

On global well-posedness for the Einstein-Maxwell-Euler system in Bondi coordinates

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ABSTRACT - We analyze the Einstein-Maxwell equations for an irrotational stiff fluid. Under the spherical symmetry assumption on the space-time, in Bondi coordinates, the considered model is reduced to a nonlinear evolution system of partial integrodifferential equations. Assuming regularity at the center of symmetry and that the matter content of the initial light cone is the so-called null dust, the characteristic initial value problem associated to the obtained system is solved globally by a contraction mapping argument. In future work we will address the issue of global well-posedness for the considered model in other physically interesting cases where the matter content of the initial light cone is not the null dust.

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1. Introduction

Since their formulation, the relativistic theories of gravitation, electromagnetism and hydrodynamics have experienced an intensive and rapidly developing research activity over the years. The system that unifies these three theories consists of the Einstein-Maxwell-Euler (EME) equations. These equations are of central importance in several areas of high-energy

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astrophysics, plasma physics and nuclear physics. We refer the reader to three excellent books by Lichnerowicz [1], Choquet-Bruhat [2] and Anile [3] for a thorough review on these equations and their interests, both on mathematical and physical view points. Here we briefly mention some areas and problems which constitute physical motivations for studying EME equations: collapse of massive rotating stars towards neutron stars or black holes, merging of binary systems, models of supernova explosions, structure and stability of neutron stars, galactic microquasars, active galactic nuclei, gamma-ray bursts, free electron laser, high-energy collisions among heavy nuclei. Apart from the fundamental in-depth mathematical analysis [1, 2, 3] of the EME equations, a couple of recent contributions can be found in [4, 5, 6, 7] and some references therein. It should be noted that few works have been devoted to the problem of global well-posedness for the EME equations. This is understandable since these equations are highly nonlinear and investigating their global well-posedness is a very difficult task. To make some progress in this direction, it is reasonable to work under some simplifying and physically interesting assumptions. Noutchequeme and Nangne [8] proved global existence of a unique solution to the EME system for a perfect fluid of pure radiation type in a Robertson-Walker space-time, by reducing the considered model to a nonlinear evolution system of ordinary differential equations (ODE). In the present investigation we address the issue of global existence and uniqueness for the EME equations, for an irrotational stiff fluid evolving in a spherically symmetric curved space-time. The investigation is conducted in the so-called Bondi coordinates so that the problem at hand is a characteristic initial value problem with initial data assigned on a light cone with tip at the center of symmetry. The importance of characteristic initial value problems (with initial data prescribed on one or more null hypersurfaces of the space-time) was amply demonstrated by Christodoulou [9] in his famous book on black holes formation. In Bondi coordinates, we reduce the problem to a nonlinear evolution system of partial integrodifferential equations (PIDE). The mathematical analysis of the obtained system faces two major difficulties: (1) the appearance of non-converging integrals of the form $\int_r^\infty \frac{ds}{s(1+u+s)^m}$ in some essential estimates, and (2) the treatment of initial data whose smallness does not necessarily ensure the existence and uniqueness of a solution for the considered problem. To overcome these difficulties we have defined an appropriate functional framework and assume that the matter content of the initial light cone is the so-called null dust. This

latter assumption amounts to assigning zero initial data for the initial value problem associated to the obtained nonlinear evolution system (see (31)), which we solve globally by a fixed point method. The present work is the generalization, to the considered EME model, of previous ones achieved by Christodoulou [10] for the Einstein-scalar field system, Chae [11] for the Einstein-Maxwell-Higgs system, Tadmor and Tchaptand [12, 13] for the Einstein-Yang-Mills-Higgs system. Some work on characteristic initial value problems without symmetry assumption can be found in [14, 15, 16].

The rest of the paper is outlined as follows. Section 2 is devoted to the derivation of the considered EME equations in Bondi coordinates. We show that the equations reduce to a nonlinear evolution system of PIDE. The main result of the work is stated and proved in section 3.

2. Electrovacuum Einstein-Maxwell-Euler equations in Bondi coordinates

2.1 – General form of the Einstein-Maxwell-Euler equations

The evolution of a charged perfect fluid is governed by the Einstein-Maxwell-Euler equations which, in local chart $(x^\alpha)_{\alpha=0,\dots,3}$ on a four dimensional space-time manifold, read

$$(1) \quad R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T_{\alpha\beta},$$

$$(2) \quad \nabla_\alpha(*F)^{\alpha\beta} = 0, \quad \nabla_\alpha F^{\alpha\beta} = J^\beta,$$

$$(3) \quad \nabla_\alpha T^{\alpha\beta} = 0.$$

(1) are the Einstein equations where $g_{\alpha\beta}$ are the components of the unknown space-time metric \mathbf{g} which, throughout the work, is assumed to have signature $(-, +, +, +)$. $R_{\alpha\beta}$ and R represent, respectively, the Ricci tensor and the scalar curvature relative to the space-time metric. $T_{\alpha\beta}$ are the components of the energy-momentum tensor, given by

$$(4) \quad T_{\alpha\beta} = (p + \rho)U_\alpha U_\beta + pg_{\alpha\beta} + g^{\lambda\mu}F_{\alpha\lambda}F_{\beta\mu} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta},$$

where p is the pressure of the fluid, ρ its mass-energy density, and \mathbf{U} its four velocity normalized to unity so that $U_\alpha U^\alpha = -1$. (2) consists, respectively, of the first and second groups of Maxwell equations, where ∇ denotes

covariant derivative w.r.t. the space-time metric. $F^{\alpha\beta}$ are the contravariant components of the unknown antisymmetric electromagnetic field (also called Faraday tensor) \mathbf{F} ; $*\mathbf{F}$ is the Hodge dual of \mathbf{F} . J^β are the components of the charge-current four-vector \mathbf{J} , which classically splits into convection part and conduction part as $J^\beta = \delta U^\beta + \sigma e^\beta$, where δ is the density of electric charge in the fluid, σ the electrical conductivity of the fluid, and \mathbf{e} is the four-vector electric field defined by $e^\alpha = U_\beta F^{\beta\alpha}$. (3) are the relativistic Euler equations.

2.2 – *Electrovacuum Einstein-Maxwell-Euler system*

It will be assumed from now on that the density of electric charge and the electrical conductivity of the fluid vanish identically:

$$(5) \quad \delta \equiv 0, \quad \sigma \equiv 0.$$

The charge-current four-vector \mathbf{J} then vanishes identically, i.e.,

$$(6) \quad \mathbf{J} \equiv 0.$$

The EME system for a fluid with zero density of electric charge and zero electrical conductivity is often referred to as electrovac EME system. It consists of twenty three equations, including equation $U_\alpha U^\alpha = -1$, with twenty two unknowns (10 $g_{\alpha\beta}$, 4 U_α , 6 $F_{\alpha\beta}$, p , ρ). An equation of state is needed to close it. Moreover it is a highly nonlinear system and, in order to obtain global solution for it, it makes sense to impose some more simplifying assumptions. As in [10, 11, 12, 13] we will work in a Bondi coordinates system $(x^\alpha) = (u, r, \theta, \varphi)$, where the spherically symmetric space-time metric takes the form

$$(7) \quad g_{\alpha\beta} dx^\alpha dx^\beta = -e^{2\nu} du^2 - 2e^{\nu+\lambda} du dr + r^2 (d\theta^2 + \sin^2\theta d\varphi^2).$$

Here ν and λ are real functions of the retarded time coordinate u and the radial coordinate r only. We assume the flow to be stiff and irrotational. Then the equation of state reads

$$(8) \quad p = \rho,$$

and the four velocity of the fluid can be written as

$$(9) \quad U_\alpha = (-\psi_{,\mu} \psi^{,\mu})^{-\frac{1}{2}} \psi_{,\alpha},$$

where ψ is a time function, i.e $\psi_{,\mu} \psi^{,\mu} < 0$; the subscript “,” denotes usual

partial derivative. Supposing that $\rho > 0$, after some algebra, it follows that

$$(10) \quad \rho = -\psi_{,\mu}\psi^{,\mu},$$

$$(11) \quad (\psi^{,\alpha})_{;\alpha} = 0,$$

where the subscript “,” denotes covariant derivative. It is worth mentioning that (10) is the relativistic counterpart of the famous Bernoulli’s law for irrotational flows in classical Fluids Mechanics, and it determines ρ (and hence p) once the potential ψ and the space-time metric g are known.

Note that equations (10) and (11) above have been obtained under the condition $\psi_{,\mu}\psi^{,\mu} < 0$; in that case, using (8), (9) and (10), the energy-momentum tensor in (4) becomes

$$(12) \quad T_{\alpha\beta} = 2\psi_{,\alpha}\psi_{,\beta} - \psi_{,\mu}\psi^{,\mu}g_{\alpha\beta} + g^{\lambda\mu}F_{\alpha\lambda}F_{\beta\mu} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta},$$

which is regular and well defined irrespective of the sign of $\psi_{,\mu}\psi^{,\mu}$. The cases $\psi_{,\mu}\psi^{,\mu} > 0$ and $\psi_{,\mu}\psi^{,\mu} = 0$, can be interpreted by arguments similar to those used in [17, 18]. When $\psi_{,\mu}\psi^{,\mu} = 0$, the energy-momentum tensor defined by (12) reduces to

$$(13) \quad T_{\alpha\beta} = 2\psi_{,\alpha}\psi_{,\beta} + g^{\lambda\mu}F_{\alpha\lambda}F_{\beta\mu} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}.$$

If $\frac{\partial\psi}{\partial r}$ and \mathbf{F} vanish identically, (13) coincides with the energy-momentum tensor of the so-called null dust matter. This will serve as the matter content of the light cone where initial data will be specified. From now on ψ is regarded as the main fluid variable, whose evolution is governed by equation (11), and (12) as the expression of the energy-momentum tensor.

We will use the following ansatz for the electromagnetic field \mathbf{F} , compatible with the ansatz (7) for the space-time metric (this facilitates calculation of the components of the energy-momentum tensor (13)):

$$(14) \quad (F_{\alpha\beta}) = \begin{pmatrix} 0 & 0 & a & b \sin \theta \\ 0 & 0 & k & l \sin \theta \\ -a & -k & 0 & 0 \\ -b \sin \theta & -l \sin \theta & 0 & 0 \end{pmatrix},$$

where $a, b, k,$ and l are functions of u and r only. Using (14), direct calculation shows that the first group of Maxwell equations in (2) are equivalent to

$$(15) \quad \dot{k} = a', \quad \dot{l} = b'.$$

Here and in the remainder of the work, \cdot denotes differentiation with respect to u and $'$ differentiation with respect to r . Similarly, using (5), (6), (7) and (14), one deduces from straightforward though lengthy computation that the second group of Maxwell equations in (2) are satisfied if and only if

$$(16) \quad a = k = 0,$$

$$(17) \quad (\dot{l} + b')e^{-v-\lambda} - l'e^{-2\lambda} = l(v' - \lambda')e^{-2\lambda}.$$

Observe that the first equation in (15) is satisfied if it is the case for (16); if the second equation in (15) is satisfied then equation (17) becomes

$$(18) \quad \dot{l} - \frac{1}{2}e^{v-\lambda}l' = \frac{1}{2}(e^{v-\lambda})'l.$$

We now handle the Einstein equations (1). As in [10, 11, 12, 13] we introduce an adapted null tetrad $(e_\alpha)_{\alpha=0,\dots,3}$ defined by $e_0 = e^{-v}\frac{\partial}{\partial u} - \frac{1}{2}e^{-\lambda}\frac{\partial}{\partial r}$, $e_1 = e^{-\lambda}\frac{\partial}{\partial r}$, (e_2, e_3) is an orthonormal frame on the unit 2-sphere. After lengthy calculation the relevant Einstein fields equations are found to be equivalent to the following:

$$(19) \quad \frac{1}{2}(\lambda' + v') - 2e^{\lambda-v}\dot{\lambda} = 2re^{2\lambda}\left(e^{-v}\dot{\psi} - \frac{1}{2}e^{-\lambda}\psi'\right)^2 + r^{-1}e^{2\lambda}\left(e^{-v}b - \frac{1}{2}e^{-\lambda}l\right)^2,$$

$$(20) \quad (e^{v-\lambda})' + r^{-1}(e^{v-\lambda} - e^{v+\lambda}) = 0,$$

$$(21) \quad \lambda' + v' = r\left[(\psi')^2 + \frac{1}{2}r^{-2}l^2\right],$$

$$(22) \quad v'' + (v' - \lambda')(v' + r^{-1}) - e^{\lambda-v}\left[(\dot{\lambda})' + (\dot{v})'\right] = 2e^{\lambda-v}\dot{\psi}\psi' - (\psi')^2,$$

$$(23) \quad e^{-v-\lambda}bl - \frac{1}{2}e^{-2\lambda}l^2 = 0.$$

Equation (11) reads

$$(24) \quad 2(\dot{\psi})' + 2r^{-1}\dot{\psi} - e^{v-\lambda}[\psi'' + \psi'(2r^{-1} + v' - \lambda')] = 0.$$

REMARK 1. (i) If v , λ , and l are known, then setting

$$(25) \quad b = \frac{1}{2}le^{v-\lambda},$$

it is easy to see that equation (23) is satisfied. Moreover the second equation $\dot{l} = b'$ in (15) is equivalent to (18) for b given by (25).

(ii) If regularity is assumed at the center, an argument similar to the one used in [10, 13] leads to the conclusion that the Einstein equations (20) and (21), together with equation (24) and (18), are equivalent to the full set of the considered electrovac EME equations.

2.3 – Reduction of the electrovac EME equations to a nonlinear evolution system

As in [10, 11, 12, 13] we define a new function $h = (r\psi)'$ to gain

$$(26) \quad \psi(u, r) = \bar{h}(u, r) := \frac{1}{r} \int_0^r h(u, s) ds, \quad \psi'(u, r) = \frac{h(u, r) - \bar{h}(u, r)}{r}.$$

In view of (26) the Einstein equation (21) reads $\lambda' + \nu' = \frac{1}{r} \left[(h - \bar{h})^2 + \frac{1}{2} l^2 \right]$, and the solution which satisfies the asymptotic condition $\lambda + \nu \rightarrow 0$, as $r \rightarrow \infty$, is

$$(27) \quad \lambda + \nu = - \int_r^{+\infty} \frac{1}{s} \left[(h - \bar{h})^2 + \frac{1}{2} l^2 \right] ds.$$

The Einstein equation (20) is integrated to give

$$(28) \quad e^{\nu-\lambda} = \frac{1}{r} \int_0^r e^{\lambda+\nu} ds.$$

We use (26) to rewrite equation (24) as

$$(29) \quad \dot{h} - \frac{1}{2} e^{\nu-\lambda} h' = \frac{1}{2r} (e^{\lambda+\nu} - e^{\nu-\lambda}) (h - \bar{h}).$$

Using the definition of \bar{l} as in (26), we recast equation (18) as

$$(30) \quad \dot{l} - \frac{1}{2} e^{\nu-\lambda} l' = \frac{1}{2r} (e^{\lambda+\nu} - e^{\nu-\lambda}) (l - \bar{l}) + \frac{1}{2r} (e^{\lambda+\nu} - e^{\nu-\lambda}) \bar{l}.$$

In view of (26), (27) and (28), equations (29) and (30) form a nonlinear evolution system of PIDE. It should be noted that, in the spaces of functions used in previous work [10, 11, 12, 13], the mathematical analysis of the model considered in the present investigation comes up against several challenges. One of these difficulties is to estimate $\int_r^\infty \frac{l^2}{s} ds$, which involves

non-converging integrals of the form $\int_r^\infty \frac{ds}{s(1+u+s)^m}$. A way around this toughness is to define a new functional framework where all the relevant quantities can be estimated appropriately. Another delicate problem is the prescription of initial data for the considered EME system. In the present work we assume that the matter content on the light cone $u = 0$ is the so-called null dust. This amounts to considering, as observed earlier in paragraph 2.2, the Cauchy problem for the aforementioned nonlinear evolution system of PIDE with zero initial data prescribed on the light cone $u = 0$. Set $W = \begin{pmatrix} h \\ l \end{pmatrix}$ and denote by I the 2×2 identity matrix. Then, in view of (29) and (30), the problem under investigation reads

$$(31) \quad \begin{aligned} DW(u, r) &= \frac{1}{2r} (g(u, r) - \tilde{g}(u, r)) [W(u, r) + (\sigma - I)\overline{W}(u, r)], \\ W(0, r) &= 0, \end{aligned}$$

where $D = \frac{\partial}{\partial u} - \frac{\tilde{g}}{2} \frac{\partial}{\partial r}$, $g = e^{\lambda+v}$, $\tilde{g} = e^{v-\lambda}$, $\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

3. Existence and uniqueness of classical solutions

We begin by introducing the spaces of functions used and some preliminary notations. For a vector function $W = \begin{pmatrix} h \\ l \end{pmatrix}$ we will use the notation $|W| := |h| + |l|$, and simply write $W \in C^1([0, \infty) \times (0, \infty))$ if h and l are both in $C^1([0, \infty) \times (0, \infty))$. Define the Banach functions spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ by

$$(32) \quad \begin{aligned} \mathcal{X} &= \{W = W(u, r) \in C^1([0, \infty) \times (0, \infty)) : \|W\|_{\mathcal{X}} < \infty\}, \\ \mathcal{Y} &= \{W = W(u, r) \in C^1([0, \infty) \times (0, \infty)) : \|W\|_{\mathcal{Y}} < \infty\}, \end{aligned}$$

where $\|W\|_{\mathcal{X}} := \sup_{u \geq 0, r > 0} \{r^{-2} [(1+r+u)^4 |W(u, r)| + (1+r+u)^5 |W'(u, r)|]\}$, $\|W\|_{\mathcal{Y}} := \sup_{u \geq 0, r > 0} [r^{-2} (1+r+u)^4 |W(u, r)|]$. It is now time to state and prove the main result of the work.

THEOREM 1. *There exists a unique global classical solution $W \in C^1([0, \infty) \times (0, \infty))$ of (31). In addition this solution fulfills the*

decay property

$$(33) \quad |W(u, r)| \leq c(1 + u + r)^{-2}, \quad |W'(u, r)| \leq c(1 + u + r)^{-3},$$

where c is a positive real constant. Moreover, the corresponding space-time is timelike and null geodesically complete toward the future.

PROOF. Throughout the proof $(c_i)_{i=1,2,\dots}$ denote positive real constants. Note that the functions spaces used here are fundamentally different from those of [10, 11, 12, 13]. The cornerstone of the proof is the standard fixed point theorem. Define the mapping $\mathcal{K} : W \mapsto w = \mathcal{K}(W)$, where w is the solution of the first order linear initial value problem

$$(34) \quad Dw = \frac{1}{2r}(g - \tilde{g})[w + (\sigma - I)\overline{W}], \quad w(0, r) = 0.$$

(i) **\mathcal{K} is a mapping from a ball of \mathcal{X} into itself.** Let $x > 0$, $W \in B_x$, where B_x denotes the closed ball, in \mathcal{X} , of radius x centered at 0. From the definition of the differential operator D in (31), the characteristic system of ODE associated to the initial value problem (34) is

$$(35) \quad \frac{dr}{du} = -\frac{1}{2}\tilde{g}, \quad \frac{dw}{du} = \frac{1}{2r}(g - \tilde{g})[w + (\sigma - I)\overline{W}],$$

with initial data $r(0) = r_0$, $w(0) = 0$. Denote by $r(u) = \gamma(u, r_0)$ the solution of the Cauchy problem

$$(36) \quad \frac{dr}{du} = -\frac{1}{2}\tilde{g}(u, r), \quad r(0) = r_0.$$

Then $r_1 = r_0 - \frac{1}{2} \int_0^{u_1} \tilde{g}(u, \gamma(u, r_0)) du$, where $r_1 = \gamma(u_1, r_0)$. Integrating the second ODE of (35) along γ we obtain

$$(37) \quad w(u_1, r_1) = \int_0^{u_1} \left\{ \exp \left(\int_u^{u_1} [N(u, r)]_\gamma dv \right) \right\} [f]_\gamma du,$$

where

$$(38) \quad N(u, r) = \frac{1}{2r}(g - \tilde{g}), \quad f(u, r) = \frac{1}{2r}(g - \tilde{g})(\sigma - I)\overline{W}.$$

We have to estimate N and f . Since $W \in B_x$, the following estimates (39), (40) and (41) are straightforward from the definition of \overline{W} in (26) and that of

the Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ in (32):

$$(39) \quad |W(u, r)| \leq \frac{xr^2}{(1+u+r)^4},$$

$$(40) \quad |(\sigma - I)\overline{W}(u, r)| \leq |\overline{W}(u, r)| \leq \frac{xr^2}{3(1+u)(1+u+r)^3},$$

$$(41) \quad |W(u, r) - \overline{W}(u, r)| \leq \frac{xr^3}{4(1+u)(1+u+r)^4}.$$

(39) and (41) imply

$$\left| \frac{1}{r} \left[(h - \bar{h})^2 + \frac{1}{2}l^2 \right] \right| \leq \frac{c_1 x^2 r^3}{(1+u)^2 (1+u+r)^6}.$$

This gives

$$(42) \quad \int_0^\infty \left| \frac{1}{r} \left[(h - \bar{h})^2 + \frac{1}{2}l^2 \right] \right| dr \leq \frac{c_2 x^2}{(1+u)^4}.$$

(42) yields

$$(43) \quad \exp \left(- \int_0^\infty \left| \frac{1}{r} \left[(h - \bar{h})^2 + \frac{1}{2}l^2 \right] \right| dr \right) \geq \exp(-c_2 x^2).$$

From (27), (28), the definition of g and \tilde{g} in (31) and the estimate (41), we get

$$(44) \quad |g(u, r) - \tilde{g}(u, r)| \leq \frac{c_3 x^2 r^4}{(1+u)^3 (1+u+r)^5}.$$

In view of (43) and (44) we gain

$$(45) \quad \tilde{g}(u, r) \geq \exp(-c_2 x^2) - c_3 x^2.$$

It is easy to see that the function $L = L(x) = \exp(-c_2 x^2) - c_3 x^2$ has a unique positive root x_0 and $L(x) \in (0, 1]$ for all $x \in [0, x_0)$. We estimate the function f defined in (38), by using (40) and (44), to get

$$(46) \quad |f(u, r)| \leq \frac{17x^3 r^2}{480(1+u)^4 (1+u+r)^5}.$$

For $r(u) = \gamma(u, r_0)$ it holds, in view of (36) and (45), that

$$(47) \quad r = r_1 + \frac{1}{2} \int_u^{u_1} \tilde{g}(s, r(s)) ds \geq r_1 + \frac{L(x)}{2} (u_1 - u).$$

Since $L(x) \in (0, 1]$ for all $x \in [0, x_0)$, (47) implies

$$(48) \quad 1 + u + r \geq 1 + u + r_1 + \frac{L(x)}{2} (u_1 - u) \geq \frac{L(x)}{2} (1 + u_1 + r_1).$$

From (46) and (48) we gain the estimate

$$(49) \quad \int_0^{u_1} |[f]_\gamma| du \leq \frac{c_4 x^3 r_1^2}{(1 + u_1 + r_1)^5 L^5(x)}.$$

In view of (44) we estimate the function N defined in (38) to get

$$(50) \quad |N(u, r)| \leq \frac{17x^2}{160}.$$

From (50) we get

$$(51) \quad \left| \int_0^{u_1} [N]_\gamma dv \right| \leq c_2 x^2.$$

Considering (37), (49) and (51) we finally arrive at the following estimate for the solution $w = \mathcal{K}(W)$ of the initial value problem (34):

$$(52) \quad \sup_{u \geq 0, r > 0} \left[r^{-2} (1 + u + r)^4 |w(u, r)| \right] \leq \frac{c_4 x^3}{L^5(x)} \exp(c_2 x^2).$$

We now estimate $\sup_{u \geq 0, r > 0} \left[r^{-2} (1 + u + r)^5 |w'(u, r)| \right]$. Set $z(u, r) = w'(u, r)$, with $z(0, r_0) = 0$. Differentiating (34) w.r.t. r yields

$$(53) \quad Dz = N_1 z + B_1 [w + (\sigma - I)\overline{W}] + N(\sigma - I)\overline{W}',$$

where $N_1 = \frac{\tilde{g}'}{2} + \frac{g - \tilde{g}}{2r}$, $B_1 = \frac{(g - \tilde{g})'}{2r} - \frac{g - \tilde{g}}{2r^2}$, $N = \frac{g - \tilde{g}}{2r}$. By employing the same calculations and arguments as earlier we gain the following estimate for the solution z of the initial value problem (53):

$$(54) \quad \sup_{u \geq 0, r > 0} \left[r^{-2} (1 + r + u)^5 |z(u, r)| \right] \leq \frac{(c_5 + c_6 x^2) x^3}{L^{10}(x)} \exp(c_7 x^2).$$

It follows from (32), (52) and (54) that $\|w\|_{\mathcal{X}} \leq \frac{(c_8 + c_6 x^2)x^3}{L^{10}(x)} \exp(c_7 x^2)$. To have $\|w\|_{\mathcal{X}} \leq x$, it suffices that $\Omega(x) \leq 1$, where $\Omega(x) = \frac{(c_8 + c_6 x^2)x^2}{L^{10}(x)} \exp(c_7 x^2)$. One sees that $\lim_{x \rightarrow 0} \Omega(x) = 0$. Hence there exists $x_1 \in (0, x_0)$ such that, for $x \in (0, x_1]$, $\Omega(x) \leq 1$. This shows that \mathcal{K} is a mapping from B_x into itself for $x \in (0, x_1]$.

(ii) **The mapping \mathcal{K} is a contraction.** Let w_1 and w_2 be two solutions of (34) with $w_j(0, r) = 0$, $w_j = \mathcal{K}(W_j)$, $W_j \in \mathcal{X}$, $j = 1, 2$. We use the following shorthand notations $g_j = g(W_j)$, $z_j = z(W_j)$, assume $\max\{\|W_1\|_{\mathcal{X}}, \|W_2\|_{\mathcal{X}}\} \leq x \leq x_1$, and set for convenience $w_1 - w_2 = \mathcal{D}$, $W_1 - W_2 = \Theta$, $\|\Theta\|_{\mathcal{Y}} = y$, $D_1 = \frac{\partial}{\partial u} - \frac{\tilde{g}_1}{2} \frac{\partial}{\partial r}$. Then we have the following nonlinear initial value problem with unknown \mathcal{D}

$$(55) \quad D_1 \mathcal{D} = M[\mathcal{D} + (\sigma - I)\overline{\Theta}] + B_2(w_2 + (\sigma - I)\overline{W_2}) + B_3 w_2', \quad \mathcal{D}(0, r) = 0,$$

where $M = \frac{g_1 - \tilde{g}_1}{2r}$, $B_2 = \frac{(g_1 - \tilde{g}_1) - (g_2 - \tilde{g}_2)}{2r}$, $B_3 = \frac{\tilde{g}_1 - \tilde{g}_2}{2}$. Using repeatedly the mean value theorem and similar tools as in the previous step we get $\|\mathcal{D}\|_{\mathcal{Y}} \leq \Xi(x)y$, where $\Xi(x) = \frac{c_9(x^2 + x^4)}{L^4(x)} \exp(c_2 x^2)$. It is easy to see that the function Ξ is strictly monotonically increasing on $(0, x_1]$ and $\lim_{x \rightarrow 0} \Xi(x) = 0$. One deduces that there exists $x_2 \in (0, x_1]$ such that $\Xi(x) < \frac{1}{2}$ for all $x \in (0, x_2]$. Thus the mapping $W \mapsto \mathcal{K}(W)$ contracts in \mathcal{Y} for $\|W\|_{\mathcal{X}} \leq x_2$. This concludes the proof of the global existence and uniqueness of classical solution for (31). The decay property (33) of the solution is a direct consequence of the definition (32) of the Banach spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$.

In view of (27) and (42) one deduces easily from the definition of g in (31) that, for each $r > 0$, $g \rightarrow 1$ if $u \rightarrow \infty$. In view of (44), this implies that, for each $r > 0$, $\tilde{g} \rightarrow 1$ if $u \rightarrow \infty$. So, as $u \rightarrow \infty$, the metric given in Bondi coordinates by (7) becomes the Minkowski metric. This completes the proof of Theorem 1. \square

REMARK 2. The present work can serve as a basis for future investigation on the issue of global existence of generalized solutions, with arbitrary initial data, for the considered electrovac EME model.

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