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Ostrowski Type Inequalities Related to the Generalized Baouendi-Grushin Vector Fields

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ABSTRACT - In this paper, we employ a new method to prove a representation formula related to the generalized Baouendi-Grushin vector fields, and then the Ostrowski type inequalities is established in the ball and bounded domain, respectively, via the representation formula and L^∞ norm of the horizontal gradient. In addition, in the same spirit, we show the Hardy inequalities with boundary term related to the generalized Baouendi-Grushin vector fields.

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1. Introduction

In 1938, A. Ostrowski [21] established the following sharp integral inequality

$$\left|\frac{1}{b-a}\int_a^b f(y)dy - f(x)\right| \leq \left(\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}(b-a)\right)\|f'\|_\infty \tag{1.1}$$

for $f \in C^1[a,b]$ and $x \in [a,b]$, which is known as the Ostrowski type inequality. Because the Ostrowski type inequality is useful in some fields, it

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was extended from intervals to rectangles and general domains in $\mathbb{R}^N$ (see [3] and [4]). Moreover, many authors improved inequality (1.1) to the higher-order derivatives case (see [5, 17, 22] and the references therein).

Recently, Lian and Yang [16] obtained the Ostrowski type inequality on the H-type group by the representation formula.

Our main interesting in this paper is to prove the Ostrowski type inequality related to the generalized Baouendi-Grushin (B-G) vector fields by an improved method. To do this, we first introduce the generalized B-G vector fields.

Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, y = (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m$, $\alpha > 0$. The generalized B-G vector fields is defined as

$$Z_i = \frac{\partial}{\partial x_i}, \quad Z_{n+j} = |x|^{\alpha} \frac{\partial}{\partial y_j} \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m). \tag{1.2}$$

Denote the horizontal gradient

$$\nabla_L = (Z_1, \ldots, Z_n, Z_{n+1} \ldots, Z_{n+m}),$$

and $div_L(u_1, u_2, \ldots, u_n, u_{n+1} \ldots, u_{n+m}) = \sum_{i=1}^{n+m} Z_i u_i$ denotes the generalized divergence. Then the second order degenerate elliptic operator and p-degenerate sub-elliptic operator are defined as

$$\mathcal{L}_\alpha = \Delta_x + |x|^{2\alpha} \Delta_y = \sum_{i=1}^{n+m} Z_i = \nabla_L \cdot \nabla_L, \quad and$$

$$\mathcal{L}_{p,\alpha} u = div_L\big(|\nabla_L u|^{p-2} \nabla_L u\big) = \nabla_L\big(|\nabla_L u|^{p-2} \nabla_L u\big), \quad p > 1,$$

respectively, where Δ_x and Δ_y are Laplace operators on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

$\mathcal{L}_\alpha$ is usually called the B-G operator. This second order operator belongs to the wide class of sub-elliptic operators introduced and studied by Franchi and Lanconelli in [13, 15]. When $x \neq 0$, $\mathcal{L}_\alpha$ is elliptic and becomes degenerate on the manifold $\{0\} \times \mathbb{R}^m$. Moreover, For $\alpha = 1$, the B-G operator can be viewed as Tricomi operator for transonic flow restricted to subsonic regions.

In particular, Caffarelli and Silvestre [6] established an equivalent definition of the fractional Laplacian Δ^s, $s \in (0, 1)$, which can be written to a nondivergence form as $\mathcal{L}_\alpha$ with $m = 1$ by the suitable change of variables. Thus it seems conveniently to study fractional Laplacian equation in some sense. Furthermore, a series of results obtained for B-G operator can be carried over to fractional Laplacian. For example, the fundamental solution of equation $\mathcal{L}_\alpha u = 0$ can apply to the fractional Laplacian problem, see [6, 19].

In recent years, the integral inequalities and partial differential equations related to the generalized B-G vector fields have been paid much attention by many scholars (see, e.g., [8, 9, 11, 10, 19, 20]).

To state our main results, we need also some notations. There exists a natural family of anisotropic dilations attached to $\mathcal{L}_\alpha$, i.e.,

$$\delta_\lambda(x,y) = (\lambda x, \lambda^{\alpha+1} y), \qquad \lambda > 0,\ (x,y) \in \mathbb{R}^{n+m}. \tag{1.3}$$

The homogeneous dimension with respect to the dilations (1.3) is $Q = n + (\alpha+1)m$.

The distance function is written as

$$d := d(x,y) = |(x,y)| = (|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2)^{\frac{1}{2(\alpha+1)}}.$$

Take $\xi = (x,y) \in \mathbb{R}^{n+m}$, $|\xi| = d(\xi)$, $x^* = \dfrac{x}{|\xi|}$, $y^* = \dfrac{y}{|\xi|^{\alpha+1}}$, $\xi^* = (x^*, y^*)$, and write $r\xi^* = \delta_r(\xi^*) = \delta_r(x^*, y^*)$. The open ball of radius R and centered at $(0,0) \in \mathbb{R}^{n+m}$ is denoted by $B_L(R) = B_L((0,0),R) = \{(x,y) \in \mathbb{R}^{n+m} | d(x,y) < R\}$. Let $\Sigma := \{(x,y) \in \mathbb{R}^{n+m} \mid d(x,y) = 1\}$ be the B-G unit sphere and $|\Sigma|$ be the volume of the Σ. Obviously, $\xi^* \in \Sigma$. For $f \in C(\overline{B_L(R)})$, let

$$\tilde{f}(r) = \frac{1}{|\Sigma|} \int_\Sigma f(r\xi^*) d\mu(\xi^*)$$

be the avenge of f over the B-G sphere, here $0 < r < R$, $d\mu$ is the surface measure on Σ. Denote $\mathcal{N}(f) = \|f - \tilde{f}\|_\infty = \|f - \tilde{f}\|_{L^\infty(B_L(R))} = \sup\limits_{\xi \in B_L(R)} |f - \tilde{f}(r)|$.

One of the main results in this paper is

THEOREM 1.1. Assume that $f \in C^1(\overline{B_L(R)})$ and $0 < \alpha < n$. Then for $\xi = r\xi^*$, there holds

$$\begin{aligned} \left| f(\xi) - \frac{1}{|B_L(R)|} \int_{B_L(R)} f(\eta) d\eta \right| &\le \mathcal{N}(f) + \frac{\Gamma\left(\frac{n-\alpha}{2(\alpha+1)}\right)\Gamma\left(\frac{Q}{2(\alpha+1)}\right)}{\Gamma\left(\frac{n}{2(\alpha+1)}\right)\Gamma\left(\frac{Q-\alpha}{2(\alpha+1)}\right)} \\ &\quad \times \left(\frac{Q}{Q+1} R - r + \frac{2r^{Q+1}}{(Q+1)R^Q} \right) \|\nabla_L f\|_\infty. \end{aligned} \tag{1.4}$$

It seems that the estimate (1.4) is not sharp since $|\nabla_L d| \neq 1$ with $(x,y) \not\equiv (x,0)$ (see next section).

Suppose that $f(\xi) = f(d(\xi))$ is the radial function. Then, we can improve the result of Theorem 1.1 as follows.

COROLLARY 1.2. Let $f \in C^1(\overline{B_L(R)})$ be a radial function. Then for $\xi = r\xi^*$, it holds

$$(1.5)\quad \left| f(\xi) - \frac{1}{|B_L(R)|} \int\limits_{B_L(R)} f(\eta) d\eta \right| \leq \mathcal{N}(f) + \left(\frac{Q}{Q+1} R - r + \frac{2r^{Q+1}}{(Q+1)R^Q} \right) \|f'\|_\infty .$$

Let $\Omega \subset \mathbb{R}^{n+m}$ be a bounded domain, and let

$$Lip(\Omega) = \{u \in C(\overline{\Omega}) : \; |u(x) - u(y)| \leq K|x - y| \quad \text{for some } K > 0 \text{ and for any } x, y \in \Omega\}.$$

We define

$$C_0^1(\Omega) = \{u \in C^1(\overline{\Omega}), \nabla_L u = 0 \text{ on } \partial\Omega\},$$

and

$$W^{1,\infty}(\Omega) := \{u \in C^1(\overline{\Omega}), \nabla_L u \in Lip(\Omega)\}.$$

If $f \in W^{1,\infty}(\Omega) \cap C_0^1(\Omega)$, we can extend the function f by zero to F on $B_L(R)$, the smallest ball centered at the origin and containing Ω. We state the following theorem.

THEOREM 1.3. Assume that $0 < \alpha < n$ and $f \in W^{1,\infty}(\Omega) \cap C_0^1(\Omega)$. Let $R = \inf\{R_0 > 0 \,|\, \Omega \subset B_L(R_0)\}$. Then for all $\xi = r\xi^* \in \Omega$ and $F \in W^{1,\infty}(B_L(R))$,

$$(1.6)\quad \begin{aligned} &\left| f(\xi) - \frac{1}{|\Omega|} \int\limits_{\Omega} f(\eta) d\eta \right| \leq \mathcal{N}(F) + \left(1 - \frac{|\Omega|}{|B_L(R)|} \right) \left| \frac{1}{|\Omega|} \int\limits_{\Omega} f(\eta) d\eta \right| \\ &+ \frac{\Gamma\left(\frac{n-\alpha}{2(\alpha+1)}\right) \Gamma\left(\frac{Q}{2(\alpha+1)}\right)}{\Gamma\left(\frac{n}{2(\alpha+1)}\right) \Gamma\left(\frac{Q-\alpha}{2(\alpha+1)}\right)} \left(\frac{Q}{Q+1} R - r + \frac{2r^{Q+1}}{(Q+1)R^Q} \right) \|\nabla_L f\|_{L^\infty(\Omega)} . \end{aligned}$$

Note that Liu and Luan [18] established the Ostrowski type inequality associated with Carnot-Carathéodory distance d_{cc} on the Grushin plane $(\alpha = 2, n = m = 1)$. Since $|\nabla_L d_{cc}| = 1$, the estimate (1.4) is sharp for Grushin plane.

A key process in the proof of our results is to establish the representation formula related to the generalized B-G vector fields. We employ a new technique to prove the representation formula which is different from Lian *et al.* [16] and Dong *et al.* [12], bases on the appropriate use of a suitable vector field and elementary calculations.

We also can obtain the following Hardy inequalities with boundary term by the above mentioned technique. It has the advantage that it allows us to compute explicit constants for the remainder term. We point out that our method is different from M. Zhu and Z. Wang [23] and Adimurthi *et al.* [1, 2].

Let $D_0^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{D^{1,p}}= \left(\int\limits_{\Omega} |\nabla_L u|^p d\xi\right)^{\frac{1}{p}}$.

THEOREM 1.4. Let $R > 0, 1<p<\infty$ and $p \neq Q$. Then for all $u \in D^{1,p}(\mathbb{R}^{n+m})$,

$$\begin{aligned}\int\limits_{B_L(R)} |\nabla_L u|^p d\xi - \left|\frac{Q-p}{p}\right|^p \int\limits_{B_L(R)} \psi_{p\alpha}\frac{|u|^p}{d^p}d\xi \\ \geq -\frac{Q-p}{p}\left|\frac{Q-p}{p}\right|^{p-2}\frac{1}{R^{p-1}}\int\limits_{\partial B_L(R)} \psi_{p\alpha}\frac{|u|^p}{|\nabla d|}dH_{Q-1},\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}\int\limits_{B_L^c(R)} |\nabla_L u|^p d\xi - \left|\frac{Q-p}{p}\right|^p \int\limits_{B_L^c(R)} \psi_{p\alpha}\frac{|u|^p}{d^p}d\xi \\ \geq \frac{Q-p}{p}\left|\frac{Q-p}{p}\right|^{p-2}\frac{1}{R^{p-1}}\int\limits_{\partial B_L(R)} \psi_{p\alpha}\frac{|u|^p}{|\nabla d|}dH_{Q-1},\end{aligned} \tag{1.8}$$

where $B_L^c(R) := \mathbb{R}^{n+m}\backslash B_L(R)$, dH_{Q-1} is the $(Q-1)$-dimensional Hausdorff measure and ∇ is usual gradient on $\mathbb{R}^{n+m}$. For further improvement of these inequalities see Sec. 4.

REMARK 1.5. The Hardy inequalities related to the generalized B-G vector fields have been established in space $C_0^\infty(\Omega)$ or $C_0^\infty(\mathbb{R}^{n+m})$, see [8, 9, 11]. In this work, we extend these inequalities to $D^{1,p}(\Omega)$ or $D^{1,p}(\mathbb{R}^{n+m})$.

This paper is organized as follows. In section 2, we give some elementary facts and provide a representation formula related to the generalized B-G vector fields for function without compact support. In section 3, we prove Theorem 1.1 and 1.3. In section 4, we derive a family of Hardy inequalities with boundary term.

2. A representation formula related to the generalized B-G vector fields

Let us note that some essential differences appearing unavoidably. We recall some known facts about the generalized B-G vector fields (see, e.g., [15, 19, 8]).

We need the following concepts. A function $u:\Omega\subset\mathbb{R}^{n+m}\to\mathbb{R}$ is said cylindrical, if $u(x,y)=u(r,s)$ (i.e., u depends only on $r=|x|$ and $s=|y|$), and in particular, if $u(x,y)=u(d(x,y))$, we call u is radial. Let $\Omega=B_L(R_2)\backslash\overline{B_L(R_1)}$ with $0\le R_1\le R_2\le+\infty$, $u\in L^1(\Omega)$. Then, the change of polar coordinates defined in [19, 10]

$$(x_1,\dots,x_n,y_1,\dots,y_m)\to(\rho,\theta,\theta_1,\dots,\theta_{n-1},\gamma_1,\dots,\gamma_{m-1})$$

allows the following formula to hold

$$\begin{aligned}\int_\Omega u(x,y)dxdy&=\int_{R_1}^{R_2}\int_\Sigma u(\delta_\rho(x,y))\rho^{Q-1}d\mu d\rho\\&=s_{n,m}\int_{R_1}^{R_2}\rho^{Q-1}u\left(\rho|\sin\theta|^{\frac{1}{\alpha+1}},\frac{\rho^{\alpha+1}}{\alpha+1}|\cos\,\theta|\right)d\rho,\end{aligned}\tag{2.1}$$

where $s_{n,m}=\left(\frac{1}{\alpha+1}\right)^m\omega_n\omega_m\int_{a_1}^{a_2}|\sin\theta|^{\frac{n}{\alpha+1}-1}|\cos\,\theta|^{m-1}d\theta$, ω_n,ω_m are the Lebesgue measure of the unitary Euclidean sphere in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a_1,a_2 depend on n and m, see [19].

Let us recall some elementary facts about distance function $d=d(x,y)$.

$$\begin{aligned}\nabla_L d=&\frac{|x|^\alpha}{d^{2\alpha+1}}(|x|^\alpha x_1,|x|^\alpha x_2,\dots,|x|^\alpha x_n,(\alpha+1)y_1,\dots,(\alpha+1)y_m),\\|\nabla_L d|^p=&\psi_{p\alpha}=\frac{|x|^{p\alpha}}{d^{p\alpha}}\le1,\quad\mathcal{L}_\alpha d=\psi_{2\alpha}\frac{Q-1}{d},\quad\mathcal{L}_{p,\alpha}d=\frac{Q-1}{d}\psi_{p\alpha},\quad p\ge1.\end{aligned}$$

It is easy to see d and $\nabla_L d$ are homogeneous of degree one and zero with respect to the dilations (1.3), respectively. From the formula of $\nabla_L d$, it holds

$$\begin{aligned}&\nabla_L(d^{2(\alpha+1)})=2(\alpha+1)d^{2\alpha+1}\nabla_L d\\&=2(\alpha+1)|x|^\alpha\big(|x|^\alpha x_1,|x|^\alpha x_2,\cdots,|x|^\alpha x_n,(\alpha+1)y_1,\cdots,(\alpha+1)y_m\big).\end{aligned}\tag{2.2}$$

Noticing $\nabla_L(d^{2(\alpha+1)})=2(\alpha+1)d^{2\alpha+1}\nabla_L d$, it is not difficult to obtain

$$|\nabla_L(d^{2(\alpha+1)})|^2=[2(\alpha+1)d^{2\alpha+1}]^2|\nabla_L d|^2=[2(\alpha+1)]^2|x|^{2\alpha}d^{2(\alpha+1)}.\tag{2.3}$$

Introduce the following two matrix

$$A_\alpha = \begin{pmatrix} I_n & O \\ O & |x|^{2\alpha} I_m \end{pmatrix}, \quad \sigma_\alpha = \begin{pmatrix} I_n & O \\ O & |x|^{\alpha} I_m \end{pmatrix},$$

where I_n and I_m are the n and m order unit matrix, respectively. Thus, we can denote

$$\mathcal{L}_\alpha u = div_L(\nabla u) = div(A_\alpha \nabla u) = div(\sigma_\alpha^T \sigma_\alpha \nabla u), \quad \text{and} \quad \nabla_L = \sigma_\alpha \nabla.$$

Let $\xi = (x, y) \in \mathbb{R}^{n+m}$, $x^* = \dfrac{x}{|\xi|}$, $y^* = \dfrac{y}{|\xi|^{\alpha+1}}$, $\xi^* = (x^*, y^*)$ and $r\xi^* = \delta_r(\xi^*)$. We have $\xi^* \in \Sigma = \{(x, y) \in \mathbb{R}^{n+m} \mid d(x, y) = 1\}$. Arguing as in [14], we can obtain the following result from the polar coordinates and dilations (1.3).

LEMMA 2.1. If $f \in L^1(\mathbb{R}^{n+m})$, then

$$\int_{\mathbb{R}^{n+m}} f(\xi) d\xi = \int_0^\infty \int_\Sigma f(\rho\xi^*) \rho^{Q-1} d\mu(\xi^*) d\rho. \tag{2.4}$$

On the other hand, by Federer's coarea formula

$$\int_{\mathbb{R}^N} f(x) dx = \int_{-\infty}^{\infty} ds \int_{g=s} \frac{f(x)}{|\nabla g(x)|} dH_{N-1},$$

where $g \in Lip(\mathbb{R}^N)$, dH_{N-1} denotes the $(N-1)$-dimensional Hausdorff measure and ∇ is usual gradient. It follows

$$\int_{B_L(R)} f(\xi) d\xi = \int_0^R d\rho \int_{\partial B_L(\rho)} \frac{f(\xi)}{|\nabla d|} dH_{Q-1} = \int_0^R d\rho \int_\Sigma \frac{f(\xi)}{|\nabla d|} \rho^{Q-1} dH_{Q-1} \circ \delta_\rho(\xi). \tag{2.5}$$

Combining (2.4) with (2.5), it yields

$$\int_\Sigma f(\rho\xi^*) d\mu = \int_\Sigma \frac{f(\xi)}{|\nabla d|} dH_{Q-1} \circ \delta_\rho(\xi) = \int_{\partial B_L(\rho)} \frac{f(\xi)}{d^{Q-1} |\nabla d|} dH_{Q-1}. \tag{2.6}$$

Using the polar coordinates and (2.1), as in [7], we have

LEMMA 2.2. *Suppose that* $\gamma > -n$, *and*

$$C_\gamma = \int_\Sigma |x^*|^\gamma d\mu.$$

Then,

$$C_\gamma = \left(\frac{1}{\alpha+1}\right)^m \frac{4\pi^{\frac{n+m}{2}}\Gamma\left(\frac{n+\gamma}{2(\alpha+1)}\right)}{\Gamma(\frac{n}{2})\Gamma\left(\frac{Q+\gamma}{2(\alpha+1)}\right)}.$$

In particular,

$$|\Sigma| = C_0 = \int_\Sigma d\mu = \left(\frac{1}{\alpha+1}\right)^m \frac{4\pi^{\frac{n+m}{2}}\Gamma\left(\frac{n}{2(\alpha+1)}\right)}{\Gamma(\frac{n}{2})\Gamma\left(\frac{Q}{2(\alpha+1)}\right)}.$$

PROOF. For $\gamma > -Q$, it follows from (2.4) that

$$\begin{aligned}
C_\gamma = \int_\Sigma |x^*|^\gamma d\mu = (Q+\gamma)\int_0^1 r^{Q+\gamma-1}dr \int_\Sigma |x^*|^\gamma d\mu \\
=&(Q+\gamma)\int_0^1\int_\Sigma r^{Q-1}|rx^*|^\gamma d\mu dr \\
=&(Q+\gamma)\int_{B_L(1)} |x|^\gamma dxdy.
\end{aligned}$$

Now, invoking (2.1), as in [19], it gets for $\gamma > -n$

$$\begin{aligned}
C_\gamma =& (Q+\gamma)\int_{B_L(1)} |x|^\gamma dxdy \\
=& (Q+\gamma)\left(\frac{1}{\alpha+1}\right)^m \omega_n\omega_m \int_0^1 \rho^{Q+\gamma-1}d\rho \int_{a_1}^{a_2} |\sin\theta|^{\frac{n+\gamma}{\alpha+1}-1}|\cos\,\theta|^{m-1}d\theta \\
=& \left(\frac{1}{\alpha+1}\right)^m \omega_n\omega_m \int_{a_1}^{a_2} |\sin\theta|^{\frac{n+\gamma}{\alpha+1}-1}|\cos\,\theta|^{m-1}d\theta \\
=& \left(\frac{1}{\alpha+1}\right)^m \frac{4\pi^{\frac{n+m}{2}}\Gamma\left(\frac{n+\gamma}{2(\alpha+1)}\right)}{\Gamma(\frac{n}{2})\Gamma\left(\frac{Q+\gamma}{2(\alpha+1)}\right)}.
\end{aligned}$$

The proof of Lemma 2.2 is finished.

LEMMA 2.3. *Assume* $0<R_1<R_2<\infty$ *and* $f\in C^1(\overline{B_L(R_2)\backslash B_L(R_1)})$. *Then*

$$\int_\Sigma f(R_2\xi^*)d\mu-\int_\Sigma f(R_1\xi^*)d\mu = \int_{B_L(R_2)\backslash B_L(R_1)} \frac{1}{d^Q(\xi)|x|^{2\alpha}}\left\langle \nabla_L f, \nabla_L\left(\frac{d^{2(\alpha+1)}(\xi)}{2(\alpha+1)}\right)\right\rangle d\xi. \tag{2.7}$$

PROOF. Let $\boldsymbol{T}$ be a C^1 vector field on $U:=B_L(R_2)\backslash B_L(R_1)$ and be specified later. For any $f\in C^1(U)$, integrating by parts, we have

$$-\int_U div_L\boldsymbol{T}fd\xi=-\int_U div\sigma_\alpha^T\boldsymbol{T}fd\xi=\int_U\langle\boldsymbol{T},\nabla_L f\rangle d\xi-\int_{\partial U}f\sigma_\alpha^T\langle\boldsymbol{T},\boldsymbol{n}\rangle dH_{Q-1}, \tag{2.8}$$

where $\boldsymbol{n}$ is the outer unit normal vector to the boundary.

Choose now $\boldsymbol{T}=\frac{1}{d^Q(\xi)|x|^{2\alpha}}\nabla_L\left(\frac{d^{2(\alpha+1)(\xi)}}{2(\alpha+1)}\right)$. From (2.2), it is easy to see $\boldsymbol{T}=\frac{\nabla_L d}{d^{Q-2\alpha-1}|x|^{2\alpha}}=\frac{\sigma_\alpha\nabla d}{d^{Q-2\alpha-1}|x|^{2\alpha}}$. We compute

$$\begin{aligned}
div\sigma_\alpha^T\boldsymbol{T} &= div\,\sigma_\alpha^T\left(\frac{\nabla_L d}{d^{Q-2\alpha-1}|x|^{2\alpha}}\right)\\
&=\frac{\mathcal{L}_\alpha d}{d^{Q-2\alpha-1}|x|^{2\alpha}}+\langle\nabla_L d,\nabla_L(d^{2\alpha+1-Q}|x|^{-2\alpha})\rangle\\
&=\frac{|x|^{2\alpha}}{d^{2\alpha}}\frac{Q-1}{dd^{Q-2\alpha-1}|x|^{2\alpha}}+(2\alpha+1-Q)|x|^{-2\alpha}d^{2\alpha-Q}\langle\nabla_L d,\nabla_L d\rangle\\
&\quad+d^{2\alpha+1-Q}\langle\nabla_L d,\nabla_L(|x|^{-2\alpha})\rangle\\
&=\frac{Q-1}{d^Q}+\frac{2\alpha+1-Q}{d^Q}-2\alpha\frac{d^{2\alpha+1-Q}|x|^\alpha}{d^{2\alpha+1}}|x|^{-2\alpha-2}|x|^{\alpha+2}\\
&=\frac{2\alpha}{d^Q}-\frac{2\alpha}{d^Q}=0.
\end{aligned} \tag{2.9}$$

Note that $\boldsymbol{n}=\frac{\nabla d}{|\nabla d|}$ and $|\nabla_L d|^2=A_\alpha\langle\nabla d,\nabla d\rangle$. Combining (2.8) and (2.9), we obtain

$$
\begin{aligned}
\int\limits_U \frac{1}{d^Q(\xi)|x|^{2\alpha}}\left\langle \nabla_L f, \nabla_L\left(\frac{d^{2(\alpha+1)}(\xi)}{2(\alpha+1)}\right)\right\rangle d\xi &= \int\limits_{\partial U} f\sigma_\alpha^T\left\langle \frac{\nabla_L d}{d^{Q-2\alpha-1}|x|^{2\alpha}}, \boldsymbol{n}\right\rangle dH_{Q-1}\\
&= \int\limits_{\partial U} f\sigma_\alpha^T\left\langle \frac{\nabla_L d}{d^{Q-2\alpha-1}|x|^{2\alpha}}, \frac{\nabla d}{|\nabla d|}\right\rangle dH_{Q-1}\\
&= \int\limits_{\partial U} \frac{f}{|\nabla d|}\,\frac{|\nabla_L d|^2}{d^{Q-2\alpha-1}|x|^{2\alpha}} dH_{Q-1}\\
&= \int\limits_{\partial B_L(R_2)} \frac{f}{d^{Q-1}|\nabla d|} dH_{Q-1} - \int\limits_{\partial B_L(R_1)} \frac{f}{d^{Q-1}|\nabla d|} dH_{Q-1}.
\end{aligned}
\tag{2.10}
$$

Invoking (2.6), we deduce the formula (2.7). Hence, this completes the proof of Lemma 2.3. □

3. Ostrowski type inequalities

In this section, we prove Theorem 1.1 and 1.3. To begin with, we need the following estimates.

LEMMA 3.1. *Let* $0<R_1<R_2<\infty, 0<\alpha<n$ *and* $f \in C^1(\overline{B_L(R_2)\backslash B_L(R_1)})$. *Then*

$$
\begin{aligned}
\left|\int\limits_\Sigma f(R_2\xi^*)d\mu - \int\limits_\Sigma f(R_1\xi^*)d\mu\right| &\le \int\limits_{B_L(R_2)\backslash B_L(R_1)} \frac{1}{d^{Q-(\alpha+1)}|x|^\alpha}|\nabla_L f|d\xi\\
&\le C_{-\alpha}|R_2-R_1|\,\|\nabla_L f\|_\infty.
\end{aligned}
\tag{3.1}
$$

PROOF. Using the pointwise Schwartz inequality, it follows from Lemma 2.3 and (2.3) that

$$
\begin{aligned}
\left|\int\limits_\Sigma f(R_2\xi^*)d\mu - \int\limits_\Sigma f(R_1\xi^*)d\mu\right| &\le \int\limits_{B_L(R_2)\backslash B_L(R_1)} \frac{1}{d^Q|x|^{2\alpha}}|\nabla_L f|\left|\nabla_L\left(\frac{d^{2(\alpha+1)}}{2(\alpha+1)}\right)\right|d\xi\\
&\le \int\limits_{B_L(R_2)\backslash B_L(R_1)} \frac{1}{d^{Q-(\alpha+1)}|x|^\alpha}|\nabla_L f|d\xi\\
&\le \|\nabla_L f\|_\infty\left|\int\limits_{B_L(R_2)\backslash B_L(R_1)} \frac{1}{d^{Q-(\alpha+1)}|x|^\alpha}d\xi\right|.
\end{aligned}
$$

Therefore, invoking Lemma 2.2, we get for $0<\alpha<n$

$$\begin{aligned}\left|\int_\Sigma f(R_2\xi^*)d\mu-\int_\Sigma f(R_1\xi^*)d\mu\right| &\le \|\nabla_L f\|_\infty\left|\int_{B_L(R_2)\setminus B_L(R_1)}\frac{1}{d^{Q-1}|x^*|^\alpha}d\xi\right|\\ &=\|\nabla_L f\|_\infty\left|\int_{R_1}^{R_2}dr\int_\Sigma |x^*|^{-\alpha}d\mu\right|\\ &=C_{-\alpha}|R_2-R_1|\|\nabla_L f\|_\infty.\end{aligned}$$

The lemma is proved. □

Proof of Theorem 1.1. Since

$$|B_L(R)|=|\Sigma|\frac{R^Q}{Q},$$

it follows from this and Lemma 2.1 that

$$\begin{aligned}\left|f(\xi)-\frac{1}{|B_L(R)|}\int_{B_L(R)}f(\eta)d\eta\right| &\le |f-\tilde f(r)|\\ &+\left|\frac{1}{|\Sigma|}\int_\Sigma f(r\xi^*)d\mu-\frac{Q}{R^Q|\Sigma|}\int_0^R\int_\Sigma f(\rho\xi^*)\rho^{Q-1}d\mu d\rho\right| \qquad (3.2)\\ &=:\mathcal{N}(f)+I.\end{aligned}$$

By Lemma 3.1 and 2.2, we have

$$\begin{aligned}I&=\left|\frac{1}{|\Sigma|}\int_\Sigma f(r\xi^*)d\mu-\frac{Q}{R^Q|\Sigma|}\int_0^R\int_\Sigma f(\rho\xi^*)\rho^{Q-1}d\mu d\rho\right|\\ &\le\frac{Q}{R^Q|\Sigma|}\left|\int_0^R\int_\Sigma f(r\xi^*)\rho^{Q-1}d\mu d\rho-\int_0^R\int_\Sigma f(\rho\xi^*)\rho^{Q-1}d\mu d\rho\right|\\ &\le\frac{QC_{-\alpha}}{R^Q|\Sigma|}\int_0^R|r-\rho|\rho^{Q-1}d\rho\|\nabla_L f\|_\infty \qquad (3.3)\\ &=\frac{QC_{-\alpha}}{R^Q|\Sigma|}\left(\frac{R^{Q+1}}{Q+1}-\frac{rR^Q}{Q}+\frac{2r^{Q+1}}{(Q+1)Q}\right)\|\nabla_L f\|_\infty\\ &=\frac{C_{-\alpha}}{|\Sigma|}\left(\frac{Q}{Q+1}R-r+\frac{2r^{Q+1}}{(Q+1)R^Q}\right)\|\nabla_L f\|_\infty\\ &=\frac{\Gamma\left(\frac{n-\alpha}{2(\alpha+1)}\right)\Gamma\left(\frac{Q}{2(\alpha+1)}\right)}{\Gamma\left(\frac{n}{2(\alpha+1)}\right)\Gamma\left(\frac{Q-\alpha}{2(\alpha+1)}\right)}\left(\frac{Q}{Q+1}R-r+\frac{2r^{Q+1}}{(Q+1)R^Q}\right)\|\nabla_L f\|_\infty.\end{aligned}$$

Combining (3.3) and (3.2), we obtain the conclusion. □

Proof of Corollary 1.2. Let $f \in C^1(\overline{B_L(R)})$ be a radial function. From Lemma 2.3 and 2.2, the following estimate holds

$$\begin{aligned}
\left| \int_{\Sigma} f(R_2\xi^*)d\mu - \int_{\Sigma} f(R_1\xi^*)d\mu \right| &\le \int_{B_L(R_2)\backslash B_L(R_1)} \frac{1}{d^Q|x|^{2\alpha}} |\nabla_L f||\nabla_L(\frac{d^{2(\alpha+1)}}{2(\alpha+1)})|d\xi \\
&\le \int_{B_L(R_2)\backslash B_L(R_1)} \frac{1}{d^{Q-(\alpha+1)}|x|^{\alpha}} |f'||\nabla_L d|d\xi \\
&\le \|f'\|_\infty \left| \int_{B_L(R_2)\backslash B_L(R_1)} \frac{1}{d^{Q-1}} d\xi \right| \\
&= C_0|R_2 - R_1|\|f'\|_\infty .
\end{aligned}$$

Similar to (3.3), we can get

$$\begin{aligned}
I =& \left| \frac{1}{|\Sigma|} \int_{\Sigma} f(r\xi^*)d\mu - \frac{Q}{R^Q|\Sigma|} \int_0^R \int_{\Sigma} f(\rho\xi^*)\rho^{Q-1}d\mu d\rho \right| \\
\le& \frac{C_0}{|\Sigma|} \left(\frac{Q}{Q+1}R - r + \frac{2r^{Q+1}}{(Q+1)R^Q} \right) \|f'\|_\infty \\
=& \left(\frac{Q}{Q+1}R - r + \frac{2r^{Q+1}}{(Q+1)R^Q} \right) \|f'\|_\infty .
\end{aligned}$$

Hence, we obtain (1.5) from the proof of Theorem 1.1. □

Proof of Theorem 1.3. Assume that $R = \inf\{R_0 > 0 \,|\, \Omega \subset B_L(R_0)\}$. Then

$$F(\xi) := \begin{cases} f(\xi), & \text{if } \xi \in \overline{\Omega}, \\ 0, & \text{if } \xi \in \overline{B_L(R)}\backslash\Omega, \end{cases}$$

and satisfies $F \in W^{1,\infty}(B_L(R)) \cap C_0^1(B_L(R))$. We can split $\left| f(\xi) - \frac{1}{|\Omega|}\int_\Omega f(\eta)d\eta \right|$ as

$$\begin{aligned}
&\left| f(\xi) - \frac{1}{|\Omega|} \int_\Omega f(\eta)d\eta \right| \\
&\le \left| F(\xi) - \frac{1}{|B_L(R)|} \int_{B_L(R)} F(\eta)d\eta \right| + \left| \frac{1}{|B_L(R)|} \int_{B_L(R)} F(\eta)d\eta - \frac{1}{|\Omega|} \int_\Omega f(\eta)d\eta \right| \qquad (3.4) \\
&=: I_1 + I_2 .
\end{aligned}$$

By the definition of R, we can obtain

$$
\begin{aligned}
I_2 &= \left| \frac{1}{|B_L(R)|} \int\limits_{B_L(R)} F(\eta)d\eta - \frac{1}{|\Omega|} \int\limits_{\Omega} f(\eta)d\eta \right| \\
&= \left| \frac{1}{|B_L(R)|} \int\limits_{\Omega} f(\eta)d\eta - \frac{1}{|\Omega|} \int\limits_{\Omega} f(\eta)d\eta \right| \\
&= \left| \frac{1}{|B_L(R)|} - \frac{1}{|\Omega|} \right| \left| \int\limits_{\Omega} f(\eta)d\eta \right| \\
&= \left(1 - \frac{|\Omega|}{|B_L(R)|} \right) \frac{1}{|\Omega|} \left| \int\limits_{\Omega} f(\eta)d\eta \right|.
\end{aligned} \tag{3.5}
$$

Further, replacing $B_L(R)$ by Ω in Theorem 1.1, it follows

$$
\begin{aligned}
I_1 &= \left| F(\xi) - \frac{1}{|B_L(R)|} \int\limits_{B_L(R)} F(\eta)d\eta \right| \\
&\le \|F - \tilde{F}\|_\infty + \frac{\Gamma\left(\frac{n-\alpha}{2(\alpha+1)}\right)\Gamma\left(\frac{Q}{2(\alpha+1)}\right)}{\Gamma\left(\frac{n}{2(\alpha+1)}\right)\Gamma\left(\frac{Q-\alpha}{2(\alpha+1)}\right)} \left(\frac{Q}{Q+1} R - r + \frac{2r^{Q+1}}{(Q+1)R^Q} \right) \|\nabla_L f\|_{L^\infty(\Omega)}.
\end{aligned} \tag{3.6}
$$

Substituting (3.5) and (3.6) into (3.4), we conclude (1.6). Theorem 1.3 is established. □

4. Hardy inequalities with boundary term

In this section, we present some Hardy inequalities with boundary term which improves the Hardy inequality established in [8, 9, 11]. Here we point out that our interests is to establish inequalities, the best constants of inequalities are not discussed (due to the optimality of the constants in the inequalities is of its independent interest).

Proof of Theorem 1.4. Let $\boldsymbol{T}$ be a C^1 vector field on $B_L(R)$ and let it be specified later. For any $u \in D^{1,p}(\mathbb{R}^{n+m})$, integrating by parts, we have

$$
\int\limits_{B_L(R)} div_L \boldsymbol{T} |u|^p d\xi = -p \int\limits_{B_L(R)} \langle \boldsymbol{T}, |u|^{p-2} u \nabla_L u \rangle d\xi + \int\limits_{\partial B_L(R)} |u|^p \sigma_\alpha^T \langle \boldsymbol{T}, \boldsymbol{n} \rangle dH_{Q-1}. \tag{4.1}
$$

Using Hölder's inequality and Young's inequality we obtain

$$
\begin{aligned}
(4.2)\qquad -p\int_{\Omega}\langle \boldsymbol{T},\nabla_L u\rangle |u|^{p-2}u\,d\xi &\le p\Big(\int_{B_L(R)} |\nabla_L u|^p d\xi\Big)^{\frac{1}{p}}\Big(\int_{B_L(R)} |\boldsymbol{T}|^{\frac{p}{p-1}}|u|^p d\xi\Big)^{\frac{p-1}{p}}\\
&\le \int_{B_L(R)} |\nabla_L u|^p d\xi + (p-1)\int_{B_L(R)} |\boldsymbol{T}|^{\frac{p}{p-1}}|u|^p d\xi.
\end{aligned}
$$

Thereby, substituting (4.2) into (4.1), it follows

$$
(4.3)\qquad \int_{B_L(R)}\big[div_L\boldsymbol{T}-(p-1)|\boldsymbol{T}|^{\frac{p}{p-1}}\big]|u|^p d\xi \le \int_{B_L(R)} |\nabla_L u|^p d\xi + \int_{\partial B_L(R)} |u|^p \sigma_\alpha^T \langle \boldsymbol{T},\boldsymbol{n}\rangle dH_{Q-1}.
$$

Choose now $\boldsymbol{T} = A|A|^{p-2}\dfrac{|\nabla_L d|^{p-2}\nabla_L d}{d^{p-1}}$, where $A = \dfrac{Q-p}{p}$ for $p \neq Q$. A direct computation gets

$$
div_L\boldsymbol{T} - (p-1)|\boldsymbol{T}|^{\frac{p}{p-1}}|u|^p = |A|^{p-1}\frac{|\nabla_L d|^p}{d^p},
$$

and note that $\boldsymbol{n} = \dfrac{\nabla d}{|\nabla d|}$. Combining these with (4.3) yields

$$
\begin{aligned}
&|A|^p \int_{B_L(R)} \psi_{p\alpha}\frac{|u|^p}{d^p}d\xi\\
&\le \int_{B_L(R)} |\nabla_L u|^p d\xi + A|A|^{p-2}\frac{1}{R^{p-1}}\int_{\partial B_L(R)} \psi_{p\alpha}\frac{|u|^p}{|\nabla d|}dH_{Q-1}.
\end{aligned}
$$

Thus, inequality (1.7) is proved.

A similar argument can prove inequality (1.8), so we omit the details here. □

In the same spirit with Theorem 1.4, we can obtain the following Hardy inequality with boundary term on bound domain.

THEOREM 4.1. *Let $1<p<Q$. If $\Omega \subset \mathbb{R}^{n+m}$ is a smooth domain with $\partial\Omega$ being bounded and $0 \notin \partial\Omega$, then there exists $R_0 = \inf\{R>0 \mid \Omega \subset B_L(R)\}$ such that for all $u \in D^{1,p}(\mathbb{R}^{n+m})$*

$$\int_{\Omega} |\nabla_L u|^p d\xi - \left|\frac{Q-p}{p}\right|^p \int_{\Omega} \psi_{p\alpha} \frac{|u|^p}{d^p} d\xi \tag{4.4}$$
$$\geq -\left|\frac{Q-p}{p}\right|^{p-1} \frac{1}{R_0^{p-1}} \int_{\partial\Omega} \psi_{p\alpha} \frac{|u|^p}{|\nabla d|} dH_{Q-1},$$

In addition, if $\Omega \subset \mathbb{R}^{n+m}$ is a bounded and star-sharped with respect to the origin, there exists $R_1 = \inf\{R > 0 \,|\, \Omega \subset B_L(R)\}$ such that for all $u \in D^{1,p}(\mathbb{R}^{n+m})$

$$\int_{\Omega^c} |\nabla_L u|^p d\xi - \left|\frac{Q-p}{p}\right|^p \int_{\Omega^c} \psi_{p\alpha} \frac{|u|^p}{d^p} d\xi \tag{4.5}$$
$$\geq \left|\frac{Q-p}{p}\right|^{p-1} \frac{1}{R_1^{p-1}} \int_{\partial\Omega} \psi_{p\alpha} \frac{|u|^p}{|\nabla d|} dH_{Q-1},$$

where $\Omega^c := \mathbb{R}^{n+m} \backslash \Omega$.

In addition, J. Dou, Q. Guo and P. Niu [11] had established the following Hardy inequalities with remainder term.

Proposition 4.2. *Let $0 \in \Omega$ be a bounded domain in $\mathbb{R}^{n+m}$ and $1<p<\infty$.*

(1). If $p \neq Q$, then there exists a positive constant $R_0 \geq \sup_{\xi\in\Omega} d(\xi)$ such that for any $R > R_0$ and all $u \in D_0^{1,p}(\Omega\backslash\{0\})$,

$$\int_{\Omega} |\nabla_L u|^p d\xi \geq \left|\frac{Q-p}{p}\right|^p \int_{\Omega} \psi_{p\alpha} \frac{|u|^p}{d^p} d\xi + \frac{p-1}{2p}\left|\frac{Q-p}{p}\right|^{p-2} \int_{\Omega} \psi_{p\alpha} \frac{|u|^p}{d^p} \left(\ln\left(\frac{R}{d}\right)\right)^{-2} d\xi. \tag{4.6}$$

In particular, if $2 \leq p<Q$, one can take $\sup_{\xi\in\Omega} d(\xi) = R_0$.

(2). If $p = Q$, then there exists $R > \sup_{\xi\in\Omega} d(\xi)$ such that for all $u \in D_0^{1,p}(\Omega\backslash\{0\})$

$$\int_{\Omega} |\nabla_L u|^p d\xi \geq \left(\frac{p-1}{p}\right)^p \int_{\Omega} \psi_{p\alpha} \frac{|u|^p}{(d \ln\left(\frac{R}{d}\right))^p} d\xi. \tag{4.7}$$

We note that a simple density argument can show that $D_0^{1,p}(\Omega\backslash\{0\}) = D_0^{1,p}(\Omega)$. Hence, inequalities (4.6) and (4.7) can be extended to the space $D^{1,p}(\Omega)$.

THEOREM 4.3. *Let $0 \in \Omega$ be a bounded domain in $\mathbb{R}^{n+m}$ with smooth boundary and $1<p<\infty$.*

(1). *If $p \neq Q$, then there exists a small positive constant M_0 such that for any $R > R_0 := e^{\frac{1}{M_0}} \sup_{\xi\in\Omega} d(\xi)$ and all $u \in D^{1,p}(\Omega)$,*

$$\begin{aligned}
&\int_\Omega |\nabla_L u|^p d\xi - \left|\frac{Q-p}{p}\right|^p \int_\Omega \psi_{p\alpha} \frac{|u|^p}{d^p} d\xi \\
&-\frac{p-1}{2p}\left|\frac{Q-p}{p}\right|^{p-2} \int_\Omega \psi_{p\alpha} \frac{|u|^p}{d^p}\left(\ln\left(\frac{R}{d}\right)\right)^{-2} d\xi \qquad (4.8)\\
&\geq C_{p,Q,R_0} \int_{\partial\Omega} \psi_{p\alpha} \frac{|u|^p}{|\nabla d|} dH_{Q-1},
\end{aligned}$$

where

$$C_{p,Q,R_0} = \\
= \begin{cases}
\dfrac{p-Q}{p}\left|\dfrac{Q-p}{p}\right|^{p-2} \dfrac{\left(1+\frac{p-1}{Q-p}M_0+aM_0^2\right)}{R_0^{p-1}} & \text{for } 1<p<2, 0<a<\dfrac{(2-p)(p-1)}{6p^2A^2}, \\
\dfrac{p-Q}{p}\left|\dfrac{Q-p}{p}\right|^{p-2} \dfrac{\left(1+\frac{p-1}{Q-p}M_0\right)}{R_0^{p-1}} & \text{for } 2 \leq p<Q, \\
\dfrac{Q-p}{p}\left|\dfrac{Q-p}{p}\right|^{p-2} \dfrac{\left(\frac{p-1}{Q-p}M_0+aM_0^2\right)}{R_0^{p-1}} & \text{for } p>Q, 0>a>\dfrac{(2-p)(p-1)}{6p^2A^2}.
\end{cases}$$

(2). *If $p = Q$, then there exists $R > \sup_{\xi\in\Omega} d(\xi)$ and the definition of M_0 as above such that for all $u \in D^{1,p}(\Omega)$*

$$\begin{aligned}
(4.9)\quad &\int_\Omega |\nabla_L u|^p d\xi - \left(\frac{p-1}{p}\right)^p \int_\Omega \psi_{p\alpha} \frac{|u|^p}{\left(d\ln\left(\frac{R}{d}\right)\right)^p} d\xi \\
&\geq -\left(\frac{p-1}{p}\right)^{p-1} \frac{M_0^{p-1}}{R_0^{p-1}} \int_{\partial\Omega} \psi_{p\alpha} \frac{|u|^p}{|\nabla d|} dH_{Q-1}.
\end{aligned}$$

PROOF. Let $\mathcal{B}(s) = -\dfrac{1}{\ln(s)}$, $s \in (0,1)$ and $A = \dfrac{Q-p}{p}$. For $R > \sup_{\xi\in\Omega} d(\xi)$, there exists a constant $M > 0$ such that

$$0 \le \mathcal{B}\left(\frac{d(\xi)}{R}\right) \le M, \quad \xi \in \Omega.$$

Furthermore,

$$\nabla_L \mathcal{B}^\gamma\left(\frac{d}{R}\right) = \gamma \frac{\mathcal{B}^{\gamma+1}\left(\frac{d}{R}\right)\nabla_L d}{d}, \quad \frac{d\mathcal{B}^\gamma\left(\frac{\rho}{R}\right)}{d\rho} = \gamma \frac{\mathcal{B}^{\gamma+1}\left(\frac{\rho}{R}\right)}{\rho}, \quad \text{for all } \gamma \in \mathbb{R},$$

and

$$\int_a^b \frac{\mathcal{B}^{\gamma+1}(s)}{s} ds = \frac{1}{\gamma}\left[\mathcal{B}^\gamma(b) - \mathcal{B}^\gamma(a)\right].$$

(1) For $p \ne Q$. Let $\boldsymbol{T}$ be a C^1 vector field on Ω and let it be specified later. For any $u \in D^{1,p}(\Omega)$, similar to (4.3), it holds

$$(4.10)\quad \int_\Omega [div_L \boldsymbol{T} - (p-1)|\boldsymbol{T}|^{\frac{p}{p-1}}]|u|^p d\xi \le \int_\Omega |\nabla_L u|^p d\xi + \int_{\partial\Omega} |u|^p \sigma_\alpha^T \langle \boldsymbol{T}, \boldsymbol{n}\rangle dH_{Q-1}.$$

We choose

$$\boldsymbol{T}(d) = A|A|^{p-2}\frac{|\nabla_L d|^{p-2}\nabla_L d}{d^{p-1}}\left(1 + \frac{p-1}{pA}\mathcal{B}\left(\frac{d}{R}\right) + a\mathcal{B}^2\left(\frac{d}{R}\right)\right),$$

where a is a parameter to be chosen later.

As in [11], take

(*i*) $a > \dfrac{(2-p)(p-1)}{6p^2A^2} > 0$ for $1 < p < 2 < Q$,

(*ii*) $a = 0$ for $2 \le p < Q$,

(*iii*) $a < \dfrac{(2-p)(p-1)}{6p^2A^2} < 0$ for $p > Q$.

Thus, it is not difficult to choose M_0(small enough) in all cases such that for $0 < \mathcal{B} \le M_0$, and then

$$(4.11)\quad div_L \boldsymbol{T} - (p-1)|\boldsymbol{T}|^{\frac{p}{p-1}} \ge |A|^p \frac{|\nabla_L d|^p}{d^p}\left(1 + \frac{p-1}{2pA^2}\mathcal{B}^2\left(\frac{d}{R}\right)\right).$$

Noting that the condition $\mathcal{B} \le M_0$ is equivalent to $R \ge R_0 := e^{\frac{1}{M_0}} \sup_{\xi\in\Omega} d(\xi)$ and $\boldsymbol{n} = \dfrac{\nabla d}{|\nabla d|}$, if $p < Q$, then

$$\begin{aligned}
&\int_{\partial\Omega} |u|^p \sigma_\alpha^T \langle \boldsymbol{T}, \boldsymbol{n}\rangle dH_{Q-1} \\
&= \int_{\partial\Omega} A|A|^{p-2} \frac{|\nabla_L d|^p}{d^{p-1}} \left(1 + \frac{p-1}{pA} \mathcal{B}\left(\frac{d}{R}\right) + a\mathcal{B}^2\left(\frac{d}{R}\right)\right) \frac{|u|^p}{|\nabla d|} dH_{Q-1} \\
&\leq A|A|^{p-2} \frac{\left(1 + \frac{p-1}{pA} M_0 + aM_0^2\right)}{R_0^{p-1}} \int_{\partial\Omega} \psi_{p\alpha} \frac{|u|^p}{|\nabla d|} dH_{Q-1},
\end{aligned} \tag{4.12}$$

and substituting (4.11) and (4.12) into (4.10), we conclude inequality (4.8). If $p > Q$, we have

$$\int_{\partial\Omega} |u|^p \sigma_\alpha^T \langle \boldsymbol{T}, \boldsymbol{n}\rangle dH_{Q-1} \leq A|A|^{p-2} \frac{\left(\frac{p-1}{pA} M_0 + aM_0^2\right)}{R_0^{p-1}} \int_{\partial\Omega} \psi_{p\alpha} \frac{|u|^p}{|\nabla d|} dH_{Q-1}, \tag{4.13}$$

where the facts $A < 0, a < 0$ is used.

(2) For $p = Q$. Choose $\boldsymbol{T}(d) = \left(\frac{p-1}{p}\right)^{p-1} \frac{|\nabla_L d|^{p-2} \nabla_L d}{d^{p-1}} \mathcal{B}^{p-1}\left(\frac{d}{R}\right)$, as in [11], we can obtain

$$div_L \boldsymbol{T} - (p-1)|\boldsymbol{T}|^{\frac{p}{p-1}} = \left(\frac{p-1}{p}\right)^p \mathcal{B}^p\left(\frac{d}{R}\right) \frac{|\nabla_L d|^p}{d^p}. \tag{4.14}$$

Notice

$$\begin{aligned}
\int_{\partial\Omega} |u|^p \sigma_\alpha^T \langle \boldsymbol{T}, \boldsymbol{n}\rangle dH_{Q-1} &= \int_{\partial\Omega} \left(\frac{p-1}{p}\right)^{p-1} \frac{|\nabla_L d|^p}{d^{p-1}} \mathcal{B}^{p-1}\left(\frac{d}{R}\right) \frac{|u|^p}{|\nabla d|} dH_{Q-1} \\
&\leq \left(\frac{p-1}{p}\right)^{p-1} \frac{M_0^{p-1}}{R_0^{p-1}} \int_{\partial\Omega} \psi_{p\alpha} \frac{|u|^p}{|\nabla d|} dH_{Q-1}.
\end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15) yields inequality (4.9). □

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