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Finite groups with some CAP-subgroups[1]

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ABSTRACT - A subgroup A of a group G is said to be a CAP-subgroup of G if for any chief factor H/K of G, there holds $H \cap A = K \cap A$ or $HA = KA$. We investigate the influence of CAP-subgroups on the structure of finite groups. Some recent results are generalized.

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1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation, as in Huppert [7]. G always denotes a finite group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$, G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$.

For a subgroup A of G, if H/K is a chief factor of G, then we will say that:

(1) *A covers H/K* if $HA = KA$;
(2) *A avoids H/K* if $H \cap A = K \cap A$;

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(3) A has *the cover and avoidance properties* in G, in brevity, A is *a CAP-subgroup* of G ([4]), if A either covers or avoids every chief factor of G.

Clearly normal subgroups are CAP-subgroups. Examples of CAP-subgroups in the universe of solvable groups are well-known. The most remarkable CAP-subgroups of a solvable group are perhaps the Hall subgroups. By an obvious consequence of the definition of supersolvable group every subgroup of supersolvable group is a CAP-subgroup. In the literature, a lot of people have investigated the influence of the CAP-subgroups of G on the structure of G, please see [3], [4], [5], [6], [9], [11], [12], [13], [14], etc. For example, in [3] the first author has gotten the following results: 1. ([3, Theorem A]) Let p be a prime, G be a p-solvable group. Suppose that all maximal subgroups of the Sylow p-subgroups of G are CAP-subgroups of G, then G is p-supersolvable; 2. ([3, Theorem C]) Suppose that G is a group and for every prime p in $\pi(G)$ and for every Sylow p-subgroup P of G, every maximal subgroup of P is a CAP-subgroup of G. Then G is supersolvable.

In this paper, we extend Ezquerro's the results at least in three aspects: first, removing the hypotheses that G is p-solvable in [3, Theorem A]; secondly, reducing the number of restricted maximal subgroups of Sylow subgroups; in third, giving the unified forms of Ezquerro's results.

Suppose that P is a p-group for some prime p. Let $\mathcal{M}(P)$ be the set of all maximal subgroups of P.

DEFINITION ([10]). *Let d_p be the smallest generator number of a p-group P, i.e., $p^{d_p} = |P/\Phi(P)|$. We consider the set $\mathcal{M}_{d_p}(P) = \{P_1, ..., P_{d_p}\}$ of all elements of $\mathcal{M}(P)$ such that*

$$\bigcap_{i=1}^{d_p} P_i = \Phi(P).$$

We know that

$$|\mathcal{M}(P)| = \frac{p^{d_p}-1}{p-1}, \qquad |\mathcal{M}_{d_p}(P)| = d_p$$

and

$$\lim_{d_p\to\infty} \frac{\frac{p^{d_p}-1}{p-1}}{d_p} = \infty,$$

so

$$|\mathcal{M}(P)| >> |\mathcal{M}_{d_p}(P)|.$$

Our main result is as follows.

MAIN RESULT. *Suppose that G is a group and p is a fixed prime number in $\pi(G)$ and P is a Sylow p-subgroup of G. Suppose that every member in $\mathcal{M}_{d_p}(P)$ is a CAP-subgroup of G. Then either P is of order p or G is p-supersolvable.*

2. Preliminaries

LEMMA 2.1. *Let N be a normal subgroup of G and A a CAP-subgroup of G. Then:*

1. *AN is a CAP-subgroup of G;*
2. *AN/N is a CAP-subgroup of G/N;*
3. *For any chief series (*) of G, A covers or avoids every chief factor of the series (*) and furthermore, the order of A is the product of the orders of the covered chief factors in the series (*).*

PROOF. (1) is given in [14, §1, Lemma 1.4]; (2) follows from (1); (3) is clear by the definition of CAP-subgroup. □

LEMMA 2.2 ([7, I, Hauptsatz 17.4]). *Suppose that N is an abelian normal subgroup of G and $N \leq M \leq G$ such that $(|N|, [G : M]) = 1$. If N is complemented in M, then N is complemented in G.*

LEMMA 2.3. *Let P be a non-cyclic Sylow p-subgroup of G and $p \in \pi(G)$. Suppose that $\Phi(P)_G = 1$ and $O_p(G) > 1$ and suppose that every member in $\mathcal{M}_{d_p}(P)$ is a CAP-subgroup of G. Then:*

1. *N_p is at most of order p for every minimal normal subgroup N of G;*
2. *every minimal normal subgroup of G contained in P is of order p;*
3. *$G = O_p(G) \rtimes M$, the semi-direct product of $O_p(G)$ with a subgroup M of G and $O_p(G)$ is a direct product of normal subgroups of G of order p.*

PROOF. (1) Suppose that N is minimal normal in G. For any $P_i \in \mathcal{M}_{d_p}(P)$, we know that either $N \leq P_i$ or $N \cap P_i = 1$. If $N \leq P_i$ for all

$P_i \in \mathcal{M}_{d_p}(P)$, then

$$N \leq \bigcap_{i=i}^{d_p} P_i = \Phi(P),$$

which is contrary to the hypotheses that $\Phi(P)_G = 1$. Hence there exists a $P_{i_0} \in \mathcal{M}_{d_p}(P)$ such that $N \cap P_{i_0} = 1$. Since P_{i_0} is maximal in P, we have N_p is at most of order p.

(2) It is a corollary of (1).

(3) Let N_1 be a minimal normal subgroup of G contained in $O_p(G)$. Then N_1 is of order p by (2) and $N_1 \cap \Phi(P) = 1$ by the hypotheses that $\Phi(P)_G = 1$. Hence there exists a maximal subgroup S_1 of P such that $N_1 \cap S_1 = 1$. By Lemma 2.2, N_1 has a complement K in G, i.e., $G = N_1K$ and $N_1 \cap K = 1$. Then $O_p(G) = N_1(O_p(G) \cap K)$. It is easy to see that $O_p(G) \cap K$ is normal in G and $P \cap K$ is a Sylow p-subgroup of K. If $O_p(G) \cap K = 1$, then our theorem holds. So assume that $O_p(G) \cap K \neq 1$. Then we can pick a minimal subgroup N_2 contained in $O_p(G) \cap K$. By (1), N_2 is of order p and there exists a maximal subgroup S_2 of P such that $N_2 \cap S_2 = 1$ by the hypothesis that $\Phi(P)_G = 1$. Then $P = N_2S_2 = S_2(O_p(G) \cap K) = S_2(P \cap K)$. Since $|(P \cap K) : (S_2 \cap K)| = |S_2(P \cap K) : S_2| = |P : S_2| = p$, $S_2 \cap K$ is a complement of N_2 in $P \cap K$. Therefore N_2 has a complement L in K by Lemma 2.2. Then $G = N_1K = (N_1 \times N_2) \rtimes L$. Continuing this process, we have finally $G = O_p(G) \rtimes M$ and $O_p(G) = N_1 \times N_2 \times \cdots \times N_r$, where N_i is a normal subgroup of G of order p. □

Let p be a prime and $n > 1$ a natural number. If p^s divides n but p^{s+1} does not divide n, we write $(n)_p = p^s$. Let t be a prime and $b > 1$ and let k be a natural number. If t, b and k satisfy that t divides $b^k - 1$ but t does not divide $b^i - 1$ for all i with $1 \leq i < k$, then k is called the order of b module t and is denoted by $exp_t(b)$.

LEMMA 2.4. *Suppose that H is a nonabelian simple group. If the Sylow r-subgroups H_r of H are of order r, where r is a prime, then the out automorphism group $Out(H)$ of H is a r'-group.*

PROOF. Suppose that, in the contrary, the order of $Out(H)$ is divided by r. Obviously $r > 2$ by [7, IV Satz 2.8]. We will conduct a contradiction by applying the classification of finite simple groups.

If H is a sporadic simple group, then by [2], $|Out(H)| \mid 2$. If H is a alternating group, then when $H = A_6$, $|Out(H)| = 2^2$; when $H \neq A_6$, $|Out(H)| = 2$. Hence by $r > 2$ and $r \mid |Out(H)|$, we may assume that H is a

Lie type simple group over $GF(q)$ with $q = p^f$. By [2], $|Out(H)| = dfg$ and so $r \mid dfg$, where the numbers d, f, g are tabulated in [2, Table 5].

Suppose that $r = p$. By the order of Lie type simple groups and $|H_r| = r$, we have $H = A_1(p)$. But when $H = A_1(p)$, $|Out(H)| = 2$, $r \nmid |Out(H)|$, a contradiction. Hence $r \neq p$.

Let $exp_r(q) = t$, then $t \mid r-1$. By [7, P.190] and [8, P.502] we have

$$(*)\ \ (q^n-1)_r = \begin{cases} (q^t-1)_r\left(\dfrac{n}{t}\right)_r, & \text{if } t \text{ divides } n; \\ 1, & \text{if } t \text{ does not divide } n. \end{cases}$$

It is well known that if $(b, r-1) = 1$, then $r \mid q^{bd}-1$ if and only if $r \mid q^d-1$. Hence

$$(**)\ \ \left(q^{nr^s}-1\right)_r = \begin{cases} (q^n-1)_r r^s, & \text{if } r \text{ divides } q^n-1; \\ 1, & \text{if } r \text{ does not divide } q^n-1. \end{cases}$$

Assume that $H={}^2A_2(q)$. If $r \mid dg$, then $r = 3$ and $r \mid q+1$. We have $q \equiv -1(mod\ r)$ and so $3 \mid q^2-q+1$. Thus

$$|H_3| = \frac{1}{3}(q^2-1)_3(q^3+1)_3 = \frac{1}{3}(q+1)_3^2(q^2-q+1)_3 \geq 3^2,$$

a contradiction. Assume that $r \nmid dg$. Then $r \mid f$. Let $f = r^s k$ with $(k, r) = 1$. Assume that $r \mid q+1$. By previous argument, we may assume that $r > 3$. Thus

$$|H_r| = (q^2-1)_r(q^3+1)_r = (q+1)_r^2(q^2-q+1)_r \geq r^2,$$

a contradiction. Hence we may assume that $r \nmid q+1$ and so $t \in \{1, 6\}$.

When $t = 1$,

$$|H_r| = (q^2-1)_r(q^3+1)_r \geq (p^f-1)_r = (p^k-1)_r r^s \geq r^{s+1},$$

a contradiction.

When $t = 6$, $(q^3-1)_r = 1$. By (**),

$$|H_r| = (q^3+1)_r = (p^{6f}-1)_r = r^s(p^{6k}-1)_r \geq r^{s+1},$$

again a contradiction.

Assume that $H = D_4(q)$. Suppose that $r \mid gd$. Since $r > 2$, we have $r = 3$. Since $3 \mid q^2-1$, by $|D_4(q)| = \dfrac{1}{(2, q-1)^2}q^6(q^4-1)^2(q^2-1)$, we have $r^3 \mid |H|$, a contradiction. Hence we may assume that $r \nmid gd$ and $r \mid f$. By (*), it is easy to obtain that $|H_r| > r$, a contradiction.

From now, we assume that $H \notin \{{}^2A_2(q), D_4(q)\}$.

Suppose that $r \mid dg$. Since $r > 2$ and $g \in \{1,2\}$, we have $r \mid d$ and H is one of simple groups $A_n(q)(n > 1)$, ${}^2A_n(q)$, $E_6(q)$ with $r = 3$, ${}^2E_6(q)$ with $r = 3$. If $H = E_6(q)$, then $r \mid q - 1$; if $H={}^2E_6(q)$, then $r \mid q + 1$; if $H = A_n(q)$, then $r \mid q - 1$ and $n \geq 2$; if $H= {}^2A_n(q)$, then $r \mid q + 1$ and $n \geq 4$, it is easy to obtain that $r^2 \mid |H|$ from (*), a contradiction.

Suppose that $r \nmid dg$, then $f = r^s k$ with $s \geq 1$ and $(k, r) = 1$. Let $exp_r(q) = c$. From the orders of Lie type simple groups, we have $q^c - 1 \mid |H|$ if c is odd or $q^{\frac{1}{2}c} + 1 \mid |H|$ if c is even.

When c is odd, by (**)

$$|H_r| \geq (q^c - 1)_r = (p^{kr^s c} - 1)_r = (p^{kc} - 1)_r r^s \geq r^{s+1},$$

a contradiction.

When c is even, by $r \nmid q^{\frac{1}{2}c} - 1$ and (**), we have

$$|H_r| \geq (q^{\frac{1}{2}c} + 1)_r = (q^c - 1)_r = (p^{kr^s c} - 1)_r = (p^{kc} - 1)_r r^s \geq r^{s+1},$$

a final contradiction.

This completes the proof of the lemma. □

3. The proof of main result

Suppose that the theorem is false and G is a counter-example with minimal order. We will derive a contradiction in several steps.

STEP 1. $O_{p'}(G) = 1$.

Denote $N = O_{p'}(G)$. If $N > 1$, we consider the factor group G/N. Obviously, PN/N is a Sylow p-subgroup of G/N, which is isomorphic to P, so PN/N has the same smallest generator number as P, i.e., d_p and so

$$\mathcal{M}_{d_p}(P/N) = \{P_1/N, ..., P_{d_p}/N\}.$$

We know that every P_i/N is also a CAP-subgroup of G/N by Lemma 2.1. Thus G/N satisfies the hypotheses of the theorem. We have that either PN/N is of order p or $G/O_{p'}(G)$ is p-supersolvable by the choice of G, it follows that either P is of order p or G is p-supersolvable, a contradiction. Thus, we have $N = O_{p'}(G) = 1$, as desired.

STEP 2. P is non-cyclic.

If P is cyclic, then the unique maximal subgroup $\Phi(P)$ of P is CAP-subgroup in G by the hypotheses. Hence either P is of order p or G is p-supersolvable by [1, Theorem 3.2], a contradiction.

STEP 3. $\Phi(P)_G = 1$, therefore, $O_p(G)$ is an elementary abelian group.

If not, take any $T \leq \Phi(P)_G$ such that $T \trianglelefteq G$. We consider the factor group G/T. Since every maximal subgroup of P contains $\Phi(P)$ and P/T has the same smallest generator number as P, so

$$\mathcal{M}_{d_p}(P/T) = \{P_1/T, ..., P_{d_p}/T\}.$$

We know that every P_i/T is also a CAP-subgroup of G/N by Lemma 2.1. Thus, G/T satisfies the hypotheses of the theorem. Hence, either P/T is of order p or G/T is p-supersolvable by the choice of G. If P/T is of order p, then P is cyclic, contrary to Step 2. Hence G/T is p-supersolvable, then G is p-supersolvable, a contradiction.

STEP 4. If N is minimal normal in G contained in P, then $|N| = p$.

By Lemma 2.3(2).

STEP 5. All minimal normal subgroups of G are contained in $O_p(G)$.

Assume that H is a minimal normal subgroup of G which is not a p-subgroup. As $O_{p'}(G) = 1$ by Step 1, we have that $p \| H|$ and H is non-abelian characteristic simple group. Then

(5.1) All $P_i \in \mathcal{M}_{d_p}(P)$ avoid the chief factor $H/1$, H is a non-abelian simple group with $|H_p| = p$.

By Lemma 2.3(1) we know that $|H_p| = p$. So H is a non-abelian simple group. Obviously H is avoided by every $P_i \in \mathcal{M}_{d_p}(P)$.

(5.2) $O_p(G) = 1$.

If $O_p(G) \neq 1$, we can pick a minimal normal subgroup N of G contained in $O_p(G)$. By Step 4 we know that N is of order p. Consider the chief series of G:

$$1 \lhd N \lhd NH \lhd \cdots \lhd G.$$

For an arbitrary $P_i \in \mathcal{M}_{d_p}(P)$, since P_i avoids HN/N, P_i must cover N by

Lemma 2.1(3). Hence $N \leq P_i$. Then

$$N \leq \bigcap_{i=i}^{d_p} P_i = \Phi(P),$$

which is contrary to Step 3.

(5.3) $C_G(H) = 1$.

Suppose that $C_G(H) \neq 1$. Now we pick a minimal normal subgroup H^* of G contained in $C_G(H)$. Then $H \cap H^* = 1$. For any $P_i \in \mathcal{M}_{d_p}(P)$, we know that P_i avoids H, P_i must cover H^* by Lemma 2.1(3). Therefore, H^* is a group of order p, which is contrary to (5.2).

(5.4) $G = PH$.

By (5.3), we know that the non-abelian simple group H is the unique minimal normal subgroup of PH. So all chief factors of PH are $H/1$ or a cyclic group of order p. By (5.1), we know that all $P_i \in \mathcal{M}_{d_p}(P)$ cover or avoid all chief factors of PH. So PH satisfies the hypothesis of the theorem. If $PH < G$, then either P is of order p or PH is p-supersolvable by the minimal choice of G. If PH is p-supersolvable, then H is p-supersolvable. But this is contrary to (5.1). Hence $G = PH$.

(5.5) Finishing the proof of (5).

By (5.3) we have $C_G(H) = 1$. Then G and G/H are isomorphic to a subgroup of $Aut(H)$ and a subgroup of $Aut(H)/Inn(H)$, respectively. This means that H_p is of order p and p divides the order of $Out(H)$. By Lemma 2.4, this is impossible.

Step 6. $G = O_p(G) \rtimes M$, the semi-direct product of $O_p(G)$ with a subgroup M of G and $O_p(G)$ is a direct product of normal subgroups of G of order p.

By Lemma 2.3(3).

Step 7. The final contradiction.

Since $N \leq Z(P)$ for any minimal normal subgroup N of G, $P \leq C_G(O_p(G))$. Since $C_G(O_p(G)) \cap M \triangleleft \langle O_p(G), M \rangle = G$, $C_G(O_p(G)) \cap M = 1$ by Step 4 and 5. Then $P \cap M = 1$. This implies that $P = P \cap O_p(G)M = O_p(G)(P \cap M) = O_p(G)$. Therefore by Step 6 we have that G is p-supersolvable, the final contradiction. □

REMARK. The authors do not know the proof without using the classification of finite simple groups.

4. Applications

We give some applications of our main result.

Suppose that p is the smallest prime dividing the order of G. We know that G is p-nilpotent if G_p is cyclic by [7, IV Satz 2.8] and p-supersolubility implies the p-nilpotency. By our main result we immediately have the following corollary.

COROLLARY 4.1. *Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if every member in $\mathcal{M}_{d_p}(P)$ is a CAP-subgroup of G.*

COROLLARY 4.2. *Suppose that P is a Sylow p-subgroup of G and $N_G(P)$ is p-nilpotent for some prime $p \in \pi(G)$. Then G is p-nilpotent if and only if every member in $\mathcal{M}_{d_p}(P)$ is a CAP-subgroup of G.*

PROOF. We only need to prove the "if" part.

By our main result we know that either P is cyclic or G is p-supersolvable. If P is cyclic, then we have $N_G(P) = C_G(P)$. Applying Burnside's p-nilpotence criterion ([7, Hauptsatz IV.2.6]), we get that G is p-nilpotent. Now suppose that G is p-supersolvable. Since the p-length of p-supersolvable groups is at most 1, we have $PO_{p'}(G)$ is normal in G. Set $\overline{G} = G/O_{p'}(G)$. Then $\overline{G} = N_{\overline{G}}(\overline{P}) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent by hypothesis. Hence G is p-nilpotent, as desired. □

Suppose that G is p-solvable. If Sylow p-subgroups of G are cyclic, then G is p-supersolvable. Therefore, immediately from our main result, we have the following corollary which is a generalization of [3, Theorem A].

COROLLARY 4.3. *Suppose that G is a p-solvable group, where p is a fixed prime number in $\pi(G)$, and P is a Sylow p-subgroup of G. Then G is p-supersolvable if every member in $\mathcal{M}_{d_p}(P)$ is a CAP-subgroup of G.*

The following is a generalization of [3, Theorem C].

THEOREM 4.4. *Suppose that G is a group. Then G is supersolvable if and only if every member in $\mathcal{M}_{d_p}(P)$ is a CAP-subgroup of G for every prime p in $\pi(G)$ and for every Sylow p-subgroup P of G*

PROOF. We only need to prove the "if" part.

By Corollary 4.3 it is sufficient to prove that G is solvable. Hence we want to prove that every chief factor of G is solvable. Suppose that L/K is an arbitrary chief factor of G. For any prime $p \in \pi(L/K)$, we know that there exists a maximal subgroup H of a Sylow p-subgroup of G such that H either covers or avoids L/K. If H covers L/K, obviously L/K is solvable. Hence assume that H avoids L/K. This implies that $|L/K|_p \leq p$. Therefore, every Sylow subgroup of L/K is of prime order. Hence L/K is solvable.

This completes the proof of Theorem 4.4. □

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