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An Identity on Partial Generalized Automorphisms of Prime Rings

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ABSTRACT - Let R be a prime ring with center $Z(R)$, $T : R \longrightarrow R$ be a non-zero partial generalized automorphism of R, L a Lie ideal of R, $s \geq 0, t \geq 0$ and $n \geq 1$ fixed integers, such that $(u^s(T(u) \circ u)u^t)^n = 0$ for all $u \in L$. If either $\mathrm{Char}(R) > n+1$ or $\mathrm{Char}(R) = 0$, then $L \subseteq Z(R)$.

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1. Introduction

The standard identity s_4 in four variables is defined as follows:

$$s_4 = \sum (- 1)^{\tau} X_{\tau(1)} X_{\tau(2)} X_{\tau(3)} X_{\tau(4)}$$

where $(- 1)^{\tau}$ is the sign of a permutation τ of the symmetric group of degree 4.

In all that follows, unless stated otherwise, R always denotes a prime ring, $Z(R)$ the center of R, Q its Martindale quotient ring. The center of Q, denoted by C, is called the extended centroid of R (we refer the reader to [1] for these objects). It is well-known that C is a field. For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for Lie commutator $xy - yx$ and Jordan commutator $xy + yx$ respectively. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \longrightarrow R$ is

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called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Starting from this definition, Brešar [2] introduced first the definition of generalized derivation: an additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. Hence, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier (i.e., an additive mapping satisfying $F(xy) = F(x)y$ for all $x, y \in R$). Basic examples are derivations and generalized inner derivations (i.e., mappings of type $x \longrightarrow ax + xb$ for some $a, b \in R$). We refer to call such mappings generalized inner derivations for the reason they present a generalization of the concept of inner derivations (i.e., mappings of the form $x \longrightarrow ax - xa$ for some $a \in R$). Zhang and Wang [20], defined first the concept of partial generalized automorphisms: an additive mapping $T : R \longrightarrow R$ is called a right (resp. left) partial generalized automorphism if there exists an automorphism $\sigma : R \longrightarrow R$ such that $T(xy) = T(x)\sigma(y)$ (resp. $T(xy) = \sigma(x)T(y)$) holds for all $x, y \in R$. If T is both left as well as a right partial generalized automorphism, then it is called a partial generalized automorphism. They proved that every partial generalized automorphism T on a dense right ideal of R can be uniquely extended to a partial generalized automorphism of Q and assume the form $T(x) = \lambda\sigma(x)$ for all $x \in Q$ for some $\lambda \in C$ and an automorphism σ of Q.

This paper is included in a line of investigation concerning the relationship between the global structure of a ring R and the behaviors of some additive mappings defined on R that satisfy certain special identities. A well-known result of Herstein [14] states that if ρ is a right idea of R such that $u^n = 0$ for all $u \in \rho$, where n is a fixed positive integers, then $\rho = 0$. In [6], Chang and Lin considered the situation when $d(u)u^n = 0$ for all $u \in \rho$, where d is a nonzero derivation of R. In [8], Dhara and De Filippis studied the case when $u^sH(u)u^t = 0$ for all $u \in L$, where L a noncommutative Lie ideal of R, H a generalized derivation of R and s, t are fixed nonnegative integers. More precisely, they proved the following: Let R be a prime ring, H a nonzero generalized derivation of R and L a noncommutative Lie ideal of R. Suppose that $u^sH(u)u^t = 0$ for all $u \in L$. Then R satisfies s_4, the standard identity in four variables.

On the other hand, in [5] Carini and De Filippis proved that if R is a prime ring with $\mathrm{Char}(R) \neq 2$ satisfying $[d(u), u]^n = 0$ for all $u \in L$, where L is a noncentral Lie ideal and d a nonzero derivation of R, then R is commutative.

De Filippis [7] generalized this result for generalized derivations. In [19], Wang studied the identity $[\sigma(u), u]^n = 0$ replacing the derivation d by an automorphism σ of R and obtained that R satisfies s_4. Recently, Dhara and Sharma [9], considered the situation $(u^{n_1}[d(u), u]u^{n_2})^{n_3} = 0$ for all $u \in L$, where n_1, n_2, n_3 are fixed nonnegative integers, L a noncentral Lie ideal and d a derivation of R and proved that $d = 0$ provided $\mathrm{Char}(R) \neq 2$. Later, Dhara [10] extended this result by replacing the derivation d with a generalized derivation F. Motivated by the previous results, we here prove a similar version for partial generalized automorphisms involving Jordan commutators. More precisely we will study the identity $(u^s(T(u) \circ u)u^t)^n = 0$ for all $u \in L$, where $s \geq 0, t \geq 0, n \geq 1$ are fixed integers, L a noncentral Lie ideal and $T \neq 0$ a partial generalized automorphism of R and proved that L is central in this case.

2. Main results

THEOREM 2.1. *Let R be a prime ring, $T : R \longrightarrow R$ be a non-zero partial generalized automorphism of R, I a two-sided ideal, $s \geq 0, t \geq 0$ and $n \geq 1$ fixed integers, such that $(u^s(T(u) \circ u)u^t)^n = 0$ for all $u \in [I, I]$. If either Char$(R) > n + 1$ or Char$(R) = 0$, then R is commutative.*

PROOF. Assume that R is not commutative, so that I is not central. In view of Zhang and Wang [20, Theorem 3.1], $T(x) = \lambda\sigma(x)$ for all $x \in R$, where $\lambda \in C$ and σ is an automorphism of R. We are given that $(u^s(T(u) \circ u)u^t)^n = 0$ for all $u \in [I, I]$. This implies that $([x, y]^s(\lambda\sigma([x, y]) \circ [x, y])[x, y]^t)^n = 0$ for all $x, y \in I$. Since T is nonzero and so $\lambda \neq 0$, then it is invertible in C, which implies that $([x, y]^s(\sigma([x, y]) \circ [x, y])[x, y]^t)^n = 0$ for all $x, y \in I$. If $\sigma = 1_R$, the identity map on R, from above equation, $2^n[x, y]^{(s+t+2)n} = 0$ and so $[x, y]^{(s+t+2)n} = 0$ for all $x, y \in I$. Then it follows from Herstein [13, Theorem 2] that R is commutative, a contradiction. Hence onward we assume that $\sigma \neq 1_R$. Since σ is an automorphism of R, we have

$$([x, y]^s([\sigma(x), \sigma(y)] \circ [x, y])[x, y]^t)^n = 0 \text{ for all } x, y \in I. \tag{2.1}$$

By Kharchenko's theorem [16], we divide the proof into two cases.

CASE 1. Let σ be Q-outer. Since either Char$(R) > n+1$ or Char$(R) = 0$, by Chuang [4, Main theorem], we arrive at $([x, y]^s([u, v] \circ [x, y])[x, y]^t)^n = 0$

for all $x, y, u, v \in I$, in particular, by letting $u = x$ and $v = y$, then $2^n[x, y]^{(s+t+2)n} = 0$ for all $x, y \in I$. By the same arguments as above, we get a contradiction.

CASE 2. If σ is Q-inner, then there exists an invertible element $b \in Q$ such that $\sigma(x) = b^{-1}xb$ for all $x \in R$. We note that $b \notin C$ since $\sigma \neq 1_R$. By Chuang [3, Theorem 2], I, R and Q satisfy the same generalized polynomial identities (or GPIs in brief), from (2.1) we have

$$([x, y]^s([b^{-1}xb, b^{-1}yb] \circ [x, y])[x, y]^t)^n = 0 \text{ for all } x, y \in I. \tag{2.2}$$

In case the center C of Q is infinite, we have $([x, y]^s([b^{-1}xb, b^{-1}yb] \circ [x, y])[x, y]^t)^n = 0$ for all $x, y \in Q \bigotimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \bigotimes_C \overline{C}$ are prime and centrally closed [12, Theorem 2.5 and Theorem 3.5], we may replace R by Q or $Q \bigotimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC = C$) which is either finite or algebraically closed and $([x, y]^s([b^{-1}xb, b^{-1}yb] \circ [x, y])[x, y]^t)^n = 0$ for all $x, y \in R$. By Martindale [18, Theorem 3], RC (and so R) is a strongly primitive ring. In light of Jacobson's theorem [15, pp.75], R is isomorphic to a dense ring of linear transformations of a vector space V. Let ${}_RV$ be a faithful irreducible left R-module with commuting division $D = End({}_RV)$. Since C is either finite or algebraically closed, we know that D must coincide with C. By the density theorem, R acts densely on V_D.

For any given $v \in V$, we want to show that v and bv are linearly D-dependent. If $bv = 0$ then v and bv are D-dependent and we are done in this case. Suppose that $bv \neq 0$, v and bv are D-independent. We consider the following two cases.

SUBCASE 1. Assume that $v, bv, b^{-1}v$ are D-independent. Then by the density of R, there exist $x, y \in R$ such that

$$xv = bv; xbv = 0; yv = 0; ybv = v.$$

From (2.2), we can see that

$$0 = ([x, y]^s([b^{-1}xb, b^{-1}yb] \circ [x, y])[x, y]^t)^n v = (-1)^{(s+t+1)n} 2^n v \neq 0$$

a contradiction.

SUBCASE 2. Otherwise, $v, bv, b^{-1}v$ are D-dependent. Since v and bv are D-independent, then $b^{-1}v = vd_1 + bvd_2$ for some $d_1, d_2 \in D$. Moreover, we claim that $d_2 \neq 0$. Indeed, if $d_2 = 0$, then $b^{-1}v = vd_1$ and $v = bvd_1$, contradicting the independence of v and bv. By the density of R, there exist $x, y \in R$ such that

$$xv = 0; xbv = v; yv = b^{-1}v = vd_1 + bvd_2; ybv = bvd_1.$$

It follows from (2.2) that

$$0 = ([x,y]^s([b^{-1}xb, b^{-1}yb] \circ [x,y])[x,y]^t)^n v = 2^n v d_2^{(s+t+2)n} \neq 0$$

again a contradiction. From the above we have proven that $bv = v\alpha_v$ for all $v \in V$, where $\alpha_v \in D$ depends on $v \in V$. In fact, it is easy to check that α_v is independent of the choice of $v \in V$. Indeed, for any $v, w \in V$, by the above arguments, there exist $\alpha_v, \alpha_w, \alpha_{v+w} \in D$ such that $bv = v\alpha_v$; $bw = w\alpha_w$; $b(v+w) = (v+w)\alpha_{v+w}$ and so $v\alpha_v + w\alpha_w = b(v+w) = (v+w)\alpha_{v+w}$. Hence $v(\alpha_v - \alpha_{v+w}) + w(\alpha_w - \alpha_{v+w}) = 0$. If v and w are D-independent, then $\alpha_v = \alpha_{v+w} = \alpha_w$ and we are done. Otherwise, v and w are D-dependent, say $v = \lambda w$ for some $\lambda \in D$. Thus $v\alpha_v = bv = b\lambda w = \lambda bw = \lambda w \alpha_w = v\alpha_w$, that is $V(\alpha_v - \alpha_w) = 0$. Since V is faithful, hence $\alpha_v = \alpha_w$.

So we conclude that there exists $\delta \in D$ such that $bv = v\delta$ for all $v \in V$. We claim that $\delta \in Z(D)$, the center of D. Indeed, for any $\beta \in D$, we have $b(v\beta) = (v\beta)\delta = v(\beta\delta)$ and on the other hand $b(v\beta) = (bv)\beta = (v\delta)\beta = v(\delta\beta)$. Therefore $V(\beta\delta - \delta\beta) = 0$ and hence $\beta\delta = \delta\beta$, which implies that $\delta \in Z(D)$. So $b \in C$, a contradiction. This completes the proof.

Remark. Let R be a prime ring and L a non-central Lie ideal of R. Then either there exists a non-zero ideal I of R such that $0 \neq [I, R] \subseteq L$ or $\mathrm{Char}(R) = 2$ and R satisfies s_4.

Proof. See [14, pp. 4-5], [11, Lemma 2] and [17, Theorem 4].

Theorem 2.2. *Let R be a prime ring with center $Z(R)$, $T : R \longrightarrow R$ be a non-zero partial generalized automorphism of R, L a Lie ideal of R, $s \geq 0, t \geq 0$ and $n \geq 1$ fixed integers, such that $(u^s(T(u) \circ u)u^t)^n = 0$ for all $u \in L$. If either* $\mathrm{Char}(R) > n+1$ *or* $\mathrm{Char}(R) = 0$, *then $L \subseteq Z(R)$.*

Proof. Assume that L is non-central. By previous Remark and since $\mathrm{Char}(R) \neq 2$, there exists a non-zero ideal I of R such that $0 \neq [I, R] \subseteq L$. By Theorem 2.1, it follows that R is commutative, which is a contradiction.

The following example demonstrates that R to be prime is essential in Theorem 2.1.

EXAMPLE 2.3. Let Z be the ring of all integers. Set $R=\left\{\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}|a,b,c\in Z\right\}$ and $I=\left\{\begin{pmatrix}0&a&b\\0&0&0\\0&0&0\end{pmatrix}|a,b\in Z\right\}$. Next, let us define a mapping $T:R\longrightarrow R$ given by $T\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&-a&b\\0&0&-c\\0&0&0\end{pmatrix}$. The fact $\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\neq 0$ implies that $\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}=0$, proving R is not prime. And it is clear that I is a non-zero ideal of R and T is a nonzero partial generalized automorphism of R. And it is easy to check that $(u^s(T(u)\circ u)u^t)^n=0$ for all $u\in[I,I]$. However R is not commutative.

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REFERENCES

[1] K. I. BEIDAR - W. S. MARTINDALE - V. MIKHALEV, *Rings with generalized identities, Monographs and Textbooks in Pure and Applied Mathematics*, 196. Marcel Dekker, Inc., New York, 1996.

[2] M. BREŠAR, *On the distance of the composition of two derivations to be the generalized derivations*, Glasgow Math. J., **33** (1991), pp. 89–93.

[3] C. L. CHUANG, *GPIs having coefficents in Utumi quotient rings*, Proc. Amer. Math. Soc., **103** (3) (1988), pp. 723–728.

[4] C. L. CHUANG, *Differential identities with automorphisms and anti automorphisms*, J. Algebra, **160** (1993), pp. 130–171.

[5] L. CARINI - V. DE FILIPPIS, *Commutators with power central values on a Lie ideal*, Pacific J. Math., **193** (2) (2000), pp. 269–278.

[6] C. M. CHANG - Y. C. LIN, *Derivations on one-sided ideals of prime rings*, Tamsui Oxf. J. Math. Sci., **17** (2) (2001), pp. 139–145.

[7] V. DE FILIPPIS, *Generalized derivations and commutators with nilpotent values on Lie ideals*, Tamsui Oxf. J. Math. Sci., **22** (2006), pp. 167–175.

[8] B. Dhara - V. De Filippis, *Notes on generalized derivations on Lie ideals in prime rings*, Bull. Korean Math. Soc., **46** (3) (2009), pp. 599–605.

[9] B. Dhara - R. K. Sharma, *Derivations with power central values on Lie ideals in prime rings*, Czech. Math. J., **58** (2008), pp. 147–153.

[10] B. Dhara, *Generalized derivations with vanishing power values on Lie ideals* (to appear).

[11] O. M. Di Vincenzo, *On the n-th centralizers of a Lie ideal*, Boll. U. M. I., **3**-A (1989), pp. 77–85.

[12] J. S. Erickson - W. S. Martindale III - J. M. Osborn, *Prime nonassociative algebras*, Pacific J. Math., **60** (1) (1975), pp. 49–63.

[13] I. N. Herstein, *Center-like elements in prime rings*, J. Algebra, **60** (1979), pp. 567–574.

[14] I. N. Herstein, *Topics in ring theory*, Univ. of Chicago Press, Chicago, 1969.

[15] N. Jacobson, *Structure of rings*, Amer. Math. Soc., Providence, RI, 1964.

[16] V. K. Kharchenko, *Generalized identities wtih automorphisms*, Algebra i Logika, **14** (1975), pp. 132–148.

[17] C. Lanski - S. Montgomery, *Lie structure of prime ring of characteristic* 2, Pacific J. Math., **42** (1) (1972), pp. 117–136.

[18] W. S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra, **12** (1969), pp. 176–584.

[19] Y. Wang, *Power-centralizing automorphisms of Lie ideals in prime rings*, Comm. Algebra, **34** (2006), pp. 609–615.

[20] X. Zhang - Y. Wang, *(Right) partial generalized automorphisms of semi-prime rings*, Northeast Math. J., **18** (3) (2002), pp. 261–265.

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