinite type morphisms of* (**K**/**Int**)*, and the arrows being commutative squares.*

(3) *There is a target functor* $U_t : (\mathbf{PrPf}) \to (\mathbf{K}/\mathbf{Int})$*, sending* $(f : X \to S)$ *to* S*.*

THEOREM 4.5.8. *The ZR functor* $\mathrm{ZR}^f(K, \cdot)$ *is the left adjoint of* U_t*.*

PROOF. The unit $\varepsilon : \mathrm{Id} \Rightarrow U_t \circ \mathrm{ZR}^f(K, \cdot)$ of the adjoint is the identity. The counit $\eta_X : \mathrm{ZR}^f(K, S) \to X$ for a proper, of profinite type morphism $f : X \to S$ is given as follows: Since $f : X \to S$ is of profinite type and proper, $\tilde{f} : \mathrm{ZR}^f(K, X) \to \mathrm{ZR}^f(K, S)$ is an isomorphism. Then, η_X is defined by

$$\mathrm{ZR}^f(K, S) \xrightarrow{\tilde{f}^{-1}} \mathrm{ZR}^f(K, X) \xrightarrow{\pi_X} X.$$

These two natural transforms ε and η give the adjoint $\mathrm{ZR}^f(K, \cdot) \dashv U_t$. □

4.6 – *The embedding problem revisited*

In this subsection, we will construct a compactification functor from the ZR functor, and characterize it by the universal property.

DEFINITION 4.6.1. *A Q-morphism $f : X \to Y$ of $\mathscr{A}$-schemes is* strict*, if for any quasi-compact open subset U of X and a section $a \in \mathscr{O}_X(U)$, there exists a quasi compact open subset V of Y and a section $b \in \mathscr{O}_Y(V)$ such that:*

(i) $U = f^{-1}V$*, and*
(ii) $f^{\#}(b) = a$*.*

In particular, an open immersion is strict.

LEMMA 4.6.2. *Consider the pushout diagram of $\mathscr{A}$-schemes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ S & \xrightarrow[\tilde{f}]{} & T \end{array}$$

namely, $T = S \coprod_X Y$. Suppose f is an open immersion (resp. strict Q-morphism). Then, $\tilde{f}$ is an open immersion (resp. strict Q-morphism).

PROOF. We will only prove for the case f is an open immersion. In this case, $C(X)_{\mathrm{cpt}}$ is a localization $(C(Y)_{\mathrm{cpt}})_W$ of $C(Y)_{\mathrm{cpt}}$ along some $W \in C(Y)_{\mathrm{cpt}}$. Since

$$C(T)_{\mathrm{cpt}} = C(S)_{\mathrm{cpt}} \times_{C(X)_{\mathrm{cpt}}} C(Y)_{\mathrm{cpt}},$$

in the category of II-rings, $C(S)_{\mathrm{cpt}}$ is the localization of $C(T)_{\mathrm{cpt}}$ along $(1, W) \in C(T)_{\mathrm{cpt}}$. Hence, the map $|\tilde{f}| : |S| \to |T|$ is an open immersion on the underlying space.

Let us show that $\tilde{f}$ is strict Q-morphism. Suppose $a \in \mathscr{O}_S(U)$ is a section of S for a quasi-compact open set U of S. Pulling back a by g gives a section $g^{\#}(a) \in \mathscr{O}_X(g^{-1}U)$. Since f is strict, there is a quasi-compact open $V \in Y$ and a section $b \in \mathscr{O}_Y(V)$ such that $f^{-1}V = g^{-1}U$ and $g^{\#}(a) = f^{\#}(b)$. Then, (U, V) gives a quasi-compact open subset of T, and $(a, b) \in \mathscr{O}_T(U, V)$ gives a section. This section (a, b) maps to a via $\tilde{f}^{\#}$, hence $\tilde{f}$ is strict. □

DEFINITION 4.6.3. (1) *Let $T \to S$ be a dominant morphism of irreducible, reduced $\mathscr{A}$-schemes, and K be the function field of T. The (classical)* Zariski-Riemann space $\mathrm{ZR}^f(T, S)$ *of $T \to S$ is defined by the pushout of the following:*

$$\begin{array}{ccc} \mathrm{ZR}^f(K, T) & \longrightarrow & \mathrm{ZR}^f(K, S) \\ \downarrow & & \vdots\downarrow \\ T & \cdots\cdots\to & \mathrm{ZR}^f(T, S) \end{array}$$

(2) *Let S be an irreducible reduced $\mathscr{A}$-scheme. We denote by* $(\mathbf{Int}/S)$ *the category of irreducible, reduced $\mathscr{A}$-schemes dominant over S, and dominant S-morphisms.*

(3) *Let $f : T \to T'$ be a morphism in* $(\mathbf{Int}/S)$*. Then f naturally induces a morphism* $\mathrm{ZR}^f(T,S) \to \mathrm{ZR}^f(T',S)$ *from the universal property of pushouts.*

PROPOSITION 4.6.4. (1) $\mathrm{ZR}^f(T,S)$ *is proper and of profinite type over S.*
(2) *$T \to \mathrm{ZR}^f(T,S)$ is a $\mathbb{Q}$-morphism (resp. open immersion) if $T \to S$ is of profinite type (resp. strongly of profinite type).*

PROOF. (1) Let K be the function field of T, and set $X = \mathrm{ZR}^f(T,S)$. Applying the ZR functor $\mathrm{ZR}^f(K,\cdot)$ to the pushout diagram of the definition of $\mathrm{ZR}^f(T,S)$ yields the following pushout diagram:

$$\begin{array}{ccccc} \mathrm{ZR}^f(K,\mathrm{ZR}^f(K,T)) & \longrightarrow & \mathrm{ZR}^f(K,\mathrm{ZR}^f(K,S)) & \xrightarrow{\simeq} & \mathrm{ZR}^f(K,S) \\ \downarrow{\scriptstyle\simeq} & & \downarrow & & \\ \mathrm{ZR}^f(K,T) & \longrightarrow & \mathrm{ZR}^f(K,X) & & \end{array}$$

This is indeed a pushout, since $\mathrm{ZR}^f(K,\cdot)$ is a left adjoint and hence preserves colimits. This shows that the right vertical arrow is also an isomorphism, which tells that $X \to S$ is proper and of profinite type.
(2) This follows from Lemma 4.6.2. Note that $\mathrm{ZR}^f(K,T) \to \mathrm{ZR}^f(K,S)$ is a strict $\mathbb{Q}$-morphism if $T \to S$ is of profinite type, or strongly of profinite type. □

This proposition shows that $\mathrm{ZR}^f(\cdot,S)$ is functor from $(\mathbf{Int}/S)$ to the full subcategory $(\mathbf{PrPf}/S)$ of $(\mathbf{Int}/S)$ consisting of irreducible reduced $\mathscr{A}$-schemes, proper and of profinite type over S.

THEOREM 4.6.5. $\mathrm{ZR}^f(\cdot,S)$ *is the left adjoint of the underlying functor* $U : (\mathbf{Int}/S) \to (\mathbf{PrPf}/S)$.

PROOF. The unit $\varepsilon_T : T \to \mathrm{ZR}^f(T,S)$ is the canonical morphism, for any $T \in (\mathbf{Int}/S)$. The counit $\eta_X : \mathrm{ZR}^f(X,S) \to X$ for a proper, of profinite type morphism $X \to S$ is defined as follows. Consider the pushout diagram:

$$\begin{array}{ccc} \mathrm{ZR}^f(K,X) & \longrightarrow & \mathrm{ZR}^f(K,S) \\ \downarrow & & \downarrow \\ X & \xrightarrow[\iota_X]{} & \mathrm{ZR}^f(X,S) \end{array}$$

Since X is proper and of profinite type over S, the upper horizontal arrow is an isomorphism, hence the lower arrow is also. Define η_X as the inverse of ι_X. These two natural transforms ε and η give the adjoint $\mathrm{ZR}^f(\cdot, S) \dashv U$. □

In particular, we have:

COROLLARY 4.6.6. *Let $X \to S$ be a separated morphism of integral $\mathscr{Q}$-schemes. Then, there exists a proper, of profinite type morphism $\overline{X} \to S$ of $\mathscr{A}$-schemes with a Q-morphism $\iota : X \to \overline{X}$. Moreover, this embedding ι of X is universal, and ι is an open immersion if X is of finite type over S.*

This is a variant of Nagata embedding ([Con]). Note that from Proposition 4.2.2, $X \to \mathrm{ZR}_S(X)$ becomes also an open immersion, if X is separated, of finite type over S.

REMARK 4.6.7. There are previous constructions of Zariski-Riemann spaces for ordinary schemes; for example, see [Tem]. We can see without difficulty that our construction coincides with them. However, the proof will be somewhat technical and takes some time to check that these are equivalent, since the definition is very different: the previous one is defined by the limit space of admissible blow ups. The comparison will be treated in the forthcoming paper.

5. Appendix: The definition of $\mathscr{A}$-schemes

In this section, we will briefly review the definition of $\mathscr{A}$-schemes, when the algebraic system σ is that of rings. For the general definition and detailed proofs, we refer to [Tak].

Before we start the definitions, we will explain the intuitive idea and the essential differences between $\mathscr{A}$-schemes and ordinary schemes.

(1) The fundamental property of ordinary schemes is that the global section functor admits the left adjoint, namely the spectrum functor:

$$\mathrm{Spec} : (\mathbf{Ring}) \rightleftarrows (\mathbf{Sch})^{\mathrm{op}} : \Gamma.$$

This enables various construction of schemes, such as fiber products. However, the construction of the co-unit $X \to \mathrm{Spec}\,\Gamma(X)$ of the above adjoint does not actually use the axiom of schemes that it is locally isomorphic to the spectrum of a ring; it just uses the property that the restriction functor corresponds to localizations: let us describe it more explicitly. Let

$\mathscr{O}_X$ be a sheaf of functions on a space X. When there is a function $f \in \mathscr{O}_X$, it determines the zero locus $\beta(f) = \{f = 0\}$ on X. This correspondence is the intuitive idea of the support morphism defined below.

When the function f is restricted to an open set V such that $V \cap \beta(f) = \emptyset$, then $f|_V$ is nowhere vanishing. Therefore, f must be invertible in $\mathscr{O}_X(V)$. This is formulated below as the property which we refer to as 'restrictions reflect localizations'.

These setups enable us to construct the counit morphism. This is why we put emphasis on these properties.

(2) On the other hand, we stick on to coherent underlying spaces, when defining $\mathscr{A}$-schemes. This is because coherent spaces have good properties in nature, and we can take limits and colimits in the category of coherent spaces. This shows that we do not have any reason to 'forget' the coherence properties, even when we consider limit and colimit spaces. We believe that this restriction is not wrong, since we have already seen in § 1 that there are various benefits because of this.

DEFINITION 5.1.8. (1) *An* idealic semiring *is a set R endowed with two operators $+$ and $\cdot$, satisfying:*

(a) *R is a commutative monoid with respect to $+$ and $\cdot$, with two unit elements 0 and 1, respectively. Further, R is idempotent with respect to $+$: $a + a = a$ for any $a \in R$.*

(b) *The distribution law holds: $(a + b)c = ac + bc$ for any elements $a, b, c \in R$.*

(c) *0 is the absorbing element with respect to the multiplication: $0 \cdot a = 0$ for any $a \in R$. 1 is the absorbing element with respect to the addition.*

Note that an idealic semiring has a natural ordering, defined by $a \leq b \Leftrightarrow a + b = b$.

(2) *An* II-ring *is an idealic semiring with idempotent multiplications. This is conventionally called a* distributive lattice*, used in Stone duality.*

(3) *The category of II-rings are denoted by* (**IIRng**).

DEFINITION 5.1.9. (1) *A topological space X is* sober*, if any irreducible closed subset Z of X has a unique generic point ξ_Z, namely, $Z = \overline{\{\xi_Z\}}$.*

(2) *A sober space is* coherent*, if it is quasi-compact, quasi-separated (namely, the intersection of any two quasi-compact open subset is again quasi-compact), and has a quasi-compact open basis. We denote by* (**Coh**) *the category of coherent spaces and quasi-compact morphisms.*

(3) *For a sober space X, $C(X)$ is the set of all closed subsets Z of X. This becomes an idealic semiring, defining the addition as taking intersections, and the multiplication as taking unions. Moreover, this semiring is complete, i.e. admits infinite summations. The category of complete II-rings is denoted by* (**IIRng**†).

(4) *For a coherent space X, $C(X)_{\mathrm{cpt}}$ is the set of all closed subsets Z of X such that $X \setminus Z$ is quasi-compact. This becomes an idealic semiring.*

The correspondence $X \mapsto C(X)_{\mathrm{cpt}}$ gives an equivalence of categories $(\mathbf{Coh})^{\mathrm{op}} \to (\mathbf{IIRng})$: the inverse is given by $R \mapsto \operatorname{Spec} R$, where $\operatorname{Spec} R$ is the set of prime ideals of R with the well known topology. This is the Stone duality.

DEFINITION 5.1.10. (1) *For a ring R, let $\alpha_1(R)$ be the set of finitely generated ideals of R, divided by the equivalence relation generated by $I \cdot I = I$. This gives a functor* (**Rng**) $\to$ (**IIRng**)*, where* (**Rng**) *is the category of rings.*

(2) *For any ring R, $\alpha_2 : R \to \alpha_1(R)$ is a multiplication-preserving map, sending $f \in R$ to the principal ideal generated by f. This map gives a natural transformation, and preserves localizations.*

(3) *Let X be a coherent space. A* (**Coh**)$^{\mathrm{op}}$*-valued (in other words,* (**IIRng**)*-valued) sheaf τ_X is defined by $U \mapsto \varinjlim_V V$, where V runs through all quasi-compact open subsets of U, and the inductive limit is taken in the category of coherent spaces,* not *in the category of topological spaces.*

We are finally in the stage of defining $\mathscr{A}$-schemes.

DEFINITION 5.1.11. *(1) An $\mathscr{A}$-scheme is a triple $(X, \mathscr{O}_X, \beta_X)$ where X is a coherent space, $\mathscr{O}_X$ is a ring valued sheaf on X, and $\beta_X : \alpha_1\mathscr{O}_X \to \tau_X$ is a morphism of* (**IIRng**)*-valued sheaves on X (which we refer to as the 'support morphism'. Here, $\alpha_1\mathscr{O}_X$ is the sheafification of $U \mapsto \alpha_1\mathscr{O}_X(U)$), satisfying the following property: for any two open subsets $U \supset V$ of X, the* restriction maps reflect localizations*, i.e. the map $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ factors through $\mathscr{O}_X(U)_Z$, where $Z = U \setminus V$ is a closed subset of U and $\mathscr{O}_X(U)_Z$ is the localization of $\mathscr{O}_X(U)$ along*

$$\{a \in \mathscr{O}_X(U) \mid \beta_X\alpha_2(a) \geq Z\}.$$

(2) *Let $X = (|X|, \mathscr{O}_X, \beta_X)$ and $Y = (|Y|, \mathscr{O}_Y, \beta_X)$ be two $\mathscr{A}$-schemes. A morphism $f : X \to Y$ of $\mathscr{A}$-schemes is a pair $f = (|f|, f^{\#})$, where $|f| : |X| \to |Y|$ is a quasi-compact morphism between underlying*

spaces, and $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$ *is a morphism of ring valued sheaves on* Y *which makes the following diagram commutative:*

$$\begin{array}{ccc} \mathscr{O}_Y & \xrightarrow{f^{\#}} & |f|_*\mathscr{O}_X \\ {\scriptstyle \beta_Y}\downarrow & & \downarrow{\scriptstyle \beta_X} \\ \tau_Y & \xrightarrow[|f|^{-1}]{} & |f|_*\tau_X \end{array}$$

(3) *The spectrum functor* $\mathrm{Spec}^{\mathscr{A}} : (\mathbf{Rng}) \to (\mathscr{A}\text{-}\mathbf{Sch})^{\mathrm{op}}$ *from the category of rings to the opposite category of* $\mathscr{A}$*-schemes, is defined as follows: for a ring* R*, the underlying space is defined by* $X = \mathrm{Spec}\, R$*. The structure sheaf* $\mathscr{O}_X$ *is the sheafification of* $U \mapsto R_Z$*, where* $Z = X \setminus U$ *is the complement closed subset of* X*, and* R_Z *is the localization along*

$$\{a \in R \mid (a) \geq Z\}.$$

The support morphism $\beta_X : \alpha_1\mathscr{O}_X \to \tau_X$ *is the canonical isomorphism. Hence we set* $\mathrm{Spec}^{\mathscr{A}} R = (X, \mathscr{O}_X, \beta_X)$.

For a homomorphism $f : A \to B$*, we have a morphism* $\mathrm{Spec}^{\mathscr{A}} B \to \mathrm{Spec}^{\mathscr{A}} A$*, as is well known.*

The spectrum functor is the left adjoint of the global section functor $\Gamma : (\mathscr{A}\text{-}\mathbf{Sch})^{\mathrm{op}} \to (\mathbf{Rng})$.

REMARK 5.1.12. (1) There are some differences in the notation with that of [Tak]: in the previous paper, the category of II-rings is denoted by (**PIIRng**). Also, the sheaf τ_X is denoted by τ'_X. This is because we are comparing them with those of sober spaces in [Tak], and hence had to distinguish the notation. However, this is not necessary in this paper.

(2) In [Tak], presheaves on a coherent space X is defined as a functor $C(X)_{\mathrm{cpt}} \to (\mathbf{Set})$. In this paper, most of the presheaves are described in a usual way, namely, we attach algebras to each *open* subsets of X. However, in some of the definitions and arguments, we describe sheaves as a functor from $C(X)_{\mathrm{cpt}}$ to simplify the argument. These two ways of descriptions are essentially the same.

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