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SE-Supplemented Subgroups of Finite Groups

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ABSTRACT - Let G be a finite group, H a subgroup of G and H_{seG} be the subgroup generated by all subgroups of H which are S-quasinormally embedded in G. Then we say that H is SE-supplemented in G if G has a subgroup T such that $HT = G$ and $H \cap T \leq H_{seG}$. We investigate the influence of SE-quasinormally embedded of some subgroups on the structure of finite groups. Our results improve and extend a series of recent results.

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KEYWORDS. S-quasinormal subgroup, S-quasinormally embedded subgroup, FE-supplemented subgroup, formation.

1. Introduction

Throughout this paper, all groups are finite and G denotes a finite group.

An interesting question in theory of finite groups is to determine the influence of the embedding properties of members of some distinguished families of subgroups of a group on the structure of the group.

Recall that a subgroup H of G is said to be *S-quasinormal*, *S-permutable*, or *$\pi(G)$-permutable* in G (Kegel [22]) if $HP = PH$ for all Sylow subgroups P of G. A subgroup H of G is said to be *S-quasinormally embedded* or *S-permutably embedded* in G (Ballester-Bolinches and Pedraza-Aguilera [6]) if each Sylow subgroup of H is also a Sylow subgroup of some

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S-quasinormal subgroup of G. A subgroup A of G is said to be *c-normal* (Wang [32]) (*c-supplemented* (Ballester-Bolinches, Wang and Guo [9])) in G if G has a normal subgroup (a subgroup, respectively) T such that $AT = G$ and $A \cap T \leq A_G$.

Let $\mathcal{F}$ be a class of groups. If $1 \in \mathcal{F}$, then we write $G^{\mathcal{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathcal{F}$. The class $\mathcal{F}$ is said to be a *formation* if either $\mathcal{F} = \emptyset$ or $1 \in \mathcal{F}$ and every homomorphic image of $G/G^{\mathcal{F}}$ belongs to $\mathcal{F}$ for every group G. A formation $\mathcal{F}$ is said to be *saturated* or *local* if $\mathcal{F}$ contains every group G with $G^{\mathcal{F}} \leq \Phi(G)$. A class $\mathcal{F}$ of groups is said to be *solubly saturated* or *Baer-local* (see [11, Chapter IV, Definition 4.9]) if $\mathcal{F}$ contains every group G with $G^{\mathcal{F}} \leq \Phi(N)$ for some soluble normal subgroup N of G.

Researches of many authors are connected with analysis of the following general question: *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Under what conditions on E then, does G belong to $\mathcal{F}$?*

We recall some recent results in this direction. If $\mathcal{F}$ is a saturated formation containing all supersoluble groups and G has a normal subgroup E such that $G/E \in \mathcal{F}$, then the following results are true:

(1) If the cyclic subgroups of E of prime order or order 4 are or S-quasinormal (Ballester-Bolinches and Pedraza-Aguilera [7], Asaad and Csörgö [2]), or c-normal (Ballester-Bolinches and Wang [8]), or c-supplemented (Ballester-Bolinches, Wang and Guo [9], Wang and Li [35]) in G, then $G \in \mathcal{F}$.

(2) If the cyclic subgroups of $F^*(E)$ of prime order or order 4 are or S-quasinormal (Li and Wang [24]), or c-normal (Wei, Wang and Li [36]), or S-quasinormally embedded (Li and Wang [25]), or c-supplemented (Wang, Wei and Li [34], Wei, Wang and Li [37]) in G, then $G \in \mathcal{F}$.

(3) If the maximal subgroups of every Sylow subgroup of E are or S-quasinormal (Asaad [1]), or c-normal (Wei [38]), or c-supplemented (Ballester-Bolinches and Guo [5]) in G, then $G \in \mathcal{F}$.

(4) If the maximal subgroups of every Sylow subgroup of $F^*(E)$ are or S-quasinormal (Li and Wang [26]), or c-normal (Wei, Wang and Li [37]), or c-supplemented (Wei, Wang and Li [34]), or S-quasinormally embedded (Li and Wang [25]) in G, then $G \in \mathcal{F}$.

(5) If E is soluble and the maximal subgroups of every Sylow subgroup of $F(E)$ are S-quasinormally embedded in G, then $G \in \mathcal{F}$ (Asaad and Heliel [3]).

In these results, $F^*(E)$ is the generalized Fitting subgroup of E, that is, the product of all normal quasinilpotent subgroups of E [21, Chapter X].

Bearing in mind the above-mentioned results, it is natural to ask:

(I) *Is it true that all the above-mentioned results can be strengthened by considering the more general case, where $\mathcal{F}$ is a Baer-local formation?*

(II) *Is it true that all the above-mentioned results can be improved by using some weaker condition?*

In this paper, we give positive answers to both these questions.

Let H_{seG} be the subgroup generated by all the subgroups of H which are S-quasinormally embedded in G. We call H_{seG} the *SE-core* of H in G.

DEFINITION 1.1. *Let H be a subgroup of G. We say that H is SE-supplemented in G if there exists some subgroup T of G such that $HT = G$ and $H \cap T \leq H_{seG}$.*

The key to solving Questions (I) and (II) are the following two results.

THEOREM 1.2. *Let E be a normal subgroup of G. If the cyclic subgroups of E of prime order or order 4 are SE-supplemented in G, then each chief factor of G below E is cyclic.*

THEOREM 1.3. *Let E be a normal subgroup of G. If the maximal subgroups of every Sylow subgroup of E are SE-supplemented in G, then each chief factor of G below E is cyclic.*

A chief factor H/K of G is called *$\mathcal{F}$-central* in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathcal{F}$.

In [31], the following result was proved.

THEOREM 1.4 [31, Theorem 3.1]. *Let $\mathcal{F}$ be any formation and E a normal subgroup of G. If each chief factor of G below $F^*(E)$ is $\mathcal{F}$-central in G, then each chief factor of G below E is $\mathcal{F}$-central in G as well.*

Base on these theorems, we may directly obtained the following results.

COROLLARY 1.5. *Let E be a normal subgroup of G. If the cyclic subgroups of $F^*(E)$ of prime order or order 4 are SE-supplemented in G, then each chief factor of G below E is cyclic.*

COROLLARY 1.6. *Let E be a normal subgroup of G. If the maximal subgroups of every Sylow subgroup of $F^*(E)$ are SE-supplemented in G, then each chief factor of G below E is cyclic.*

It is clear that if $\mathcal{F}$ is a Baer-local formation containing all supersoluble groups and G has a cyclic normal subgroup E such that $G/E \in \mathcal{F}$, then $G \in \mathcal{F}$ (see Lemma 2.17 below). Hence from Theorems 1.2, 1.3 and 1.4, we also directly get the following

THEOREM 1.7. *Let $\mathcal{F}$ be a Baer-local formation containing all supersoluble groups and G has normal subgroups $X \leq E$ such that $G/E \in \mathcal{F}$. Suppose that the cyclic subgroups of X of prime order or order 4 are SE-supplemented in G. If either $X = E$ or $X = F^*(E)$, then $G \in \mathcal{F}$ and each chief factor of G below E is cyclic.*

THEOREM 1.8. *Let $\mathcal{F}$ be a Baer-local formation containing all supersoluble groups and G has normal subgroups $X \leq E$ such that $G/E \in \mathcal{F}$. Suppose that the maximal subgroups of every Sylow subgroup of X are SE-supplemented in G. If either $X = E$ or $X = F^*(E)$, then $G \in \mathcal{F}$ and each chief factor of G below E is cyclic.*

It is easy to see that all S-quasinormal subgroups, S-quasinormally embedded subgroups, c-normal subgroups, and c-supplemented subgroups are all SE-supplemented in G. But the converse is not true (see the example in Section 5). Hence Theorems 1.7 and 1.8 give affirmative answers to Questions (I) and (II) and consequently, a large number of known results (for example, the above results in (1)-(5)) are generalized.

All unexplained notations and terminologies are standard. The reader is referred to [11], [4] and [16] if necessary.

2. Preliminaries

LEMMA 2.1 [22]. *Let G be a group and $H \leq K \leq G$.*

(1) *If H is S-quasinormal in G, then H is S-quasinormal in K.*
(2) *Suppose that H is normal in G. Then K/H is S-quasinormal in G if and only if K is S-quasinormal in G.*
(3) *If H is S-quasinormal in G, then H is subnormal in G.*

From Lemma 2.1 (3) we get

LEMMA 2.2. *If H is an S-quasinormal subgroup of G and H is a p-group for some prime p, then $O^p(G) \leq N_G(H)$.*

LEMMA 2.3. *Suppose that A, B are subgroups of G.*

(1) *If A is S-quasinormal in G, then $A \cap B$ is S-quasinormal in B* [10].

(2) *If A and B are S-quasinormal in G, then $A \cap B$ is S-quasinormal in G* [22].

(3) *If A is S-quasinormal in G, then A/A_G is nilpotent* [10].

LEMMA 2.4 [6]. *Let G be a group and $H \leq K \leq G$.*

(1) *If H is S-quasinormally embedded in G, then H is S-quasinormally embedded in K.*

(2) *If H is normal in G and E is an S-quasinormally embedded subgroup of G, then EH is S-quasinormally embedded in G and EH/H is S-quasinormally embedded in G/H.*

LEMMA 2.5. *Suppose that H is an S-quasinormally embedded subgroup of G. If $H \leq O_p(G)$ for some prime p, then H is S-quasinormal in G.*

PROOF. Suppose that H is a Sylow p-subgroup of some S-quasinormal subgroup E of G. Then $H = O_p(G) \cap E$. Hence by Lemma 2.3(2), H is S-quasinormal in G.

LEMMA 2.6. *Suppose that N is a normal subgroup of G and $H \leq K \leq G$. Then:*

(1) $H_{seG} \trianglelefteq H$.

(2) $H_{seG} \leq H_{seK}$.

(3) $H_{seG}N/N \leq (HN/N)_{se(G/N)}$.

(4) *If* $(|N|, |H|) = 1$, *then* $H_{seG}N/N = (HN/N)_{se(G/N)}$.

(5) $(H_{seG})^x = (H^x)_{seG}$, *for any* $x \in G$.

PROOF. Let L be an S-quasinormally embedded subgroup of G contained in H, q be a prime dividing $|L|$, Q a Sylow q-subgroup of L and E an S-quasinormal subgroup of G such that $Q \in \mathrm{Syl}_q(E)$.

(1) Let $x \in H$. Then $L^x \leq H$. If R is a Sylow q-subgroup of L^x, then $R = {Q_1}^x$ for some Sylow q-subgroup Q_1 of L. Without loss of generality, we may assume that $Q_1 = Q$. Obviously, $Q^x \in \mathrm{Syl}_q(E^x)$ and E^x is an S-quasinormal subgroup of G. Hence L^x is an S-quasinormally embedded subgroup of G. This implies that (1) holds.

(2) By Lemma 2.3(1), $E \cap K$ is an S-quasinormal subgroup of K. Since $Q \leq E \cap K$, Q is a Sylow q-subgroup of $E \cap K$. Hence $L \leq H_{seK}$ and so $H_{seG} \leq H_{seK}$.

(3) Clearly $LN/N \leq HN/N$ and LN/N is an S-quasinormally embedded subgroup of G/N by Lemma 2.4(2). Hence $H_{seG}N/N \leq (HN/N)_{se(G/N)}$.

(4) In view of (3), we only need to prove that $(HN/N)_{se(G/N)} \leq H_{seG}N/N$. Let V/N be an S-quasinormally embedded subgroup of G/N such that $V/N \leq HN/N$. Then $V = V \cap HN = N(V \cap H)$. We now show that $U = V \cap H$ is S-quasinormally embedded in G. Let p be an arbitrary prime dividing $|U|$ and $P \in \mathrm{Syl}_p(U)$. Then $P \in \mathrm{Syl}_p(V)$ since $(|N|, |H|) = 1$. Hence $PN/N \in \mathrm{Syl}_p(V/N)$. Let W/N be an S-quasinormal subgroup of G/N such that $PN/N \in \mathrm{Syl}_p(W/N)$. Then $PN/N = W_pN/N$ for some Sylow p-subgroup W_p of W. Hence $PN = W_pN$. Since P and W_p are all Sylow p-subgroups of PN, $P = (W_p)^n$ for some $n \in N$. By Lemma 2.1(2), W is S-quasinormal in G. Then, clearly, W^n is S-quasinormal in G. Now since $P = (W_p)^n \in \mathrm{Syl}_p(W^n)$, we see that U is S-quasinormally embedded in G. Therefore $U \leq H_{seG}$. It follows that $V/N = UN/N \leq H_{seG}N/N$ and consequently $(HN/N)_{se(G/N)} \leq H_{seG}N/N$.

(5) This is evident.

Lemma 2.7. *Let H be an SE-supplemented subgroup of G and N a normal subgroup of G.*

(1) *If $H \leq K \leq G$, then H is SE-supplemented in K.*
(2) *If $N \leq H$, then H/N is SE-supplemented in G/N.*
(3) *If $(|N|, |H|) = 1$, then HN/N is SE-supplemented in G/N.*
(4) *H^x is SE-supplemented in G, for all $x \in G$.*

Proof. Let T be a subgroup of G such that $HT = G$ and $H \cap T \leq H_{seG}$.

(1) Since $K = H(T \cap K)$ and $(T \cap K) \cap H = T \cap H \leq H_{seG} \leq H_{seK}$ by Lemma 2.6(2), (1) holds.

(2) Since $(H/N)(NT/N) = G/H$ and $(H/N) \cap (NT/N) = N(H \cap T)/N \leq H_{seG}N/N \leq (H/N)_{se(G/N)}$ by Lemma 2.6(3), H/N is SE-supplemented in G/N.

(3) Since $(|N|, |H|) = 1, N \leq T$. Hence $(T/N) \cap (HN/N) = N(T \cap H)/N \leq NH_{seG}/N \leq (NH/N)_{se(G/N)}$. This shows that HN/N is SE-supplemented in G/N.

(4) Since $HT = G$, we have $H^xT^x = G$. On the other hand, since $H \cap T \leq H_{seG}$, $H^x \cap T^x = (H \cap T)^x \leq (H_{seG})^x = (H^x)_{seG}$ by Lemma 2.6(5). Hence H^x is SE-supplemented in G.

Recall that for a class $\mathcal{F}$ of groups, a group G is said to be a minimal non-$\mathcal{F}$-group if $G \notin \mathcal{F}$ but every proper subgroup of G belongs to $\mathcal{F}$.

The following result about minimal non-$\mathcal{F}$-groups is useful in our proofs.

LEMMA 2.8 [28, VI, Theorem 25.4]. *Let $\mathcal{F}$ be a saturated formation and G a minimal non-$\mathcal{F}$-group such that $G^{\mathcal{F}}$ is soluble. Then:*

(a) *$G^{\mathcal{F}}$ is a p-group for some prime p and $G^{\mathcal{F}}$ is of exponent p or exponent 4 (if P is a non-abelian 2-group).*

(b) *$G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, and $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a non-$\mathcal{F}$-central in G.*

LEMMA 2.9 [12, Theorem 2.4]. *Let P be a group and α a p'-automorphism of P.*

(1) *If $[\alpha, \Omega_2(P)] = 1$, then $\alpha = 1$.*

(2) *If $[\alpha, \Omega_1(P)] = 1$ and either p is odd or P is abelian, then $\alpha = 1$.*

We use $\mathcal{A}(p-1)$ to denote the formation of all abelian groups of exponent dividing $p-1$. The symbol $Z_{\mathcal{U}}(G)$ denotes the largest normal subgroup of G such that every chief factor of G below $Z_{\mathcal{U}}(G)$ is cyclic ($Z_{\mathcal{U}}(G) = 1$ if G has no such non-identity normal subgroups).

LEMMA 2.10 [31, Lemma 2.2]. *Let E be a normal p-subgroup of G. If $E \leq Z_{\mathcal{U}}(G)$, then*

$$(G/C_G(E))^{\mathcal{A}(p-1)} \leq O_p(G/C_G(E)).$$

We use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$ when P is not a non-abelian 2-group; otherwise, let $\Omega(P) = \Omega_2(P)$.

The following lemma may be proved based on some results in [23] on f-hypercentral action (see [28, Chapter II] or [11, Chapter IV, Section 6]). For reader's convenience, we give a direct proof.

LEMMA 2.11. *Let P be a normal p-subgroup of G. If either $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$ or $\Omega \leq Z_{\mathcal{U}}(G)$, then $P \leq Z_{\mathcal{U}}(G)$.*

PROOF. Let $C = C_G(P)$ and H/K be any chief factor of G below P. Then $O_p(G/C_G(H/K)) = 1$ by [40, Appendix C, Corollary 6.4].

Suppose that $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$. Then by Lemma 2.10, $(G/C_G(P/\Phi(P)))^{\mathcal{A}(p-1)}$ is a p-group. Hence $(G/C)^{\mathcal{A}(p-1)}$ is a p-group by Theorem 1.4 in [13, Chapter 5]. This implies that $G/C_G(H/K) \in \mathcal{A}(p-1)$ and so $|H/K| = p$ by [40, Chapter 1, Theorem 1.4]. Therefore $P \leq Z_{\mathcal{U}}(G)$.

Now assume that $\Omega \leq Z_{\mathcal{U}}(G)$. Then $(G/C_G(\Omega))^{\mathcal{A}(p-1)}$ is a p-group by Lemma 2.10. Hence $(G/C)^{\mathcal{A}(p-1)}$ is a p-group by Lemma 2.9. Thus we also have $P \leq Z_{\mathcal{U}}(G)$.

LEMMA 2.12 [2, Lemma 4]. *Let P be a p-subgroup of G, where $p > 2$. Suppose that all subgroups of P of order p are S-quasinormal in G. If a is a p'-element of $N_G(P) \setminus C_G(P)$, then a induces in P a fixed-point-free automorphism.*

LEMMA 2.13 [17, Theorem 3.1]. *Let A, B, E be normal subgroups of G. Suppose that $G = AB$. If $E \leq Z_{\mathcal{U}}(A) \cap Z_{\mathcal{U}}(B)$ and $(|G : A|, |G : B|) = 1$, then $E \leq Z_{\mathcal{U}}(G)$.*

LEMMA 2.14 [39]. *Let G be a group and $A \leq G$.*

(1) *If A is subnormal in G and A is a π-subgroup of G, then $A \leq O_\pi(G)$.*
(2) *If A is subnormal in G and A is nilpotent, then $A \leq F(G)$.*

The following lemma is well known.

LEMMA 2.15. *Let $A, B \leq G$ and $G = AB$.*

(1) *$G_p = A_pB_p$ for some Sylow p-subgroups G_p, A_p and B_p of G, A and B, respectively.*
(2) *$G = AB^x$ for all $x \in G$.*

LEMMA 2.16 [28, Chapter I, Lemma 4.1]. *Let Q be an irreducible automorphism group of an elementary abelian p-group P of order p^n. Then Q is cyclic with $|Q| \mid p^n - 1$ and n is the smallest positive integer such that $|Q|$ divides $p^n - 1$.*

Following Doerk and Hawkes [11], we use $C^p(G)$ to denote the intersection of the centralizers of all abelian p-chief factors of G ($C^p(G) = G$ if G has no such chief factors).

For every function f of the form

$$(*) \qquad f : \mathbb{P} \cup \{0\} \to \{\text{group formations}\},$$

we put, following [30], $CLF(f) = \{G \text{ is a group} \mid G/G_{\mathcal{S}} \in f(0)$ and $G/C^p(G) \in f(p)$ for any prime $p \in \pi(\mathrm{Com}(G))\}$. Here $G_{\mathcal{S}}$ denotes the $\mathcal{S}$-radical of G (that is, the largest normal soluble subgroup of G); $\mathrm{Com}(G)$ denotes the class of all abelian groups A such that $A \simeq H/K$ for some composition factor H/K of G.

It is well known that a formation $\mathcal{F}$ is a Bear-local formation if and only if there exists a function f of the form (*) such that $\mathcal{F} = CLF(f)$ (see, for example, [30, Theorem 1]).

LEMMA 2.17. *Let $\mathcal{F}$ be a Baer-local formation containing all supersoluble groups and E a subgroup of G such that $G/E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.*

PROOF. Without loss of generality, we may assume that E is a minimal normal subgroup of G. Then $|E| = p$ for some prime p. By [30, Theorem 1], $\mathcal{F} = CLF(f)$ for some function of the form (*). It is clear that the group $H = E \rtimes (G/C_G(E))$ is supersoluble. Hence $H \in \mathcal{F}$ and thereby $G/C_G(E) \in f(p)$. Let $C^p/E = C^p(G/E)$. Then $C^p(G) = C^p \cap C_G(E)$ by [11, Chapter A, Theorem 3.2]. But since $G/E \in \mathcal{F}$, $G/C^p \simeq (G/E)/C^p(G/E) \in f(p)$ and consequently $G \in \mathcal{F}$.

3. Proof of Theorem 1.2

Theorem 1.2 is a special case of the following theorem when $\pi_i = \pi(E)$, the set of all prime divisors of $|E|$.

THEOREM 3.1. *Let E be a normal subgroup of G, $p_1 < p_2 < \dots < p_n$ the set of all prime divisors of $|E|$ and $\pi_i = \{p_1, p_2, \dots, p_i\}$. Suppose that for each $p \in \pi_i$ and for any Sylow p-subgroup P of E, the cyclic subgroups of P of prime order or order 4 are SE-supplemented in G. Then E has a normal Hall π_i'-subgroup $E_{\pi_i'}$ and each chief factor of G between E and $E_{\pi_i'}$ is cyclic.*

PROOF. Suppose that this theorem is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. Let $p = p_1$ be the smallest prime dividing $|E|$ and P a Sylow p-subgroup of E. Let $Z = Z_{\mathcal{U}}(G)$ and $C = C_G(P)$. We proceed via the following steps.

(1) *E is p-nilpotent.*

Without loss of generality, we may assume that $i = 1$.

If $E \neq G$, then the hypothesis is true for (E, E) by Lemma 2.7(1). Hence E is p-nilpotent by the choice of (G, E). Now assume that $E = G$ and G is not p-nilpotent. Then G has a p-closed Schmidt subgroup $H = H_p \rtimes H_q$ [20, Chapter IV, Theorem 5.4]. We may assume that $H_p \leq P$. By Lemma 2.8, $H_p/\Phi(H_p)$ is a non-central chief factor of H and H_p is a group of exponent p

or exponent 4 (if $p = 2$ and H_p is non-abelian). Hence $|H_p/\Phi(H_p)| > p$ since p is the smallest prime dividing $|H|$.

Let $\Phi = \Phi(H_p)$, X/Φ be a minimal subgroup of H_p/Φ, $x \in X \setminus \Phi$ and $L = \langle x \rangle$. Then $|L| = p$ or $|L| = 4$. Hence L is SE-supplemented in G. Suppose that $L_{seG} \neq L$. Then for some proper subgroup T of G we have $LT = G$. Hence $H = L(H \cap T)$ and $H \cap T \neq H$. Because $\Phi \leq \Phi(H)$, we have $(H \cap T)\Phi < H$. Since the maximal subgroup of L is contained in Φ, $|H : (H \cap T)\Phi| = p$. Hence $|H_p/\Phi| = |H : (H \cap T)\Phi| = p$. This contradiction shows that $L_{seG} = L$ is S-quasinormally embedded in G since L is cyclic. Hence L is S-quasinormally embedded in H by Lemma 2.4(1). Then by Lemma 2.5, L is S-quasinormal in H. It follows from Lemma 2.1 that $L\Phi/\Phi = X/\Phi$ is S-quasinormal in H/Φ. This shows that every minimal subgroup of H_p/Φ is S-quasinormal in H/Φ and hence $|H_p/\Phi| = p$ by Lemma 2.11 in [29], a contradiction. Hence E is p-nilpotent.

(2) *Let $E_{p'}$ be a Hall p'-subgroup of E. Then $E_{p'}$ is normal in G and the hypothesis holds for $(G, E_{p'})$ and for $(G/E_{p'}, E/E_{p'})$.*

By (1), $E_{p'}$ is characteristic in E. Hence $E_{p'}$ is normal in G. Clearly, the hypothesis holds for $(G, E_{p'})$. By Lemma 2.7(3), the hypothesis also holds for $(G/E_{p'}, E/E_{p'})$.

(3) *$E = P$ is not a minimal normal subgroup of G.*

Suppose that $E \neq P$. Then $E_{p'} \neq 1$. Hence every chief factor of $G/E_{p'}$ below $E/E_{p'}$ is cyclic by the choice of (G, E). On the other hand, the minimality of (G, E) implies that $E_{p'}$ has a normal Hall π_i'-subgroup V and each chief factor of G between $E_{p'}$ and V is cyclic. Hence V is a normal Hall π_i'-subgroup of E and each chief factor of G between E and V is cyclic, which contradicts the choice of (G, E). Hence $E = P$. Suppose that P is a minimal normal subgroup of G. Then every minimal subgroup L of P is S-quasinormal in G. Indeed, since L is SE-supplemented in G, there exists some subgroup T of G such that $LT = G$ and $L \cap T \leq L_{seG}$. Suppose that $L \cap T = 1$. Then $T \cap P$ is normal in G and $|P : (T \cap P)| = p$. It follows that $T \cap P = 1$ and so $|E| = |P| = p$. Consequently, $E \leq Z$. This contradiction shows that $L \leq T$. Hence $L = L_{seG}$ is S-quasinormally embedded in G. Then by Lemma 2.5, we see that every minimal subgroup of P is S-quasinormal in G. Therefore $|P| = p$ by Lemma 2.11 in [29], a contradiction. Hence (3) holds.

(4) *G has a non-identity normal subgroup $R \leq P$ such that P/R is a non-cyclic chief factor of G, $R \leq Z$ and $V \leq R$ for any normal subgroup $V \neq P$ of G contained in P.*

Let P/R be a chief factor of G. Then $R \neq 1$ by (3) and the hypothesis holds for (G, R). Therefore $R \leq Z$ and so P/R is not cyclic by the choice of $(G, P) = (G, E)$. Now let $V \neq P$ be any normal subgroup of G contained in P. Then $V \leq Z$. If $V \nleq R$, then from the G-isomorphism $P/R = VR/R \simeq V/V \cap R$ we have $P \leq Z$, which contradicts the choice of $(G, E) = (G, P)$. Hence $V \leq R$.

(5) $P \leq O^p(G)$.

Suppose that $P \nleq O^p(G)$. Then from the G-isomorphism $O^p(G)P/O^p(G) \simeq P/O^p(G) \cap P$, we see that G has a cyclic chief factor of the form P/V, where $O^p(G) \cap P \leq V$, which contradicts (4).

(6) $\Omega(P) = P$.

If $\Omega(P) < P$, then by (4), $\Omega \leq Z$. Hence $P \leq Z$ by Lemma 2.11, which contradicts the choice of (G, P).

(7) *There is a prime* $q \neq p$ *such that* q *divides* $|G : C|$.

Let G_p be a Sylow p-subgroup of G. Suppose that $|G : C| = p^n$. Then any chief factor of G_p below P is a chief factor of G, which implies that $P \leq Z$.

The final contradiction.

By (6), $\Omega(P) = P$. Let $V_1, V_2, \ldots, V_t$ be the set of all cyclic subgroups of P of order p and order 4 (if P is a non-abelian 2-group). We may assume that $P/R = (V_1R/R)(V_2R/R) \cdots (V_tR/R)$ and V_iR/R is a group of order p for all $i = 1, 2, \ldots, t$. Suppose that for some i we have $V_iT = G$, where $T \neq G$. Then $P = V_i(T \cap P)$, where $T \cap P \neq P$. Let $N = N_G(T \cap P)$. It is clear that $|P : T \cap P|$ is either p or 4. Hence either $N = G$ or $|G : N| = 2$ and $NP = G$. In the former case, G has a cyclic chief factor $P/T \cap P = V_i(T \cap P)/T \cap P \simeq V_i/V_i \cap T \cap P$. In the second case, G has a cyclic chief factor $P/P \cap N$. But in view of (4), both these cases are impossible. Hence by Lemma 2.5, $V_1, V_2, \ldots, V_t$ are S-quasinormal subgroups of G. This shows that if Q is a every Sylow subgroup Q of G, then $V_iQ = QV_i$ for every $i \leq t$ and so V_i is subnormal in V_iQ by Lemma 2.1(3). Consequently, V_i is normal in V_iQ. Suppose that $p = 2$. Then V_iQ is nilpotent and so $Q \leq C_G(V_i)$. Therefore $O^p(G) \leq C_G(P/R)$. This implies $C_G(P/R) = G$, which contradicts (4). Hence $p > 2$. We claim that $O^p(G) \neq G$. Indeed, if $O^p(G) = G$, then V_1R/R is normal in G/R by Lemma 2.2 and so $P/R = V_1R/R$ is cyclic, a contradiction. Next we show that $O^q(G) \neq G$ for some prime $q \neq p$. Assume that $O^q(G) = G$ for all primes $q \neq p$. Then for every chief factor H/K of G of order p we have $C_G(H/K) = G$. In particular, $L \leq Z(G)$ for every minimal normal subgroup L of G contained in R. This implies that $C_P(a) \neq 1$ for every $a \in G$. Hence by (4) and Lemma 2.12, G/C is a p-group, which contradicts (4). Thus $O^q(G) \neq G$ for some prime $q \neq p$. By the choice of G, we have $P \leq Z_{\mathcal{U}}(O^p(G))$ and $P \leq Z_{\mathcal{U}}(O^q(G))$. Thus, by (5) and Lemma 2.13, we obtain that $P \leq Z$. The final contradiction completes the proof.

4. Proof of Theorem 1.3

Theorem 1.3 is a special case of the following theorem when $\pi_i = \pi(E)$.

THEOREM 4.1. *Let E be a normal subgroup of G, $p_1 < p_2 < \ldots < p_n$ the set of all prime divisors of $|E|$ and $\pi_i = \{p_1, p_2, \ldots, p_i\}$. Suppose that for each $p \in \pi_i$, the maximal subgroups of any Sylow p-subgroup of E are SE-supplemented in G. Then E has a normal Hall π_i'-subgroup $E_{\pi_i'}$ and each chief factor of G between E and $E_{\pi_i'}$ is cyclic.*

PROOF. Assume that this theorem is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Let $p = p_1$ be the smallest prime dividing $|E|$ and P a Sylow p-subgroup of E. Let $Z = Z_{\mathcal{U}}(G)$. We proceed the proof via the following steps.

(1) *E is p-nilpotent.*

We may consider, without loss of generality, that $i = 1$. Assume that E is not p-nilpotent. Then:

(a) $E = G$.

Indeed, if $E < G$, then $|E| + |E| < |G| + |E|$. Hence the hypothesis is true for (E, E) by Lemma 2.7(1). The choice of (G, E) implies that E is p-nilpotent, a contradiction.

(b) $O_{p'}(G) = 1$.

Let $D = O_{p'}(G)$. By Lemma 2.7(3), the hypothesis is true for $(G/D, ED/D)$. Hence, if $D \neq 1$, then G/D is p-nilpotent by the choice of (G, E). Therefore G is p-nilpotent, a contradiction.

(c) *If $P \leq V < G$, then V is p-nilpotent.*

In fact, by Lemma 2.7(1), the hypothesis holds for V. Hence V is p-nilpotent by the choice of G.

(d) $O_{p'}(L) = 1$ *for all S-quasinormal subgroups L of G.*

By Lemma 2.1(3), L is subnormal in G. It follows that $O_{p'}(L)$ is subnormal in G. Hence $O_{p'}(L) \leq O_{p'}(G) = 1$ by Lemma 2.14(1).

(e) *If N is an abelian minimal normal subgroup of G, then G/N is p-nilpotent.*

In view of (b), N is a p-group and so $N \leq P$. Thus the hypothesis is true for $(G/N, E/N)$ by Lemma 2.7(2). The choice of (G, E) implies that G/N is p-nilpotent.

(f) *G is p-soluble.*

In view of (e), we need only to show that G has an abelian minimal normal subgroup. Suppose that this is false. Then $p = 2$ by the Feit-Thompson odd theorem. By Lemmas 2.1(3) and 2.14(1), we see that every non-identity subgroup of P is not S-quasinormal in G. Hence for every non-identity S-quasinormally embedded in G subgroup $L \leq V$, where V is a maximal subgroup of P, and for any S-quasinormal subgroup W of G such that $L \in \mathrm{Syl}_2(W)$ we have $L \neq W$. Moreover, $W_G \neq 1$. Indeed, if $W_G = 1$, then W is nilpotent by Lemma 2.3(3). Hence $O_{2'}(W) \neq 1$, which contradicts (d). Note also that for any minimal normal subgroup N of G we have $NP = G$ (otherwise, N is 2-nilpotent by (c), a contradiction). It follows that N is the unique minimal normal subgroup of G. Therefore $N \leq W$ (since $W_G \neq 1$) and consequently $N \cap P = N \cap L$.

Now we show that $V_{seG} \neq 1$ for any maximal subgroup V of P. In fact, suppose that $V_{seG} = 1$ and let T be a subgroup of G such that $VT = G$ and $V \cap T \leq V_{seG} = 1$. Then T is a complement of V in G. This induces that T is 2-nilpotent since the order of a Sylow 2-subgroup of T is equal to 2. We may, therefore, assume that $T = N_G(H_1)$ for some Hall $2'$-subgroup H_1 of G. It is clear that $H_1 \leq N$. By [15], any two Hall $2'$-subgroups of N are conjugate in N. By Frattini Argument, $G = NT$. Then $P = (P \cap N)(P \cap T^x)$ for some $x \in G$ by Lemma 2.15(1). Let $T_1 = T^x = N_G({H_1}^x)$. It is clear that $P \cap T_1 \neq P$. Hence we can choose a maximal subgroup V_1 in P containing $P \cap T_1$. By the hypothesis, there exists a subgroup T_2 such that $G = V_1T_2$, where $V_1 \cap T_2 \leq (V_1)_{seG}$. If $(V_1)_{seG} = 1$, then as above, we have that T_2 is 2-nilpotent and we may assume that $T_2 = N_G(H_2)$ for some Hall $2'$-subgroup H_2 of G. By [15] again, we have $((H_1)^x)^y = H_2$ for some $y \in G$. Therefore, $G = VT = VT_1 = V_1T_2 = V_1{T_1}^y = V_1T_1$ by Lemma 2.15(2) and $P = V_1(P \cap T_1) = V_1$. This contradiction shows that $(V_1)_{seG} \neq 1$. Let L be any non-identity S-quasinormally embedded subgroup of G contained in V_1 and W be an S-

quasinormal subgroup of G such that $L \in \mathrm{Syl}_2(W)$. Then $L \cap N = P \cap N$, which implies $P = (P \cap N)(P \cap T_1) = (L \cap N)(P \cap T_1) \leq V_1$, a contradiction.

Therefore for every maximal subgroup V of P we have $(V_1)_{seG} \neq 1$. But then from above, we know that $N \cap P \leq V$. Hence $N \cap P \leq \Phi(P)$ and so N is 2-nilpotent by [20, Chapter IV, Theorem 4.7], a contradiction. Hence, (f) holds.

The final contradiction for (1).

Let N be any minimal normal subgroup of G. Then in view of (b) and (f), N is a p-group and so G/N is p-nilpotent by (e). This implies that N is the unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Hence G is a primitive group and thereby $N = C_G(N) = F(G)$ by [11, Chapter A, Theorem 17.2]. Let M be a maximal subgroup of G such that $G = N \rtimes M$. Let $M_p \in \mathrm{Syl}_p(M)$ and V be a maximal subgroup of P such that $M_p \leq V$. Then $VM \neq G$ and so $VM^x \neq G$ for all $x \in G$ by Lemma 2.15(2). Since V is SE-supplemented in G, there is a subgroup T of G such that $VT = G$ and $V \cap T \leq V_{seG}$. Suppose that $V_{seG} = 1$. Then T is a complement of V in G. It follows that $|T_p| = p$, where $T_p \in \mathrm{Syl}_p(T)$. Then T is p-nilpotent since p is the smallest prime dividing $|G|$. Hence $T_{p'} \trianglelefteq T$, where $T_{p'}$ is a Hall p'-subgroup of T. Since G is p-soluble, any two Hall p'-subgroups of G are conjugate. Therefore there is an element $x \in G$ such that $T_{p'} \leq M^x$. If $T_p \leq M^x$, then $T \leq M^x$ and $G = VT = VM^x$, a contradiction. Hence $T_p \nleq M^x$. But $G/N \simeq M^x \leq N_G(T_{p'})$ (since G/N is p-nilpotent) and $T_p \leq N_G(T_{p'})$. Therefore $G = \langle M^x, T_p \rangle = N_G(T_{p'})$, which contradicts (b). Hence $V_{seG} \neq 1$. Let $L \neq 1$ be an S-quasinormally embedded subgroup of G such that $L \leq V$ and W an S-quasinormal subgroup of G such that $L \in \mathrm{Syl}_p(W)$. Suppose that $L = W$. Then by Lemma 2.2, $N \leq L^G = L^{PT_{p'}} = L^P \leq V$, a contradiction. Hence $L \neq W$. Then in view of (b) and Lemma 2.1(3), we have $W_G \neq 1$. This implies that $N \leq L \leq V$ and so $V = VN = P$. This final contradiction shows that (1) holds.

(2) $E = P$.

See (3) in the proof of Theorem 1.3.

(3) *If N is a minimal normal subgroup of G contained in P, then $P/N \leq Z_{\mathcal{U}}(G/N)$, N is the only minimal normal subgroup of G contained in P and $|N| > p$.*

Indeed, by Lemma 2.7(2), the hypothesis holds on G/N for any minimal normal subgroup N of G contained in P. Hence $P/N \leq Z_{\mathcal{U}}(G/N)$ by the choice of $(G,E)=(G,P)$. If $|N|=p$, $P\leq Z_{\mathcal{U}}(G)$, a contradiction. If G has two minimal normal subgroups R and N contained in P, then $NR/R\leq P/R$ and from the G-isomorphism $RN/N\simeq N$ we have $|N|=p$, a contradiction. Hence, (3) holds.

(4) $\Phi(P)\neq 1$.

Suppose that $\Phi(P)=1$. Then P is an elementary abelian p-group. Let N_1 be any maximal subgroup of N. We show that N_1 is S-quasinormal in G. Let B be a complement of N in P and $V=N_1B$. Then V is a maximal subgroup of P. Hence V is SE-supplemented in G. Let T be a subgroup of G such that $G=TV$ and $T\cap V\leq V_{seG}$. If $T=G$, then $V=V_{seG}$ is S-quasinormal in G by Lemma 2.5. Hence $V\cap N=V_{seG}\cap N=N_1B\cap N=N_1(B\cap N)=N_1$ is S-quasinormal in G by Lemma 2.3(2). Now assume that $T\neq G$. Then $1\neq T\cap P<P$. Since $G=VT=PT$ and P is abelian, $T\cap P$ is normal in G. Hence $N\leq T\cap P\leq T$ and consequently $N_1\leq N\cap T\cap V\leq N\cap V_{seG}\leq N$. Clearly, $N\nleq V$. Hence $N_1=N\cap V_{seG}$. By Lemma 2.5 and since the subgroup generated by all S-quasinormal in G subgroup of V is also S-quasinormal in G (cf. [29, Lemma 2.8(1)]), we see that V_{seG} is S-quasinormal in G. Thus by Lemma 2.3(2), N_1 is S-quasinormal in G. This shows that every maximal subgroup of N is S-quasinormal in G. Hence some maximal subgroup of N is normal in G by Lemma 2.11 in [29]. This contradiction shows that $\Phi(P)\neq 1$.

The final contradiction.

By (4), $\Phi(P)\neq 1$. Let N be a minimal normal subgroup of G contained in $\Phi(P)$. Then the hypothesis is still true for G/N. Hence $P/N\leq Z_{\mathcal{U}}(G/N)$ by the choice of (G,E). This means that $P/\Phi(P)\leq Z_{\mathcal{U}}(G/\Phi(P))$. Then by Lemma 2.11, we obtain that $P\leq Z$. This final contradiction completes the proof.

5. Final remarks

In Section 1, we have seen that a large number of known results follow from our results. Now we consider some further applications.

1. A group G is said to be *quasisupersoluble* [18] if for every its non-cyclic chief factor H/K and every $x\in G$, x induces an inner automorphism on H/K. It is cleat that every supersoluble group is quasisupersoluble. Moreover, in [18] it is proved that the class of all quasisupersoluble groups

is a Baer-local formation. Hence from Theorems 1.7 and 1.8 we also obtain the following results.

THEOREM 5.1. *Let G be a group with normal subgroups $X \leq E$ such that G/E is quasisupersoluble. Suppose that the cyclic subgroups of X of prime order or order 4 are SE-supplemented in G. If either $X = E$ or $X = F^*(E)$, then G is quasisupersoluble.*

THEOREM 5.2. *Let G be a group with normal subgroups $X \leq E$ such that G/E is quasisupersoluble. Suppose that the maximal subgroups of every Sylow subgroup of X are SE-supplemented in G. If either $X = E$ or $X = F^*(E)$, then G is quasisupersoluble.*

2. Recall that a subgroup H of G is said to be *weakly S-permutable (S-supplemented) in* G [29] if there are a subnormal subgroup (a subgroup, respectively) T G and an S-quasinormal subgroup H_{sG} of G contained in H such that $HT = G$ and $H \cap T \leq H_{sG}$.

The following example shows that in general the set of SE-supplemented subgroups of a group is wider than the set of all its S-supplemented subgroups and the set of all its S-quasinormally embedded subgroups. Consequently, the set of SE-supplemented subgroups of a group is also wider than the set of all S-quasinormal subgroups, the set of weakly S-permutable subgroups, the set of all c-normal subgroups and the set of all c-supplemented subgroups since these subgroups are either S-supplemented or S-quasinormally embedded in G.

EXAMPLE 5.3. Let Ly be the Lyons simple group. Then $|Ly| = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$. Hence in view of [14] there is a group D with minimal normal subgroup N such that $C_D(N) = N \leq O_{67}(D)$, $D/N \simeq Ly$ and $N \leq \Phi(D)$. Let Q be a group of order 17. Let $G = D \wr Q = K \rtimes Q$, where K is the base group of the regular wreath product G. Then $P = \Phi(K) = N^\natural$ (we use here the terminology in [11, Chapter A]). Moreover, in view of [11, Chapter A, Proposition 18.5], P is the only minimal normal subgroup of G. It is clear also that $|P| > 67^2$.

Since P is an elementary abelian 67-group, then in view of Maschke's theorem, $P = P_1 \times P_2 \times \ldots \times P_t$, where P_i is a minimal normal subgroup of PQ for all $i = 1, 2, \ldots, t$. Suppose that $Q \leq C_G(P_i)$ for all $i = 1, 2, \ldots, t$. Then $Q \leq C_G(P)$. Hence $PQ = P \times Q = C_G(P)$ is normal in G and so Q is normal in G. This contradiction shows that for some i we have $C_Q(P_i) = 1$.

Hence $S := P_i rtimes Q = Q^S$. Since Q is a Sylow 17-subgroup of G, it is S-quasinormally embedded in G. Hence $S = S_{seG}$ and consequently S is SE-supplemented in G. Suppose that P_i is an S-quasinormally embedded subgroup of G and let V be an S-quasinormal subgroup of G such that $P_i \in \mathrm{Syl}_p(V)$. Since 17 divides $67 + 1$ and does not divide $67 - 1$, $|P_i| = 67^2$ by Lemma 2.16. By Lemma 2.1(3), V is subnormal in G and V/V_G is nilpotent by Lemma 2.3(2). Suppose that $V_G = 1$. Then V is a subnormal nilpotent subgroup of G. It follows from Lemma 2.14(2) that $V \leq P$. Thus $V = P_i$ is S-quasinormal in G. It is clear that $O^{67}(G) = G$. By Lemma 2.2, we see that P_i is normal in G. This induces that $P = P_i$ and so $|P| = 67^2$, a contradiction. Therefore $V_G \neq 1$ and so $P \leq V_G$. But then $P \leq P_i$, a contradiction again. Thus P_i is not an S-quasinormally embedded subgroup of G. Similarly one can proved that any maximal subgroup of P_i is not an S-quasinormally embedded subgroup of G. Hence S is not S-quasinormally embedded in G and if L is any non-identity S-quasinormally embedded subgroup of G contained in S, then $L = Q^x$ for some $x \in S$. Moreover, obviously, S has no non-identity S-quasinormal in G subgroups, that is, $S_{sG} = 1$. Now we show that S is not S-complemented in G. Indeed, if S is S-complemented in G, then S has a complement T in G since $S_{sG} = 1$. Clearly, $T \leq K$. Hence $K = K \cap TS = T(K \cap S) = TP_i$. But since $P_i \leq P = \Phi(K)$, we obtain $T = K$, which implies $T \cap S \neq 1$. This contradiction shows that S is not S-supplemented in G.

Base on the above, we also see that the results in [29] in the case where the subgroup D in [29, Theorems] is of prime order or 4 can be obtained by our results in this paper.

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