A Note on the First Rigid Cohomology Group for Geometrically Unibranch Varieties

Nobuo Tsuzuki

To Professor Francesco Baldassarri on the occasion of his 60th birthday, with much gratitude.

Abstract - We prove a vanishing of the first rigid cohomology group for geometrically unibranch varieties with supports in a proper closed subset and apply it to the full faithfulness problem of the restriction functors of overconvergent isocrystals. As an application, we prove that the first rigid cohomology group is pure of weight 1 for proper and geometrically unibranch varieties over a finite field. We also establish a comparison result of rigid cohomology groups between a geometrically unibranch variety and its normalization.

1. Introduction

By the recent development one can understand the rigid cohomology for smooth varieties well. However, it is still difficult to understand it for singular varieties. In this note we study the rigid cohomology for mildly singular varieties. We prove a vanishing of rigid cohomology groups of degree ≤ 1 for geometrically unibranch varieties with supports in a proper closed subset and apply it to the full faithfulness problem of the restriction functors of overconvergent isocrystals (see the full faithfulness problem in [31, Conjecture 1.2.1]). We calculate weights of Frobenius eigenvalues of the first rigid cohomology groups for geometrically unibranch varieties over finite fields. We also establish a com-

(*) Indirizzo dell'A.: Mathematical Institute, Tohoku University, Aramaki Aza-Aoba 6-3, Aobaku Sendai 980-8578, Japan.

E-mail : tsuzuki@math.tohoku.ac.jp

parison theorem of rigid cohomology groups between a geometrically
unibranch variety and its normalization, more generally for universal
homeomorphisms. These results are p-adic analogues of the comparison
theorem of inverse images for universal homeomorphisms in etale co-
homology [1, VIII, 1].

1.1 – *Results*

First we recall the definition of the notion "geometrically uni-
branch". Let us fix notation. For a scheme T over a spectrum $\mathrm{Spec}\, k$ of a
field k and an extension l of k, we put $T_l = T \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, l$. For a scheme
T, T_{red} (resp. T_{nor}) denotes the reduced closed subscheme associated to
T (resp. the normalization of disjoint sum of the reduced closed sub-
schemes of irreducible components of T in the total function field). We
use the same notation R_{red} and R_{nor} for the associated rings of a ring R
as above.

A scheme T is said to be geometrically unibranch if, for any point t of T,
(a) $\mathcal{O}_{T_{\mathrm{red}},t}$ is integral, (b) the normalization $(\mathcal{O}_{T_{\mathrm{red}},t})_{\mathrm{nor}}$ of $\mathcal{O}_{T_{\mathrm{red}},t}$ is local, and
(c) the residue field of $(\mathcal{O}_{T_{\mathrm{red}},t})_{\mathrm{nor}}$ is a purely inseparable extension of the
residue field of $\mathcal{O}_{T_{\mathrm{red}},t}$ [19, 0, 23.2.1]. If T is a normal scheme over $\mathrm{Spec}\, k$ for
a field k, then T_l is geometrically unibranch for any field extension l of k
[19, IV, Proposition 6.15.6]. Moreover, if T is geometrically unibranch, then
the canonical morphism $T_{\mathrm{nor}} \to T$ is a universal homeomorphism, that is,
the morphism of associated topological spaces of any base change is a
homeomorphism [19, IV, Definition 2.4.2, Corollaire 18.12.11].

Let K be a complete discrete valuation field of mixed characteristics
$(0, p)$, $\mathcal{V}$ the integer ring of K, and k the residue field of $\mathcal{V}$. Let $j_X : X \to \overline{X}$
be an open immersion of separated schemes locally of finite type over
$\mathrm{Spec}\, k$ (note that $\overline{X}$ might not be proper), Z a closed k-subscheme of X,
$U = X \setminus Z$ with the open immersion $j_U : U \to \overline{X}$, and M a convergent
isocrystal M on X/K overconvergent along $\overline{X} \setminus X$ (for simplicity, we say an
overconvergent isocrystal on $(X, \overline{X})/K$ later). We denote the rigid coho-
mology for an overconvergent isocrystal M on $(X, \overline{X})/K$ with supports in Z
by $H^r_{\mathrm{rig}\, Z}((X, \overline{X})/K, M)$ (resp. $H^r_{\mathrm{rig}\, Z}(X/K, M)$ if $\overline{X}$ is proper), whose defi-
nition is recalled briefly in section 2.1. Then there is a canonical long exact
sequence

$$
\begin{aligned}
\cdots \;\to\; & H^r_{\mathrm{rig}\, Z}((X, \overline{X})/K, M) \;\to\; H^r_{\mathrm{rig}}((X, \overline{X})/K, M) \to H^r_{\mathrm{rig}}((U, \overline{X})/K, j_U^\dagger M) \\
\to\; & H^{r+1}_{\mathrm{rig}\, Z}((X, \overline{X})/K, M) \;\to\; \qquad\qquad \cdots,
\end{aligned}
$$

which is called an excision sequence [7, Proposition 2.5]. One of the main results of this note is the following vanishing of rigid cohomology groups. The vanishing in the smooth case is known at least if $\overline{X}$ is proper over $\operatorname{Spec} k$ [7, Théorème 3.8] [30, Theorem 4.1.1] (see the detail for a general $\overline{X}$ in section 2.2).

THEOREM 1.1.   (1) *If U has a nonempty intersection with each connected component of X, then $H^0_{\operatorname{rig} Z}((X, \overline{X})/K, M) = 0$.*

(2) *Let $X = \cup_\lambda X_\lambda$ be the irreducible decomposition of X, and put $X_{\lambda,\mu} = X_\lambda \cap X_\mu, U_\lambda = U \cap X_\lambda$ and $U_{\lambda,\mu} = U \cap X_{\lambda,\mu}$. Suppose that*

(i) *U_λ is dense in X_λ for any λ,*

(ii) *$U_{\lambda,\mu}$ is dense in $X_{\lambda,\mu}$ for any λ and μ if $X_{\lambda,\mu} \neq \emptyset$,*

(iii) *X_λ is geometrically unibranch for any λ.*

Then the vanishing $H^1_{\operatorname{rig} Z}((X, \overline{X})/K, M) = 0$ holds.

Note that each of the three conditions (i)-(iii) in the Theorem 1.1 (2) above is preserved by the base extension X_l for any extension l over k (see [19, IV, Proposition 6.15.7] for the condition (iii)). We prove the theorem above for a smooth X (to be self-contained) in Theorem 2.9 and for a general X in sections 3 and 4.

By the excision sequence above and Theorem 2.13 and Corollary 2.14 we have

COROLLARY 1.2.   *With the same hypotheses of Theorem 1.1 (2), the restriction map*

$$H^0_{\operatorname{rig}}((X, \overline{X})/K, M) \to H^0_{\operatorname{rig}}((U, \overline{X})/K, j^\dagger_U M)$$

is bijective, and the restriction map

$$H^1_{\operatorname{rig}}((X, \overline{X})/K, M) \to H^1_{\operatorname{rig}}((U, \overline{X})/K, j^\dagger_U M)$$

is injective. In particular, the restriction functor from the category of overconvergent isocrystals on $(X, \overline{X})/K$ to the category of overconvergent isocrystals on $(U, \overline{X})/K$ is fully faithful.

COROLLARY 1.3.   *With the same hypotheses of Theorem 1.1 (2), suppose that K admits a lift σ of Frobenius. Then the restriction functor from the category of overconvergent F-isocrystals on $(X, \overline{X})/K$ with respect to σ to the category of overconvergent F-isocrystals on $(U, \overline{X})/K$ with respect to σ is fully faithful.*

When X is smooth over $\operatorname{Spec} k$, the full faithfulness results in Corollaries 1.2 and 1.3 were proved by a local study of p-adic differential equations by K. Kedlaya in [21, Theorem 5.2.2] (see also [31, Theorem 4.1.1]).

In etale cohomology, the similar property of Corollary 1.2 for a normal scheme X follows from the surjectivity of the natural homomorphism $\pi_1^{\mathrm{alg}}(U) \to \pi_1^{\mathrm{alg}}(X)$ of algebraic fundamental groups. The following theorem is a p-adic analogue of the comparison of inverse images for universal homeomorphisms in etale cohomology [1, VIII, Corollaire 1.2, Examples 1.3 (c)].

THEOREM 1.4.   *With the notation as above, suppose X is geometrically unibranch. Let $g : X_{\mathrm{nor}} \to X$ be the canonical structure morphism, $\overline{X_{\mathrm{nor}}}$ a compactification of X_{nor} over $\overline{X}$ (take the normalization of $\overline{X}$, for example), W be the inverse image of Z in X_{nor}, and g^*M the inverse image of M as an overconvergent isocrystal on $(X_{\mathrm{nor}}, \overline{X_{\mathrm{nor}}})/K$. Then the natural homomorphism*

$$g^* : H^r_{\mathrm{rig}\,Z}((X,\overline{X})/K, M) \to H^r_{\mathrm{rig}\,W}((X_{\mathrm{nor}}, \overline{X_{\mathrm{nor}}})/K, g^*M)$$

is an isomorphism for any r.

As an application, we have a following weight result for geometrically unibranch varieties. The author ignores the corresponding result in the l-adic theory in literatures.

THEOREM 1.5.   *Suppose k is a finite field of p^s elements and that K admits a lift σ of Frobenius such that $\sigma^s = \mathrm{id}_K$. If X is proper and geometrically unibranch over $\operatorname{Spec} k$, then $H^1_{\mathrm{rig}}(X/K)$ is pure of weight 1. Here $H^r_{\mathrm{rig}}(X/K)$ denotes the rigid cohomology of X/K for the unit convergent F-isocrystal $\mathcal{O}_{]X[}$ on X/K.*

1.2 – Ideas and plan

In section 2 we recall some results on rigid cohomology, cohomological descent for proper hypercoverings [13] [32] [33] (see Propositions 2.5, 2.6 and Theorem 2.7), and alterations. Especially, the long exact sequence

$(*)$
$$0 \to H^0_{\mathrm{rig}\,Z}((X,\overline{X})/K, M) \to H^0_{\mathrm{rig}\,W_0}((Y_0, \overline{Y}_0)/K, M_0) \to \ker(d_1^{1,0})$$
$$\to H^1_{\mathrm{rig}\,Z}((X,\overline{X})/K, M) \to H^1_{\mathrm{rig}\,W_0}((Y_0, \overline{Y}_0)/K, M_0)$$

where $d_1^{1,0} : H_{\mathrm{rig}\,W_1}^0((Y_1, \overline{Y_1})/K, M_1) \to H_{\mathrm{rig}\,W_2}^0((Y_2, \overline{Y_2})/K, M_2)$, induced from the spectral sequence with respect to a proper hypercovering $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$ in Corollary 2.8 is one of the important tools we are going to use (see the notation in Theorem 2.7).

In section 3 we study the connectedness of tubes for connected schemes. The connectedness in Lemma 3.1 plays an essential role to prove Theorem 1.1. We also show the invariance of rigid cohomology groups of degree 0 for proper surjective morphisms whose geometric fibers are connected in Theorem 3.6. This is one of the keys of this note.

In section 4 we prove Theorem 1.1 (2). In order to prove it we take a regular alteration $\overline{Y} \to \overline{X}$ (Theorem 2.15) and consider Stein factorization of $\overline{Y} \to \overline{X}$. Starting from a smooth $Y \subset \overline{Y}$, we apply the long exact sequence ($*$) above with respect to the Čech hypercovering associated to each proper surjective morphism of the Stein factorization. For the first morphism we use the invariance of rigid cohomology groups of degree 0 for proper surjective morphisms with geometric connected fibers, and for the second morphism we use a certain density of a fiber product which follows from the geometrical unibranchness of varieties in Proposition 4.3.

In section 5 we prove the coincidence of the categories of overconvergent isocrystals and its rigid cohomology groups by inverse images for a separated universal homeomorphism $Y \to X$ of finite type (Theorem 5.2). Theorem 1.4 follows from these more general comparison results. A key is to show that the constant bi-simplicial scheme gives a resolution both of X and of Y as proper hypercoverings (Proposition 5.1). Then we can apply A. Shiho's proper descent of overconvergent isocrystals [29, Proposition 7.3] and proper cohomological descent of rigid cohomology [32, Theorem 2.3.1]. In the case of convergent isocrystals the proper descent and the above equivalence of categories was studied by A. Ogus [26, Theorem 4.6, Corollary 4.10]. These comparisons are p-adic analogues of those in etale cohomology [1, VIII, Théorème 1.1, Corollaire 1.2, Remarques 1.4].

In section 6 we study Frobenius eigenvalues of $H_{\mathrm{rig}}^1(X/K)$ for geometrically unibranch X and prove Theorem 1.5 in Theorem 6.3.

In section 7 we give an example such that the natural homomorphism $H_{\mathrm{rig}}^0((X, \overline{X})/K, M) \to H_{\mathrm{rig}}^0((U, \overline{X})/K, j_U^\dagger M)$ is not surjective in the case where X is not geometrically unibranch. Hence, for the full faithfulness of the restriction functor $j_U^\dagger$, some conditions on X are needed in general. We also give an example such that the restriction functor $j_U^\dagger$ is not essentially surjective in the case where X is normal.

2. Preparations

Let $j_X : X \to \overline{X}$ be an open immersion of separated schemes locally of finite type over Spec k, Z a closed subscheme of X, $U = X \setminus Z$ with a canonical open immersion $j_U : U \to \overline{X}$, and M an overconvergent isocrystal on $(X, \overline{X})/K$. Let L be an extension of K as complete discrete valuation fields and l the residue field of L. For a commutative diagram

$$
\begin{array}{ccc}
Y & \overset{j_Y}{\to} & \overline{Y} \\
g \downarrow & & \downarrow \\
X & \underset{j_X}{\to} & \overline{X}
\end{array}
$$

over Spec k such that j_Y is an open immersion of separated schemes of locally of finite type over Spec l, we denote by $g^* M$ the inverse image of M as an overconvergent isocrystal on $(Y, \overline{Y})/L$.

2.1 – *Fundamental tools in rigid cohomology.*

We recall several fundamental tools, which we will use later, in rigid cohomology.

We recall the definition of the rigid cohomology $H^q_{\mathrm{rig}\, Z}((X, \overline{X})/K, M)$ on $(X, \overline{X})/K$ with supports in Z for an overconvergent isocrystal M on $(X, \overline{X})/K$, which is defined by P. Berthelot [5, Section 2]. In the case where $\overline{X}$ is embedded as a closed subscheme into a separated formal scheme $\mathcal{P}$ locally of finite type over Spf $\mathcal{V}$ which is smooth around X, it is defined by the hypercohomology

$$
H^r_{\mathrm{rig}\, Z}((X, \overline{X})/K, M) := \mathbb{R}^r \Gamma(]\overline{X}[_{\mathcal{P}}, \underline{\Gamma}^\dagger_Z(M \otimes_{j^\dagger_X \mathcal{O}_{]\overline{X}[_{\mathcal{P}}}} j^\dagger_X \Omega^\bullet_{]\overline{X}[_{\mathcal{P}}/K})).
$$

Here $j^\dagger_X$ is an exact functor of overconvergent sections along $\overline{X} \setminus X$ and $\underline{\Gamma}^\dagger_Z$ is an exact functor of overconvergent sections with supports in Z defined by the kernel of the epimorphism $j^\dagger_X \to j^\dagger_U$ on the category of sheaves of Abelian groups on the tube $]\overline{X}[_{\mathcal{P}}$ of $\overline{X}$ in Raynaud's generic fiber $\mathcal{P}_K$ associated to $\mathcal{P}$. By definition we have $]\overline{X}[_{\mathcal{P}} =]\overline{X}_{\mathrm{red}}[_{\mathcal{P}}$ and the identity

$$
H^r_{\mathrm{rig}\, Z_{\mathrm{red}}}((X_{\mathrm{red}}, \overline{X}_{\mathrm{red}})/K, M) = H^r_{\mathrm{rig}\, Z}((X, \overline{X})/K, M).
$$

REMARK 2.1.   (1) When one begins with the pair $(X, \overline{X})$, one can find, locally on $\overline{X}$, a closed embedding of $\overline{X}$ into a formal scheme $\mathcal{P}$ over Spf $\mathcal{V}$ as

above, and can define overconvergent isocrystals and rigid cohomology on $(X, \overline{X})/K$ by the simplicial construction using an open covering of $\overline{X}$ in general [7, 1.5 Remarque 4] [13, Section 10.5, 10.6]. All the definitions do not depend on any choices.

(2) When one considers the "full" rigid cohomology, i.e., over-convergent along "full boundaries" (namely "$\overline{X}$" is proper over Spec k), one can find, locally on X, an open immersion $j_X : X \to \overline{X}$ for a proper scheme $\overline{X}$ over Spec k with an embedding into a formal scheme $\mathcal{P}$ over Spf $\mathcal{V}$ which is smooth around X. Then one can define an over-convergent isocrystal and rigid cohomology on X/K by the simplicial construction using an open covering of X as in the previous remarks. Note that, for a separated scheme X of finite type over Spec k, there always exists a completion $\overline{X}$ by Nagata's compactification [22] [23]. In the full boundaries case we use the notation $H^r_{\mathrm{rig}\,Z}(X/K, M)$ instead of $H^r_{\mathrm{rig}\,Z}((X, \overline{X})/K, M)$.

In this note we always begin with a pair $(X, \overline{X})$, however, the results also hold in the case of overconvergence along full boundaries.

PROPOSITION 2.2 (Finite scalar extensions) [7, Proposition 1.8]. *For a finite extension L of K, let l (resp. $f : X_l \to X$) denote the residue field of L (resp. the extension of base fields). Then, for any r, we have a canonical isomorphism*

$$H^r_{\mathrm{rig}\,Z_l}((X_l, \overline{X}_l)/L, f^*M) \cong H^r_{\mathrm{rig}\,Z}((X, \overline{X})/K, M) \otimes_K L.$$

REMARK 2.3. One can find base change theorems of rigid cohomology for an arbitrary scalar extension L of K in [33, Theorem 8.1.1] and [4, Corollary 5.5.2].

PROPOSITION 2.4 [7, Proposition 2.5]

(1) *If Z is a disjoint sum of closed subschemes Z_1 and Z_2 of X, then $H^r_{\mathrm{rig}\,Z}((X, \overline{X})/K, M) \cong H^r_{\mathrm{rig}\,Z_1}((X, \overline{X})/K, M) \oplus H^r_{\mathrm{rig}\,Z_2}((X, \overline{X})/K, M)$ for any r.*

(2) (Excision sequence) *If Z_1 is a closed subscheme of Z over Spec k, then there is a canonical long exact sequence*

$$\cdots \to H^r_{\mathrm{rig}\,Z_1}((X, \overline{X})/K, M) \to H^r_{\mathrm{rig}\,Z}((X, \overline{X})/K, M) \to H^r_{\mathrm{rig}\,Z\backslash Z_1}((X\backslash Z_1, \overline{X})/K, j^\dagger_{X\backslash Z_1}M)$$
$$\to H^{r+1}_{\mathrm{rig}\,Z_1}((X, \overline{X})/K, M) \to \quad \cdots.$$

We recall some spectral sequences which we will use later (see [33, Sections 2, 5, 7]).

PROPOSITION 2.5 (Čech spectral sequence for open coverings)

(1) *Let* $\{\overline{Y}_\lambda\}_{\lambda \in \Lambda}$ *be an open covering of* $\overline{X}$ *over* $\operatorname{Spec} k$ *indexed by a totally order set* Λ, *put* $\overline{Y}_{\lambda_0\lambda_1\cdots\lambda_q} = \cap_{i=0}^q \overline{Y}_{\lambda_i}$, $Y_{\lambda_0\lambda_1\cdots\lambda_q} = X \cap \overline{Y}_{\lambda_0\lambda_1\cdots\lambda_q}$ *with the structure morphism* $f_{\lambda_0\lambda_1\cdots\lambda_q} : Y_{\lambda_0\lambda_1\cdots\lambda_q} \to X$, *and* $W_{\lambda_0\lambda_1\cdots\lambda_q} = Z \cap \overline{Y}_{\lambda_0\lambda_1\cdots\lambda_q}$. *Then there is a spectral sequence*

$$E_1^{qr} = \bigoplus_{\lambda_0 < \lambda_1 < \cdots < \lambda_q} H^r_{\operatorname{rig} W_{\lambda_0\lambda_1\cdots\lambda_q}}((Y_{\lambda_0\lambda_1\cdots\lambda_q}, \overline{Y}_{\lambda_0\lambda_1\cdots\lambda_q})/K, f^*_{\lambda_0\lambda_1\cdots\lambda_q}M) \Rightarrow$$
$$H^{q+r}_{\operatorname{rig} Z}((X, \overline{X})/K, M).$$

(2) *Let* $\{Y_\lambda\}_{\lambda \in \Lambda}$ *be an open covering of* X *over* $\operatorname{Spec} k$ *indexed by a totally order set* Λ, *put* $Y_{\lambda_0\lambda_1\cdots\lambda_q} = \cap_{i=0}^q Y_{\lambda_i}$ *with the structure morphism* $f_{\lambda_0\lambda_1\cdots\lambda_q} : Y_{\lambda_0\lambda_1\cdots\lambda_q} \to X$, *and* $W_{\lambda_0\lambda_1\cdots\lambda_q} = Z \cap Y_{\lambda_0\lambda_1\cdots\lambda_q}$. *Then there is a spectral sequence*

$$E_1^{qr} = \bigoplus_{\lambda_0 < \lambda_1 < \cdots < \lambda_q} H^r_{\operatorname{rig} W_{\lambda_0\lambda_1\cdots\lambda_q}}((Y_{\lambda_0\lambda_1\cdots\lambda_q}, \overline{X})/K, f^*_{\lambda_0\lambda_1\cdots\lambda_q}M) \Rightarrow$$
$$H^{q+r}_{\operatorname{rig} Z}((X, \overline{X})/K, M).$$

PROPOSITION 2.6 (Mayer-Vietoris spectral sequence). *Suppose* $\overline{X}$ *is separated of finite type over* $\operatorname{Spec} k$, *and let* $\{\overline{Y}_\lambda\}$ *be a finite closed covering of* $\overline{X}$ *over* $\operatorname{Spec} k$. *Then there is a spectral sequence*

$$E_1^{qr} = \bigoplus_{\lambda_0 < \lambda_1 < \cdots < \lambda_q} H^r_{\operatorname{rig} W_{\lambda_0\lambda_1\cdots\lambda_q}}((Y_{\lambda_0\lambda_1\cdots\lambda_q}, \overline{Y}_{\lambda_0\lambda_1\cdots\lambda_q})/K, f^*_{\lambda_0\lambda_1\cdots\lambda_q}M) \Rightarrow$$
$$H^{q+r}_{\operatorname{rig} Z}((X, \overline{X})/K, M),$$

where $\overline{Y}_{\lambda_0\lambda_1\cdots\lambda_q} = \cap_{i=0}^q \overline{Y}_{\lambda_i}$, $Y_{\lambda_0\lambda_1\cdots\lambda_q} = X \cap \overline{Y}_{\lambda_0\lambda_1\cdots\lambda_q}$ *with the structure morphism* $f_{\lambda_0\lambda_1\cdots\lambda_q} : Y_{\lambda_0\lambda_1\cdots\lambda_q} \to X$, *and* $W_{\lambda_0\lambda_1\cdots\lambda_q} = Z \cap Y_{\lambda_0\lambda_1\cdots\lambda_q}$.

A simplicial scheme $T_\bullet$ over S is a contravariant functor from the standard simplicial category to the category of schemes over S, and denotes it by $T_\bullet \to S$. A simplicial scheme $T_\bullet \to S$ is said to be a proper hypercovering if the canonical morphism $T_0 \to S$ is proper surjective and if the canonical morphism $T_{q+1} \to \operatorname{cosk}_q(T_\bullet^{(q)})_{q+1}$ is proper surjective for any q. Here $T_\bullet^{(q)}$ is the truncation of $T_\bullet$ until the q-th stage (q-truncation) and cosk_q is the right adjoint of the q-truncation functor. We often use a spectral sequence with respect to a Čech hypercovering associated to a proper surjective morphism, which is defined as follows. A Čech hypercovering $T_\bullet \to S$ associated to a proper surjective morphism $T \to S$ consists of its q-th stage $T_q = T \times_S \cdots \times_S T$ (the fiber product of $q+1$ copies of T over S) and standard face and degeneracy morphisms. Indeed,

$\operatorname{cosk}_q(T_\bullet^{(q)})_{q+1} = T_{q+1}$ in this case. Hence, the associated Čech hypercovering $T_\bullet \to S$ is a proper hypercovering.

THEOREM 2.7 (Spectral sequence for proper hypercoverings) [32, Theorem 4.5.1]. *Suppose that $\overline{X}$ is separated of finite type over $\operatorname{Spec} k$. Let $\overline{g}_\bullet : \overline{Y}_\bullet \to \overline{X}$ be a proper hypercovering over $\operatorname{Spec} k$, $g_\bullet : Y_\bullet \to X$ (resp. $W_\bullet \to Z$) the induced proper hypercovering by the inverse image, and M_q the inverse image of M as an overconvergent isocrystal on $(Y_q, \overline{Y}_q)/K$ by g_q. Then there is a spectral sequence of rigid cohomology with supports*

$$E_1^{qr} = H_{\operatorname{rig} W_q}^r((Y_q, \overline{Y}_q)/K, M_q) \Rightarrow H_{\operatorname{rig} Z}^{q+r}((X, \overline{X})/K, M).$$

COROLLARY 2.8. *With the notation as in Theorem 2.7, there is a long exact sequence*

$$0 \to H_{\operatorname{rig} Z}^0((X, \overline{X})/K, M) \to E_1^{0,0} \to \ker(d_1^{1,0} : E_1^{1,0} \to E_1^{2,0}) \to$$
$$H_{\operatorname{rig} Z}^1((X, \overline{X})/K, M) \to E_1^{0,1}.$$

Note that all the properties of rigid cohomology above are functorial in M, $(X, \overline{X})$, Z and hypercoverings $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$. Moreover, when M is an overconvergent F-isocrystal, natural homomorphisms between rigid cohomology groups commute with Frobenius.

2.2 – Cohomological purity

In the case where X is smooth over $\operatorname{Spec} k$, Theorem 1.1 has been already known as cohomological purity by the Gysin map at least when $\overline{X}$ is proper ([7, Théorème 3.8] [30, Theorem 4.1.1]). In general the author can not find a direct proof in literature.

THEOREM 2.9. *Suppose that X is a smooth scheme separated of finite type over $\operatorname{Spec} k$ and denote the least codimension of generic points of Z in X by e. Then*

$$H_{\operatorname{rig} Z}^r((X, \overline{X})/K, M) = 0$$

for $r < 2e$.

PROOF. Suppose $e \geq 1$. Replacing k by a finite extension, we may assume that $\overline{X}$ is geometrically integral by Propositions 2.2 and 2.6.

After replacing k by a finite extension of k again, there is an open smooth subscheme of the reduced closed subscheme Z_{red}. Hence we may assume that (X, Z) is a smooth pair such that Z is pure of codimension e by Propositions 2.2 and 2.4. Since (X, Z) is a smooth pair, Z is, locally on X, an intersection of all irreducible components of a strict normal crossing divisor of X [16, II, Théorème 4.10]. Hence we may assume that there is a strict normal crossing divisor $D = \cup_{i=1}^{e} D_i$ of X, where D_i is an irreducible component, such that $Z = \cap_{i=1}^{e} D_i$ by Proposition 2.5. We denote the Zariski closure of D_i and Z in $\overline{X}$ by $\overline{D}_i$ and $\overline{Z}$, respectively, and put $\iota_i : D_1 \cap D_2 \cap \cdots \cap D_i \to X$ to be the closed immersion.

Applying the Čech spectral sequences with respect to a certain open covering of $\overline{X}$ and a certain open covering of X in Proposition 2.5, we may assume that $\overline{X}$ is affine and X is an open subscheme which is defined by $\overline{g} \neq 0$ in $\overline{X}$ for some $\overline{g} \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$.

LEMMA 2.10. *With the notation as above, there is an affine formal scheme $\mathcal{X} = \mathrm{Spf}\, \mathcal{A}$ of finite type over $\mathrm{Spf}\, \mathcal{V}$ such that $\overline{X}$ is an irreducible component of $(\mathcal{X} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\, k)_{\mathrm{red}}$ and $\mathcal{X}$ is smooth over $\mathrm{Spf}\, \mathcal{V}$ around X.*

PROOF. Let B be a smooth $\mathcal{V}$-algebra of finite type such that $X \cong \mathrm{Spec}\, B \otimes_{\mathcal{V}} k$. Such a ring B exists by [15, Théorème 6]. Let C be a polynomial ring of finite variables over $\mathcal{V}$ with a $\mathcal{V}$-algebra surjective homomorphism $C \to \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$ and a $\mathcal{V}$-algebra homomorphism $C \to B$ such that the diagram

$$
\begin{array}{ccc}
C & \to & \Gamma(\overline{X}, \mathcal{O}_{\overline{X}}) \\
\downarrow & & \downarrow \\
B & \to & \Gamma(X, \mathcal{O}_X)
\end{array}
$$

is commutative. Fix a lift $g \in C$ of the element $\overline{g}$. Since $\overline{g}$ is invertible in $\Gamma(X, \mathcal{O}_X)$, we may assume that the image of g in B is invertible. Replacing C by a $\mathcal{V}$-polynomial ring adding variables which goes to generators of B multiplied by suitable powers of g, we may assume that the $\mathcal{V}$-algebra homomorphism $C[1/g] \to B$ is also surjective. Let $(f_1, \cdots, f_r)$ be the kernel of the surjection $C[1/g] \to B$ and choose a nonnegative integer n such that all $g^n f_i$'s are included in C. Define $\mathcal{A}$ by a p-adic completion of $C/(g^n f_1, \cdots, g^n f_r)$ and put $\mathcal{X} = \mathrm{Spf}\, \mathcal{A}$. Since X is open and $\overline{X}$ is closed in $\mathcal{X} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\, k$, $\overline{X}$ is an irreducible component of $(\mathcal{X} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\, k)_{\mathrm{red}}$. Moreover, $\mathcal{X}$ is smooth over $\mathrm{Spf}\, \mathcal{V}$ around X by construction. $\square$

Let us continue the proof of Theorem 2.9. Since the Zariski closure of X in $(\mathcal{X} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\,k)_{\mathrm{red}}$ is $\overline{X}$, we can replace $\overline{X}$ by $(\mathcal{X} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\,k)_{\mathrm{red}}$. Applying the Čech spectral sequence with respect to a certain open covering of X in Proposition 2.5, we may assume that each irreducible component D_i of D is defined by a single equation $h_i = 0$ in X for an element $h_i \in \mathcal{A}$. Then the desired vanishing follows from the lemma below since $H^r_{\mathrm{rig}\,Z}(Z/K, \iota_e^* M) = H^r_{\mathrm{rig}}(Z/K, \iota_e^* M) = 0$ for $r < 0$. $\square$

LEMMA 2.11. *With the notation above, we have a sequence of isomorphisms*

$$
\begin{aligned}
H^r_{\mathrm{rig}\,Z}((X, \overline{X})/K, M) &\cong H^{r-2}_{\mathrm{rig}\,Z}((D_1, \overline{D}_1)/K, \iota_1^* M) \cong \cdots \\
&\cong H^{r-2e}_{\mathrm{rig}\,Z}((D_1 \cap D_2 \cap \cdots \cap D_e, \overline{D}_1 \cap \overline{D}_2 \cap \cdots \cap \overline{D}_e)/K, \iota_e^* M).
\end{aligned}
$$

PROOF. Let us put $Y = D_1$, $\overline{Y} = \overline{D}_1$, $\iota = \iota_1 : Y \to X$, $h = h_1$, $\mathcal{Y} = \mathrm{Spf}\,\mathcal{A}/(h)$, and denote Raynaud's generic fiber of $\mathcal{Y}$ by $\mathcal{Y}_K$. Then there is a natural closed immersion $\overline{Y} \to \mathcal{Y}$ such that $\mathcal{Y}$ is smooth around Y, and $]\overline{Y}[_{\mathcal{Y}}$ is a closed rigid analytic subspace in $]\overline{X}[_{\mathcal{X}}$ which is defined by $h = 0$. We denote the closed immersion by $\iota_K :]\overline{Y}[_{\mathcal{Y}} \to]\overline{X}[_{\mathcal{X}}$. Then ι_K is an affinoid morphism. Moreover, for any open subscheme T in X, we have (i) if V is a strict neighborhood of $]T[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$, then $V \cap]\overline{Y}[_{\mathcal{Y}}$ is a strict neighborhood of $]T \cap Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$, and (ii) if W is a strict neighborhood of $]T \cap Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$, then there is a strict neighborhood V of $]T[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ such that $W \supset V \cap]\overline{\mathcal{Y}}[_{\mathcal{Y}}$ (use standard strict neighborhoods in [6, Examples 1.2.4] to prove (i) and (ii)). It implies that there exists a natural transformation $j_T^\dagger \to i_{K*} j_{T \cap Y}^\dagger$.

Let $j_X^\dagger \Omega^\bullet_{]\overline{X}[_{\mathcal{X}}/K}(\log \mathcal{Y}_K)$ be a logarithmic de Rham complex of $]\overline{X}[_{\mathcal{X}}$ over K, which is overconvergent along $\overline{X} \setminus X$ with a logarithmic pole along the divisor $h = 0$. Since $\mathcal{X}$ is smooth around X and (X, Y) is a smooth pair, there is a strict neighborhood V of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ such that $V \cap]\overline{Y}[_{\mathcal{Y}}$ is a smooth divisor of the smooth rigid analytic space V. Hence the logarithmic de Rham complex $j^\dagger \Omega^\bullet_{]\overline{X}[_{\mathcal{X}}/K}(\log \mathcal{Y}_K)$ is a complex of sheaves of locally free $j_X^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules of finite rank. By taking residues with respect to dh/h we have homomorphisms

$$
M \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j_X^\dagger \Omega^\bullet_{]\overline{X}[_{\mathcal{X}}/K}(\log \mathcal{Y}_K) \to \iota_{K*}(\iota^* M \otimes_{j_Y^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}} j_Y^\dagger \Omega^\bullet_{]\overline{Y}[_{\mathcal{Y}}/K})[-1]
$$

$$
j_{X \setminus Z}^\dagger M \otimes_{j_{X \setminus Z}^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j_{X \setminus Z}^\dagger \Omega^\bullet_{]\overline{X}[_{\mathcal{X}}/K}(\log \mathcal{Y}_K) \to \iota_{K*}(j_{Y \setminus Z}^\dagger \iota^* M \otimes_{j_{Y \setminus Z}^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j_{Y \setminus Z}^\dagger \Omega^\bullet_{]\overline{Y}[_{\mathcal{Y}}/K})[-1]
$$

of complexes of $j_X^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules and of $j_{X \setminus Z}^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, respectively, where $[a]$ means the a-th shift of the complexes, such that the natural

diagram

$$
\begin{array}{ccc}
M \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_X^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K}(\log \mathcal{Y}_K) & \rightarrow & \iota_{K*}(\iota^* M \otimes_{j_Y^\dagger \mathcal{O}_{]\overline{Y}[_\mathcal{Y}}} j_Y^\dagger \Omega^\bullet_{]\overline{Y}[_\mathcal{Y}/K})[-1] \\
\downarrow & & \downarrow \\
j_{X\backslash Z}^\dagger M \otimes_{j_{X\backslash Z}^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_{X\backslash Z}^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K}(\log \mathcal{Y}_K) & \rightarrow & \iota_{K*}(j_{Y\backslash Z}^\dagger \iota^* M \otimes_{j_{Y\backslash Z}^\dagger \mathcal{O}_{]\overline{Y}[_\mathcal{Y}}} j_{Y\backslash Z}^\dagger \Omega^\bullet_{]\overline{Y}[_\mathcal{Y}/K})[-1]
\end{array}
$$

is commutative. These diagrams fit into the commutative diagram with short exact rows

$$
\begin{array}{ccccc}
0 & \rightarrow & M \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_X^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K} & \rightarrow & M \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_X^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K}(\log \mathcal{Y}_K) \\
& & \downarrow & & \downarrow \\
0 & \rightarrow & j_{X\backslash Z}^\dagger M \otimes_{j_{X\backslash Z}^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_{X\backslash Z}^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K} & \rightarrow & j_{X\backslash Z}^\dagger M \otimes_{j_{X\backslash Z}^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_{X\backslash Z}^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K}(\log \mathcal{Y}_K) \\
& & & & \rightarrow \quad \iota_{K*}(\iota^* M \otimes_{j_Y^\dagger \mathcal{O}_{]\overline{Y}[_\mathcal{Y}}} j_Y^\dagger \Omega^\bullet_{]\overline{Y}[_\mathcal{Y}/K})[-1] \quad \rightarrow \quad 0 \\
& & & & \downarrow \\
& & & & \rightarrow \quad \iota_{K*}(j_{Y\backslash Z}^\dagger \iota^* M \otimes_{j_{Y\backslash Z}^\dagger \mathcal{O}_{]\overline{Y}[_\mathcal{Y}}} j_{Y\backslash Z}^\dagger \Omega^\bullet_{]\overline{Y}[_\mathcal{Y}/K})[-1] \quad \rightarrow \quad 0,
\end{array}
$$

where both first arrows in horizontal rows are natural inclusions. The natural inclusions

$$
M \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_X^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K}(\log \mathcal{Y}_K) \quad \rightarrow \quad j_{X\backslash Z}^\dagger M \otimes_{j_{X\backslash Z}^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_{X\backslash Z}^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K}(\log \mathcal{Y}_K)
$$

$$
\rightarrow \quad j_{X\backslash Y}^\dagger M \otimes_{j_{X\backslash Y}^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_{X\backslash Y}^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K}
$$

induce isomorphisms

$$
\mathbb{R}\Gamma(]\overline{X}[_\mathcal{X}, M \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_X^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K}(\log \mathcal{Y}_K))
$$

$$
\cong \mathbb{R}\Gamma(]\overline{X}[_\mathcal{X}, j_{X\backslash Z}^\dagger M \otimes_{j_{X\backslash Z}^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_{X\backslash Z}^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K}(\log \mathcal{Y}_K))
$$

$$
\cong \mathbb{R}\Gamma(]\overline{X}[_\mathcal{X}, j_{X\backslash Y}^\dagger M \otimes_{j_{X\backslash Y}^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_{X\backslash Y}^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K})
$$

by [10, Theorem 1.1.1, Remarks 1.1.2.4] since all exponents of differential operator along Y are 0. If T is an open subscheme of Y, then $\mathbb{R}\iota_{K*}E = \iota_{K*}E$ for any coherent sheaf of $j_T^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}$-modules [13, Theorem 5.2.1]. Hence we have an isomorphism

$$
\mathbb{R}\Gamma(]\overline{X}[_\mathcal{X}, \underline{L}_Z^\dagger(M \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j_X^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/K})) \cong \mathbb{R}\Gamma(]\overline{Y}[_\mathcal{Y}, \underline{L}_Z^\dagger(M \otimes_{j_Y^\dagger \mathcal{O}_{]\overline{Y}[_\mathcal{Y}}} j_Y^\dagger \Omega^\bullet_{]\overline{Y}[_\mathcal{Y}/K}))[-2].
$$

Applying this isomorphism for smooth pairs $(X, D_1), (D_1, D_1 \cap D_2), \cdots,$ $(D_1 \cap D_2 \cap \cdots \cap D_{e-1}, D_1 \cap D_2 \cap \cdots \cap D_e)$ successively, we have obtained the assertion. $\square$

REMARK 2.12.   (1) The comparison between logarithmic and usual rigid cohomologies were discussed in [2, Theorem A1], [3, Theorem 3.1], [27, Theorem 2.4.9], [30, Theorem 3.5.1] and so on. In each of them the locally

freeness of $\mathrm{sp}_* M$ is assumed, where $\mathrm{sp} : \mathcal{P}_K \to \mathcal{P}$ is the specialization morphism. In [10, Theorem 1.1.1] they do not assume the locally freeness.

(2) The Gysin isomorphism (Lemma 2.11) was studied in more general contexts when $\overline{X}$ is proper in [7, Théorème 3.8] and [30, Theorem 4.1.1].

(3) In this paper we need only the vanishing of $H^0_{\mathrm{rig}Z}(X/K, M)$ and $H^1_{\mathrm{rig}Z}(X/K, M)$ for a smooth X. These follows from Kedlaya's fully faithful theorem of the restriction functor from the category of overconvergent isocrystals on $(X, \overline{X})/K$ to that of overconvergent isocrystals on $(X \setminus Z, \overline{X})/K$ [21, Theorem 5.2.1] and the comparison theorem between hom (resp. ext) groups and rigid cohomologies in the next section 2.3.

2.3 – Hom and Ext

We recall the relation between the Hom group (resp. the extension group (Yoneda's Ext group)) of the category of overconvergent isocrystals and the rigid cohomology group H^0_{rig} (resp. H^1_{rig}).

THEOREM 2.13. *Let M and N be overconvergent isocrystals on $(X, \overline{X})/K$ and $M^\vee$ a dual of M.*

(1) [6, Proposition 2.2.7] *There is a natural isomorphism*

$$\mathrm{Hom}(M, N) \cong H^0_{\mathrm{rig}}((X, \overline{X})/K, M^\vee \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} N).$$

Here Hom *means a K-vector space of homomorphisms in the category of overconvergent isocrystals on $(X, \overline{X})/K$.*

(2) [12, Proposition 1.2.2] *There is a natural isomorphism*

$$\mathrm{Ext}(M, N) \cong H^1_{\mathrm{rig}}((X, \overline{X})/K, M^\vee \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} N).$$

Here Ext *means a Yoneda's Ext group in the category of overconvergent isocrystals on $(X, \overline{X})/K$.*

By the definition of overconvergent F-isocrystals and [6, Proposition 2.2.7] we have

COROLLARY 2.14. *Suppose that K has a Frobenius σ. Let M and N be overconvergent F-isocrystals on $(X, \overline{X})/K$ with respect to σ, $M^\vee$ be a dual of M, and*

$$C(1 - F; M, N) =$$
$$\mathrm{Cone}(M^\vee \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} N \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} j_X^\dagger \Omega_{]\overline{X}[/K} \xrightarrow{1-F} M^\vee \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} N \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} j_X^\dagger \Omega^\bullet_{]\overline{X}[/K})$$

where F is the Frobenius on the de Rham complex induced from that of $M^\vee \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} N$.

(1) *There is a natural isomorphism*

$$\mathrm{Hom}(M, N) \cong H^0(]\overline{X}[, C(1 - F; M, N)[-1]),$$

where Hom *means a K^σ-space of homomorphisms in the category of overconvergent F-isocrystals on $(X, \overline{X})/K$ and $K^\sigma = \{a \in K \,|\, \sigma(a) = a\}$.*

(2) *There is a natural isomorphism*

$$\mathrm{Ext}(M, N) \cong H^1(]\overline{X}[, C(1 - F; M, N)[-1]),$$

where Ext *means Yoneda's Ext group in the category of overconvergent F-isocrystals on $(X, \overline{X})/K$.*

Moreover there is an exact sequence of K^σ-spaces:

$$0 \to \mathrm{Hom}(M, N) \to H^0_{\mathrm{rig}}((X, \overline{X})/K, M^\vee \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} N) \overset{1-F}{\to} H^0_{\mathrm{rig}}((X, \overline{X})/K, M^\vee \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} N)$$

$$\to \mathrm{Ext}(M, N) \to H^1_{\mathrm{rig}}((X, \overline{X})/K, M^\vee \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} N) \overset{1-F}{\to} H^1_{\mathrm{rig}}((X, \overline{X})/K, M^\vee \otimes_{j_X^\dagger \mathcal{O}_{]\overline{X}[}} N).$$

2.4 – Alteration.

We recall a theorem of the existence of regular alteration by A.J. de Jong [20, Theorem 4.1 and its proof 4.5]. The following statement is what we need in this note.

THEOREM 2.15. *Suppose that X is separated of finite type over $\mathrm{Spec}\, k$. Then there is a projective surjective morphism $g : Y \to X$ such that Y is regular and g is generically finite. Moreover, there exist a purely inseparable finite extension l of k and a projective surjective and generically etale morphism $Y' \to X_l$ such that Y' is smooth over $\mathrm{Spec}\, l$.*

3. Connectedness.

3.1 – Connectedness and rigid cohomology groups of degree 0

A rigid analytic space W is connected if there is no admissible covering $\{V_\lambda\}_{\lambda \in \Lambda}$ of W such that $(\cup_{\lambda \in \Lambda_1} V_\lambda) \cap (\cup_{\lambda \in \Lambda_2} V_\lambda) = \emptyset$, $\cup_{\lambda \in \Lambda_1} V_\lambda \neq \emptyset$ and $\cup_{\lambda \in \Lambda_2} V_\lambda \neq \emptyset$ for a disjoint sum $\Lambda = \Lambda_2 \coprod \Lambda_2$ [9, 9.1.1]. An affinoid variety is connected if and only if it is connected with respect to the Zariski topology [9, 9.1.4, Proposition 8].

LEMMA 3.1. *Let T be a separated scheme of finite type over $\operatorname{Spec} k$ with a closed immersion $T \to \mathcal{P}$ of separated formal schemes of finite type over $\operatorname{Spf} \mathcal{V}$ such that $\mathcal{P}$ is smooth around T.*

(1) *Suppose that k is algebraically closed. $]T[_{\mathcal{P}}$ is connected if and only if the de Rham cohomology groups $H^0(]T[_{\mathcal{P}}, \Omega^\bullet_{]T[_{\mathcal{P}}/K})$ of degree 0 is a K-vector space of dimension 1.*

(2) *If T is connected, then the tube $]T[_{\mathcal{P}}$ is a connected rigid analytic space.*

The de Rham cohomology group $H^r(]T[_{\mathcal{P}}, \Omega^\bullet_{]T[_{\mathcal{P}}/K})$ in the theorem above is just the rigid cohomology group $H^r_{\mathrm{rig}}((T,T)/K, \mathcal{O}_{]T[})$ for the unit convergent isocrystal $\mathcal{O}_{]T[}$.

PROOF. (1) Since a constant function is horizontal with respect to connection, the dimension of $H^0(]T[_{\mathcal{P}}, \Omega^\bullet_{]T[_{\mathcal{P}}/K})$ over K is at least the cardinal of connected components of $]T[_{\mathcal{P}}$. Hence, if $]T[_{\mathcal{P}}$ is not connected, then $\dim_K H^0(]T[_{\mathcal{P}}, \Omega^\bullet_{]T[_{\mathcal{P}}/K}) > 1$.

Suppose that $]T[_{\mathcal{P}}$ is connected. Let t be a closed point of T. Since $\mathcal{P}$ is smooth over the spectrum of the algebraically closed field k, the completion $\widehat{\mathcal{O}_{\mathcal{P},t}}$ along t is isomorphic to the ring $\mathcal{V}[[x_1, \cdots, x_n]]$ of formal power series in n variables over $\mathcal{V}$ for some n. Hence the tube $]t[_{\mathcal{P}}$ is an open unit ball of dimension n over K, and $H^0(]t[_{\mathcal{P}}, \Omega^\bullet_{]t[_{\mathcal{P}}/K}) = K$. Let s be an element of the kernel of the restriction homomorphism $H^0(]T[_{\mathcal{P}}, \Omega^\bullet_{]T[_{\mathcal{P}}/K}) \to H^0(]t[_{\mathcal{P}}, \Omega^\bullet_{]t[_{\mathcal{P}}/K})$. The homomorphism $\mathcal{O}_{]T[_{\mathcal{P}}} \to \mathcal{O}_{]T[_{\mathcal{P}}}$ of convergent isocrystal on $(T,T)/K$ induced by the above section s (Theorem 2.13) has a nontrivial kernel. Since the category of overconvergent isocrystals is Abelian [6, Corollaire 2.2.10] and a convergent isocrystal is locally free by [6, Proposition 2.2.3], the kernel should be $\mathcal{O}_{]T[_{\mathcal{P}}}$ by the connectedness of $]T[_{\mathcal{P}}$. Hence s is 0 and we have $\dim_K H^0(]T[_{\mathcal{P}}, \Omega^\bullet_{]T[_{\mathcal{P}}/K}) = 1$.

(2) It is sufficient to show the connectedness of $]T[_{\mathcal{P}} =]T_{\mathrm{red}}[_{\mathcal{P}}$ for each irreducible component of T. For any extension L of K as complete discrete valuation fields such that $\mathcal{W}$ is the ring of integers of L and the reside field of $\mathcal{W}$ is l, the canonical morphism $T_l \to T$ is surjective and the canonical morphism $]T_l[_{\mathcal{P}_{\mathcal{W}}} \to]T[_{\mathcal{P}}$ is continuous and surjective. Here $\mathcal{P}_{\mathcal{W}} = \mathcal{P} \times_{\operatorname{Spf} \mathcal{V}} \operatorname{Spf} \mathcal{W}$ as formal schemes. Hence we may assume that k is algebraically closed and T is integral.

First we assume that T is smooth over $\operatorname{Spec} k$. We may assume that T is affine. Then there is an affine integral scheme $\widehat{T}$ which is smooth of

finite type over $\operatorname{Spec} \mathcal{V}$ by [15, Théorème 6]. Since the affinoid ring of the p-adic completion $\mathcal{T}$ of $\widehat{T}$ is an integral domain, so is the affinoid ring of the affinoid variety $]T[_{\mathcal{T}} = \mathcal{T}_K$ and hence $]T[_{\mathcal{T}}$ is connected by the remark just before Lemma 3.1. Consider a natural commutative diagram

$$
\begin{array}{ccccc}
& & T & & \\
& \swarrow & \downarrow & \searrow & \\
\mathcal{P} & \stackrel{\mathrm{pr}_1}{\leftarrow} & \mathcal{P} \times_{\operatorname{Spf} \mathcal{V}} \mathcal{T} & \stackrel{\mathrm{pr}_2}{\rightarrow} & \mathcal{T}
\end{array}
$$

of formal $\mathcal{V}$-schemes such that T is embedded into $\mathcal{P} \times_{\operatorname{Spf} \mathcal{V}} \mathcal{T}$ diagonally. Since the rigid cohomology is independent of choices of embeddings, the morphism pr_1 and pr_2 induce isomorphisms

$$
H^0(]T[_{\mathcal{P}}, \Omega^\bullet_{]T[_{\mathcal{P}}/K}) \stackrel{\cong}{\to} H^0(]T[_{\mathcal{P} \times_{\operatorname{Spf} \mathcal{V}} \mathcal{T}}, \Omega^\bullet_{]T[_{\mathcal{P} \times_{\operatorname{Spf} \mathcal{V}} \mathcal{T}}/K}) \stackrel{\cong}{\leftarrow} H^0(]T[_{\mathcal{T}}, \Omega^\bullet_{]T[_{\mathcal{T}}/K}).
$$

Applying (1), we have $\dim_K H^0(]T[_{\mathcal{P}}, \Omega^\bullet_{]T[_{\mathcal{P}}/K}) = \dim_K H^0(]T[_{\mathcal{T}}, \Omega^\bullet_{]T[_{\mathcal{T}}/K}) = 1$. Hence $]T[_{\mathcal{P}}$ is connected again by (1).

Now we prove the assertion in general cases. By applying the existence of regular alteration (Theorem 2.15), there is a projective surjective morphism $X \to T$ such that X is integral and smooth over k. Since X is projective over T, there is a smooth projective formal $\mathcal{V}$-scheme $\mathcal{Q}$ over $\mathcal{P}$ with a closed immersion $X \to \mathcal{Q}$ such that the diagram

$$
\begin{array}{ccc}
X & \to & \mathcal{Q} \\
\downarrow & & \downarrow \\
T & \to & \mathcal{P}
\end{array}
$$

is commutative. Since $\mathcal{Q}$ is smooth over $\operatorname{Spf} \mathcal{V}$ around X, $]X[_{\mathcal{Q}}$ is connected. The continuity and surjectivity of the canonical morphism $]X[_{\mathcal{Q}} \to]T[_{\mathcal{P}}$ imply that $]T[_{\mathcal{P}}$ is connected. $\square$

Let $j_X : X \to \overline{X}$ be an open immersion of separated schemes locally of finite type over $\operatorname{Spec} k$. Let us denote by $H^r_{\mathrm{rig}}((X, \overline{X})/K)$ the rigid cohomology on $(X, \overline{X})/K$ for the unit overconvergent isocrystal $j^\dagger_X \mathcal{O}_{]\overline{X}[}$.

PROPOSITION 3.2. *Suppose that k is algebraically closed. Then the dimension of $H^0_{\mathrm{rig}}((X, \overline{X})/K)$ as a K-vector space is the cardinal of connected components of X.*

PROOF. It is sufficient to prove $\dim_K H^0_{\mathrm{rig}}((X, \overline{X})/K) = 1$ when X is connected. Let $\{X_\lambda\}$ be an open covering of X and suppose $\dim_K H^0_{\mathrm{rig}}((X_\lambda, \overline{X})/K) = 1$ for any λ. Then $H^0_{\mathrm{rig}}((X_\lambda, \overline{X})/K)$ is generated by the identity homomorphism $j^\dagger_{X_\lambda} \mathcal{O}_{]\overline{X}[} \to j^\dagger_{X_\lambda} \mathcal{O}_{]\overline{X}[}$ by Theorem 2.13. By gluing and Proposition 2.5, $H^0_{\mathrm{rig}}((X_\lambda, \overline{X})/K)$ is generated by the identity homomorphism $j^\dagger_{X_\lambda} \mathcal{O}_{]\overline{X}[} \to j^\dagger_{X_\lambda} \mathcal{O}_{]\overline{X}[}$ since X is connected. Hence we may assume that $\overline{X}$ is affine and of finite type over $\mathrm{Spec}\, k$. Then one can find a closed immersion $\overline{X} \to \mathcal{P}$ over $\mathrm{Spf}\, \mathcal{V}$ such that $\mathcal{P}$ is a separated formal scheme of finite type over $\mathrm{Spf}\, \mathcal{V}$ which is smooth around X. Since a constant function is an overconvergent section, the assertion follows from Lemma 3.1. $\qquad\square$

COROLLARY 3.3. *Let U be an open subscheme of X such that $U \cap Y \neq \emptyset$ for each connected component Y of X. Then the restriction map*

$$H^0_{\mathrm{rig}}((X, \overline{X})/K) \to H^0_{\mathrm{rig}}((U, \overline{X})/K)$$

is bijective.

EXAMPLE 3.4. Let $K = \mathbb{Q}_p(\pi)$ with $\pi^{p-1} = -p$, $k = \mathbb{F}_p$, $X = \overline{X} = \mathrm{Spec}\, k[x, y]/(xy)$, $X_1 = \{x = 0\} \subset X$, $X_2 = \{y = 0\} \subset X$, $U = X \setminus X_1$ and embed X in $\mathcal{P} = \widehat{\mathbb{A}}^2_{\mathcal{V}}$ with coordinates x, y. We define a convergent isocrystal M on $(X, X)/K$ by a free $\mathcal{O}_{]X[_\mathcal{P}}$-module of rank 1 with a connection

$$\nabla(1) = -\pi dy.$$

Then, $M|_{]X_1[_\mathcal{P}}$ is a convergent isocrystal on $(X_1, X_1)/K$ which is Dwork's isocrystal corresponding to an additive character, and $M|_{]X_2[_\mathcal{P}}$ is isomorphic to the unit object $\mathcal{O}_{]X_2[_\mathcal{P}}$ on $(X_2, X_2)/K$. Hence $H^0_{\mathrm{rig}}((X, X)/K, M) = 0$ and $H^0_{\mathrm{rig}}((U, X)/K, j^\dagger_U M) = K$. It means that the hypothesis of Corollary 3.3 is not enough for the coincidence between $H^0_{\mathrm{rig}}((X, \overline{X})/K, M)$ and $H^0_{\mathrm{rig}}((U, \overline{X})/K, j^\dagger_U M)$ in the case with coefficients. $\qquad\square$

3.2 – Rank of overconvergent isocrystals.

If one fixes a closed embedding $\overline{X} \to \mathcal{P}$ over $\mathrm{Spf}\, \mathcal{V}$ such that $\mathcal{P}$ is smooth around X, then any overconvergent isocrystal on $(X, \overline{X})/K$ is a locally free $j^\dagger_X \mathcal{O}_{]\overline{X}[_\mathcal{P}}$-module of finite rank on the tube $]\overline{X}[_\mathcal{P}$ [6, Proposition 2.2.3]. It is independent of the choice of embeddings. Moreover, the notion of rank depends only on X:

PROPOSITION 3.5.   *Let X be a separated scheme locally of finite type over* $\operatorname{Spec} k$. *Then the rank of a convergent isocrystal on $(X, X)/K$ is stable on each connected component of X.*

PROOF.   Since the problem is local on X, we may assume that X has a closed embedding into a smooth formal scheme over $\operatorname{Spf} \mathcal{V}$. Since any convergent isocrystal is locally free on the tube, the assertion follows from Lemma 3.1 (2).   □

3.3 – *Proof of Theorem* 1.1 (1).

We may assume that X is connected and U is nonempty. Take $s \in H^0_{\mathrm{rig}}((X, \overline{X})/K, M)$ such that $j^\dagger_U(s) = 0$, then we will prove $s = 0$. Since s determines a homomorphism $\eta : j^\dagger_X \mathcal{O}_{]\overline{X}[} \to M$ as overconvergent isocrystals on $(X, \overline{X})/K$ by Theorem 2.13, the kernel $\ker(\eta)$ is an overconvergent isocrystal. Indeed, the category of overconvergent isocrystals is Abelian [6, Corollaire 2.2.10]. By the hypothesis of s we have $\ker(\eta)|_{]U[} \cong \mathcal{O}_{]U[}$. The connectedness of X implies that the restriction map $\ker(\eta)|_{]X[} \to \mathcal{O}_{]X[}$ is an isomorphism since $\ker(\eta)|_{]X[}$ is of rank 1 on X by Proposition 3.5. Hence $s|_{]X[} = 0$. Since the restriction functor from the category of overconvergent isocrystals on $(X, \overline{X})/K$ to the category of overconvergent isocrystals on $(X, X)/K$ is faithful [6, Corollaire 2.2.9], we have $s = 0$. This completes a proof of Theorem 1.1 (1).   □

3.4 – *Invariance of rigid cohomology groups of degree* 0.

THEOREM 3.6.   *Let*

$$
\begin{array}{ccc}
Y & \xrightarrow{j_Y} & \overline{Y} \\
g \downarrow & & \downarrow \overline{g} \\
X & \xrightarrow[j_X]{} & \overline{X}
\end{array}
$$

be a commutative diagram of separated schemes locally of finite type over $\operatorname{Spec} k$ *such that j_X and j_Y are open immersions, g is proper surjective and $\overline{g}$ is proper on the Zariski closure of Y in $\overline{Y}$. Suppose that the geometric fiber $g^{-1}(x)$ is connected for any geometric point x of X which is above a closed*

point of X. Then, for any overconvergent isocrystal M on $(X, \overline{X})/K$, the natural morphism

$$g^* : H^0_{\mathrm{rig}}((X, \overline{X})/K, M) \to H^0_{\mathrm{rig}}((Y, \overline{Y})/K, g^*M)$$

is bijective.

PROOF. We may assume that $\overline{g}$ is proper surjective and $\overline{g}^{-1}(X) = Y$. The problem is local on X and $\overline{X}$ by Čech spectral sequence (Proposition 2.5). Hence we may assume that $\overline{X}$ is of finite type over $\mathrm{Spec}\, k$. Let $\overline{g}_\bullet : \overline{Y}_\bullet \to \overline{X}$ be a Čech hypercovering associated to the proper surjective morphism $\overline{g} : \overline{Y} \to \overline{X}$ and put $Y_\bullet = \overline{g}_\bullet^{-1}(X)$ with the structure morphism $g_\bullet : Y_\bullet \to X$.

Applying the spectral sequence to the hypercovering $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$ (Theorem 2.7), we have

$$H^0_{\mathrm{rig}}((X, \overline{X})/K, M) = \ker(H^0_{\mathrm{rig}}((Y_0, \overline{Y}_0)/K, M_0) \xrightarrow{d_1^{00}} H^0_{\mathrm{rig}}((Y_1, \overline{Y}_1)/K, M_1))$$

where $\mathrm{pr}_i : \overline{Y}_1 \to \overline{Y}_0 = \overline{Y}$ is the i-th projection for $i = 0, 1$ and $d_1^{00} = \mathrm{pr}_1^* - \mathrm{pr}_0^*$. In order to obtain a desired natural identification, we need $d_1^{00} = 0$. Hence, it is sufficient to prove that the morphism

$$\Delta^* : H^0_{\mathrm{rig}}((Y_1, \overline{Y}_1)/K, M_1) \to H^0_{\mathrm{rig}}((Y, \overline{Y})/K, g^*M)$$

induced by the diagonal $\Delta : \overline{Y} \to \overline{Y}_1$ is injective because $\mathrm{pr}_0 \circ \Delta = \mathrm{pr}_1 \circ \Delta$.

Let $s \in H^0_{\mathrm{rig}}((Y_1, \overline{Y}_1)/K, M_1)$ be a horizontal section of M_1 on $]\overline{Y}_1[$ such that $\Delta^*(s) = 0$. Then s determines a homomorphism $\eta : j^\dagger_{Y_1} \mathcal{O}_{]\overline{Y}_1[} \to M_1$ as overconvergent isocristals on $(Y_1, \overline{Y}_1)/K$ by Theorem 2.13. Since the category of overconvergent isocrystals is Abelian [6, Corollaire 2.2.10], the kernel $\ker(\eta)$ of η is an overconvergent subisocrystal of $j^\dagger_{Y_1} \mathcal{O}_{]\overline{Y}_1[}$. In order to prove $s = 0$, it is sufficient to prove the rank of the restriction $\ker(\eta)|_{]Y_1[}$ of $\ker(\eta)$ on $(Y_1, Y_1)/K$ is always 1 on Y_1 since the restriction functor from the category of convergent isocrystals on $(Y_1, \overline{Y}_1)/K$ to the category of overconvergent isocrystals on $(Y_1, Y_1)/K$ is faithful [6, Corollaire 2.2.9]. Since the canonical morphism $\overline{Y}_{1,\overline{k}} \to \overline{Y}_1$, where $\overline{k}$ is an algebraic closure of k, is surjective, we have only to prove the assertion in the case where k is algebraically closed.

Let x be a closed point of X, and Y_x (resp. Y_{1x}) the fiber of $g : Y \to X$ (resp. $g_1 : Y_1 \to X$) at x. Since the intersection $Y_{1x} \cap \Delta(Y)$ is nonempty, the rank of $\ker(\eta)|_{]Y_{1x} \cap \Delta(Y)[}$ is 1 by the hypothesis of s. Since Y_x is connected, so is also Y_{1x} and $\Delta(Y)$ meets any connected component of Y_1. Hence the rank of $\ker(\eta)|_{]Y_1[}$ is 1 on Y_1 by Proposition 3.5. This completes a proof. $\square$

4. Proof of Theorem 1.1 (2)

4.1 – *The case of proper coverings with geometrically connected fibers*

Let us fix the situation in this and the next sections, 4.1 and 4.2. Let $U \subset X \subset \overline{X}$ be a sequence of open immersions of separated integral schemes of finite type over $\operatorname{Spec} k$ with $Z = X \setminus U$ (U might be empty in section 4.1), and $\overline{g} : \overline{Y} \to \overline{X}$ a proper and surjective morphism of integral schemes over $\operatorname{Spec} k$, Y, V and W and the inverse images of X, U and Z in $\overline{Y}$, respectively. We denote the structure morphism by $g : Y \to X$. Let $\overline{g}_\bullet : \overline{Y}_\bullet \to \overline{X}$ be a Čech hypercovering induced from the proper surjective morphism $\overline{g} : \overline{Y} \to \overline{X}$, and $Y_\bullet$, (resp. $V_\bullet$, resp. $W_\bullet$) the inverse image of X (resp. U, resp. Z) in $\overline{Y}_\bullet$ by $\overline{g}_\bullet$ and denote the structure morphism by $g_\bullet : Y_\bullet \to X$. Consider a spectral sequence

$$E_1^{qr} = H^r_{\operatorname{rig} W_q}((Y_q, \overline{Y}_q)/K, M_q) \Rightarrow H^{q+r}_{\operatorname{rig} Z}((X, \overline{X})/K, M)$$

with respect to the proper hypercovering $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$ in Theorem 2.7.

PROPOSITION 4.1. *Suppose that each geometric fiber $g^{-1}(x)$ is connected for any geometric point x of X which is above a closed point of X. Then $E^{0,0}_\infty = E^{0,0}_1$ and $E^{q,0}_\infty = E^{q,0}_2 = 0$ for any $q \geq 1$. In particular, the natural morphism $g^* : H^1_{\operatorname{rig} Z}((X, \overline{X})/K, M) \to H^1_{\operatorname{rig} W}((Y, \overline{Y})/K, g^* M)$ is injective. Moreover, if furthermore $H^1_{\operatorname{rig} W}((Y, \overline{Y})/K, g^* M) = 0$, then we have a vanishing $H^1_{\operatorname{rig} Z}((X, \overline{X})/K, M) = 0$.*

PROOF.   What we want is to prove that the edge homomorphism

$$E_1^{q,0} = H^0_{\operatorname{rig} W_q}((Y_q, \overline{Y}_q)/K, M_q) \xrightarrow{d_1^{q,0}} E_1^{q+1,0} = H^0_{\operatorname{rig} W_{q+1}}((Y_{q+1}, \overline{Y}_{q+1})/K, M_{q+1}).$$

is 0 if q is even and is bijective if q is odd. Note that

$$d_1^{q,0} = \operatorname{pr}^*_{1,2,\cdots,q,q+1} - \operatorname{pr}^*_{0,2,\cdots,q,q+1} + \cdots + (-1)^{q+1} \operatorname{pr}^*_{0,1,2,\cdots,q}$$

where $\operatorname{pr}_? : \overline{Y}_{q+1} \to \overline{Y}_q$ is the ?-th projection for $? = (1, 2, \cdots, q, q+1), \cdots,$ $(0, 1, 2, \cdots, q)$. Since the natural commutative diagram

$$
\begin{array}{ccccccc}
0 & \to & H^0_{\operatorname{rig} W_q}((Y_q, \overline{Y}_q)/K, M_q) & \to & H^0_{\operatorname{rig}}((Y_q, \overline{Y}_q)/K, M_q) & \to & H^0_{\operatorname{rig}}((V_q, \overline{Y}_q)/K, M_q) \\
& & d_1^{q,0} \downarrow & & \widetilde{d}^{q,0}_{X,1} \downarrow & & \downarrow \widetilde{d}^{q,0}_{U,1} \\
0 & \to & H^0_{\operatorname{rig} W_{q+1}}((Y_{q+1}, \overline{Y}_{q+1})/K, M_{q+1}) & \to & H^0_{\operatorname{rig}}((Y_{q+1}, \overline{Y}_{q+1})/K, M_{q+1}) & \to & H^0_{\operatorname{rig}}((V_{q+1}, \overline{Y}_{q+1})/K, M_{q+1})
\end{array}
$$

arising from the spectral sequences of rigid cohomology groups on $(X, \overline{X})$ and $(U, \overline{X})$ with respect to the proper hypercoverings $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$ and

$(V_\bullet, \overline{Y}_\bullet) \to (U, \overline{X})$ has exact horizontal rows, it is sufficient to prove that $\widetilde{d}_{X,1}^{q,0}$ and $\widetilde{d}_{U,1}^{q,0}$ are 0-maps if q is even and are bijective if q is odd. If one denotes by $\Delta_q : \overline{Y} \to \overline{Y}_q$ (resp. $\Delta_{q+1} : \overline{Y} \to \overline{Y}_{q+1}$) the diagonal morphism, one has $\mathrm{pr}_? \circ \Delta_{q+1} = \Delta_q$ for any ?. Since $g : Y \to X$ is proper with a connected fiber at any geometric point above a closed point of X by our hypothesis, so are $g_q : Y_q \to X$ and $g_{q+1} : Y_{q+1} \to X$. Hence, by applying Theorem 3.6, both Δ_q^* and Δ_{q+1}^* are bijective, all $\mathrm{pr}_?^* : H_{\mathrm{rig}}^0((Y_q, \overline{Y}_q)/K, M_q) \to H_{\mathrm{rig}}^0((Y_{q+1}, \overline{Y}_{q+1})/K, M_{q+1})$ for ? coincide with each other and are bijective. It implies that $\widetilde{d}_{X,1}^{q,0}$ is a 0-map if q is even and is bijective if q is odd. The same holds for $\widetilde{d}_{U,1}^{q,0}$. This completes a proof. $\square$

4.2 – The case of finite coverings

Keep the notation as in the previous section 4.1.

PROPOSITION 4.2. *Suppose that U is not empty, $\overline{X}$ is geometrically integral, the function field $k(\overline{Y})$ of $\overline{Y}$ is a finite separable extension of the function field $k(\overline{X})$ of $\overline{X}$, and Y is the integral closure of X in $k(\overline{Y})$. If X is geometrically unibranch, then $E_1^{q,0} = H_{\mathrm{rig}W_q}^0((Y_q, \overline{Y}_q)/K, M_q) = 0$ for any $q \geq 0$. In particular, the natural homomorphism*

$$g^* : H_{\mathrm{rig}\,Z}^1((X, \overline{X})/K, M) \to H_{\mathrm{rig}W}^1((Y, \overline{Y})/K, g^*M)$$

is injective. Moreover, if furthermore $H_{\mathrm{rig}W}^1((Y, \overline{Y})/K, g^*M) = 0$, then we have a vanishing $H_{\mathrm{rig}\,Z}^1((X, \overline{X})/K, M) = 0$.

Proposition 4.2 above follows from Theorem 1.1 (1) and Lemma 4.3 below.

LEMMA 4.3. *V_q is dense in Y_q.*

LEMMA 4.4. *Let Y_q' be the integral closure of X in $k(\overline{Y})^{\otimes_{k(\overline{X})} q+1}$ for a nonnegative integer q, where $k(\overline{Y})^{\otimes_{k(\overline{X})} q+1}$ is a tensor product of $q+1$ copies of $k(\overline{Y})$ over $k(\overline{X})$. Then the natural morphism $h_q : Y_q' \to Y_q$ is surjective.*

PROOF OF LEMMA 4.3 ASSUMING LEMMA 4.4. Since $k(\overline{Y})^{\otimes_{k(\overline{X})} q+1}$ is a finite direct product of finite separable extensions of $k(\overline{X})$, Y_q' is a finite disjoint union of integral closures of X in some finite separable extensions of $k(\overline{X})$. Hence, if V_q' is the inverse image of the nonempty U in Y_q', the topological

closure $V_q'^{\text{cl}}$ of V_q' in Y_q' is Y_q' itself. Since $h_q(V_q'^{\text{cl}})$ is included in the topological closure $h_q(V_q')^{\text{cl}}$ of $h_q(V_q')$ in Y_q and h_q is surjective by Lemma 4.4, $V_q = h_q(V')$ is dense in Y_q.   □

PROOF OF LEMMA 4.4.   First we reduce the assertion to the case where k is algebraically closed and X is normal. Let $\overline{k}$ be an algebraic closure of k. Consider a natural commutative diagram

$$
\begin{array}{ccccc}
Y_q' & \longleftarrow & & & Y_{q,\overline{k}}' \\
\downarrow & & & & \downarrow \\
Y_q & \leftarrow & Y_{q,\overline{k}} & \leftarrow & ((Y_{\overline{k}})_{\text{nor}})^{q+1} \\
\downarrow & & \downarrow & & \downarrow \\
X & \leftarrow & X_{\overline{k}} & \leftarrow & (X_{\overline{k}})_{\text{nor}}
\end{array}
$$

where $Y_{q,\overline{k}}'$ is the integral closure of $X_{\overline{k}}$ in the tensor product $\overline{k}(\overline{Y})^{\otimes_{\overline{k}(\overline{X})} q+1}$ of $q+1$ copies of $\overline{k}(\overline{Y})$ over $\overline{k}(\overline{X})$ and $((Y_{\overline{k}})_{\text{nor}})^{q+1}$ is a fiber product of $q+1$ copies of $(Y_{\overline{k}})_{\text{nor}}$ over $(X_{\overline{k}})_{\text{nor}}$. Remember that X is geometrically integral and $\overline{k}(\overline{Y})$ is a finite separable extension over $\overline{k}(\overline{X})$ by our hypothesis. Hence $Y_{q,\overline{k}}'$ (resp. $\overline{k}(\overline{Y})^{\otimes_{\overline{k}(\overline{X})} q+1}$) is a finite disjoint union of integral schemes such that each component is isomorphic to each other (resp. a disjoint sum of finite separable extensions over $\overline{k}(\overline{X}_{\overline{k}})$ such that each direct summand is isomorphic to each other).

LEMMA 4.5.   *The morphism $((Y_{\overline{k}})_{\text{nor}})^{q+1} \to Y_{q,\overline{k}}$ in the diagram above is surjective.*

PROOF.   Let $y_0, y_1, \cdots, y_q$ be points in $Y_{\overline{k}}$ such that all $y_0, y_1, \cdots, y_q$ go to x in $X_{\overline{k}}$ by $Y_{\overline{k}} \to X_{\overline{k}}$. Since $X_{\overline{k}}$ and $Y_{\overline{k}}$ are again geometrically unibranch [19, IV, Propositions 6.15.6, 6.15.7], there exist unique points $y_0', \cdots, y_q' \in (Y_{\overline{k}})_{\text{nor}}$ and $x' \in (X_{\overline{k}})_{\text{nor}}$ such that $y_0', \cdots, y_q'$ and x' are above $y_0, \cdots, y_q$ and x, respectively. The uniqueness implies that all $y_0', \cdots, y_q'$ go to x' by $(Y_{\overline{k}})_{\text{nor}} \to (X_{\overline{k}})_{\text{nor}}$. Hence, we have the surjectivity.   □

Since $Y_{q,\overline{k}} \to Y_q$ is faithfully flat, it is surjective. In order to prove the surjectivity of $h_q : Y_q' \to Y_q$, we have only to prove the surjectivity of $(Y_{q,\overline{k}})' \to ((Y_{\overline{k}})_{\text{nor}})^{q+1}$. Hence, we may assume that k is algebraically closed and X is normal.

For a scheme S of finite type over $\operatorname{Spec} k$, denote by S_0 the set of closed points of S. Then the residue field of $s \in S_0$ is k because k is algebraically

closed. Since a finite morphism is closed, we have $h_q((Y_q')_0) \subset (Y_q)_0$. We have only to show $h_q((Y_q')_0) = (Y_q)_0$ since the subset of closed points is dense in the case of schemes of finite type over a field. The claim $h_q((Y_q')_0) = (Y_q)_0$ and hence Lemma 4.4 follow from the lemma below. $\square$

LEMMA 4.6. *Let $(A, \mathfrak{m})$ be a normal local domain with a fraction field E such that $k = A/\mathfrak{m}$ is algebraically closed, F a finite separable extension of E, and B the integral closure of A in F. For a nonnegative integer q, let $B^{\otimes_A q+1}$ be a tensor product of $q + 1$ copies of B over A, $F^{\otimes_E q+1}$ a tensor product of $q + 1$ copies of F over E, C_q the integral closure of A in $F^{\otimes_E q+1}$, and $h : \operatorname{Spec} C_q \to \operatorname{Spec} B^{\otimes_A q+1}$ the structure morphism induced by the natural A-algebra homomorphism $B^{\otimes_A q+1} \to F^{\otimes_E q+1}$. Then $h((\operatorname{Spec} C_q)_0) = (\operatorname{Spec} B^{\otimes_A q+1})_0$.*

PROOF. Let F' be a Galois closure of the finite separable extension F/E, B' the integral closure of A in F', and C_q' the integral closure of A in $(F')^{\otimes_E q+1}$. Consider the natural commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} C_q' & \to & \operatorname{Spec} (B')^{\otimes_A q+1} \\
\downarrow & & \downarrow \\
\operatorname{Spec} C_q & \to & \operatorname{Spec} B^{\otimes_A q+1}.
\end{array}
$$

Since the natural morphism $\operatorname{Spec} B' \to \operatorname{Spec} B$ is surjective and since the surjectivity is preserved by any base change, it is sufficient to prove the assertion for the morphism $\operatorname{Spec} C_q' \to \operatorname{Spec} (B')^{\otimes_A q+1}$. Hence we may assume that the extension F/E is Galois.

Suppose that the extension of F/E is Galois with Galois group G. Then B is stable under the action of G. Let $\mathfrak{n}_1 = \mathfrak{n}, \mathfrak{n}_2, \cdots, \mathfrak{n}_r$ be all maximal ideals of B. Then $\mathfrak{m} = \mathfrak{n}_i \cap A$ for any i and $(\operatorname{Spec} B)_0 = \{\sigma(\mathfrak{n}) \mid \sigma \in G\}$ [24, Theorem 4.7.7]. Since $B/\mathfrak{m}B$ is a finite k-algebra, we have $(B/\mathfrak{m}B)_{\mathrm{red}} = \prod_{i=1}^{r} B/\mathfrak{n}_i$. The surjectivity $((B/\mathfrak{m}B)_{\mathrm{red}})^{\otimes_k q+1} \to ((B/\mathfrak{m}B)^{\otimes_k q+1})_{\mathrm{red}}$ implies that $B^{\otimes_A q+1}$ has exactly r^{q+1} maximal ideals. Moreover, any maximal ideal of $B^{\otimes_A q+1}$ is generated by $\sum_{j=0}^{q} \iota_j(\sigma_j(\mathfrak{n}))$ for some $\sigma_0, \cdots, \sigma_q \in G$, where $\iota_j : B \to B^{\otimes_A q+1}$ is an inclusion into j-th component of the tensor product. Say $\mathfrak{n}_{\sigma_0, \cdots, \sigma_q}$ to be the maximal ideal generated by $\sum_{j=0}^{q} \iota_j(\sigma_j(\mathfrak{n}))$.

Since F/E is Galois, we have $F^{\otimes_E q+1} = \prod_{\rho_1, \cdots, \rho_q \in G} F_{\rho_1, \cdots, \rho_q}$ where $F_{\rho_1, \cdots, \rho_q} = F$ and the identification onto the $(\rho_1, \cdots, \rho_q)$-component is given

by $\quad F^{\otimes_E q+1} \to F_{\rho_1,\cdots,\rho_q} \ (a_0 \otimes a_1 \otimes \cdots \otimes a_q \mapsto a_0 \rho_1^{-1}(a_1) \cdots \rho_q^{-1}(a_q))$. Then $C_q = \prod\limits_{\rho_1,\cdots,\rho_q \in G} B_{\rho_1,\cdots,\rho_q}$ where $B_{\rho_1,\cdots,\rho_q} = B$ in $F_{\rho_1,\cdots,\rho_q}$. Let $\sigma(\mathfrak{n})_{\rho_1,\cdots,\rho_q}$ be a maximal ideal of C_q corresponding to the maximal ideal $\sigma(\mathfrak{n})$ of $(\rho_1,\cdots,\rho_q)$-component $B_{\rho_1,\cdots,\rho_q} = B$, then $(\mathrm{Spec}\, C_q)_0 = \{\sigma(\mathfrak{q})_{\rho_1,\cdots,\rho_q} \mid \sigma, \rho_1,\cdots,\rho_q \in G\}$. If we denote the $(\rho_1,\cdots,\rho_q)$-component of $h^{\#} : B^{\otimes_A q+1} \to C_q$ by $h^{\#}_{\rho_1,\cdots,\rho_q} : B^{\otimes_A q+1} \to B_{\rho_1,\cdots,\rho_q}$, then we have

$$(h^{\#}_{\rho_1,\cdots,\rho_q})^{-1}(\sigma(\mathfrak{n})_{\rho_1,\cdots,\sigma_q}) = \mathfrak{n}_{\sigma,\rho_1\sigma,\cdots,\rho_q\sigma}.$$

Therefore, $h((\mathrm{Spec}\, C_q)_0) = (\mathrm{Spec}\, B^{\otimes_A q+1})_0$. $\qquad\square$

4.3 – First reduction

Let us start to prove Theorem 1.1 (2). Let us take an open covering $\{\overline{Y}_\lambda\}_{\lambda \in \Lambda}$ of $\overline{X}$ over $\mathrm{Spec}\, k$ by a totally order set Λ such that $\overline{Y}_\lambda$ is separated of finite type over $\mathrm{Spec}\, k$, and consider a Čech spectral sequence associated to the covering as in Proposition 2.5. Since the hypotheses (i) - (iii) in Theorem 1.1 (2) is local conditions, each $(U \cap Y_{\lambda_0 \cdots \lambda_q}, Y_{\lambda_0 \cdots \lambda_q})$ satisfies the hypotheses (i) - (iii) if $Y_{\lambda_0 \cdots \lambda_q} \neq \emptyset$. Hence, Theorem 1.1 (1) implies $E_1^{q,0} = 0$ for any q. In order to prove $H_Z^1((X, \overline{X})/K, M) = 0$, it is sufficient to prove $E_1^{0,1} = 0$. Therefore, we may assume that $\overline{X}$ is separated of finite type over $\mathrm{Spec}\, k$ and X is dense in $\overline{X}$.

Let $\overline{X} = \cup_\lambda \overline{X}_\lambda$ be the irreducible decomposition over $\mathrm{Spec}\, k$. If one puts $X_\lambda = X \cap \overline{X}_\lambda$, then $X_\lambda \neq \emptyset$ for all η and $X = \cup_\lambda X_\lambda$ is the irreducible decomposition since X is dense in $\overline{X}$. Replacing k by a finite extension, we may assume that all irreducible components are geometrically irreducible by Proposition 2.2 and [19, IV, Proposition 6.15.7] (the stability of the condition (iii) by extension of base fields). Consider the Mayer-Vietoris spectral sequence in Proposition 2.6 which is associated to the irreducible decomposition. By our hypothesis (ii) Theorem 1.1 (1) implies $E_1^{1,0} = 0$. In order to prove $H^1_{\mathrm{rig}\, Z}((X, \overline{X})/K, M) = 0$, it is sufficient to prove $E_1^{0,1} = 0$. Hence we may assume that $\overline{X}$ is geometrically irreducible. Note that the rigid cohomology is determined only by the reduced structures. Therefore, we may assume that $\overline{X}$ is geometrically integral, U is nonempty, and X is geometrically unibranch.

REMARK 4.7. If $\overline{X}$ is separated of finite type over $\mathrm{Spec}\, k$, one can relax the hypothesis (ii) of Theorem 1.1 (2) to the hypothesis:

(ii)' $U_{\lambda,\mu}$ has a nonempty intersection with each connected component of $X_{\lambda,\mu}$ if $X_{\lambda,\mu} \neq \emptyset$.

Because the "of finite type" hypothesis is required to apply the Mayer-Vietoris spectral sequence (Proposition 2.6).

4.4 – *Applying alteration*

Let us continue the proof of Theorem 1.1 (2). Since we may replace the base field k by a suitable finite extension by Proposition 2.2 and the stability of geometrical unibranchness by extension of base fields [19, IV, Proposition 6.15.7], we may assume that there is a proper surjective and generically etale morphism $\overline{g} : \overline{Y} \to \overline{X}$ over $\operatorname{Spec} k$ such that $\overline{Y}$ is geometrically integral and smooth over $\operatorname{Spec} k$ by Theorem 2.15.

By applying the Stein factorization to $\overline{g} : \overline{Y} \to \overline{X}$ [18, III, 4.3], the morphism $\overline{g}$ is a composite of two morphisms

$$\overline{g} : \overline{Y} \to \overline{Y}' \to \overline{X}$$

such that

(i) $\overline{Y}'$ is the integral closure of $\overline{X}$ in the function field $k(\overline{Y})$ such that $\overline{Y}' \to \overline{X}$ is finite [24, Theorem 4.7.7] and generically etale;

(ii) $\overline{Y} \to \overline{Y}'$ is proper birational.

We denote by Y', V' and W' (resp. Y, V and W) the inverse images of X, U and Z in $\overline{Y}'$ (resp. $\overline{Y}$), respectively, and by $g' : Y' \to X$ and $g : Y \to X$ the structure morphisms. Since Y is smooth and V is dense in Y, we have $H^1_{\mathrm{rig}W}((Y, \overline{Y})/K, g^*M) = 0$ by Theorem 2.9. Since any geometric fiber of $\overline{Y} \to \overline{Y}'$ is connected by Zariski's main theorem [18, III, Théorème 4.3.1], we have $H^1_{\mathrm{rig}W'}((Y', \overline{Y}')/K, (g')^*M) = 0$ by Proposition 4.1. Finally, applying Proposition 4.2, we have the desired vanishing $H^1_{\mathrm{rig}Z}((X, \overline{X})/K, M) = 0$.

This completes a proof of Theorem 1.1 (2). □

5. Comparison for universal homeomorphisms

5.1 – *Resolution by proper hypercoverings*

In this section 5.1 the base scheme S is locally Noetherian. Let X and Y be separated schemes locally of finite type over S, $g : Y \to X$ a uni-

versal homeomorphism of finite type, $g_\bullet : Y_\bullet \to X$ the Čech hypercovering induced from the proper morphism $g : Y \to X$, $C(Y)_\bullet$ (resp. $D(Y)_{\bullet\bullet}$) a simplicial scheme (resp. a bi-simplicial scheme) such that $C(Y)_q = Y$ for any q (resp. $D(Y)_{q,r} = Y$ for any q and r) and any face map (resp. any degeneracy map) is the identity map id_Y. In other words, $C(Y)_\bullet$ (resp. $D(Y)_{\bullet\bullet}$) is the constant simplicial (resp. bi-simplicial) scheme. Since the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\mathrm{id}_Y} & Y \\
\Delta_q \downarrow & & \downarrow \Delta_r \\
Y_q & \xrightarrow{\eta} & Y_r,
\end{array}
$$

where Δ_q and Δ_r are the diagonal morphisms, is commutative for any face or degeneracy map η, the first projection

$$
D(Y)_{\bullet\bullet} \to Y_\bullet
$$

is a morphism as a bi-simplicial object over a simplicial scheme over S, that is,

$$
\begin{array}{ccc}
D(Y)_{q,\bullet} & \xrightarrow{\eta_\bullet} & D(Y)_{r,\bullet} \\
\downarrow & & \downarrow \\
Y_q & \xrightarrow{\eta} & Y_r,
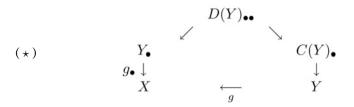\end{array}
$$

is a morphism of simplicial schemes for any morphism $\eta : Y_q \to Y_r$ in the standard simplicial category.

PROPOSITION 5.1. *With the notation as above, the diagonal closed immersion $\Delta_q : Y \to Y_q$ is surjective. In particular, $D(Y)_{q,\bullet} \to Y_q$ is a Čech hypercovering associated to the proper surjective morphism $Y \to Y_q$ for any q.*

PROOF. By definition of universal homeomorphisms Δ_q is a homeomorphism by induction on q. Since Δ_q is a closed immersion, the fiber product of finite copies of Y over Y_q is isomophic to Y by the diagonal morphism. Hence $D(Y)_{q,\bullet} \to Y_q$ is a Čech hypercovering associated to $\Delta_q : Y \to Y_q$. $\square$

By Proposition 5.1 we have a commutative diagram

$$(\star) \quad \begin{array}{ccc} & D(Y)_{\bullet\bullet} & \\ & \swarrow \qquad \searrow & \\ Y_{\bullet} & & C(Y)_{\bullet} \\ g_{\bullet} \downarrow & & \downarrow \\ X & \underset{g}{\longleftarrow} & Y \end{array}$$

of diagram of schemes over S, where $D(Y)_{\bullet\bullet} \to Y_{\bullet}$ (resp. $D(Y)_{\bullet\bullet} \to C(Y)_{\bullet}$) is the first (resp. second) projection, such that all of $D(Y)_{q,\bullet} \to Y_q, D(Y)_{\bullet,r} \to C(Y)_r, C(Y)_{\bullet} \to Y$, and $Y_{\bullet} \to X$ are proper hypercoverings.

5.2 – *Comparison theorem*

Now we return to the case of characteristic $p > 0$. Let S be a separated scheme locally of finite type over $\operatorname{Spec} k$, and $\mathfrak{S}$ a separated formal scheme locally of finite type over $\operatorname{Spf} \mathcal{V}$. Let

$$\begin{array}{ccc} Y & \overset{j_Y}{\longrightarrow} & \overline{Y} \\ g \downarrow & & \downarrow \overline{g} \\ X & \underset{j_X}{\longrightarrow} & \overline{X} \end{array}$$

be a commutative diagram of separated scheme locally of finite type over S with structure morphisms $f : X \to S$ and $h : Y \to S$ such that both j_X and j_Y are open immersions, g is of finite type, and $\overline{g}$ is proper on the Zariski closure of Y in $\overline{Y}$. When one is given $X, \overline{X}$ and Y, one can find $\overline{Y}$ if $\overline{X}$ is quasi-compact by the compactification in [14, Theorem 4.1]. The definition of overconvergent isocrystals on $(X, \overline{X})/\mathfrak{S}_K$ and its relative rigid cohomology is found in [13, section 10].

THEOREM 5.2. *Suppose that $g : Y \to X$ is a universal homeomorphism.*

(1) *The inverse image functor g^* from the category of overconvergent isocrystals on $(X, \overline{X})/\mathfrak{S}_K$ to the category of overconvergent isocrystals on $(Y, \overline{Y})/\mathfrak{S}_K$ gives an equivalence of categories. When $\mathcal{V}$ and $\mathfrak{S}$ admit compatible lifts σ and $\sigma_{\mathfrak{S}}$ of Frobenius, then the similar equivalence holds for overconvergent F-isocrystals with respect to $\sigma_{\mathfrak{S}}$.*

(2) *The natural homomorphism*

$$g^* : \mathbb{R}^r f_{\mathrm{rig}\,\mathfrak{S}_K *} M \to \mathbb{R}^r h_{\mathrm{rig}\,\mathfrak{S}_K *} g^* M$$

between relative rigid cohomology sheaves is an isomorphism for any overconvergent isocrystal M on $(X, \overline{X})/\mathfrak{S}_K$. In particular, if $S = \operatorname{Spec} k$ and $\mathfrak{S} = \operatorname{Spf} \mathcal{V}$, the natural homomorphism

$$g^* : H^r_{\mathrm{rig}Z}((X, \overline{X})/K, M) \to H^r_{\mathrm{rig}W}((Y, \overline{Y})/K, g^*M)$$

is an isomorphism for any closed subscheme Z of X with $W = g^{-1}(Z)$.

In order to prove (1) of the theorem above we review Shiho's proper descent for overconvergent isocrystals [29, Proposition 7.3]. An overconvergent isocrystal $M_\bullet$ on $(Y_\bullet, \overline{Y}_\bullet)/\mathfrak{S}_K$ is a collection of an overconvergent isocrystal M_q on $(Y_q, \overline{Y}_q)/\mathfrak{S}_K$ for all q such that there is an isomorphism $u(\eta) : \eta^* M_r \xrightarrow{\cong} M_q$ for any morphism $\eta : X_q \to X_r$ in the standard simplicial category which satisfies the conditions (i) $u(\xi\eta) = u(\eta)\eta^*(u(\xi))$ and (ii) $u(\mathrm{id}_{X_q}) = \mathrm{id}_{M_q}$. An overconvergent isocrystal on $M_{\bullet\bullet}$ (resp. $M_\bullet$) on $(D(Y)_{\bullet\bullet}, D(\overline{Y})_{\bullet\bullet})/\mathfrak{S}_K$ (resp. $(C(Y)_\bullet, C(\overline{Y})_\bullet)/\mathfrak{S}_K$) is also defined similarly. Because the descent data [28, Section 5] is involved in the conditions (i) and (ii). Hence, Shiho's proper descent is as follows by Proposition 5.1.

THEOREM 5.3 [29, Proposition 7.3]. *Both inverse image functors*

$$\begin{pmatrix} \text{overconvergent isocrystals} \\ \text{on } (X, \overline{X})/\mathfrak{S}_K \end{pmatrix} \quad \to \quad \begin{pmatrix} \text{overconvergent isocrystals} \\ \text{on } (Y_\bullet, \overline{Y}_\bullet)/\mathfrak{S}_K \end{pmatrix}$$

$$\to \quad \begin{pmatrix} \text{overconvergent isocrystals} \\ \text{on } (D(Y)_{\bullet\bullet}, D(\overline{Y})_{\bullet\bullet})/\mathfrak{S}_K \end{pmatrix}$$

are equivalences of categories.

Shiho assumed that the hypercovering is a "strongly" Čech proper hypercovering, that is, $\overline{g}^{-1}(X) = Y$ and $\overline{g}$ is proper and surjective. But one can relax the hypothesis because the category of overconvergent isocrystals does not depend on the choice of compactifications. More precisely, if one takes another $\overline{X}'$ with a commutative diagram

$$\begin{array}{ccc}
 & & \overline{X}' \\
 & \overset{j'_X}{\nearrow} & \downarrow h \\
X & \underset{j_X}{\longrightarrow} & \overline{X}
\end{array}$$

such that j'_X is an open immersion and h is proper, then the category of overconvergent isocrystals on $(X, \overline{X})/\mathfrak{S}_K$ and that on $(X, \overline{X}')/\mathfrak{S}_K$ are

equivalent (resp. the natural homomorphism $\mathbb{R}^r f_{\mathrm{rig}\, \mathfrak{S}_{K*}} M \to \mathbb{R}^r f'_{\mathrm{rig}\, \mathfrak{S}_{K*}} M'$ of relative rigid cohomology sheaves is an isomorphism, where M' is the inverse image of M and the direct image of f' means the one for $(X, \overline{X}')$ over $\mathfrak{S}_K$) [6, Théorème 2.3.5] [13, Proposition 10.2.11] (resp. [13, Proposition 10.5.3]).

Now we prove Theorem 5.2. Since the equivaleces of categories of overconvergent isocrystals on $(Y, \overline{Y})/\mathfrak{S}_K$, $(C(Y)_\bullet, C(\overline{Y})_\bullet)/\mathfrak{S}_K$ and $(D(Y)_{\bullet\bullet}, D(\overline{Y})_{\bullet\bullet})/\mathfrak{S}_K$ is clear. Hence, the assertion (1) follows Theorem 5.3 and the commutativity of the diagram $(\star)$ in section 5.1. The assertion (2) follows from the proper cohomological descent of rigid cohomology [32, Theorem 2.1.3] and the above remark of the independence of the choices of compactifications. For the rigid cohomology with supports, the excision sequence in Proposition 2.4 is also applied. $\qquad\qquad\square$

REMARK 5.4. One can prove the comparison as above by this method (see [16, IX, Théorème 4.10]) in any reasonable cohomology theory furnishing the expected properties including proper descent of sheaves and proper cohomological descent, for example, Betti cohomology, etale cohomology and so on.

6. Application to Frobenius eigenvalues

Let K be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers with residue field k of p^s elements. Suppose that K admits a Frobenius endomorphism σ, i.e., a continuous lift of the absolute Frobenius endomorphism $a \mapsto a^p$ on k such that the s-th iteration σ^s of σ is the identity on K. We fix an isomorphism between an algebraic closure $\overline{K}$ of K and the field $\mathbb{C}$ of complex numbers. For integers $w_1 \leq w_2$, $[w_1, w_2]$ denotes the set of integers w with $w_1 \leq w \leq w_2$.

Let V be a K-vector space of finite dimension with a K-linear bijective endomorphism $\Phi : V \to V$. We say that (V, Φ) is mixed of weights in $[w_1, w_2]$ for some integers $w_1 \leq w_2$ if, for any eigenvalue α of Φ, (i) α is an algebraic number and (ii) there is an integer $w \in [w_1, w_2]$ such that the complex absolute values of all conjugations of α over $\mathbb{Q}$ are $p^{sw/2}$. We say that (V, Φ) is pure of weight w if (V, Φ) is mixed of weights in $[w, w]$.

Let X be a separated scheme of finite type over $\mathrm{Spec}\, k$, $H_{\mathrm{rig}}^r(X/K)$ the r-th rigid cohomology group for the unit overconvergent isocrystal $j_X^\dagger \mathcal{O}_{]\overline{X}[}$ where $\overline{X}$ is a compactification of X over $\mathrm{Spec}\, k$. Let $F : H_{\mathrm{rig}}^r(X/K) \to$

$H^r_{\mathrm{rig}}(X/K)$ be the Frobenius endomorphism which is induced by the absolute Frobenius on X. Then $H^r_{\mathrm{rig}}(X/K)$ is a finite dimensional K-vector space and F is a σ-linear bijective endomorphism [32, Theorem 5.1.1]. The s-th iteration F^s of F is K-linear.

THEOREM 6.1 ([11] Theorem 1.2 for proper cases and Section 2 for open cases). *Suppose that X is smooth and purely of dimension d over $\operatorname{Spec} k$. Then $(H^r_{\mathrm{rig}}(X/K), F^s)$ is mixed of weights in $[r, \min\{2r, 2d\}]$. Moreover, if X is proper over $\operatorname{Spec} k$, then $(H^r_{\mathrm{rig}}(X/K), F^s)$ is pure of weight r.*

For a general X we do not know even the vanishing of usual rigid cohomology group $H^r_{\mathrm{rig}}(X/K)$ for $r > 2\dim X$. We only know that $H^r_{\mathrm{rig}}(X/K) = 0$ for sufficient large r [33, Proposition 6.4.1]. Note that the weights for rigid cohomology with compact supports was already studied in [25, Theorem 15.11].

THEOREM 6.2 [32, Corollary 5.2.3 for proper cases]. *If X is of dimension $\leq d$, then $(H^r_{\mathrm{rig}}(X/K), F^s)$ is mixed of weights in $[0, \min\{2r, 2d\}]$. Moreover, if X is proper over $\operatorname{Spec} k$, then $(H^r_{\mathrm{rig}}(X/K), F^s)$ is mixed of weights in $[0, r]$.*

PROOF. One can construct a proper hypercovering $Y_\bullet \to X$ (resp. when X is proper over $\operatorname{Spec} k$) such that each Y_q is smooth (resp. proper smooth) over $\operatorname{Spec} k$ of dimension $\leq d$ and one has a spectral sequence

$$E^{qr}_1 = H^r_{\mathrm{rig}}(Y_q/K) \Rightarrow H^{q+r}_{\mathrm{rig}}(X/K)$$

in Theorem 2.7 (see [33, Theorem 7.1.2]). The assertion follows from Theorem 6.1.   □

In geometrically unibranch cases the weights of Frobenius in the first rigid cohomology is close to that for smooth cases.

THEOREM 6.3. *Suppose that X is geometrically unibranch. Then $(H^1_{\mathrm{rig}}(X/K), F^s)$ is mixed of weights in $[1, 2]$. Moreover, if X is proper over $\operatorname{Spec} k$, then $(H^1_{\mathrm{rig}}(X/K), F^s)$ is pure of weight 1.*

In the next lemma the base field k is arbitrary of characteristic $p > 0$.

LEMMA 6.4. *Let $g : Y \to X$ be a proper surjective morphism of schemes separated of finite type over $\operatorname{Spec} k$. Suppose that X is geometrically uni-*

branch. Then the natural homomorphism

$$g^* : H^1_{\mathrm{rig}}(X/K) \to H^1_{\mathrm{rig}}(Y/K)$$

is injective.

We will give an example which tells us the natural homomorphism $g^* : H^1_{\mathrm{rig}}(X/K) \to H^1_{\mathrm{rig}}(Y/K)$ is not bijective in Example 7.2.

PROOF OF LEMMA 6.4. Note that we may replace the base field k by a finite extension of k by Proposition 2.2. Let $\overline{X}$ be a compactification of X over $\mathrm{Spec}\,k$. Since X is geometrically unibranch, one can deal with the problem on each irreducible component of X. Hence, we may assume that X is geometrically integral.

Taking the reduced closed subscheme of an irreducible component of Y, we may assume that Y is also integral by Propositions 2.2 and 2.6. By the functoriality of inverse images on rigid cohomology groups we may assume that Y is normal and there exists a compactification $\overline{Y}$ of Y over $\overline{X}$ such that $\overline{Y}$ is also a normal integral scheme by Nagata's compactification [22] [23]. Let

$$\overline{Y} \to \overline{Y}' \to \overline{X}$$

be the Stein factorization of $\overline{Y} \to \overline{X}$. Then $\overline{Y}'$ is normal and is finite over $\overline{X}$ by the normality of $\overline{Y}$, and $\overline{Y} \to \overline{Y}'$ has a connected fiber at each geometric point of $\overline{Y}'$. Put Y' to be the inverse image of X in $\overline{Y}'$. If we consider a spectral sequence

$$E_1^{qr} = H^r_{\mathrm{rig}}(\widetilde{Y}_q/K) \Rightarrow H^{q+r}_{\mathrm{rig}}(Y'/K),$$

with respect to the Čech proper hypercovering $\widetilde{Y}_\bullet \to Y'$ associated to $Y \to Y'$, then we have $E^{q0}_\infty = E^{q0}_2 = 0$ for $q \geq 1$ by the proof of Proposition 4.1. Thus $H^1_{\mathrm{rig}}(Y'/K)$ is a subspace of $H^1_{\mathrm{rig}}(Y/K)$. Hence, we may assume that $Y \to X$ is generically finite.

Since $\overline{X}$ is geometrically integral, there exists an open dense subscheme of X which is smooth over $\mathrm{Spec}\,k$. By applying the existence of regular alteration in Theorem 2.15, there is a proper surjective and generically finite morphism $Y' \to Y$ such that Y' is smooth over $\mathrm{Spec}\,k$ if we replace k by a finite extension again. Hence we may assume that Y is smooth over $\mathrm{Spec}\,k$. Hence there exists an open dense affine subscheme U of X which is smooth over $\mathrm{Spec}\,k$ such that the inverse image V of U in Y is finite and flat. By comparing rigid cohomology with Monsky-Washnitzer cohomology and using a retraction map of Monsky-Washnitzer cohomology with re-

spect to the finite flat morphism $V \to U$, the natural homomorphism $H^1_{\mathrm{rig}}(U/K) \to H^1_{\mathrm{rig}}(V/K)$ is injective [7, Proposition 3.6].

If we put $Z = X \setminus U$ and $W = Y \setminus V$, then there is a natural commutative diagram

$$
\begin{array}{ccccc}
H^1_{\mathrm{rig}\,Z}(X/K) & \to & H^1_{\mathrm{rig}}(X/K) & \to & H^1_{\mathrm{rig}}(U/K) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\mathrm{rig}\,W}(Y/K) & \to & H^1_{\mathrm{rig}}(Y/K) & \to & H^1_{\mathrm{rig}}(V/K)
\end{array}
$$

with exact rows, and $H^1_{\mathrm{rig}\,Z}(X/K) = 0$ by Theorem 1.1 (2). Therefore, $g^* : H^1_{\mathrm{rig}}(X/K) \to H^1_{\mathrm{rig}}(Y/K)$ is injective.   □

PROOF OF THEOREM 6.3.   By Theorem 2.15 there is a proper surjective morphism $g : Y \to X$ such that Y is smooth over $\operatorname{Spec} k$ since k is perfect. Hence the assertion follows from Theorem 6.1 and Lemma 6.4.   □

7. Examples

7.1 – *An example in non geometrically unibranch case*

EXAMPLE 7.1 Let p be an odd prime number, $K = \mathbb{Q}_p$, $k = \mathbb{F}_p$, and $\sigma = \mathrm{id}_{\mathbb{Q}_p}$ the Frobenius endomorphism. Let $\overline{X} = X$ be a plane curve defined by $x_0 x_2^2 = x_1^3 + x_1^2 x_0$ in the projective plane $\mathbb{P}^2_k$ over $\operatorname{Spec} k$, where x_0, x_1, x_2 are homogeneous coordinates, and $Z = [1 : 0 : 0] \in X$. Then $U = X \setminus Z \cong \mathbb{G}_{\mathrm{m},k}$. Let $Y = \overline{Y}$ be the normalization of X. Then Y is isomorphic to the projective line and the inverse image W of Z in Y consists of two distinct k-rational points s, t. Hence X is not unibranch.

Using the Poincaré duality with respect to U [8, Théorème 3.4] and the excision sequence of rigid cohomology with compact support

$$0 \to H^0_{\mathrm{rig},c}(U/K) \to H^0_{\mathrm{rig}}(X/K) \to H^0_{\mathrm{rig}}(Z/K) \to H^1_{\mathrm{rig},c}(U/K) \to \cdots$$

(see [7, Remark 3.9]), we have

$$
H^r_{\mathrm{rig}}(X/K) = \begin{cases} K(0) & \text{if } r = 0, 1, \\ K(-1) & \text{if } r = 2, \\ 0 & \text{otherwise.} \end{cases}
$$

Here $K(a)$ means the a-th Tate twist of F-isocrystal on $\operatorname{Spec} k/K$, i.e., Frobenius acts by multiplying by p^{-a}. By applying the excision sequence of

rigid cohomology and by using the argument of weights, we have

$$H^r_{\mathrm{rig}Z}(X/K) = \begin{cases} K(0) & \text{if } r = 1, \\ K(-1) \oplus K(-1) & \text{if } r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let M be an extension of $\mathcal{O}_{]X[}$ by $\mathcal{O}_{]X[}$ which corresponds to a nontrivial element of the image of $H^1_{\mathrm{rig}Z}(X/K) \to H^1_{\mathrm{rig}}(X/K)$. Then M is a convergent isocrystal on X/K, which is not split on X/K, by Theorem 2.13. Since $j^\dagger_U M$ is split on $(U, X)/K$, the natural restriction map

$$H^0_{\mathrm{rig}}(X/K, M) \to H^0_{\mathrm{rig}}(U/K, j^\dagger_U M)$$

is not surjective. $\qquad\square$

The reason why the proof of Theorem 1.1 (2) does not work in the case above is as follows. By taking an irreducible decomposition, there is a surjective closed immersion $\widetilde{Y}_1 = Y \coprod (s, t) \coprod (t, s) \to Y_1 = Y \times_X Y$, where Y is embedded diagonally into Y_1. Hence, $V_1 = V \times_U V \cong U$ is not dense in Y_1. Moreover, there is a proper hypercovering $\widetilde{Y}_\bullet \to X$ which is defined by

$$\widetilde{Y}_q = Y \coprod \coprod_{\substack{a \in \{s,t\}^{q+1} \\ a \neq (s,\cdots,s),(t,\cdots,t)}} a.$$

Hence, we have $H^1_{\mathrm{rig}Z}(X/K) = E^{1,0}_\infty = E^{1,0}_2 = K(0)$. One can also find an F-isocrystal M on X/K such that M is not split on X/K and $j^\dagger_U M$ is split on $(U, X)/K$ as an overconvergent F-isocrystal by Corollary 2.14.

7.2 – *An example of a normal case of dimension* 2

We give an example such that the restriction functor $j^\dagger_U$ from the category of overconvergent isocrystals on X/K to the category of overconvergent isocrystals on U/K is not essentially surjective.

EXAMPLE 7.2 Let p be a prime number with $p \neq 3$, $K = \mathbb{Q}_p$, $k = \mathbb{F}_p$, and $\sigma = \mathrm{id}_{\mathbb{Q}_p}$ the Frobenius endomorphism. Let $\overline{X} = X$ be a surface defined by $x_3^3 = x_1 x_2 (x_1 - x_2)$ in the projective space $\mathbb{P}^3_k$ over $\mathrm{Spec}\, k$ of dimension 3, where x_0, x_1, x_2, x_3 are homogeneous coordinates, $Z = [1 : 0 : 0 : 0] \in X$, and $U = X \setminus Z$. Then X is normal since X is a complete intersection with the only singular point Z of codimension 2 by Serre's criterion of normality [19, IV, Théorème 5.8.6].

Let us take an open covering $U = U_1 \cup U_2$, where U_i is defined by $x_i \neq 0$. Then U_1 (resp. U_2) is a product of $\mathbb{A}_k^1$ and $E \setminus \{P\}$ (resp. $E \setminus \{Q\}$), where $\mathbb{A}_k^1$ is the affine line, E is an elliptic curve defined by the homogeneous equation $z^3 = xy(x - y)$, and P_1 (resp. P_2) is a k-rational point $[0 : 1 : 0]$ (resp. $[1 : 0 : 0]$) of E. Moreover, $U_1 \cap U_2$ is a product of $\mathbb{A}_k^1$ and $E \setminus \{P_1, P_2\}$. Since $H^1_{\mathrm{rig}}(U_i/K) \cong H^1_{\mathrm{rig}}(E \setminus \{P_i\}/K) \cong H^1_{\mathrm{rig}}(E)$ for $i = 1, 2$ and

$$0 \to H^1_{\mathrm{rig}}(E) \to H^1_{\mathrm{rig}}(E \setminus \{P_1, P_2\}/K) \to K(-1) \to 0$$

is exact by the Künneth formula [8, Théorème 3.2] and the excision sequence in Proposition 2.4, we have

$$H^r_{\mathrm{rig}}(U/K) = \begin{cases} K(0) & \text{if } r = 0, \\ H^1_{\mathrm{rig}}(E/K) & \text{if } r = 1, \\ K(-1) & \text{if } r = 2, \\ 0 & \text{otherwise} \end{cases}$$

by the Čech spectral sequence associated to the covering above in Proposition 2.5. Using the Poincaré duality with respect to U [8, Théorème 2.4] and the excision sequence of rigid cohomology with compact support, we have

$$H^r_{\mathrm{rig}}(X/K) = \begin{cases} K(0) & \text{if } r = 0, \\ 0 & \text{if } r = 1, \\ K(-1) & \text{if } r = 2, \\ H^1_{\mathrm{rig}}(E/K)(-1) & \text{if } r = 3, \\ K(-2) & \text{if } r = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the natural restriction map

$$H^1_{\mathrm{rig}}(X/K) \to H^1_{\mathrm{rig}}(U/K)$$

is not surjective. Since a nontrivial element of $H^1_{\mathrm{rig}}(U/K)$ corresponds to an overconvergent connection on an extension of the unit object by the unit object by Theorem 2.13, the restriction functor $j_U^\dagger$ is not essentially surjective.

Let $\widetilde{g} : \widetilde{X} \to X$ be a desingularization obtained by a blowing up along the reduced closed point Z. Then the exceptional divisor $\widetilde{Z} = \widetilde{g}^{-1}(Z)$ is isomorphic to the elliptic curve E. Applying an excision sequence for $\widetilde{X}$ and

$U = \widetilde{X} \setminus \widetilde{Z}$, we have

$$
H^r_{\mathrm{rig}}(\widetilde{X}/K) = \left\{
\begin{array}{ll}
K(0) & \text{if } r = 0, \\
H^1_{\mathrm{rig}}(E/K) & \text{if } r = 1, \\
K(-1) \oplus K(-1) & \text{if } r = 2, \\
H^1_{\mathrm{rig}}(E/K)(-1) & \text{if } r = 3, \\
K(-2) & \text{if } r = 4, \\
0 & \text{otherwise}
\end{array}
\right.
$$

by the weight reason. Indeed, $H^2_{\mathrm{rig}}(X/K)(1)$ is generated by the cycle classes corresponding to $\widetilde{Z}$ and the strict transform $\widetilde{Z}'$ ($\cong \mathbb{P}^1_k$) of $X \setminus U_1$. Hence the natural homomorphism $\widetilde{g}^* : H^1_{\mathrm{rig}}(X/K) \to H^1_{\mathrm{rig}}(\widetilde{X}/K)$ is not bijective. $\qquad\square$

In etale cases A. Grothendieck's purity, $\pi_1^{\mathrm{alg}}(X \setminus Z) \xrightarrow{\cong} \pi_1^{\mathrm{alg}}(X)$, holds for a pair (X, Z) consisting of a locally complete intersection X and a closed subscheme Z of codimension ≥ 3 [17, X, Théorème 3.4]. The case of the second example does not satisfy the assumption of Grothendieck's purity theorem because Z is of codimension 2.

REFERENCES

[1] M. ARTIN - A. GROTHENDIECK - J. L. VERDIER, *Theorie des topos et cohomologie etale des schemas (SGA4)*, Lecture Notes in Math., 269, 270, and 305, Springer-Verlag, Heidelberg, 1972/3.

[2] F. BALDASSARRI - P. BERTHELOT, *On Dwork cohomology for singular hypersurfaces*, Geometric aspects of Dwork theory. Vol. I, II, pp. 177–244, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[3] F. BALDASSARRI - B. CHIARELLOTTO, *Algebraic versus rigid cohomology with logarithmic coefficients*, Barsotti Symposium in Algebraic Geometry, Perspect. Math., **15**, Academic press, San Diego, CA (1994), pp. 11–50.

[4] V. G. BERKOVICH, *Integration of one-forms on p-adic analytic spaces*, Annals of Math. Studies 162, Princeton University Press, Princeton, NJ, 2007.

[5] P. BERTHELOT, *Géométrie rigide et cohomologie des variétés algébriques de caractéristique p*, Introductions aux cohomologies *p*-adiques (Luminy, 1984). Mém. Soc. Math. France (N.S.) No. **23**, 3 (1986), pp. 7–32.

[6] P. BERTHELOT, *Cohomologie rigide et cohomologie rigide à supports propres*, Première partie, preprint, 1996.

[7] P. BERTHELOT, *Finitude et pureté cohomologique en cohomologie rigide*, With an appendix in English by Aise Johan de Jong. Invent. Math. **128**, no. 2 (1997), pp. 329–377.

[8] P. BERTHELOT, *Dualité de Poincaré et formule de Künneth en cohomologie rigide*, C.R. Acad. Sc. Paris Sér. I Math. **325**, no. 5 (1997), pp. 493–498.

[9] S. BOSCH - U. GÜNTZER - R. REMMERT, *Non-archimedean analysis*, Grundlehren der Mathematischen Wissenschaften **261**, Springer-Verlag, Berlin, 1984.

[10] D. CARO - N. TSUZUKI, *Overholonomicity of overconvergent F-isocrystals over smooth varieties*, Annals of Math. **176**, no. 2 (2012), pp. 747–813.

[11] B. CHIARELLOTTO - B. LE STUM, *Sur la pureté de la cohomologie cristalline*, C. R. Acad. Sci. Paris Sér. I Math. **326**, no. 8 (1998), pp. 961–963.

[12] B. CHIARELLOTTO - B. LE STUM, *F-isocristaux unipotents*, Compositio Math. **116**, no. 8 (1999), pp. 81–110.

[13] B. CHIARELLOTTO - N. TSUZUKI, *Cohomological descent of rigid cohomology for étale coverings*, Rend. Sem. Mat. Univ. Padova **109** (2003), pp. 63–215.

[14] B. CONRAD, *Deligne's note on Nagata compactifications*, J. Ramanujan Math. Soc. **22**, no. 3 (2007), pp. 205–257.

[15] R. ELKIK, *Solutions d'équations à coefficients dans un anneau hensélien*, Ann. Sci. École Norm. Sup. **6** (1973), pp. 553–603.

[16] A. GROTHENDIECK, *Revétements étale et groupe fondemental (SGA1)*, Lecture Notes in Math. **224**, Springer-Verlag, Heidelberg, 1971.

[17] A. GROTHENDIECK, *Cohomologie locale des faisceaux coherents et theoremes de Lefschetz locaux et globaux (SGA2)*, Augmenté d'un exposé par Michéle Raynaud. Séminaire de Géométrie Algébrique du Bois-Marie, 1962. Advanced Studies in Pure Mathematics, Vol. 2. North-Holland Publishing Co., Amsterdam; Masson & Cie, Éditeur, Paris, 1968.

[18] A. GROTHENDIECK - J. DIEUDONNÉ, *Éléments de géométrie algébrique III*, Étude cohomologique des faisceaux cohérents, Inst. Hautes Études Sci. Publ. Math. **11** (1961), pp. 5–167; **17** (1963), pp. 5–91.

[19] A. GROTHENDIECK - J. DIEUDONNÉ, *Éléments de géométrie algébrique IV*, Étude locale des schémas et des morphismes de schémas, Inst. Hautes Études Sci. Publ. Math. **20** (1964), pp. 5–259; **24** (1965), pp. 5–231; **32** (1967), pp. 5–361.

[20] A. J. DE JONG, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. **83** (1996), pp. 51–93.

[21] K. KEDLAYA, *Semistable reduction for overconvergent F-isocrystals*, I: Unipotence and logarithmic extensions, Compositio Math. **143**, no. 5 (2007), pp. 1164–1212.

[22] M. NAGATA, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto, **2** (1962), pp. 1–10.

[23] M. NAGATA, *A generalization of the imbedding problem of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **3** (1963), pp. 89–102.

[24] M. NAGATA, *Field theory*, Monographs and textbooks in pure and applied mathematics **40**, Marcel Dekker Inc. New York and Basel, 1977.

[25] Y. NAKKAJIMA, *Weight filtration and slope filtration on the rigid cohomology of a variety in characteristic $p > 0$*, to appear in Mém. Soc. Math. France.

[26] A. OGUS, *F-isocrystals and de Rham cohomology II – convergent isocrystals*, Duke Math. J. **51**, no. 4 (1984), pp. 765–850.

[27] A. SHIHO, *Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology*, J. Math. Sci. Univ. Tokyo, **9**, no. 1, (2002), pp. 1–163.

[28] A. SHIHO, *Relative log convergent cohomology and relative rigid cohomology I*, preprint, 2007. http://arxiv.org/abs/0707.1742v2

[29] A. SHIHO, *Relative log convergent cohomology and relative rigid cohomology II*, preprint, 2007. http://arxiv.org/abs/0707.1743v2

[30] N. TSUZUKI, *On the Gysin isomorphism of rigid cohomology*, Hiroshima Math. J. **29**, no. 3 (1999), pp. 479–527.

[31] N. TSUZUKI, *Morphisms of F-isocrystals and the finite monodromy theorem for unit-root F-isocrystals*, Duke Math. J. **111**, no. 3 (2002), pp. 385–418.

[32] N. TSUZUKI, *Cohomological descent of rigid cohomology for proper coverings*, Invent. Math. **151**, no. 1 (2003), pp. 101–133.

[33] N. TSUZUKI, *Cohomological descent in rigid cohomology*, Geometric aspects of Dwork theory. Vol. I, II, pp. 931–981, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.