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# On the (non-)Contractibility of the Order Complex of the Coset Poset of an Alternating Group

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ABSTRACT - Let $\mathrm{Alt}_k$ be the alternating group of degree $k$. In this paper we prove that the order complex of the coset poset of $\mathrm{Alt}_k$ is non-contractible for a big family of $k \in \mathbb{N}$, including the numbers of the form $k = p + m$ where $m \in \{3, \ldots, 35\}$ and $p > k/2$. In order to prove this result, we show that $P_G(-1)$ does not vanish, where $P_G(s)$ is the Dirichlet polynomial associated to the group $G$. Moreover, we extend the result to some monolithic primitive groups whose socle is a direct product of alternating groups.

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## 1. Introduction

For a finite group $G$, let $\mathcal{C}(G)$ be the set of (right) cosets of all proper subgroups of G, partially ordered by inclusion. Let $\Delta = \Delta(C(G))$ be the order complex of $\mathcal{C}(G)$, so the $k$-dimensional faces of $\Delta$ are chains of length $k$ from $\mathcal{C}(G)$. The study of $\Delta$ was initiated in a paper of K. S. Brown (see [2]), who attributed therein to S. Bouc the observation that there exists a connection between the reduced Euler characteristic $\tilde{\chi}(\mathcal{C}(G))$ and the Dirichlet polynomial $P_G(s)$ of $G$, defined by

$$P_G(s) = \sum_{n=1}^{\infty} \frac{a_n(G)}{n^s}, \quad \text{where} \quad a_n(G) = \sum_{H \leq G, |G:H|=n} \mu_G(H).$$

Here $\mu_G$ is the Möbius function of the subgroup lattice of $G$, which is

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defined inductively by $\mu_G(G) = 1$, $\mu_G(H) = -\sum_{K>H} \mu_G(K)$. The counterpart of the Dirichlet polynomial is called the probabilistic zeta function of $G$ (see [1] and [6]).

In particular, Brown ([2], § 3) showed that

$$P_G(-1) = -\tilde{\chi}(\mathcal{C}(G)).$$

It is a well-known fact that if $\Delta(\mathcal{C}(G))$ is contractible, then its reduced Euler characteristic $\tilde{\chi}(\mathcal{C}(G))$ is zero. Hence, if $P_G(-1) \neq 0$, then the simplicial complex associated to the group $G$ is non-contractible.

Moreover, in [2], Brown conjectured the following.

CONJECTURE 1. *If $G$ is a finite group, then $P_G(-1) \neq 0$. Hence the order complex of the coset poset of $G$ is non-contractible.*

The conjecture was proved for some families of groups, as the following theorem shows.

THEOREM 2. *Let $G$ be a finite group.*

(1) *If $G$ is soluble, then $P_G(-1) \neq 0$ (see* [2]*).*
(2) *If $G$ is a classical group which does not contain non-trivial graph automorphisms, then $P_G(-1) \neq 0$ (*[9, Theorem 2]*).*
(3) *If $G$ is a Suzuki simple group or a Ree simple group, then $P_G(-1) \neq 0$ (*[7]*).*

Moreover, in our PhD thesis (see [8, Theorem 7.1]), we proved Conjecture 1 for a class of monolithic primitive groups with socle isomorphic to a direct product of copies of a simple classical group with some conditions on the rank and the number of factors of the direct product.

In this paper we show the following.

THEOREM 3. *Let $G$ be $\mathrm{Sym}_k$ or $\mathrm{Alt}_k$. Let $p$ be a prime number such that $\frac{k}{2} < p < k-2$.*

(1) *If $k - p \leq 35$, then $P_G(-1) \neq 0$.*
(2) *If $p > 2((k-p)!)^3$, then $P_G(-1) \neq 0$.*

A proof of this result proceeds as follows (the general strategy is the same which we employed in [9]). Let $r$ be a prime number and let $f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$ be a Dirichlet polynomial. We denote by $f^{(r)}(s)$ the Dirichlet polynomial

$$\sum_{(n,r)=1} \frac{a_n}{n^s}.$$

Let $p$ be a prime number such that $\frac{k}{2} < p < k-2$. We have that

$$P_G(s) = P_G^{(p)}(s) + \sum_{p|k} \frac{a_k(G)}{k^s}.$$

The first summand $P_G^{(p)}(s)$ collects the contribution given by the subgroups $H$ of $X$ such that $H$ contains a Sylow $p$-subgroup, which are intransitive subgroups of $G$ (see [10, Lemma 9]). In [10, Proposition 12] an explicit formula for the Dirichlet polynomial $P_G^{(p)}(s)$ is given. So, a careful analysis of the value $P_G^{(p)}(-1)$, shows that $|P_G^{(p)}(-1)|_p = p$ in many cases (see Section 3), where $|a|_p$ is the greatest power of $p$ that divides the integer number $a$ (see Section 2 for a more precise definition).

Since $n$ divides $a_n(G)$ for $n \in \mathbb{N} - \{0\}$ (see Lemma 5), we have that

$$\left| \sum_{p|n} a_n(G)n \right|_p \geq p^2,$$

hence

$$|P_G(-1)|_p = |P_G^{(p)}(-1) + \sum_{p|n} a_n(G)n|_p = p$$

whenever $|P_G^{(p)}(-1)|_p = p$. So we conclude $P_G(-1) \neq 0$.

With a little more work we can obtain a more general result. Let us note that if $N$ is a normal subgroup of a finite group $G$, then $P_G(s) = P_{G/N}(s)P_{G,N}(s)$ (see [2]), where

$$P_{G,N}(s) = \sum_{n \geq 1} \frac{a_n(G,N)}{n^s}, \quad \text{with } a_n(G,N) = \sum_{\substack{H \leq G,\, |G:H|=n, \\ NH=G,}} \mu_G(H).$$

Thus, if $1 = N_0 < N_1 < \ldots < N_l = G$ is a chief series of $G$, applying the above formula repeatedly, we obtain

$$P_G(s) = \prod_{i=0}^{l-1} P_{G/N_i, N_{i+1}/N_i}(s).$$

If the chief factor $N_{i+1}/N_i$ is abelian, by a result of [4], we have that

$$P_{G/N_i, N_{i+1}/N_i}(s) = 1 - \frac{c_i}{|N_i|^s}$$

where $c_i$ is the number of complements of $N_{i+1}/N_i$ in $G/N_i$. Hence $P_{G/N_i,N_{i+1}/N_i}(-1) \neq 0$.

If the chief factor $N_{i+1}/N_i$ is non-abelian, then there exists a monolithic primitive group $L_i$ such that

$$P^{(r)}_{L_i,\mathrm{soc}(L_i)}(s) = P^{(r)}_{G/N_i,N_{i+1}/N_i}(s)$$

for each prime divisor $r$ of the order of the socle $\mathrm{soc}(L_i)$ of $L_i$ (see, for example, [3, Proposition 16]).

The above discussion suggests that in order to prove Conjecture 1, we can focus our attention on the monolithic primitive groups. In particular, here we prove the following.

THEOREM 4. *Let $G$ be a primitive monolithic group with socle $N$ isomorphic to a direct product of $n$ copies of the alternating group* $\mathrm{Alt}_k$ *with $k \geq 8$. Let $p$ be a prime number such that $\frac{k}{2} < p < k-2$. If $p > 2n((k-p)!)^{2n+1}$, then $P_G(-1) \neq 0$.*

## 2. Some useful results and definitions

Let $a$ be an integer and let $r$ be a prime number. We denote by $|a|_r$ the $r$-part of $a$, i.e. $|a|_r = r^i$ where $i \in \mathbb{N}$ such that $r^i$ divides $a$ but $r^{i+1}$ does not divide $a$. Moreover, we set $|0|_r = 0$.

If $b = c/d$ is a rational number for some $c \in \mathbb{Z}, d \in \mathbb{N} - \{0\}$, then we let $|b|_r = \frac{|c|_r}{|d|_r}$.

We record here an important result on the Möbius function of the subgroup lattice of $G$.

LEMMA 5 ([5], Theorem; 4.5). *Let $A$ be a finite group and $B$ a subgroup of $A$. The index $|N_A(B) : B|$ divides $\mu_A(B)|A : BA'|$.*

In particular, if $X$ is an almost simple group with socle $G$ and $H$ is a subgroup of $G$, then Lemma 5 yields $|G : H|$ divides $\mu_G(H)|G : N_G(H)|$. So, in particular, $n$ divides $a_n(X, G)$.

We can say some more words on the Dirichlet polynomial of a monolithic primitive group $L$ with non-abelian socle $N$. Assume that $S$ is a simple component of $L$, define $X = N_L(S)/C_L(S)$ and $n = |L : N_L(S)|$. We have that $N \cong S^n$. Since $S \cong \mathrm{soc}(X)$, assume that $S \leq X$. The following result shows a connection between the Dirichlet polynomials $P_{L,N}(s)$ and $P_{X,S}(s)$.

THEOREM 6 (See [11, Theorem 5]).

$$P_{L,N}^{(r)}(s) = P_{X,S}^{(r)}(ns - n + 1)$$

*for each prime divisor r of the order of S.*

Let $X$ be $\mathrm{Sym}_k$ or $\mathrm{Alt}_k$ with $k \geq 8$. A key role in our paper is played by the Dirichlet polynomial $P_{X,S}^{(p)}(s)$, which can be expressed by an explicit formula.

PROPOSITION 7 (See [10, Proposition 12]). *Let $k \geq 8$. Let $X$ be either* $\mathrm{Alt}_k$ *or* $\mathrm{Sym}_k$, *and let $p$ be a prime number such that* $\frac{k}{2} < p < k - 2$. *Then*

$$P_{X,S}^{(p)}(s) = \sum_{\omega \in \Psi_k : k_l \geq p} \frac{\mu(\omega)}{\nu(\omega)\iota(\omega)^{s-1}}.$$

*Here $\Psi_k$ is the set of partitions of k, i.e. non-decreasing sequences of positive integers whose sum is k, we have $\omega = (k_1, \ldots, k_l) \in \Psi_k$ for some $l \leq k, k_1, \ldots, k_l \in \mathbb{N} - \{0\}$ such that $k_1 + \ldots + k_l = k$, and we define*

$$\mu(\omega) = (-1)^{l-1}(l-1)!, \quad \iota(\omega) = \frac{k!}{\prod_{i=1}^{l} k_i!}, \quad \nu(\omega) = \prod_{i=1}^{k} \omega_i!,$$

*where $\omega_i = |\{j : k_j = i\}|$.*

## 3. The Dirichlet polynomial $P^{(p)}(s)$

Let $X = \mathrm{Sym}_k$ or $\mathrm{Alt}_k$ for $k \geq 8$. For this section, we let $P(s) = P_{X,\mathrm{soc}(X)}(s)$. The aim of this section is to find the value $|P^{(p)}(1-n)|_p$ for $n \geq 2$ a natural number.

Let $p$ be a prime number such that $\frac{k}{2} < p < k - 2$. By Proposition 7, we have that

$$P^{(p)}(1-n) = \sum_{\omega \in \Psi_k : k_l \geq p} \frac{\mu(\omega)}{\nu(\omega)\iota(\omega)^{-n}} = \sum_{\omega = (k_1, \ldots, k_l) \in \Psi_k : k_l \geq p} \frac{(-1)^{l-1}(l-1)!(k!)^n}{\prod_{i=1}^{k} \omega_i! (\prod_{i=1}^{l} k_i!)^n} =$$

$$= \sum_{j=0}^{k-p} g(j,n) \left( \frac{k!}{(k-j)!} \right)^n,$$

where

$$g(j,n) = \sum_{\omega=(k_1,\dots,k_{l-1})\in\Psi_j} \frac{(-1)^{l-1}(l-1)!}{\prod_{i\geq 1}\omega_i!\left(\prod_{i=1}^{l-1} k_i!\right)^n}$$

for $j \geq 0$ (the set $\Psi_0$ consists of the empty partition $\omega = (\,)$, so $g(0,n) = 1$).

Consider the polynomial

$$f_n(x) = g(0,n) + \sum_{j=1}^{k-p} g(j,n)\left(\prod_{i=0}^{j-1}(x+k-p-i)\right)^n$$

in $\mathbb{Q}[x]$. Clearly $f_n(p) = P^{(p)}(1-n)$. We want to show that $x$ divides $f_n(x)$ in $\mathbb{Q}[x]$, but $x^2$ does not divide $f_n(x)$ in $\mathbb{Q}[x]$. This implies that $|P^{(p)}(1-n)|_p = p$ if the coefficient $\alpha_1$ of $x$ in $f_n(x)$ is such that $|\alpha_1|_p = 1$ (see the proof of Theorem 11).

First of all, we prove the following formula, which is very useful in order to find some properties of $g(j,n)$.

Proposition 8. *If $m \geq 1$, then*

$$\sum_{j=0}^{m} \frac{g(j,n)}{((m-j)!)^n} = 0$$

Proof. Let us rewrite the sum in another way. Let $\omega = (k_1,\dots,k_l) \in \Psi_m$. For each $h \in \{1,\dots,l\}$ there exists $\omega^h = (k_1,\dots,k_{h-1},k_{h+1},\dots,k_l) \in \Psi_{m-k_h}$. Clearly $\omega$ appears in the term $g(m,n)$ and $\omega^h$ appears in the term $g(m-k_h,n)$. In particular, the contribution of $\omega^h$ in $g(m-k_h,n)$ is

$$\frac{(-1)^{l-1}(l-1)!}{\nu(\omega^h)\left(\prod_{i\neq h} k_i!\right)^n}.$$

Moreover, since $\nu(\omega) = |i : k_i = k_h|\nu(\omega^h)$ we have that the previous expression becomes

$$\frac{(-1)^{l-1}(l-1)!|i : k_i = k_h|}{\nu(\omega)\left(\prod_{i\neq h} k_i!\right)^n}.$$

Furthermore, it is clear that if $(k_1,\dots,k_{l'}) = \tau \in \Psi_{m'}$ for some $m' < m$, then there exists a unique $\omega \in \Psi_m$ such that $\tau = \omega^h$ for some

$h \in \{1, \ldots, l'+1\}$: indeed, there exists $h \in \{1, \ldots, l'+1\}$ such that $k_{h-1} \leq m - m' \leq k_h$ (with the convention that $k_0 = 0$ and $k_{l'+1} = k$) and so $\omega = (k_1, \ldots, k_{h-1}, m - m', k_h, \ldots, k_{l'})$.

Let $H_\omega = \{h \in \{1, \ldots, l\} : h = 1 \text{ or } k_{h-1} < k_h\}$ (so $\{k_1, \ldots, k_l\} = \{k_h : h \in H_\omega\}$ and if $h_1, h_2 \in H_\omega$, then $k_{h_1} = k_{h_2}$ if and only if $h_1 = h_2$). We get

$$\sum_{j=0}^{m} \frac{g(j,n)}{((m-j)!)^n} = \sum_{(k_1,\ldots,k_l)=\omega \in \Psi_m} \left( \frac{(-1)^l l!}{\nu(\omega)\left(\prod_{i=1}^{l} k_i!\right)^n} + \sum_{h \in H_\omega} \frac{(-1)^{l-1}(l-1)!|i : k_i = k_h|}{\nu(\omega)\left(\prod_{i \neq h} k_i!\right)^n} \frac{1}{(k_i!)^n} \right)$$

$$= \sum_{(k_1,\ldots,k_l)=\omega \in \Psi_m} \frac{(-1)^l (l-1)!(l - \sum_{h \in H_\omega} |i : k_i = k_h|)}{\nu(\omega)\left(\prod_{i=1}^{l} k_i!\right)^n} = 0$$

since $\sum_{h \in H_\omega} |i : k_i = k_h| = l$ by definition. □

Thanks to the recursive formula we obtained for $g(j, n)$, we can prove the following inequalities.

Proposition 9. *We have that*

$$0 < (-1)^{m-1} g(m-1, n)/2 < (-1)^m g(m, n) \leq (-1)^{m-1} g(m-1, n) \leq 1$$

*for* $m \geq 1$ *and* $n \geq 2$.

Proof. Let us prove the proposition by induction on $m$. If $m = 1$, the claim holds by definition of $g(m, n)$. Assume that $m > 1$. By induction, it is enough to prove that

$$(-1)^m g(m, n)/2 < (-1)^{m+1} g(m+1, n) \leq (-1)^m g(m, n)$$

By Proposition 8, we have that

$$g(m+1, n) = -\sum_{j=0}^{m} \frac{g(j, n)}{((m+1-j)!)^n},$$

hence

$$(-1)^{m+1} g(m+1, n) = (-1)^m g(m, n) + \sum_{j=0}^{m-1} (-1)^{m-j} \frac{(-1)^j g(j, n)}{((m+1-j)!)^n}. \quad (\dagger)$$

Let $g'(j,n) = (-1)^{m-j}\dfrac{(-1)^j g(j,n)}{((m+1-j)!)^n}$. Now, by induction we have

$$-(-1)^j g(j,n) < -(-1)^{j-1} g(j-1,n)/2 < 0$$

for $1 \le j \le m$. Hence, for $m-j$ odd, we get

$$g'(j,n) + g'(j-1,n) < (-1)^{j-1} g(j-1,n)\frac{2-(m-j+2)^n}{2((m-j+2)!)^n} \le 0$$

for $n \ge 1$. This implies that

$$\sum_{j=0}^{m-1}(-1)^{m-j}\frac{(-1)^j g(j,n)}{((m+1-j)!)^n} = \alpha g'(0,n) + \sum_{1\le j\le m-1,\, m-j \text{ odd}} g'(j,n) + g'(j-1,n) < 0$$

where $\alpha = 0$ if $m$ is even, $\alpha = 1$ otherwise (when $m$ is odd, note that $g'(0,n) < 0$). So this proves that $(-1)^{m+1} g(m+1,n) \le (-1)^m g(m,n)$.

Now, by (†), it remains to prove that

$$(-1)^{m+1} g(m+1,n) - (-1)^m g(m,n)/2 = (-1)^m g(m,n)/2 + \sum_{j=0}^{m-1}(-1)^{m-j}\frac{(-1)^j g(j,n)}{((m+1-j)!)^n} > 0.$$

Again, by induction we have that $(-1)^j g(j,n)/2 < (-1)^{j+1} g(j+1,n)$ for $0 \le j \le m-1$. Hence, for $m-j$ even, we have that

$$g'(j,n) + g'(j-1,n) > (-1)^j g(j,n)\frac{(m+2-j)^n - 2}{((m+2-j)!)^n} \ge 0$$

for $n \ge 1$. Moreover, we get

$$(-1)^m g(m,n)/2 + g'(m-1,n) > (-1)^m g(m,n)\frac{2^{n-2}-1}{2^{n-1}} \ge 0$$

for $n \ge 2$. This implies that

$$(-1)^m g(m,n)/2 + \sum_{j=0}^{m-1}(-1)^{m-j}\frac{(-1)^j g(j,n)}{((m+1-j)!)^n} = \beta g'(0,n) + (-1)^m g(m,n)/2 + g'(m-1,n) + \sum_{1\le j\le m-2,\, m-j \text{ even}} g'(j,n) + g'(j-1,n) > 0$$

where $\beta = 1$ if $m$ is even, $\beta = 0$ otherwise (when $m$ is even, note that $g'(0,n) > 0$). So this proves that $(-1)^{m+1} g(m+1,n) > (-1)^m g(m,n)/2$. □

Now, we are ready to prove the main result of this section.

PROPOSITION 10. *Let $n \geq 2$. We have that $x$ divides $f_n(x)$ in $\mathbb{Q}[x]$, but $x^2$ does not divide $f_n(x)$ in $\mathbb{Q}[x]$.*

Let $m = k - p$. To prove that $x$ divides $f_n(x)$ is enough to show that

$$f_n(0) = \sum_{j=0}^{m} g(j,n)\left(\frac{m!}{(m-j)!}\right)^n = 0,$$

which follows from Proposition 8. In order to prove that $x^2$ does not divide $f_n(x)$ we may show that the coefficient of $x$ in $f_n(x)$ does not vanish. It is easy to realize that the coefficient of $x$ in $f_n(x)$ is:

$$\sum_{j=0}^{m} ng(j,n)\left(\frac{m!}{(m-j)!}\right)^n \sum_{i=0}^{j-1} \frac{1}{m-i} \qquad (\dagger)$$

By Proposition 8, we can subtract

$$\sum_{j=0}^{m} ng(j,n)\left(\frac{m!}{(m-j)!}\right)^n \sum_{i=0}^{m-1} \frac{1}{m-i} = 0$$

from (†), hence proving that (†) does not vanish is equivalent to prove that

$$\sum_{j=0}^{m-1} g(j,n)\left(\frac{m!}{(m-j)!}\right)^n \sum_{i=j}^{m-1} \frac{1}{m-i} \neq 0.$$

Let

$$h_{a,b}(m,n) = \sum_{j=a}^{b} g(j,n)\left(\frac{m!}{(m-j)!}\right)^n \sum_{i=j}^{m-1} \frac{1}{m-i}.$$

Let $0 \leq j \leq m-2$. Note that

$$\left(\frac{m!}{(m-j-1)!}\right)^n \sum_{i=j+1}^{m-1} \frac{1}{m-i} \geq 2\left(\frac{m!}{(m-j)!}\right)^n \sum_{i=j}^{m-1} \frac{1}{m-i} > 0$$

since

$$(m-j)^n \sum_{i=j+1}^{m-1} \frac{1}{m-i} \geq 2 \sum_{i=j}^{m-1} \frac{1}{m-i} > 0.$$

Hence, by Proposition 9 we get

$$(-1)^{j+1}g(j+1,n)\left(\frac{m!}{(m-j-1)!}\right)^n\sum_{i=j+1}^{m-1}\frac{1}{m-i} > \\ (-1)^j g(j,n)\left(\frac{m!}{(m-j)!}\right)^n\sum_{i=j}^{m-1}\frac{1}{m-i},$$

i.e. $(-1)^{j+1}h_{j,j+1}(m,n) > 0$. This implies that $h_{0,m-1}(m,n) < 0$ if $m$ is even, and $h_{3,m-1}(m,n) > 0$ if $m$ is odd. Since $g(0,n) = -g(1,n) = 1$ and $g(2,n) = 1 - 2^{-n}$, it is easy to see that $h_{0,2}(m,n) > 0$ (for $n \geq 2$ and $m \geq 3$). Thus we conclude that

$$h_{0,m-1}(m,n) = h_{0,2}(m,n) + h_{3,m-1}(m,n) > 0$$

if $m$ is odd. The proof is complete. □

We can finally prove the main theorem.

THEOREM 11. *Let $k \geq 8$ and let $p$ be a prime number such that $k/2 < p < k-2$. Assume that $p > n((k-p)!)^{n+1}$. Then*

$$|P^{(p)}(-n+1)|_p = p.$$

PROOF. Let $m = k-p$. Let $f_n(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_t x^t$ for some $\alpha_i = a_i/b_i \in \mathbb{Q}$ with $(a_i, b_i) = 1$ (if $a_i = 0$, let $b_i = 1$) and $t \in \mathbb{N}$.

By the proof of Proposition 10 we have that $\alpha_1 = -nh_{0,m-1}(m,n)$. Moreover, we have seen that $(-1)^{m-1}h_{0,m-1}(m,n) > 0$ and $(-1)^{m-1}h_{0,m-2}(m,n) < 0$, hence $|h_{m-1,m-1}(m,n)| > |h_{0,m-1}(m,n)|$. So we have

$$|\alpha_1| \leq n|h_{m-1,m-1}(m,n)| = n|g(m-1,n)|(m!)^n \leq n(m!)^n$$

being $|g(m-1,n)| \leq 1$ (by Proposition 9).

Now, we want to show that $|P^{(p)}(-n+1)|_p = p$. Note that since $p > k-p$ and $b_i$ divides $(k-p)! = m!$ (by definition of $f_n(x)$), we have that $|\alpha_i|_p = |a_i|_p$. By Proposition 10 we have that $\alpha_0 = 0$, hence $|P^{(p)}(-n+1)|_p = |f_n(p)|_p \geq p$. For a contradiction, assume that $|P^{(p)}(-n+1)|_p > p$. Then $|f_n(p)|_p \geq p^2$, hence $|\alpha_1|_p \geq p$, so

$$|a_1| \geq |\alpha_1|_p \geq p > n(m!)^{n+1} \geq |\alpha_1|m! \geq |\alpha_1||b_1| = |a_1|,$$

a contradiction. □

When the number $k-p$ is small, we know explicitly the coefficient of $x$ in $f_n(x)$. For example, we have that $|P^{(p)}(-n+1)|_p = p$ if $p$ does not appear in Table 1 (for $k-p \leq 7$ and $n \in \{2,4,6\}$). Further calculations can be done, but the coefficient of $x$ in $f_n(x)$ grows very quickly and it has very large prime factors (for example, if $k-p=35$, the coefficient of $x$ in $f_2(x)$ has a prime divisor of 50 digits).

TABLE 1. Exceptions for $p$.

| $k-p$ | $n=2$ | $n=4$ | $n=6$ |
|---|---|---|---|
| 3 | 23 | 31, 53 | 109, 617 |
| 4 | 677 | 1047469 | 19, 31, 5051 |
| 5 | 71, 103 | 643, 148579 | 3889, 595523689 |
| 6 | 7, 13 | 23, 269, 89714671 | 2357, 4584299, 35430211 |
| 7 | 863897 | 181, 2005732476817 | 70353197, 1633443829219 |

However, there are many prime numbers between $k/2$ and $k-2$ if $k$ is big enough. For example, we have the following.

PROPOSITION 12. *Let $8 \leq k \leq 10^6$. If $k \neq 13$, then there exists a prime number $p$ such that $|P^{(p)}(-1)|_p = p$.*

## 4. The main result

We can now prove the main result.

THEOREM 13. *Let $G$ be a monolithic primitive group with socle isomorphic to $\mathrm{Alt}_k^n$ for $k \geq 8$. Let $p$ be a prime number such that $k/2 < p < k-2$.*

(1) *If $p > 2n((k-p)!)^{2n+1}$, then $P_{G,\mathrm{soc}(G)}(-1) \neq 0$.*
(2) *If $n=1$ and $k-p \leq 35$, then $P_{G,\mathrm{soc}(G)}(-1) \neq 0$.*
(3) *If $n=1$ and $8 \leq k \leq 10^7$, then $P_{G,\mathrm{soc}(G)}(-1) \neq 0$.*

PROOF. By Theorem 6, we have that

$$|P^{(p)}_{G,\mathrm{soc}(G)}(-1)|_p = |P^{(p)}_{X,\mathrm{soc}(X)}(1-2n)|_p,$$

where $X = N_G(S)/C_G(S)$ and $S$ is a simple component of $G$. By Theorem 11, if $p > 2n((k-p)!)^{2n+1}$, then $|P^{(p)}_{X,\mathrm{soc}(X)}(1-2n)|_p = p$. By Lemma 5, we have that $l$ divides $a_l(G, \mathrm{soc}(G))$ for $l \geq 1$, hence

$$|P_{G,\mathrm{soc}(G)}(-1) - P^{(p)}_{G,\mathrm{soc}(G)}(-1)|_p \geq p^2,$$

thus we get the claim.

If $n = 1$ and $k - p \leq 35$, the coefficient of $x$ in $f_n(x)$ is known, so by direct computation, there exists a prime number $p'$ (not necessarily different from $p$) such that $k/2 < p' < k - 2$ such that $|P^{(p')}_{X,\mathrm{soc}(X)}(-1)|_{p'} = p'$ (except when $k = 13$). Arguing as above, we get the claim (using a direct computation for $k = 13$). Finally, if $n = 1$ and $8 \leq k \leq 10^6$, arguing as above, by Proposition 12 we have the claim. □

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