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The Behavior of Rigid Analytic Functions Around Orbits of Elements of $\mathbb{C}_p$

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ABSTRACT - Given a prime number p and the Galois orbit $O(x)$ of an element x of $\mathbb{C}_p$, the topological completion of the algebraic closure of the field of p-adic numbers, we study the behavior of rigid analytic functions around orbits of elements of $\mathbb{C}_p$.

Introduction

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic valuation. Let $O(x)$ denote the orbit of an element $x \in \mathbb{C}_p$, with respect to the Galois group $G = Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$. We are interested in the behavior of rigid analytic functions defined on $E(x) = (\mathbb{C}_p \cup \{\infty\}) \setminus O(x)$, the complement of $O(x)$. The paper consists of five sections. The first one contains notations and some basic results. In the second section we study the zeros of rigid analytic functions, which are not rational, around finite sets of $\mathbb{C}_p$ and, in particular, around orbits of algebraic elements of $\mathbb{C}_p$, see Theorem 1 and Corollary 1. The next section is concerned with the transcendental case. One has a similar result of a theorem of Barsky, see Theorem 2, and then we prove that if all the points of $O(x)$ are singular points for a rigid

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analytic function F then one has $\liminf_{z\to x} |F(z)| = 0$, see Theorem 3. In section four we give a few examples of rigid analytic functions on $E(x)$ with and without zeros. The last section is concerned with some remarks related to zeros on a corona.

1. Background material

Let p be a prime number and $\mathbb{Q}_p$ the field of p-adic numbers endowed with the p-adic absolute value $|\cdot|$, normalized such that $|p| = 1/p$. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and denote by the same symbol $|\cdot|$ the unique extension of $|\cdot|$ to $\overline{\mathbb{Q}}_p$. Further, denote by $(\mathbb{C}_p, |\cdot|)$ the completion of $(\overline{\mathbb{Q}}_p, |\cdot|)$ (see [Am], [Ar]). Let $G = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ endowed with the Krull topology. The group G is canonically isomorphic with the group $Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$, of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$, see [APZ2]. We shall identify these two groups. For any $x \in \mathbb{C}_p$ denote $O(x) = \{\sigma(x) : \sigma \in G\}$ the orbit of x, and let $\widetilde{\mathbb{Q}_p[x]}$ be the closure of the ring $\mathbb{Q}_p[x]$ in $\mathbb{C}_p$.

For any closed subgroup H of G denote $Fix(H) = \{x \in \mathbb{C}_p : \sigma(x) = x$ for all $\sigma \in H\}$. Then $Fix(H)$ is a closed subfield of $\mathbb{C}_p$. Denote $H(x) = \{\sigma \in G : \sigma(x) = x\}$. Then $H(x)$ is a closed subgroup of G, and $Fix(H(x)) = \widetilde{\mathbb{Q}_p[x]}$.

The map $\sigma \rightsquigarrow \sigma(x)$ from G to $O(x)$ is continuous, and it defines a homeomorphism from $G/H(x)$ (endowed with the quotient topology) to $O(x)$ (endowed with the induced topology from $\mathbb{C}_p$) (see [APZ1]). In such a way $O(x)$ is a closed compact and totally disconnected subspace of $\mathbb{C}_p$, and the group G acts continuously on $O(x)$: if $\sigma \in G$ and $\tau(x) \in O(x)$ then $\sigma \star \tau(x) := (\sigma\tau)(x)$.

Now, if X is a compact subset of $\mathbb{C}_p$ then by an open ball in X we mean a subset of the form $B(x, \varepsilon) \cap X$, $x \in \mathbb{C}_p$, $\varepsilon > 0$ where $B(x, \varepsilon) = \{y \in \mathbb{C}_p : |y - x| < \varepsilon\}$. Let us denote by $\Omega(X)$ the set of subsets of X which are open and compact. It is easy to see that any $D \in \Omega(X)$ can be written as a finite union of open balls in X, any two disjoint.

DEFINITION 1. *By a distribution on X with values in $\mathbb{C}_p$ we mean a map $\mu : \Omega(X) \to \mathbb{C}_p$ which is finitely additive, that is, if $D = \bigcup_{i=1}^{n} D_i$ with $D_i \in \Omega(X)$ for $1 \le i \le n$ and $D_i \cap D_j = \emptyset$ for $1 \le i \ne j \le n$, then $\mu(D) = \sum_{i=1}^{n} \mu(D_i)$. (See also [MS].)*

The norm of μ is defined by $\|\mu\| := \sup\{|\mu(D)| : D \in \Omega(X)\}$. If $\|\mu\| < \infty$ we say that μ is a measure on X.

DEFINITION 2. *Let D be a infinite subset of $\mathbb{P}^1(\mathbb{C}_p)$. A function $f : D \to \mathbb{C}_p$ is said to be rigid analytic (or Krasner analytic) on D provided that f is a uniform limit with respect to the topology of uniform convergence on D of a sequence of rational functions on D. (See also [FP], [Am].) We denote by $\mathcal{A}(D, \mathbb{C}_p)$ the set of all rigid analytic functions defined on D.*

The set $X \subset \mathbb{C}_p$ is said G-equivariant provided that $\sigma(x) \in X$ for any $x \in X$ and any $\sigma \in G$. ($X = O(x)$ is such an example.)

DEFINITION 3. *Let X be a G-equivariant compact subset of $\mathbb{C}_p$ and μ a distribution on X with values in $\mathbb{C}_p$. We say that μ is G-equivariant if $\mu(\sigma(B)) = \sigma(\mu(B))$ for any ball B in X and any $\sigma \in G$.*

REMARK. On a Galois orbit $O(x)$ there exists a unique G-equivariant probability distribution with values in $\mathbb{Q}_p$, namely the Haar distribution π_x, see [APVZ1] and [APZ2].

According to [APZ2], a rigid analytic function defined on a subset D of $\mathbb{P}^1(\mathbb{C}_p)$ is said to be *equivariant* if for any $z \in D$ one has $O(z) \subset D$ and $f(\sigma(z)) = \sigma(f(z))$ for all $\sigma \in G$.

Let $\mathcal{A}^G(D, \mathbb{C}_p)$ be the set of equivariant rigid analytic functions on D with values in $\mathbb{C}_p$ and $\mathcal{A}_0^G(D, \mathbb{C}_p)$ the subset of those that vanish at ∞ when $\infty \in D$.

2. The algebraic case

Let S be a finite set in $\mathbb{C}_p$ and let $F : \mathbb{P}^1(\mathbb{C}_p) \setminus S \to \mathbb{C}_p$ be a rigid analytic function which is not rational.

We study the zeros of such functions around S. In particular, one considers S of the form $O(\alpha) = \{\sigma(\alpha) : \sigma \in G = Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)\}$ the orbit of $\alpha \in \overline{\mathbb{Q}}_p$. Let us recall the analogue of a classical result of Picard.

PROPOSITION 1 (see [AR]). *Let $F = \sum_{-\infty}^{\infty} a_n X^n \in \mathbb{C}_p[[X, X^{-1}]]$ be a Laurent series that has infinitely many coefficients $a_n \neq 0$ for $n < 0$ and such that $R_F^- := \limsup_{n\to\infty} |a_{-n}|^{1/n} = 0$ i.e. 0 is an essential isolated* *singularity for F. Then there are infinitely many critical radii r_i strictly decreasing to* 0 *and for each* $\varepsilon > 0$, $y \in \mathbb{C}_p$, *the equation* $F(x) = y$ *has infinitely many solutions* $0 < |x| < \varepsilon$.

For F as above, we define $R_F^+ := \sup\{r \geq 0 : |a_n| r^n \to 0,\ n \to \infty\} = 1/\limsup_{n\to\infty} |a_n|^{1/n}$, by the Hadamard formula, see [AR]. One has the following result.

THEOREM 1. *Let S be a finite set in $\mathbb{C}_p$ and let $F : \mathbb{P}^1(\mathbb{C}_p) \setminus S \to \mathbb{C}_p$ be a rigid analytic function. If F is not rational then F has infinitely many zeros in any neighborhood of at least one point of S.*

PROOF. For the sake of simplicity we suppose $F(\infty) = 0$. Let us consider the Mittag-Leffler decomposition

$$F(z) = \sum_{a\in S} F_a(z) = \sum_{a\in S} \sum_{n\geq 1} \frac{b_n^{(a)}}{(z-a)^n}, \tag{1}$$

where $F_a(z) = \sum\limits_{n\geq 1} \dfrac{b_n^{(a)}}{(z-a)^n}$ and $\dfrac{|b_n^{(a)}|}{\varepsilon^n} \to 0$, $n \to \infty$, for any positive real number ε and any $a \in S$. It is easy to see that this condition is further equivalent to $|b_n^{(a)}|^{1/n} \to 0$, when $n \to \infty$. Because F is not rational there exists $e \in S$ such that F_e is not rational. In such a way from (1) one has

$$F(z) = F_e(z) + H(z), \tag{2}$$

where $H(z) = \sum\limits_{\substack{a\in S \\ a\neq e}} \sum\limits_{n\geq 1} \dfrac{b_n^{(a)}}{(z-a)^n}$. We have $y := H(e)$ is well defined and denote $A = |y| = |H(e)|$. It is clear that $R_{F_e}^- = \limsup |b_n^{(e)}|^{1/n} = 0$ and $R_{F_e}^+ = \sup\{r \geq 0 : |b_{-n}^{(e)}| r^n \to 0,\ n \to \infty\} = \infty$. (We note here that $b_{-n}^{(e)} = 0$, for any $n \geq 0$, by (1).) Because F_e is not rational we have that e is an essential isolated singularity for F_e. By Proposition 1 we have that F_e has infinitely many critical radii r_i strictly decreasing to 0 and for each $y \in \mathbb{C}_p$ the equation $F_e(x) = y$ has infinitely many solutions $0 < |x - e| < \varepsilon$. One knows that $\limsup_{z\to e} |F_e(z)| = \infty$. Then there exists $\varepsilon_A > 0$ such that if $0 < \varepsilon < \varepsilon_A$ and $|z - e| \leq \varepsilon < \varepsilon_A$ we have $|H(z)| < A + 1$. Let $z_0 \in B(e, \varepsilon)$ such that $|F_e(z_0)| > A + 1$. In fact one has

$$\lim_{i\to\infty} \sup_{z\in S(e,r_i)} |F_e(z)| = \infty = \limsup_{z\to e} |F_e(z)|,$$

where $S(e, r_i)$ is the sphere centered at e and of radius r_i. There exists

$z_1 \in S(e, |z_0 - e|)$, $|z_0 - e| = r_i$ a critical radius, such that $|F_e(z_1)| \neq |F_e(z_0)|$ because $|F_e|$ is not constant on the spheres of radii r_i, which are the critical radii, and F_e has zeros on these spheres. It is clear that $|F(z_0)| = |F_e(z_0)|$. Two cases may appear:

(i) $|F_e(z_1)| < |F_e(z_0)|$ so one has $|F(z_1)| \leq \max\{|F_e(z_1)|, A+1\} < |F_e(z_0)| = |F(z_0)|$.

(ii) $|F(z_1)| = |F_e(z_1)| > |F_e(z_0)| = |F(z_0)|$.

This implies that $|F|$ is not constant on $S(e, r_i)$ so F has zeros on this sphere. The proof is now complete. □

Corollary 1. *Let $\alpha \in \overline{\mathbb{Q}}_p \setminus \mathbb{Q}_p$ be an algebraic element and let $F : \mathbb{P}^1(\mathbb{C}_p) \setminus O(\alpha) \to \mathbb{C}_p$ be a rigid analytic function. If F is not rational then F has infinitely many zeros in any neighborhood of at least one point of $O(\alpha)$.*

Remark. If $F : \mathbb{P}^1(\mathbb{C}_p) \setminus O(\alpha) \to \mathbb{C}_p$ is a rigid analytic function that is equivariant and is not rational, then it takes any values in any neighborhood of $O(\alpha)$, so all the conjugates of α are essential singular points of F.

In what follows we give another proof of Theorem 1 without the analogue of the classical result of Picard, by exploiting the fact that F_e given above has a Weierstrass product and it is uniquely determined by its family of zeros. So, let F be as in Theorem 1. Let us consider the Mittag-Leffler decomposition

$$F(z) = \sum_{a \in S} f_a(z) = \sum_{a \in S} \sum_{n \geq 0} \frac{b_n^{(a)}}{(z-a)^n}, \tag{3}$$

where $f_a(z) = \sum_{n \geq 0} \frac{b_n^{(a)}}{(z-a)^n}$ and $\frac{|b_n^{(a)}|}{\varepsilon^n} \to 0$, $n \to \infty$, for any positive real number ε and any $a \in S$. This condition is equivalent to having $|b_n^{(a)}|^{1/n} \to 0$, when $n \to \infty$. Suppose $F(\infty) \neq 0$ and $F(0) \neq 0$. It is easy to see that there exists $e \in S$ such that f_e is not rational. In such a way f_e has the following decomposition (see [L])

$$f_e(z) = c \prod_{n \geq 1} \left(1 - \frac{a_n}{z-e}\right) \tag{4}$$

where c is a nonzero constant, $a_n \in \mathbb{C}_p$ such that $|a_1| \geq |a_2| \geq \cdots \geq |a_n| \geq \ldots$, $\lim_{n\to\infty} |a_n| = 0$ and each $|a_i|$ appears in the decomposition of f_e

only a finite number of times. Let us consider

$$F = f_e + g,\; g = \sum_{a\in S\setminus\{e\}} f_a. \tag{5}$$

One has $\limsup_{z\to e} |f_e(z)| = \infty$ and denote $A = |g(e)| < \infty$. There exists $\varepsilon_A > 0$ such that if $|z - e| < \varepsilon_A$ then $|g(z)| < A + 1$. We have the following result that implies Theorem 1.

PROPOSITION 2. *Let f_e be as above and let $\varepsilon > 0$. Then there exists $z_0 \in \mathbb{C}_p$ such that:*

1) $|z_0 - e| < \varepsilon$ *and* $|f_e(z_0)| > A + 1$.
2) *there exists* $z_1 \in \mathbb{C}_p$, $|z_0 - z_1| < |z_0 - e|$ *and* $|f_e(z_1)| < |f_e(z_0)|$.

Let us see that Proposition 2 implies Theorem 1. We choose a positive real number ε such that $\varepsilon < \varepsilon_A$ and $\varepsilon < \inf\{|e - a| : a \neq e,\ a \in S\}$. Because $|f_e(z_0)| > A + 1$ and $|g(z_0)| < A + 1$ one has $|F(z_0)| = |f_e(z_0)| > A + 1$. From $|z_0 - z_1| < |z_0 - e| < \varepsilon$ we have $|z_1 - e| < \varepsilon$ so $|g(z_1)| < A + 1 < |f_e(z_0)|$ and by $|f_e(z_1)| < |f_e(z_0)|$ one obtains $|F(z_1)| < |F(z_0)|$. On the other hand there are no elements of $S \setminus \{e\}$ in the closed ball $B[z_0, |z_1 - z_0|]$ and even in the open ball $B(z_0, \varepsilon)$. But the function F is defined on $B(z_0, \varepsilon)$ and $|F|$ is not constant on this open ball so F has zeros in $B(e, \varepsilon)$, see also [AR].

Now, let us give the proof of Proposition 2. As we know, the sequence $(|a_n|)_{n\geq 1}$ is decreasing and its limit is zero. We choose z_0 of the form $z_0 = e + a_s(1 + u)$ with s sufficiently large and $|u| < 1$, $u \neq 0$ that will be chosen later. One has

$$f_e(z) = c \prod_{i=1}^{\infty} \left(1 - \frac{a_i}{z - e}\right) = c \prod_{i=1}^{\infty} \frac{z - e - a_i}{z - e}, \tag{6}$$

so

$$f_e(z_0) = c \prod_{i=1}^{\infty} \frac{a_s(1 + u) - a_i}{a_s(1 + u)}. \tag{7}$$

It is clear that $|a_s(1 + u)| = |a_s|$. Now, if $|a_i| < |a_s|$ then $|a_s(1 + u) - a_i| = |a_s(1 + u)| = |a_s|$, and using (7) we have

$$|f_e(z_0)| = |c| \prod_{\substack{i\geq 1\\ |a_i|\geq|a_s|}} \left|\frac{a_s(1 + u) - a_i}{a_s(1 + u)}\right|. \tag{8}$$

Let us give an explanation about the form of z_0 as above, $z_0 = e + a_s(1 + u)$, with $|u| < 1$. If there exists z_1 that satisfies $|z_0 - z_1| < |z_0 - e|$ and $|f_e(z_1)| < |f_e(z_0)|$ then $|e - z_1| = |e - z_0|$ so on the sphere $S(e, |e - z_0|)$ the function $|f_e(z)|$ is not constant and this implies that $|e - z_0|$ is the same with one of the moduli $|a_s|$, $s \geq 1$. In such a way one has $z_0 = e + a_s t$, $|t| = 1$ and $z_1 = z_0 + a_s t'$, $|t'| < 1$, i.e. $z_1 = e + a_s(t + t')$. Denote $|a_s| = r_s$. Let us compute $|f_e(z_0)|$ and $|f_e(z_1)|$. We factorize

$$f_e(z) = c \prod_{i=1}^{\infty} \frac{z - e - a_i}{z - e} = h_s(z) H_s(z), \tag{9}$$

where

$$h_s(z) = \prod_{|e+a_j| = r_s} \frac{z - e - a_j}{z - e} \tag{10}$$

and

$$H_s(z) = \prod_{|e+a_j| \neq r_s} \frac{z - e - a_j}{z - e}. \tag{11}$$

In (10) the product has a finite number of factors. Let us see that $|H_s|$ is constant on the sphere $S(e, r_s)$, because if $|H_s|$ is not constant then H_s has zeros on this sphere that is not the case. The inequality $|f_e(z_1)| < |f_e(z_0)|$ is equivalent to $|h_s(z_1)| < |h_s(z_0)|$ that is equivalent to $\left|\frac{h_s(z_1)}{h_s(z_0)}\right| < 1$. Because $|z_0 - e| = |z_1 - e|$ it results that

$$\left|\frac{h_s(z_1)}{h_s(z_0)}\right| = \prod_{|e+a_j| = r_s} \left|\frac{z_1 - e - a_j}{z_0 - e - a_j}\right| = \prod_{|e+a_j| = r_s} \left|1 - \frac{z_0 - z_1}{z_0 - e - a_j}\right| < 1.$$

There exists j and a_j with $|e + a_j| = r_s$ such that $\left|1 - \frac{z_0 - z_1}{z_0 - e - a_j}\right| < 1$, which means $|z_0 - z_1| = |z_0 - e - a_j| < |e - z_0|$, so $\left|\frac{z_0 - e - a_j}{z_0 - \alpha}\right| = \left|1 - \frac{a_j}{z_0 - e}\right| < 1$. Again, one obtains $\left|\frac{a_j}{z_0 - e}\right| = 1$ so $z_0 - e = a_j(1 + w)$, with $|w| < 1$, for some a_j with $|a_j| = r_s = |a_s|$ i.e. $z_0 = e + a_j + a_j w$, with $|w| < 1$. Denote $\rho = \min\{|z_0 - e - a_j| : |e + a_j| = r_s\}$. We take z_1 such that $|z_0 - z_1| \leq \rho$ and for at least a j such that $|z_0 - e - a_j| = \rho$ one has $\left|\frac{z_0 - z_1}{z_0 - e - a_j} - 1\right| = \left|\frac{-z_1 + e + a_j}{z_0 - e - a_j}\right| < 1$. This implies $|f_e(z_1)| < |f_e(z_0)|$ and also $|z_0 - z_1| \leq \rho = |z_0 - e - a_j| < |z_0 - e|$. So $z_1 = z_0 + a_s t'$ will be determined such that for some a_j with $|z_0 - e - a_j| = \rho$, $\left|\frac{z_0 - z_1}{z_0 - e - a_j} - 1\right| < 1$. For this $\frac{z_0 - z_1}{z_0 - e - a_j}$ must have the residual image equal to $\overline{1}$. This implies

$$\overline{\left(\frac{z_1 - z_0}{z_0 - e - a_j}\right)} = \overline{\left(\frac{a_s}{z_0 - e - a_j}\right)} \times \overline{t'} = \overline{1}, \tag{12}$$

so the condition that determine z_1 is

$$\overline{t'} = \overline{\left(\frac{z_0 - e - a_j}{a_s}\right)}. \tag{13}$$

In such a way z_1 is determined by z_0. It remains to find z_0, $z_0 = e + a_s + a_s u$, with $|u| < 1$, such that $|f_e(z_0)|$ is big enough ($> A + 1$) and $|e - z_0| = |a_s|$ with $|a_s|$ small enough.

If in the decompositon of $|f_e|$ as in (8) one has $|a_i| > |a_s|$, then $|a_s(1+u) - a_i| = |a_i|$ so $\left|\frac{a_s(1+u) - a_i}{a_s(1+u)}\right| = \frac{|a_i|}{|a_s|} > 1$. This implies

$$|f_e(z_0)| = M_s \times M'_s \tag{14}$$

where $M_s = |c| \prod_{|a_i| > |a_s|} \left|\frac{a_i}{a_s}\right|$ and $M'_s = \prod_{|a_i| = |a_s|} \left|\frac{a_s(1+u) - a_i}{a_s(1+u)}\right|$

1) We see that M_s is large enough as soon as a_s is small enough that means s is large enough.

2) One has $\left|\frac{a_s(1+u) - a_i}{a_s(1+u)}\right| = \left|1 - \frac{a_i}{a_s(1+u)}\right| = \left|(1+u) - \frac{a_i}{a_s}\right|$ where $w = 1 + u$ is also a unit in $U_1(\mathbb{C}_p)$.

Now, let $\delta \in (0, 1)$ and denote

$$U_\delta(\mathbb{C}_p) = \{z : z = 1 + y,\ |y| < \delta\}. \tag{15}$$

Then $U_\delta(\mathbb{C}_p) \subset U_1(\mathbb{C}_p)$ and the factor group $U_1(\mathbb{C}_p)/U_\delta(\mathbb{C}_p)$ is an infinite group (because of the residual field of $\mathbb{C}_p$ that is $\overline{\mathbb{F}}_p$ i.e. infinite). Denote $m_s = \#\{a_i : |a_i| = |a_s|\}$. Because $U_1(\mathbb{C}_p)/U_{\delta/m_s}(\mathbb{C}_p)$ is infinite we can find $w = 1 + u \in U_1(\mathbb{C}_p)$ such that $\left|(1+u) - \frac{a_i}{a_s}\right| \geq \frac{\delta}{m_s}$, for each $1 \leq i \leq m_s$. From this one obtains

$$\prod_{|a_i| = |a_s|} \left|\frac{a_s(1+u) - a_i}{a_s(1+u)}\right| \geq \delta,$$

so by (14) we have

$$|f_e(z_0)| \geq |c| \prod_{|a_i|>|a_s|} \left|\frac{a_i}{a_s}\right| \delta. \tag{16}$$

Finally it is clear that for s large enough, by (16) one has $|f_e(z_0)| > A+1$, and the proof is done.

3. The transcendental case

Let $x \in \mathbb{C}_p$. For any $\varepsilon > 0$ denote $B(x,\varepsilon) = \{y \in \mathbb{C}_p : |x-y| < \varepsilon\}$ and $B[x,\varepsilon] = \{y \in \mathbb{C}_p : |x-y| \leq \varepsilon\}$. For any $\delta > \varepsilon > 0$ also denote $E(x,\varepsilon,\delta) = \{y \in B(x,\delta) : |y-t| \geq \varepsilon, \text{ for all } t \in O(x) \cap B(x,\delta)\}$ and $E[x,\varepsilon,\delta] = \{y \in B[x,\delta] : |y-t| \geq \varepsilon, \text{ for all } t \in O(x) \cap B(x,\delta)\}$. The complement of $E(x,\varepsilon,\delta)$ in $B(x,\delta)$ is denoted by $V(x,\varepsilon,\delta)$. Both sets $E(x,\varepsilon,\delta)$ and $V(x,\varepsilon,\delta)$ are open and closed, and one has: $\cap_\varepsilon V(x,\varepsilon,\delta) = O(x) \cap B(x,\delta)$. Denote $E(x,\delta) = \cup_\varepsilon E(x,\varepsilon,\delta) = B(x,\delta) \setminus O(x)$ and $E[x,\delta] = \cup_\varepsilon E[x,\varepsilon,\delta] = B[x,\delta] \setminus O(x)$. For any $x \in \mathbb{C}_p$ and $\varepsilon > 0$ denote $H(x,\varepsilon) = \{\sigma \in G : |\sigma(x) - x| < \varepsilon\}$ and $H[x,\varepsilon] = \{\sigma \in G : |\sigma(x) - x| \leq \varepsilon\}$. Let S_ε (respectively $\overline{S}_\varepsilon$) be a complete system of representatives for the right cosets of G with respect to $H(x,\varepsilon)$ (respectively $H[x,\varepsilon]$). Then $E(x,\varepsilon,\delta) = \cup_{\sigma \in S_\varepsilon} B(\sigma(x),\varepsilon) \cap B(x,\delta)$.

(When $\delta = \infty$ we drop δ in the above notations and obtain the standard notations and results as in [APVZ1].)

Let δ be a fixed positive real number and let $F : E[x,\delta] \to \mathbb{C}_p$ be a rigid analytic function such that the set of real numbers $\{\varepsilon \|F\|_{E[x,\varepsilon,\delta]}\}_{\delta > \varepsilon > 0}$ is bounded. (Here $\|F\|_{E[x,\varepsilon,\delta]} = \sup_{z \in E[x,\varepsilon,\delta]} |F(z)|$.)

By the Mittag-Leffler Theorem (see [FP]) one can write:

$$F(z) = F_1^{(\varepsilon)}(z) + F_2^{(\varepsilon)}(z)$$

where

$$F_1^{(\varepsilon)}(z) = \sum_{\substack{\sigma \in S_\varepsilon \\ \sigma(x) \in B(x,\delta)}} \sum_{n \geq 1} \frac{a_{n,\sigma}^{(\varepsilon)}}{(z - \sigma(x))^n}, \; a_{n,\sigma}^{(\varepsilon)} \in \mathbb{C}_p, \; \frac{|a_{n,\sigma}^{(\varepsilon)}|}{\varepsilon^n} \to 0,$$

and

$$F_2^{(\varepsilon)}(z) = \sum_{\substack{\sigma \in S_\varepsilon \\ \sigma(x) \in B(x,\delta)}} \sum_{n \geq 0} b_{n,\sigma}^{(\varepsilon)} (z - \sigma(x))^n, \; b_{n,\sigma}^{(\varepsilon)} \in \mathbb{C}_p, \; |b_{n,\sigma}^{(\varepsilon)}| \delta^n \to 0,$$

for any $z \in E(x,\varepsilon,\delta)$. It is easy to see that $F_2^{(\varepsilon)}(z) = F_2^{(\varepsilon')}(z)$ for any $0 < \varepsilon' < \varepsilon < \delta$, so $F_2^{(\varepsilon)}(z)$ has analytic continuation on the entire $B[x,\delta]$, which we denote it by $F_2(z)$. Denote $F_1(z) = F(z) - F_2(z)$.

One also has

$$F_1^{(\varepsilon)}(z) = \sum_{\substack{\sigma\in S_\varepsilon \\ \sigma(x)\in B(x,\delta)}} F_{1,\sigma}^{(\varepsilon)}(z),$$

where $F_{1,\sigma}^{(\varepsilon)}(z) = \sum_{n\geq 1} \dfrac{a_{n,\sigma}^{(\varepsilon)}}{(z-\sigma(x))^n}$, $\dfrac{|a_{n,\sigma}^{(\varepsilon)}|}{\varepsilon^n} \to 0$. By Cauchy's inequalities, one obtains

$$(17) \qquad |a_{n,\sigma}^{(\varepsilon)}| \leq \varepsilon^n \|F_1\|_{E[x,\varepsilon,\delta]} = \varepsilon^n \|F - F_2\|_{E[x,\varepsilon,\delta]} \leq \varepsilon^n \max\{\|F\|_{E[x,\varepsilon,\delta]}, \|F_2\|_{E[x,\varepsilon,\delta]}\},\ n\geq 1.$$

The Mittag-Leffler's decomposition is unique, so, for any $0<\varepsilon'<\varepsilon$ one has

$$F_{1,\sigma}^{(\varepsilon)}(z) = \sum_{\substack{\tau\in S_{\varepsilon'},\hat\tau=\hat\sigma \\ \sigma(x),\tau(x)\in B(x,\delta)}} F_{1,\tau}^{(\varepsilon')}(z), \quad z\in E[x,\varepsilon,\delta]$$

(here $\hat\tau = \hat\sigma$ means $\tau\in\sigma H(x,\varepsilon)$) and so for any $\sigma(x)\in B(x,\delta)$

$$(18) \qquad \sum_{n\geq 1} \frac{a_{n,\sigma}^{(\varepsilon)}}{(z-\sigma(x))^n} = \sum_{\substack{\tau\in S_{\varepsilon'},\hat\tau=\hat\sigma \\ \tau(x)\in B(x,\delta)}} \sum_{m\geq 1} \frac{a_{m,\tau}^{(\varepsilon')}}{(z-\tau(x))^m}.$$

Since $|z-\sigma(x)|\geq\varepsilon$, and $|\sigma(x)-\tau(x)|<\varepsilon$, it follows that $|z-\tau(x)| = |z-\sigma(x)|$ and so:

$$(19) \qquad \begin{aligned} \frac{a_{m,\tau}^{(\varepsilon')}}{(z-\tau(x))^m} &= \frac{a_{m,\sigma}^{(\varepsilon')}}{(z-\sigma(x))^m\left(1-\dfrac{\tau(x)-\sigma(x)}{z-\sigma(x)}\right)^m} \\ &= \frac{a_{m,\tau}^{(\varepsilon')}}{(z-\sigma(x))^m}\sum_{k\geq 0}\binom{m+k-1}{k}\left[\frac{\tau(x)-\sigma(x)}{z-\sigma(x)}\right]^k. \end{aligned}$$

If we denote $m+k=n$, then by identifying the coefficients of the terms of degree n in (18) and (19) one obtains:

$$(20) \qquad a_{n,\sigma}^{(\varepsilon)} = \sum_{\substack{\tau\in S_{\varepsilon'},\hat\tau=\hat\sigma \\ \tau(x)\in B(x,\delta)}} \sum_{k=0}^{n-1}\binom{n-1}{k} a_{n-k,\tau}^{(\varepsilon')}(\tau(x)-\sigma(x))^k,$$

where $n\geq 1$.

Now for any $n\geq 1$ one defines a sequence $\{\mu_{n,\varepsilon}\}_{n,\varepsilon}$ of measures on $O(x)\cap B[x,\delta]$ by the equality

$$(21) \qquad \mu_{n,\varepsilon} = \sum_{\sigma\in S_\varepsilon} a_{n,\sigma}^{(\varepsilon)}\cdot\delta_{\sigma(x)},$$

where δ_y denotes the Dirac measure concentrated at $y\in\mathbb{C}_p$.

By (20) one obtains, for $n = 1$:

$$a_{1,\sigma}^{(\varepsilon)} = \sum_{\substack{\tau \in S_{\varepsilon'}, \hat{\tau} = \hat{\sigma} \\ \tau(x) \in B(x,\delta)}} a_{1,\tau}^{(\varepsilon')}, \quad 0 < \varepsilon' \leq \varepsilon.$$

This equality further implies that for any ball B of radius δ', $\varepsilon \leq \delta' < \delta$, one has $\mu_{1,\varepsilon}(B) = \mu_{1,\varepsilon'}(B)$ whenever $\varepsilon' \leq \varepsilon$. Then by (17) and the Banach-Steinhaus Theorem (see [R]) there results that the mapping

$$B \rightsquigarrow \mu_1(B) = \lim_{\varepsilon} \mu_{1,\varepsilon}(B)$$

(where B runs over all the open balls of $O(x) \cap B[x, \delta]$) defines a p-adic measure on $O(x) \cap B[x, \delta]$. One says that $\mu = \mu_1$ is the *measure associated* to the rigid analytic function F. Here we remark that if $\|F\|_{E[x,\varepsilon,\delta]}$ is uniformly bounded for any $0 < \varepsilon < \delta$ one has $\mu = 0$.

Furthermore, for $n = 2$, by (20) one obtains

$$a_{2,\sigma}^{(\varepsilon)} = \sum_{\substack{\tau \in S_{\varepsilon'}, \hat{\tau} = \hat{\sigma} \\ \tau(x) \in B(x,\delta)}} [a_{2,\tau}^{(\varepsilon')} + a_{1,\tau}^{(\varepsilon')}(\tau(x) - \sigma(x))].$$

If B is an open ball of $O(x) \cap B[x, \delta]$ of radius δ', by the previous equality and (21) there results that

$$\mu_{2,\varepsilon}(B) - \mu_{2,\varepsilon'}(B) = \sum_{\substack{\sigma \in S_\varepsilon \\ \sigma(x) \in B}} \left[\sum_{\substack{\tau \in S_{\varepsilon'}, \hat{\tau} = \hat{\sigma} \\ \tau(x) \in B(x,\delta)}} a_{1,\tau}^{(\varepsilon')}(\tau(x) - \sigma(x)) \right]$$

and so by (17) one has:

$$|\mu_{2,\varepsilon}(B) - \mu_{2,\varepsilon'}(B)| \leq \varepsilon\varepsilon' \|F\|_{E[x,\varepsilon',\delta]} \leq \varepsilon M$$

where $M = \sup_{\delta > \varepsilon > 0} \varepsilon \|F\|_{E[x,\varepsilon,\delta]} < \infty$, by hypothesis. Then by (17) and the Banach-Steinhaus Theorem, there exists a measure μ_2 on $O(x) \cap B[x, \delta]$ defined by

$$\mu_2(B) = \lim_{\varepsilon} \mu_{2,\varepsilon}(B),$$

for all the open balls B of $O(x) \cap B[x, \delta]$. In the same manner for all $n \geq 3$ one can define a measure μ_n on $O(x)$ by:

$$\mu_n(B) = \lim_{\varepsilon} \mu_{n,\varepsilon}(B).$$

Next, by an easy computation it follows that for all $n \geq 2$, one has $\|\mu_{n,\varepsilon}\| \leq M\varepsilon^{n-1}$, and so $\mu_n = 0$ for all $n \geq 2$. In what follows we shall prove that

$$F_1(z) = \int_{O(x)\cap B[x,\delta]} \frac{1}{z-t}\, d\mu(t), \quad z \in E[x,\delta]. \tag{22}$$

Indeed, with the above notations and using (17) one has:

$$\begin{aligned} \left| F_1(z) - \int_{O(x)\cap B[x,\delta]} \frac{1}{z-t}\, d\mu_{1,\varepsilon}(t) \right| &= \left| F_1(z) - \sum_{\substack{\sigma\in S_\varepsilon \\ \sigma(x)\in B(x,\delta)}} \frac{a_{1,\sigma}^{(\varepsilon)}}{z-\sigma(x)} \right| \\ &\le \left| \sum_{\substack{\sigma\in S_\varepsilon \\ \sigma(x)\in B(x,\delta)}} \sum_{n\ge 2} \frac{a_{n,\sigma}^{(\varepsilon)}}{(z-\sigma(x))^n} \right|, \end{aligned} \tag{23}$$

for any $z \in E(x,\delta)$. If $d = dist(z, O(x)\cap B(x,\delta)) > 0$, for any $0<\varepsilon<d$ from (23) and (17) one has

$$\left| F_1(z) - \int_{O(x)\cap B[x,\delta]} \frac{1}{z-t}\, d\mu_{1,\varepsilon}(t) \right| \le \sup_{n\ge 2} \frac{1}{d} \times \left(\frac{\varepsilon}{d}\right)^{n-1} \times \varepsilon \|F\|_{E[x,\varepsilon,\delta]}, \tag{24}$$

which tends to zero when ε tends to zero.

By the definition of $\mu = \mu_1$ (see (21)) one has

$$\int_{O(x)\cap B[x,\delta]} \frac{1}{z-t}\, d\mu(t) = \lim_{\varepsilon\to 0} \int_{O(x)\cap B[x,\delta]} \frac{1}{z-t}\, d\mu_{1,\varepsilon}(t).$$

By (24) and the analytic continuation principle one obtains (22). Finally one has the following result, which is similar to a theorem of Barsky (see [B] and [APVZ1]):

THEOREM 2. *Let x be a transcendental element of $\mathbb{C}_p$ and let δ be a fixed positive real number. Let $F : E[x,\delta] \to \mathbb{C}_p$ be a rigid analytic function such that the set of real numbers $\{\varepsilon\|F\|_{E[x,\varepsilon,\delta]}\}_{\delta>\varepsilon>0}$ is bounded. There exists a unique p-adic measure μ_F on $O(x)\cap B[x,\delta]$ such that F has the following integral representation*

$$F(z) = G(z) + \int_{O(x)\cap B[x,\delta]} \frac{1}{z-t}\, d\mu_F(t), \quad z \in E[x,\delta], \tag{25}$$

where $G(z)$ is an entire function on $B[x,\delta]$, and the representation of F is unique.

REMARK. It is easy to see that if F has a singular point $t \in O(x) \cap B[x,\delta]$ then $\|F\|_{E[x,\varepsilon,\delta]} \to \infty, \varepsilon \to 0$ so one has $\limsup_{z\to t} |F(z)| = \infty$ and moreover F is transcendental over $\mathbb{C}_p(z)$. Indeed, if the reverse is true one has $\mu_F = 0$ and F has no singular points on $O(x) \cap B[x,\delta]$, which is false.

One has the following result.

THEOREM 3. *Let x be a transcendental element of $\mathbb{C}_p$ and let δ be a fixed positive real number. Let $F : E[x,\delta] \to \mathbb{C}_p$ be a rigid analytic function such that all the points of $O(x) \cap B[x,\delta]$ are singular points of F. Then* $\liminf_{z\to x} |F(z)| = 0$.

PROOF. Let us suppose the reverse. One has $\liminf_{z\to x} |F(z)| > 0$ so there exists $M > 0$ and $\delta' > 0$ such that $|F(z)| > M$ for any $z \in B[x,\delta'] \setminus O(x)$. For the sake of simplicity, let us suppose $\delta = \delta'$. (We can work with the restriction of F to $E[x,\delta']$.) By definition, F is a limit (in uniform convergence on each $E[x,\varepsilon,\delta]$) of rational functions of the form

$$F(z) = \lim_{n\to\infty} \frac{P_n^{(\varepsilon)}(z)}{Q_n^{(\varepsilon)}(z)}, \tag{26}$$

without poles in $E[x,\varepsilon,\delta]$. Fix $0 < \varepsilon < \delta$ and let n_M be a natural number such that

$$\left\| F - \frac{P_n^{(\varepsilon)}}{Q_n^{(\varepsilon)}} \right\|_{E[x,\varepsilon,\delta]} < M$$

for any $n \geq n_M$. Because $|F(z)| > M$, for any $z \in E[x,\varepsilon,\delta]$ one obtains $\left| \frac{P_n^{(\varepsilon)}(z)}{Q_n^{(\varepsilon)}(z)} \right| > M$, for any $z \in E[x,\varepsilon,\delta]$, so F is a uniform limit of rational functions without zeros and poles in $E[x,\varepsilon,\delta]$. Denote $G_n^{(\varepsilon)}(z) = \frac{P_n^{(\varepsilon)}(z)}{Q_n^{(\varepsilon)}(z)}$. One has

$$\left\| \frac{1}{F} - \frac{1}{G_n^{(\varepsilon)}} \right\|_{E[x,\varepsilon,\delta]} \leq \|F - G_n^{(\varepsilon)}\|_{E[x,\varepsilon,\delta]} \times \left\| \frac{1}{F G_n^{(\varepsilon)}} \right\|_{E[x,\varepsilon,\delta]} < \frac{1}{M^2} \|F - G_n^{(\varepsilon)}\|_{E[x,\varepsilon,\delta]}$$

for any $n \geq n_M$ so $\frac{1}{F}$ is rigid analytic on $E[x,\delta]$. Two cases may appear:

(I) The function $\frac{1}{F}$ has analytic continuation on $O(x) \cap B[x,\delta]$ and then $\frac{1}{F} = 0$ on $O(x) \cap B[x,\delta]$. Since x is transcendental over $\mathbb{Q}_p$, $O(x)$ has limit points inside $B[x,\delta]$ (see [APZ1]), and this forces $\frac{1}{F}$ to be identically zero, which is a contradiction.

(II) $\frac{1}{F}$ has singular points on $O(x) \cap B[x, \delta]$. Let us suppose that x is a singular point. As in the above considerations one obtains a sequence $(y_n)_{n\geq 1}$ such that $y_n \to x$ and $\left|\frac{1}{F}(y_n)\right| \to \infty$ i.e. $|F(y_n)| \to 0$ that is also false. □

REMARK. The hypothesis that x is transcendental over $\mathbb{Q}_p$ is really needed in the statement of Theorem 3. Situation is completely different if x is algebraic. Indeed, if $x = \alpha$ is algebraic over $\mathbb{Q}_p$, one can take $F = 1/P_\alpha$, where P_α is the minimal polynomial of α over $\mathbb{Q}_p$, and then the conclusion of the theorem is obviously false for such F.

COROLLARY 2. *If F is as in the above Theorem and $a \in \mathbb{C}_p$ then there exists a sequence $z_n \in \mathbb{C}_p \setminus O(x)$ such that* $\lim_{n\to\infty} F(z_n) = a$.

PROOF. We apply Theorem 3 for $F_a = F - a$. □

REMARK. One could say that the points of $O(x) \cap B[x, \delta]$ are essential singular points for F.

COROLLARY 3. *Let $x \in \mathbb{C}_p$ be a transcendental element and δ be a fixed positive real number. Denote $D = \cup_{\sigma\in G} B(\sigma(x), \delta) \setminus O(x)$. Let $F \in \mathcal{A}^G(D, \mathbb{C}_p)$ be a rigid analytic function that is equivariant and such that x is a singular point of F, i.e. F does not have analytic continuation to $\cup_{\sigma\in G} B(\sigma(x), \delta)$. Then* $\limsup_{z\to x} |F(z)| = \infty$ *and* $\liminf_{z\to x} |F(z)| = 0$. *Moreover, for any $a \in \mathbb{Q}_p$ there exists a sequence $(z_n)_{n\geq 1} \in \mathbb{C}_p \setminus O(x)$ such that* $\lim_{n\to\infty} F(z_n) = a$, *so all the points of $O(x)$ are essential singular points for F.*

4. A few examples of rigid analytic functions with and without zeros

1) In this section we give an example of a rigid analytic function such that it has no zeros on $\mathbb{P}^1(\mathbb{C}_p) \setminus O(x)$. For, let $x \in \mathbb{C}_p \setminus \overline{\mathbb{Q}}_p$, and let $(\varepsilon_n)_{n\geq 1}$ be a strictly decreasing sequence with limit 0. Denote $H_n = \{\sigma \in G : |\sigma(x) - x| < \varepsilon_n\}$ and $S_n = (H_n/H_{n+1})_{\text{left}}$ be a complete system of representatives on the left for H_n/H_{n+1}. Let us define the following sequence of rational functions

$$f_n(z) = \prod_{i=1}^{n} \prod_{\sigma\in S_i} \left(1 - \frac{x - \sigma(x)}{z - \sigma(x)}\right) = \prod_{i=1}^{n} \prod_{\sigma\in S_i} \frac{z - x}{z - \sigma(x)},\ n \geq 1. \tag{27}$$

For any $\delta > 0$ the sequence $(f_n)_{n\geq 1}$ is uniform convergent on $\mathbb{C}_p \setminus \cup_{\sigma\in G} B(\sigma(x), \delta)$. It is easy to see that

$$f_{n+1}(z) = f_n(z) \prod_{\sigma\in S_{n+1}} \left(1 - \frac{x-\sigma(x)}{z-\sigma(x)}\right) = f_n(z)h_n(z),$$

where $h_n(z) = \prod_{\sigma\in S_{n+1}} \frac{z-x}{z-\sigma(x)}$. One has $f_{n+1}(z) - f_n(z) = f_n(z)(h_n(z)-1)$. For any $\delta > 0$ there exists n_δ such that $\varepsilon_n < \delta$ for any $n > n_\delta$. In such a way one has $|f_n(z)| = |f_{n_\delta}(z)|$ as soon as $\operatorname{dist}(z, O(x)) \geq \delta$. Moreover

$$\begin{aligned} h_n(z) - 1 &= \prod_{\sigma\in S_{n+1}} \left(1 - \frac{x-\sigma(x)}{z-\sigma(x)}\right) - 1 \\ &= -\sum_{\sigma\in S_{n+1}} \frac{x-\sigma(x)}{z-\sigma(x)} + \sum_{\sigma,\tau\in S_{n+1}} \frac{x-\sigma(x)}{z-\sigma(x)}\cdot\frac{x-\tau(x)}{z-\tau(x)} - \cdots \\ &\quad + (-1)^{\#S_{n+1}} \prod_{\sigma\in S_{n+1}} \frac{x-\sigma(x)}{z-\sigma(x)}. \end{aligned} \tag{28}$$

By (28) one obtains

$$|h_n(z) - 1| \leq \sup_{\sigma\in S_{n+1}} \left|\frac{x-\sigma(x)}{z-\sigma(x)}\right| \leq \frac{\varepsilon_{n+1}}{\delta}, \tag{29}$$

for any $n > n_\delta$ and any z with $\operatorname{dist}(z, O(x)) \geq \delta$. Clearly, (29) implies $\lim_{n\to\infty} |h_n(z) - 1| = 0$ uniformly in z. We have

$$|f_{n+1}(z) - f_n(z)| = |f_{n_\delta}(z)|\cdot|h_n(z)-1| \leq |f_{n_\delta}(z)|\cdot\frac{\varepsilon_{n+1}}{\delta}, \tag{30}$$

that tends to 0 as soon as $n \to \infty$ uniformly in z, $\operatorname{dist}(z, O(x)) \geq \delta$. By (27)–(30) one defines

$$f(z) = \lim_{n\to\infty} f_n(z), \tag{31}$$

that is a rigid analytic function on $\mathbb{P}^1(\mathbb{C}_p) \setminus O(x)$. Moreover x is a singular point of f, so f does not have analytic continuation to the entire $O(x)$.

It remains to prove that f has no zeros in $\mathbb{C}_p \setminus O(x)$. For, let $z \in \mathbb{C}_p \setminus O(x)$ such that $f(z) = \lim_{n\to\infty} f_n(z) = 0$. Let us suppose $\delta = |z - x|$. If $\varepsilon_n < \delta$ one has $\left|\frac{x-\sigma(x)}{z-\sigma(x)}\right| < 1$ so $\left|1 - \frac{x-\sigma(x)}{z-\sigma(x)}\right| = 1$ and $|h_{n-1}(z)| = 1$. It follows that for any $n > n_\delta$

$$|f_n(z)| = |f_{n_\delta}(z)|,$$

where $n_\delta = \max\{k : \varepsilon_k \geq \delta\}$. It is clear that $|f_n(z)|$ cannot be small enough as soon as $|z-x| = \delta$, with δ fixed, which means that f has no zeros.

2) Now, we give an example of a rigid analytic function that has infinitely many zeros around $O(x)$. For, let us consider the same notations as above and let $(\alpha_n)_{n\geq 1}$ be a sequence from $\overline{\mathbb{Q}}_p$ such that $|x-\alpha_n| \sim \varepsilon_n$ for any $n \geq 1$, via the theorem of Ax-Sen, see [Ax]. Then we define

$$F_n(z) = \prod_{i=1}^{n} \prod_{\sigma\in S_i} \left(1 - \frac{\alpha_i - \sigma(x)}{z-\sigma(x)}\right) = \prod_{i=1}^{n} \prod_{\sigma\in S_i} \frac{z-\alpha_i}{z-\sigma(x)},\ n \geq 1. \tag{32}$$

Using the same arguments as above it is easy to see that $(F_n(z))_{n\geq 1}$ is uniformly convergent on $\mathbb{C}_p \setminus \cup_{\sigma\in G} B(\sigma(x), \delta)$, for any $\delta > 0$, and denote its limit by $F(z)$, which is rigid analytic on $\mathbb{P}^1(\mathbb{C}_p) \setminus O(x)$ and has $(\alpha_n)_{n\geq 1}$ as the set of zeros.

3) Another example of rigid analytic function that has zeros in every neighborhood of any of its singular points is the trace series of a transcendental element of $\mathbb{C}_p$, for more details see page 38 of [APZ2].

REMARK. The above examples satisfy the following properties: $\limsup_{z\to x} |f(z)| = \infty$ and $\liminf_{z\to x} |f(z)| = 0$.

5. Some remarks related to zeros on a corona

Let $f : \mathbb{P}^1(\mathbb{C}_p) \setminus O(x) \to \mathbb{C}_p$ be an arbitrary rigid analytic function that is equivariant with respect to G and $f(\infty) = 0$. For any positive real number ε consider the Mittag-Leffler decomposition of f on $E'(x,\varepsilon) := \mathbb{P}^1(\mathbb{C}_p) \setminus \cup_{\sigma\in\mathcal{S}_\varepsilon} B(\sigma(x), \varepsilon)$, where $\mathcal{S}_\varepsilon$ is a complete system of representatives of G with respect to the subgroups $H(x,\varepsilon) = \{\sigma \in G : |\sigma(x) - x| < \varepsilon\}$:

$$f(z) = \sum_{\sigma\in\mathcal{S}_\varepsilon} \sum_{n\geq 1} \frac{a_{n,\sigma}^{(\varepsilon)}}{(z-\sigma(x))^n}, \tag{33}$$

where $a_{n,\sigma}^{(\varepsilon)} = \sigma(a_{n,e}^{(\varepsilon)})$ and e is the neutral element of G, see [APVZ1]. In what follows we consider ε_n the fundamental sequence of all distances on $O(x)$, more precisely the image of the following function: $d : O(x) \to \mathbb{R}_+$, $d(\sigma(x)) = |\sigma(x) - x|$, which is continuous and its image is a strictly decreasing sequence $(\varepsilon_n)_{n\geq 1}$ with limit 0. Fix $\varepsilon > 0$. There exists a unique m such that $\varepsilon_m \leq \varepsilon < \varepsilon_{m-1}$, $m \geq 1$. (One can consider $\varepsilon_0 = \infty$.)

Denote $y = \dfrac{1}{z-x}$. Then $|z-x| \geq \varepsilon$ if and only if $|y| \leq \dfrac{1}{\varepsilon}$. We have the following identity

$$\left(\sum_{i\geq 0} a^i\right)^n = \sum_{k\geq 0}\binom{n+k-1}{k} a^k,\ |a|<1. \tag{34}$$

For $a = \dfrac{z-x}{\sigma(x)-x}$, $\sigma \neq e$, the condition $|a|<1$ means that $|z-x|< \min_{\sigma\in\mathcal{S}_\varepsilon,\sigma\neq e}|\sigma(x)-x|$, which means that for a fixed positive real number ε with $\varepsilon_m \leq \varepsilon < \varepsilon_{m-1}$ one has $\varepsilon_m \leq \varepsilon \leq |z-x| < \varepsilon_{m-1}$, so $\dfrac{1}{\varepsilon_{m-1}} < |y| \leq \dfrac{1}{\varepsilon}$. Using (34) in the above conditions one obtains

$$\begin{aligned}\frac{1}{(z-\sigma(x))^n} &= \frac{1}{(x-\sigma(x))^n\left(1-\frac{z-x}{\sigma(x)-x}\right)^n}\\ &= \frac{1}{(x-\sigma(x))^n}\sum_{k\geq 0}\binom{n+k-1}{k}\left[\frac{z-x}{\sigma(x)-x}\right]^k \\ &= \sum_{k\geq 0}\frac{(-1)^k\binom{n+k-1}{k}}{(x-\sigma(x))^{n+k}}\times(z-x)^k.\end{aligned} \tag{35}$$

Let us define $\widetilde{f}(y) := f(z)$ with $y = \dfrac{1}{z-x}$ as above. From (33) and (35) one has

$$\widetilde{f}(y) = \sum_{n\geq 1} a_{n,e}^{(\varepsilon)} y^n + \sum_{k\geq 1} b_k^{(\varepsilon)} y^{-k} + b_0^{(\varepsilon)}, \tag{36}$$

where

$$b_k^{(\varepsilon)} = \sum_{\substack{\sigma\in\mathcal{S}_\varepsilon\\ \sigma\neq e}}\sum_{n\geq 1}\frac{(-1)^k a_{n,\sigma}^{(\varepsilon)}\binom{n+k-1}{k}}{(x-\sigma(x))^{n+k}},\ k\geq 0. \tag{37}$$

Because f is equivariant by (37) we have the following upper bound

$$|b_k^{(\varepsilon)}| \leq \sup_{n\geq 1}\frac{|a_{n,e}^{(\varepsilon)}|}{\varepsilon_{m-1}^{n+k}} = \frac{1}{\varepsilon_{m-1}^k}\cdot\sup_{n\geq 1}\left(\frac{|a_{n,e}^{(\varepsilon)}|}{\varepsilon_{m-1}^n}\right) \leq \frac{1}{\varepsilon_{m-1}^k}\cdot A(f,\varepsilon), \tag{38}$$

where $A(f,\varepsilon) = \sup_{n\geq 1}\left(\dfrac{|a_{n,e}^{(\varepsilon)}|}{\varepsilon^n}\right)$. The inequalities (38) give us

$$R_{\tilde{f}}^{-} := \lim \sum_{k\to\infty}|b_k^{(\varepsilon)}|^{1/k} \leq \frac{1}{\varepsilon_{m-1}}. \tag{39}$$

Now, from Cauchy's inequalities

$$|a_{n,e}^{(\varepsilon)}| \leq \varepsilon^n \|f\|_{E'(x,\varepsilon)} \tag{40}$$

one obtains $|a_{n,e}^{(\varepsilon)}|^{1/n} \leq \varepsilon \|f\|_{E'(x,\varepsilon)}^{1/n}$ so

$$R_{\tilde{f}}^{+} := \frac{1}{\limsup_{n\to\infty} |a_{n,e}^{(\varepsilon)}|^{1/n}} \geq \frac{1}{\varepsilon} > \frac{1}{\varepsilon_{m-1}} \geq R_{\tilde{f}}^{-}, \tag{41}$$

which means that $\widetilde{f}(y)$ is a Laurent series on the corona $R_{\tilde{f}}^{-} < |y| < R_{\tilde{f}}^{+}$. In such a way the zeros of $\widetilde{f}(y)$ on the above corona are the same as the zeros of $f(z)$ in the corona $\varepsilon_m \leq \varepsilon \leq |z-x| < \varepsilon_{m-1}$.

REMARK. In the previous examples 1)–3) from Paragraph 4, $\tilde{f}$ associated to f has no zeros in the above corona. Moreover, in example 2) the zeros of $\tilde{f}$ are precisely on $|z| = R_{\tilde{f}}^{+}$ or $|z| = R_{\tilde{f}}^{-}$. In the case $x \in \overline{\mathbb{Q}}_p$ it is easy to see that $R_{\tilde{f}}^{+} = \infty$ so the finite zeros of $\tilde{f}$ are precisely on $|z| = R_{\tilde{f}}^{-}$.

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