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On a Divisibility Problem

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ABSTRACT - We prove that there are no integers $n \geq 2$ and $k \geq 2$ such that n^k divides $\varphi(n^k) + \sigma_k(n)$. For $k = 2$ this settles a conjecture of Adiga and Ramaswamy.

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1. Introduction

We are concerned with the classical number theoretic functions $\varphi(n)$ and $\sigma_\alpha(n)$. If $n \geq 1$ is an integer, then $\varphi(n)$ denotes the number of positive integers not execeeding n which are relatively prime to n. This function is known as the Euler totient. And, $\sigma_\alpha(n)$ denotes the sum of the αth powers of the divisiors of n. Here, α is a real or complex parameter. The main properties of these and other arithmetical functions can be found, for example, in [2].

Nicol [6] and Zhang [8] were the first who studied the divisibility problem

$$n \big| \big(\varphi(n) + \sigma(n)\big). \tag{1}$$

Here, as usual, $\sigma = \sigma_1$. As each prime number n satisfies relation (1), this has infinitely many solutions. Let $\omega(n)$ denote the number of distinct prime factors of n. If $\omega(n) \geq 2$, then the study of problem (1) is quite

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involved. Nicol showed that the solutions of (1) are not square-free and conjectured that they are all even. He also established that if $n = 2^k \cdot 3 \cdot p$, where p is a prime number of the form $p = 2^{k-2} \cdot 7 - 1$ and $k \geq 2$ is an integer, then n is a solution. Zhang proved that there are no solutions of the form $p^a \cdot q$, where p and q are distinct primes and a is a positive integer. All cases $\omega(n) = 2$ and $\omega(n) = 3$ are settled in [5]. Also, in [5] the authors proved that for any fixed integer $n \geq 2$ there are only finitely many odd composite solutions with $\omega(n) = m$, where $m \geq 2$ is a fixed integer. They also obtained an asymptotic upper bound for the number of composite solutions.

Motivated by the methods and results published in [5], Harris [3], Yang [7], Jin and Tang [4] provided theorems for $\omega(n) = 4$ as well as related results.

In 2008, Adiga and Ramaswamy [1] investigated an analogue of problem (1):

$$n^2 | \big(\varphi(n^2) + \sigma_2(n)\big). \tag{2}$$

They proved that for any $n \geq 2$ and $\omega(n) \leq 3$ there is no solution. Moreover, they conjectured that there is no integer $n \geq 2$ satisfying (2).

In this note we study the following more general divisibility problem. Let $k \geq 2$ be a fixed integer. Do there exist integers $n \geq 2$ such that

$$n^k | \big(\varphi(n^k) + \sigma_k(n)\big) \tag{3}$$

is valid? In the next section, we show that the answer to this question is "no". For $k = 2$ this settles the conjecture stated by Adiga and Ramaswamy.

2. Lemmas and main Result

In order to solve the divisibility problem (3) we need three auxiliary results. The first two lemmas offer properties of φ and σ_k, whereas the third lemma provides an inequality involving the Weierstrass product $\prod_{j=1}^{n} (1 - x_j)$.

LEMMA 1. *Let $n \geq 2$ and $k \geq 2$ be integers. If* (3) *is solvable, then we have*

$$\varphi(n^k) + \sigma_k(n) = 2 \cdot n^k. \tag{4}$$

PROOF. We obtain

$$\sigma_k(n) = \sum_{d|n} d^k = \sum_{d|n} \left(\frac{n}{d}\right)^k = n^k \cdot \sum_{d|n} \frac{1}{d^k} \tag{5}$$

and

$$\sum_{d|n} \frac{1}{d^k} \leq \sum_{d \leq n} \frac{1}{d^k} < \sum_{d=1}^{\infty} \frac{1}{d^k} = \zeta(k),$$

where ζ denotes the Riemann zeta function. Let

$$A(n,k) = \frac{\varphi(n^k) + \sigma_k(n)}{n^k}.$$

Since $\varphi(n^k) < n^k$, we get $A(n,k) < \zeta(k) + 1 \leq \zeta(2) + 1 = 2.64\ldots$. On the other hand, using $\sigma_k(n) > n^k$ yields $A(n,k) > 1$. Thus, $1 < A(n,k) < 3$. Since $A(n,k)$ is an integer, we conclude that (4) holds. □

LEMMA 2. *For all integers $n \geq 2$ and $k \geq 2$ we have*

$$\frac{\sigma_k(n)}{n^k} \leq \frac{\sigma_2(n)}{n^2} < \prod_{p|n,\, p\ prime} \frac{1}{1 - 1/p^2}. \tag{6}$$

PROOF. From (5) it follows that $\sigma_k(n)/n^k$ is decreasing with respect to k. This leads to the first inequality in (6). Let $n = \prod_{j=1}^{r} p_j^{a_j}$ be the prime factorization of n. Then,

$$\frac{\sigma_2(n)}{n^2} = \prod_{j=1}^{r} \frac{p_j^{2a_j+2} - 1}{p_j^{2a_j} \cdot (p_j^2 - 1)} = \prod_{j=1}^{r} \left(p_j^2 \cdot \frac{1 - 1/p_j^{2a_j+2}}{p_j^2 - 1} \right) < \prod_{p|n} \frac{p^2}{p^2 - 1}.$$

This settles the second inequality in (6). □

LEMMA 3. *Let $x_j \in [0, 1/(j+1)]$ for $j = 1, \ldots, r$. Then we have*

$$\prod_{j=1}^{r} (1 - x_j) + \prod_{j=1}^{r} (1 - x_j^2)^{-1} \leq 2.$$

The sign of equality holds if and only if $x_1 = \ldots = x_r = 0$.

PROOF. We define

$$F(x_1, \ldots, x_r) = \prod_{j=1}^{r} (1 - x_j) + \prod_{j=1}^{r} (1 - x_j^2)^{-1}.$$

Moreover, let

$$M = \{(x_1, \ldots, x_r) \in \mathbf{R}^r \,|\, 0 \le x_j \le 1/(j+1)\,(j = 1, \ldots, r)\}$$

and

$$\max_{(x_1,\ldots,x_r)\in M} F(x_1, \ldots, x_r) = F(c_1, \ldots, c_r).$$

It suffices to show that

$$F(c_1, \ldots, c_r) \le 2$$

with equality if and only if $c_1 = \ldots = c_r = 0$.

We use induction on r. If $r = 1$, then $0 \le c_1 \le 1/2$ and

$$2 - F(c_1) = \frac{c_1}{1 - c_1^2}\left(\frac{1}{2}\sqrt{5} + \frac{1}{2} + c_1\right)\left(\frac{1}{2}\sqrt{5} - \frac{1}{2} - c_1\right) \ge 0.$$

The sign of equality holds if and only if $c_1 = 0$.

Next, we assume that the assertion is true for $r - 1$. We define for $j \in \{1, \ldots, r\}$ and $t \in [0, 1/(j+1)]$:

$$G(t) = F(c_1, \ldots, c_{j-1}, t, c_{j+1}, \ldots, c_r).$$

Then,

$$\max_{0\le t\le 1/(j+1)} G(t) = G(c_j). \tag{7}$$

If $0 < c_j < 1/(j+1)$, then there exists a number $\lambda \in (0, 1)$ such that

$$c_j = \lambda \cdot 0 + (1 - \lambda) \cdot \frac{1}{j+1}.$$

Since

$$G''(t) = \frac{6t^2 + 2}{(1 - t^2)^3} \prod_{i=1, i\ne j}^{r} (1 - c_i^2)^{-1} > 0,$$

we conclude that G is strictly convex on $[0, 1/(j+1)]$. Hence, we obtain

$$G(c_j) < \lambda G(0) + (1 - \lambda) G(1/(j+1))$$

$$\le \max\{G(0), G(1/(j+1))\} \le \max_{0\le t\le 1/(j+1)} G(t).$$

This contradicts (7). Thus,

$$c_j \in \{0, 1/(j+1)\} \quad \text{for} \quad j = 1, \ldots, r.$$

We consider two cases.

CASE 1. All numbers $c_1, \dots, c_r$ are different from 0.
Then, $c_j = 1/(j+1)$ $(j = 1, \dots, r)$ and we get

$$F(c_1, \dots, c_r) = \frac{1}{r+1} + \prod_{j=1}^{r} \left(1 + \frac{1}{j(j+2)}\right) = 2 - \frac{r}{(r+1)(r+2)} < 2.$$

CASE 2. At least one of the numbers $c_1, \dots, c_r$ is equal to 0.
Let $c_k = 0$ with $k \in \{1, \dots, r\}$. We set

$$y_j = c_j \quad \text{for} \quad j = 1, \dots, k-1$$
$$y_j = c_{j+1} \quad \text{for} \quad j = k, \dots, r-1.$$

Then we have

$$0 \le y_j \le \frac{1}{j+1} \quad \text{for} \quad j = 1, \dots, r-1.$$

Using the induction hypothesis gives

$$F(c_1, \dots, c_r) = F(y_1, \dots, y_{r-1}) \le 2$$

with equality if and only if $y_1 = \dots = y_{r-1} = 0$, that is, $c_1 = \dots = c_{k-1} = c_{k+1} = \dots = c_r = 0$. □

We are now in a position to prove our main result.

THEOREM. *There are no integers $n \ge 2$ and $k \ge 2$ satisfying relation* (3).

PROOF. Using the known product representation $\varphi(n^k)/n^k = \prod_{p|n} (1 - 1/p)$ as well as Lemma 2 and the prime factorization $n = \prod_{j=1}^{r} {p_j}^{a_j}$ we obtain

$$\text{(8)} \quad \frac{\varphi(n^k)}{n^k} + \frac{\sigma_k(n)}{n^k} < \prod_{p|n} \left(1 - \frac{1}{p}\right) + \prod_{p|n} \frac{1}{1 - 1/p^2} = \prod_{j=1}^{r} (1 - x_j) + \prod_{j=1}^{r} (1 - x_j^2)^{-1}$$

with $x_j = 1/p_j$. Let $p_1 < p_2 < \dots < p_r$. Then, $p_j \ge j + 1$ for $j = 1, \dots, r$. Applying Lemma 3 reveals that the sum on the right-hand side of (8) is less than 2. Hence,

$$\varphi(n^k) + \sigma_k(n) < 2 \cdot n^k.$$

From Lemma 1, we conclude that (3) has no solution. □

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